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INFINITESIMAL RIGIDITY OF EUCLIDEAN SUBMANIFOLDS

by M. DAJCZER and L. RODRIGUEZ

An isometric immersion $f: M^n \to \mathbb{R}^N$ of a connected n-dimensional Riemannian manifold into the N-dimensional Euclidean space is called *rigid* if any other isometric immersion $g: M^n \to \mathbb{R}^N$ differs from f by a rigid motion of $\mathbb{R}^N$. A less general rigidity theory deals with the notion of isometric variations. An *isometric variation* is a smooth map $F: I \times M \to \mathbb{R}^N$, $I = [0,1]$, such that $F(0,x) = f(x)$ and for each $t \in I$, $F_t(x) = F(t,x)$ is an isometric immersion. The simplest example of an isometric variation is given by a smooth family of rigid motions of $\mathbb{R}^N$, i.e., $F(t,x) = C(t)f(x) + v(t)$ where $C(t)$ is an orthogonal transformation of $\mathbb{R}^N$ and $v(t)$ is a vector. Such an isometric variation is called *trivial*. If there exists a non-trivial isometric variation we say that f is *deformable*.

It is easy to see ([GR], p. 53) that the variation vector field $Z = \left.\dfrac{\partial F}{\partial t}\right|_{t=0}$ of F verifies

$$(1) \qquad \langle \tilde{\nabla}_X Z, X \rangle = 0$$

for all $X \in TM$, where $\tilde{\nabla}$ denotes the standard connection in $\mathbb{R}^N$. If F is trivial as above, then $Z = C'(0)f + v'(0)$ where $C'(0)$ is skew-symmetric. Conversely, given $Z = Cf + v$, where C is a skew-symmetric matrix and v is a constant vector, we have that $F(t,x) = e^{tC}f(x) + tv$ is an isometric variation through rigid motions, i.e., a trivial variation.

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In this paper we deal with a third rigidity notion which is a linearized version of the second rigidity notion. We say that a smooth vector fields Z defined along f with values in $\mathbb{R}^N$ is an *infinitesimal isometric variation* if it verifies (1). An alternative definition of Z is as the variation vector field of a variation which is isometric only to first order (see [GR], p. 50). Observe that for any infinitesimal isometric variation Z, the variation $G(t,x) = f(x) + tZ(x)$ is isometric to first order to f with $\frac{\partial G}{\partial t} = Z$. We say that Z is *trivial* if it can be written as $Z = Bf + v$ where B is a skew-symmetric linear transformation in $\mathbb{R}^N$ and v is a vector. If an immersion only admits trivial infinitesimal isometric variations we say that it is *infinitesimally rigid ;* otherwise we say it is *flexible*. The following provide generally non-trivial examples of infinitesimal isometric variations.

Example. — For any normal vector ψ let A_ψ denote the second fundamental form in the direction of ψ. Suppose an immersion has a normal vector field η such that at each point $A_\eta = 0$. Since $\tilde{\nabla}_X\eta = -A_\eta X + \nabla^\perp_X\eta = \nabla^\perp_X\eta$, where $\nabla^\perp$ denotes the connection of the normal bundle, we have that $Z = \eta$ satisfies condition (1). In fact, more generally, we can take $Z = Y + \eta$, where Y is any Killing vector field on M.

Example. — Let Y be a conformal Killing vector field on a manifold M^n, i.e., $\langle \nabla_X Y, X\rangle = c(x)\langle X,X\rangle$, for every $x \in M$, and $f : M^n \to \mathbb{R}^{n+p}$ an isometric immersion. Suppose that $\xi(x)$ is a normal umbilical direction which is never totally geodesic, i.e., $A_\xi X = \lambda(x)X$ where λ never vanishes. Then, if we take $Z = Y + \frac{c(x)}{\lambda(x)}\xi$, we have that

$$\begin{aligned}\langle \tilde{\nabla}_X Z, X\rangle &= \langle \tilde{\nabla}_X Y, Z\rangle + \left\langle \tilde{\nabla}_X \frac{c(x)}{\lambda(x)}\xi, X\right\rangle \\ &= \langle \nabla_X Y, X\rangle - \frac{c(x)}{\lambda(x)}\langle A_\xi X, X\rangle = 0,\end{aligned}$$

and Z is an infinitesimal isometric variation. Observe that if M admits an isometric immersion into S^{n+p-1}, then the composition with the inclusion into $\mathbb{R}^{n+p}$ has always a normal umbilical vector field.

The basic question we consider here is whether the notion of infinitesimal rigidity differs from the notion of isometric rigidity. Some of our results point in the direction that local or global conditions that imply rigidity also imply infinitesimal rigidity. These types of results will be treated in Section 1. Another question of rigidity concerns the existence of infinitesimal isometric variations which satisfy further conditions. One interesting case is the situation of having infinitesimally the same Gauss map. In [DG] it is shown that the only possible examples of isometric variations having the same Gauss map are the minimal immersions of Kaehler manifolds. We show, in Section 2, that the only flexible examples are also of this type. We conclude the section classifying pairs of immersions (not necessarily isometric) which have the same Gauss map and make a right angle (see [DR2]).

Finally, in Section 3, we use the theory of infinitesimal isometric variations to obtain a two-parameter family of immersions which are not rigid.

1. Infinitesimal rigidity.

The following is our basic result.

THEOREM 1. — *Let Z be an infinitesimal isometric variation of an isometric immersion $f : M^n \to \mathbb{R}^N$ of a connected manifold M^n. Consider the maps $G_t : M^n \to \mathbb{R}^N$, $t \in \mathbb{R}$, defined as*

$$G_t(x) = f(x) + tZ(x). \tag{2}$$

a) *For all $t \in \mathbb{R}$, G_t is an immersion and G_t and G_{-t} induce the same metric.*

b) *If f is substantial and, for some time $t_0 \neq 0$, G_{t_0} and G_{-t_0} are congruent in $\mathbb{R}^N$ then Z is trivial.*

Proof. — Observe that for any tangent vector X,

$$\|G_{t*}X\|^2 = \|X\|^2 + t^2 \|\tilde{\nabla}_X Z\|^2,$$

since by hypothesis $\langle X, \tilde{\nabla}_X Z \rangle = 0$. This proves a).

If G_{t_0} and G_{-t_0} are congruent for some $t_0 \neq 0$, then there exists a fixed orthogonal transformation T of $\mathbb{R}^N$ and a constant vector $w \in \mathbb{R}^N$ such that

$$f + t_0 Z = T(f - t_0 Z) + w.$$

Differentiating with respect to a tangent vector X, we obtain

$$X + t_0\tilde{\nabla}_X Z = T(X - t_0\tilde{\nabla}_X Z),$$

or

$$(3) \qquad t_0(T+I)\tilde{\nabla}_X Z = (T-I)X.$$

If $T + I$ is invertible, then

$$(4) \qquad \tilde{\nabla}_X Z = BX,$$

where $t_0 B = (T+I)^{-1}(T-I)$, and B is skew-symmetric by (4). Since $BX = \tilde{\nabla}_X(Bf)$, it follows that $\tilde{\nabla}_X(Z - Bf) = 0$, showing that Z is trivial.

It remains to show that $T + I$ is invertible. Suppose that on the contrary there exists a vector $\eta \in \mathbb{R}^N$ such that $T_\eta = -\eta$. For any arbitrary tangent vector X, we have using (3),

$$\begin{aligned} 2\langle X,\eta\rangle &= \langle TX, T\eta\rangle + \langle X,\eta\rangle = -\langle TX,\eta\rangle + \langle X,\eta\rangle \\ &= \langle (T-I)X,\eta\rangle = -t_0(\langle T\tilde{\nabla}_X Z,\eta\rangle + \langle \tilde{\nabla}_X Z,\eta\rangle) \\ &= -t_0(\langle \tilde{\nabla}_X Z, T^{-1}\eta\rangle + \langle \tilde{\nabla}_X Z,\eta\rangle) = 0. \end{aligned}$$

Since X is arbitrary and η is constant we have that the immersion is not substantial, which is a contradiction. This concludes the proof of b). □

Recall that Allendoerfer [A] proved that any isometric immersion into $\mathbb{R}^N$ with type number $\tau \geqslant 3$ is rigid. Here we obtain the following infinitesimal version of this result.

THEOREM 2. — *Let $f : M^n \to \mathbb{R}^N$ be an isometric immersion with type number $\tau(x) \geqslant 3$ for all $x \in M$. Then f is infinitesimally rigid.*

Proof. — We first show that any infinitesimal isometric variation is locally trivial. By theorem 1a), the immersions G_t and G_{-t} are isometric and their type number is still greater than or equal to 3 for small t when restricted to a small open set; consequently, they are locally congruent and, by Theorem 1b), the variation field Z is locally trivial, because $\tau(x) \geqslant 3$ implies that the immersion is substantial.

As we saw in the proof of part b) of Theorem 1, if Z is trivial in two open sets U_1 and U_2 we have by (2) that $\tilde{\nabla}_X Z = B_1 X$ in U_1 and

$\tilde{\nabla}_X Z = B_2 X$ in U_2, where B_1 and B_2 are skew-symmetric matrices. If the open sets intersect then $(B_1 - B_2)|_{T_xM} = 0$, for all $x \in U_1 \cap U_2$. Since f is substantial on $U_1 \cap U_2$ the affine tangent spaces T_yM, $y \in U_1 \cap U_2$ generate all of $\mathbb{R}^N$ showing that $B_1 = B_2$. We conclude that the variation is globally trivial. □

Remarks. – 1) The assumption $\tau \geqslant 3$ can be replaced by other algebraic assumptions which imply rigidity (see [CD]).

2) Goldstein and Ryan [GR] showed that the inclusion $S^n \subset \mathbb{R}^{n+1}$ is infinitesimally rigid.

3) Any infinitesimally rigid immersion is equivariant.

By a theorem of Sacksteder [Sa], if f, $g : M^n \to \mathbb{R}^{n+1}$, $n \geqslant 3$, are two isometric immersions of a compact manifold as Euclidean hypersurfaces then f and g are congruent on each connected component of the set of non-totally geodesic points of f, or on any open connected totally geodesic set of f. The following is an infinitesimal version of Sacksteder's theorem. Observe that the condition on the totally geodesic sets is different from Sacksteder's; we can allow totally geodesic sets that separate as long as they have empty interior. This last condition is necessary as Example 1 shows.

THEOREM 3. – *Let $f : M^n \to \mathbb{R}^{n+1}$, $n \geqslant 3$, be an isometric immersion of a compact manifold such that there are no open sets where f is totally geodesic. Then f is infinitesimally rigid.*

Proof. – Suppose that Z is an infinitesimal isometric variation. As above consider the immersions G_t and G_{-t}. We first claim that there is no open set U where any one of them is totally geodesic. Otherwise, by Sacksteder's theorem, so would be the other, and consequently f, since $f = 1/2(G_t + G_{-t})$. Fixing a time $t \neq 0$, let D denote the set of totally geodesic points of G_t and let $\{U_i\}$ denote the family of open connected subsets of the dense subset $M \backslash D$. By Sacksteder's theorem we have that restricted to each U_i, G_t and G_{-t} are congruent. Consequently, by Theorem 1, we have that there exists a skew-symmetric linear transformation $B_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that Z restricted to U_i is given by

$$Z = B_i f + v_i,$$

where v_i is a vector in $\mathbb{R}^{n+1}$. For each $x \in M$, we can define a unique linear map C_x in $\mathbb{R}^{n+1}$ with the following properties:

1) if X and Y are in T_xM let $\langle C_xX, Y\rangle = \langle \tilde{\nabla}_X Z, Y\rangle$,

2) if η is a normal vector to the immersion f then $\langle C_xX, \eta\rangle = \langle \tilde{\nabla}_X Z, \eta\rangle$,

3) C_x is skew-symmetric.

Observe now that on each U_i, $\tilde{\nabla}_X Z = B_i X$. Consequently, we have that at any point $x \in U_i$, C_x and B_i are skew-symmetric matrices which coincide on $f_* T_x M$. Now two skew-symmetric matrices that coincide in a hyperplane must be equal. Because Z is differentiable, we have that the map $x \mapsto C_x$ is also differentiable. However, since this map is locally constant in the open dense set $M \backslash B$, it must be globally constant, i.e., all the matrices B_i are equal to a fixed matrix B. Finally, $Z = Bf + v_i$ on each U_i. Therefore, $v = Z - Bf$ is locally constant in $M \backslash B$, thus constant in M. □

2. Infinitesimal parallel variations.

We say that Z is an *infinitesimal parallel isometric variation* if it satisfies (1) and

$$\langle \tilde{\nabla}_X Z, \eta\rangle = 0, \tag{5}$$

for all $X \in TM$ and all $\eta \in TM^\perp$. See Yano [Y] for comments on this notion.

The following theorem classifies completely all immersions which admit infinitesimal parallel isometric variations (even trivial ones).

THEOREM 4. — *Let $f : M^n \to \mathbb{R}^N$, $n \geqslant 3$, be a locally irreducible substantial isometric immersion of a connected manifold. If f admits an infinitesimal parallel isometric variation Z which is not constant, then M is Kaehler, and either*

i) *f is holomorphic and $Z = cJf$ for some complex structure J of $\mathbb{R}^N$ and constant c; here Z is trivial;*

or

ii) *f is minimal and $Z = c\tilde{f} + v$, where $\tilde{f}$ is the conjugate member of the associated family of minimal immersions of f, c is a constant and v is a constant vector.*

Proof. — As in the proof of Theorem 1 consider the isometric immersions G_t and G_{-t}. They have the same Gauss map since $G_{t*}X = f_*X + t\tilde{\nabla}_X Z$ and by the hypothesis, $\tilde{\nabla}_X Z$ is tangent. From [DG] and [DR1] we have that either G_t and G_{-t} are congruent or they are real Kaehlerian associated minimal immersions.

If they are Kaehlerian associated minimal immersions then

$$\begin{aligned} G_t(x) &= f(x) + tZ(x) = \varphi \\ G_{-t}(x) &= f(x) - tZ(x) = \cos\theta\varphi + \sin\theta\bar{\varphi}, \end{aligned}$$

where $\bar{\varphi}$ is the conjugate minimal immersion of φ and θ is a constant. Hence

$$\begin{aligned} Z &= \frac{1}{2t}[(1-\cos\theta)\,\varphi - \sin\theta\bar{\varphi}] \\ &= h[\cos\hat{\theta}\varphi + \sin\hat{\theta}\bar{\varphi}], \end{aligned}$$

for some angle $\hat{\theta}$ and constant h. Consequently, Z is homothetic to $\cos\hat{\theta}\varphi + \sin\hat{\theta}\bar{\varphi}$, an associated minimal immersion of f, since f can be written as

$$\begin{aligned} f &= \frac{1}{2}[(1+\cos\theta)\,\varphi + \sin\theta\bar{\varphi}] \\ &= h[\cos\hat{\theta}\varphi + \sin\hat{\theta}\bar{\varphi}], \end{aligned}$$

for some angle $\tilde{\theta}$ and constant h.

In case G_t and G_{-t} are congruent, by Theorem 1 we have that Z is trivial, that is, $Z = Bf + v$ where B is a skew-symmetric matrix. We want to show now that either B is a multiple of a fixed complex structure J, or that it is zero. If it is zero we are done. If it is proportional to a complex structure J then we have by (5) that for any tangent vector X and normal vector η,

$$c\langle J(f_*X), \eta\rangle = \langle\bar{\nabla}_X Z, \eta\rangle = 0,$$

showing that TM is invariant by J and consequently holomorphic.

Let us show now that B^2 is a negative multiple of the identity matrix. First observe that condition (5) becomes

$$\langle BX, \eta\rangle = 0,$$

which implies that B, and thus B^2, leaves invariant the tangent and normal spaces of the immersion f. Using the fact that B is constant we obtain by the Gauss formula that

$$\begin{aligned} (6) \qquad \nabla_Y BX + \alpha(Y,BX) &= \tilde{\nabla}_Y BX = B(\tilde{\nabla}_Y X) \\ &= B(\nabla_Y X) + B(\alpha(X,Y)). \end{aligned}$$

For a normal direction η the η-component of formula (6) gives us

$$\langle BX, A_\eta Y\rangle = \langle B(\alpha(Y,X)), \eta\rangle,$$

which is equivalent to

$$\langle X, BA_\eta Y\rangle = \langle A_{B\eta} Y, X\rangle$$

because B is skew-symmetric. Since this is true for all $X,\ Y \in TM$ we have that

$$A_{B\eta} = BA_\eta .$$

Using that $A_{B\eta}$ is symmetric, we have that

$$BA_\eta = (BA_\eta)^t = A_\eta B^t = -A_\eta B$$

and

$$A_\eta B^2 = B^2 A_\eta .$$

Since both A_η and B^2 are symmetric and commute they have the same eigenvectors.

Because B preserves the tangent planes, observe that formula (6) gives us that the tensor B restricted to TM is parallel, and thus also B^2, i.e.,

$$\nabla_Y B^2 X = B^2(\nabla_Y X)$$

for all $X,\ Y \in TM$. Therefore all the eigenspaces of the symmetric matrix B^2 are parallel and since M^n is locally irreducible, B^2 is a multiple of the identity. If it is a positive multiple then we would have that B is also, which is a contradiction since it is skew-symmetric. Thus B must be proportional to a complex structure. □

Remark. — Of course the condition of locally irreducible is necessary since otherwise we could use a different homothety on each factor.

As a corollary we get a result on pairs of immersions (not necessarily isometric) which *make a right angle* at every point, i.e., which satisfy for each tangent vector $X \in TM$,

$$\langle f_* X, g_* X \rangle = 0. \tag{7}$$

THEOREM 5. — *Let f, $g : M^n \to \mathbb{R}^{n+p}$ be two immersions which have the same Gauss map and which make a right angle. Assume that f is substantial and locally irreducible. Then M is a Kaehler manifold with the metric induced by f, and either f is holomorphic and $g = cJf + v$ for some complex structure J of $\mathbb{R}^N$, constant c and vector v, or f is minimal and $g = c\bar{f} + v$ where $\bar{f}$ is the conjugate minimal immersion of f.*

Proof. — Since $g_* X = Xg = \tilde{\nabla}_{Xg}$, then condition (7) implies condition (1) for $Z = g$. Since they have the same Gauss map then we have also that condition (5) is satisfied and we can apply the previous theorem to conclude the proof since $Z = g$ cannot be a constant vector. □

3. Families of deformable immersions.

Let F be an isometric variation of the isometric immersion f: $M^n \to \mathbb{R}^N$. In this case we can make use of the theory of infinitesimal isometric variations to produce new immersions of M^n into $\mathbb{R}^N$, with different induced metrics, which are also not isometrically rigid. We will make use of the following.

PROPOSITION 6. — *Assume that the isometric variation F has trivial variation vector field $Z_t = \dfrac{\partial F}{\partial t}$ for all $t \in I$. Then F_t is trivial.*

Proof. — By assumption $Z_t = B_t F_t + v_t$ for all $t \in I$. For any pair of points x, $y \in M$, we have

$$\begin{aligned}\frac{d}{dt}\|F_t(x) - F_t(y)\|^2 &= 2\langle F_t(x) - F_t(y), Z_t(x) - Z_t(y)\rangle \\ &= 2\langle F_t(x) - F_t(y), B_t(F_t(x) - F_t(y))\rangle = 0.\end{aligned}$$

Consequently, $\|F_t(x) - F_t(y)\| = \|F_0(x) - F_0(y)\|$, for all $t \in I$, and x, $y \in M$. □

THEOREM 7. — *Let $f: M^n \to \mathbb{R}^N$ be a deformable isometric immersion. Then there exists a two-parameter family of immersions $F(t,s,x): J \times \mathbb{R}_+ \times M^n \to \mathbb{R}^N$ which are not rigid.*

Proof. — Let $F(t,x): I \times M^n \to \mathbb{R}^N$ be a non-trivial variation of f. Now by Proposition 6 there exists an interval $J \subset I$ such that the variation field $Z_t = \dfrac{\partial F}{\partial t}$ is not trivial for all $t \in J$. Define

$$F(t,s,x) = F(t,x) + sZ_t(x).$$

As in Theorem 1, we conclude that these immersions are not rigid.

□

Observe that the new metrics induced by the $F(t,s,x)$ are different from the old one because, for any $Y \in TM$ such that $\tilde{\nabla}_Y Z_t \neq 0$, we have

$$\|F_* X\|^2 = \|X\|^2 + s^2 \|\tilde{\nabla}_X Z_t\|^2 > \|X\|^2.$$

We now consider the special case $N = n + 1$, and show that the new metrics are quite different from the original ones.

PROPOSITION 8. — *Let Z be a non-trivial variation vector field of an isometric variation of a hypersurface $f: M^n \to \mathbb{R}^{n+1}$. If the metric induced by the immersion $F = f + sZ$, $s \neq 0$ is conformal to the metric induced by f, then f is a minimal real Kaehler hypersurface.*

Proof. — If both metrics are conformal there exists a smooth function $\phi \geqslant 0$ such that

$$\|\tilde{\nabla}_X Z\|^2 = \phi(x)\|X\|^2,$$

for all $X \in TM$. For any open connected subset $U \subset M$ where $\phi > 0$ we have that f and Z are two conformal immersions which make a right angle. It follows from the theorem of [DR2] that f is a minimal real Kaehler hypersurface and that Z is conformal to the associate minimal hypersurface by a constant conformal factor, i.e., ϕ is constant on U. The result follows. □

Recall that real Kaehler hypersurfaces have been classified in [DG].

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