ROBERT F. COLEMAN

Vectorial extensions of Jacobians

Annales de l'institut Fourier, tome 40, nº 4 (1990), p. 769-783

<http://www.numdam.org/item?id=AIF_1990__40_4_769_0>

Ann. Inst. Fourier, Grenoble
40, 4 (1990), 769-783.

VECTORIAL EXTENSIONS OF JACOBIANS

by Robert F. COLEMAN

In this paper, we make explicit the connection between the theory of the universal vectorial extension as developed in *Universal Extensions and One Dimensional Crystalline Cohomology* [MaMe] and the theory of generalized jacobians as developed in *Groupes Algébriques et Corps de Classes* [S]. In particular, we prove those results needed in [C1] to verify the equality of the Mazur-Tate height [MaT] and the height defined in [CG].

A generalized jacobian of a curve represents isomorphism classes of invertible sheaves algebraically equivalent to zero (which we will abbreviate as *AEZ*) with a trivialization along a closed subscheme. The universal vectorial extension of its jacobian represents isomorphism classes of *AEZ* invertible sheaves with a connection. Naturally enough, the bridge between these two schemes is a scheme which represents isomorphism classes of *AEZ* invertible sheaves with a connection and a « horizontal trivialization » along a closed subscheme. We will make this notion precise in Section 2.

1. Over an algebraically closed field.

Suppose X is a smooth proper curve defined over an algebraically closed field K. If ω is a differential on X over K (i.e. a section of the stalk of $\Omega^1_{X/K}$ at the generic point), $(\omega)_0$ will denote the divisor of zeros of ω and $\mathrm{Res}\,(\omega)$ will denote the element

$$\sum_{P \in X(K)} \mathrm{Res}_P\,(\omega)P$$

Key-words : Generalized Jacobians - Vectorial extensions - Connections.
A.M.S. Classification : 14H40.

of $\mathrm{Div}(X) \otimes K$. We will say ω is of the *third kind* if it has at worst simple poles and $\mathrm{Res}(\omega)$ is in the image of $\mathrm{Div}(X)$. We will say ω is of the *second kind* if $\mathrm{Res}(\omega) = 0$. A *logarithmic differential* is one of the form $d\log f$ where f is an element of $K(X)^*$.

Let G denote the universal vectorial extension of the jacobian J of X. Then, as is well known, when K is a field of characteristic zero, $G(K)$ is canonically isomorphic to the the group of differentials of the third kind modulo the subgroup of logarithmic differentials. More generally, $G(K)$ is canonically isomorphic to the group of pairs (E,η) where E is a divisor of degree zero and η is a differential of the third kind such that

$$\mathrm{E} \equiv \mathrm{Res}(\eta) \bmod \mathrm{char}(K)$$

modulo the subgroup of pairs of the form $((f), d\log(f))$. We will call the pairs (E,η), as above, *differential pairs of the third kind* and the pairs $((f), d\log(f))$, *logarithmic pairs.*

Alternatively, let $\mathscr{D}$ denote the sheaf on X such that for each open U, $\mathscr{D}(U)$ is equal to those divisors on X supported on $X - U$. Consider the modified log de Rham complex $\mathcal{O}_X^* \to \Omega_X^1 \oplus \mathscr{D}$, where the map sends a section f of $\mathcal{O}_X^*$ to the pair $(df/f, (f))$. Then we can restate the above remark as, $G(K)$ is canonically isomorphic to the subgroup of $\mathbf{H}^1(\mathcal{O}_X^* \to \Omega_X^1 \oplus \mathscr{D})$ consisting of those elements whose image in $\mathbf{H}^1(\mathcal{O}_X^*)$ have degree zero.

Let D be an effective divisor on X and J_D the generalized jacobian of X with modulus D as defined in [S], No. 7 § 1.6. Then $J_D(K)$ is isomorphic to the group of divisors of degree zero disjoint from $|D|$ (the support of D regarded as a closed subscheme of X) modulo the group of principal divisors of the form (f) where $f \in K(X)$ such that f is regular on D and $f \equiv 1 \bmod D$. Now J_D is an extension of J by a group which is the product of a torus T_D and a unipotent group U_D. Whence $G_D =: J_D/T_D$ is an extension of J by U_D. More explicitly, $\mathcal{O}_{|D|}$ injects naturally into $\mathcal{O}_D$ and $G_D(K)$ is isomorphic to the group of divisors of degree zero disjoint from D modulo the subgroup of principal divisors of the form (f) where f is regular on D and the image of f in $\mathcal{O}_D(D)$ lies in the image of $\mathcal{O}_{|D|}(|D|)^*$.

Let $p(K) = \mathrm{char}(K)$ if $\mathrm{char}(K) > 0$ and ∞ otherwise. If $\mathrm{ord}_P D \leqslant p(K)$ for all $P \in X(K)$ then U_D is a vector group (as we will see below) and in this case we will call G_D the generalized vectorial jacobian of modulus D. Hence, by the universal property of G, there exists a canonical morphism from G to G_D. We will now describe this morphism on points. Let $D(-1) = D - |D|$. Suppose (E,η) is a differential pair of the third kind. Then because $\mathrm{ord}_P D \leqslant \mathrm{char}(K)$ there exists an $f \in K(X)$ such that

$$(\eta - df/f)_0 \geqslant D(-1).$$

THEOREM 1.1. — *The image of the class in $G(K)$ of (E,η) in $G_D(K)$ is the class of $E - (f)$.*

We will generalize and prove this for curves over an arbitrary base in the following sections. We will also prove generalizations of the follow propositions.

PROPOSITION 1.2. — *Let g be the genus of X. Suppose $D(-1)$ is a non-special effective divisor of degree g and $\mathrm{ord}_P(D) \leqslant p(K)$ for all $P \in X(K)$. Then the morphism from G to G_D is an isomorphism.*

Let $D(-1)$ be a non-special effective divisor of degree g. Suppose Q is a point in $(X-|D|)(K)$. Then it follows from the previous theorem and proposition that for each point in $(X-|D|)(K)$ there exists a unique differential pair of the third kind $(P-Q, \omega_{D,P,Q})$ such that $(\omega_{D,P,Q})_0 \geqslant D(-1)$.

PROPOSITION 1.3. — *There exists a unique morphism of K-schemes α: $X - |D| \to G$ which takes $P \in (X-|D|)(K)$ to the class of $(P-Q, \omega_{D,P,Q})$. Moreover, the pullback under this morphism of the space of invariant differentials on G is the space of differentials ω of the second kind on X such that $(\omega) \geqslant -D$.*

From this it follows that

COROLLARY 1.3.1. — *Suppose (E,η) is a differential pair of the third kind on X such that $(\eta)_0 \geqslant D(-1)$. Then the point in $G(K)$ represented by this pair is equal to*

$$\sum_G \alpha(E).$$

Remark. — It is easy to formulate the analogous results over an arbitrary field and to deduce them from these.

2. Over an arbitrary base.

Let S be a scheme. If T and S' are S-schemes, $T_{S'}$ will denote $T\times_{S'}S'$ and Ω^1_T will denote $\Omega^1_{T/S}$. If $\mathscr{S}$ is a quasi-coherent sheaf on S, then we let $V(\mathscr{S})$ denote the vector group $\operatorname{Spec} S(\mathscr{S})$ where $S(\mathscr{S})$ is the symmetric $\mathcal{O}_S$-algebra on $\mathscr{S}$. The scheme $V(\mathscr{S})$ is naturally a group scheme over S, but it has more structure. Namely, $V(\mathcal{O}_S)$ is a ring scheme over S and $V(\mathscr{S})$ is a $V(\mathcal{O}_S)$ module over S. Any S scheme V which is a $V(\mathcal{O}_S)$ module is called a vector group over S and every such V is of the form $V(\mathscr{S})$ where $\mathscr{S} = \mathscr{H}om_{V(\mathcal{O}_S)}(V, V(\mathcal{O}_S))$. We will also let $\mathscr{S}$ denote the functor on S schemes $f: S' \to S$ defined by $\mathscr{S}(S') = (f^*\mathscr{S})(S')$. Suppose X is an S scheme and T is a closed subscheme of X. Then $\mathscr{I}_T$ will denote the sheaf of ideals of T in $\mathcal{O}_X$, the structure sheaf of X.

Let $\pi: X \to S$ be a projective geometrically integral smooth curve over S. Let $\iota: Y \to X$ be a closed subscheme of X, finite and faithfully flat over S. We will define some functors on the category of S-schemes. For an S-scheme S',

$\mathscr{P}_{X,Y}(S')$ will denote the group of isomorphism classes of pairs $(\mathscr{L},\alpha)$ where $\mathscr{L}$ is an invertible sheaf on $X_{S'}$ and $\alpha: \mathcal{O}_{Y_{S'}} \to \iota^*\mathscr{L}$ is an isomorphism.

$\mathscr{U}_Y = \pi*(\mathcal{O}_Y^*)/\mathcal{O}_S^*$ so that $\mathscr{U}_Y(S')$ is canonically isomorphic to the subgroup of $\mathscr{P}_{X,Y}(S')$ consisting of isomorphism classes of pairs $(\mathscr{L},\alpha)$ such that $\mathscr{L}$ is isomorphic to $\mathcal{O}_{X_{S'}}$.

$\mathrm{Pic}_X(S')$ will denote the group of isomorphism classes of invertible sheaves on X.

Now both $\mathscr{P}_{X,Y}$ and $\mathscr{U}_Y$ are sheaves on the $fppf$ site over S, whereas Pic_X is a presheaf on this site. We let $\mathscr{P}\!ic_X$ denote the associated sheaf. All these sheaves are representable by group schemes over S. See Proposition 4 in [O] for $\mathscr{P}_{X,Y}$, (in the case in which $\mathscr{I}_Y$ is invertible this also follows from Theorem 1.4 of [C-C]). In particular, $\mathscr{P}\!ic_X$ is the Picard scheme of X over S and $\mathscr{P}\!ic^0_{X/S}$, the connected component of the identity of $\mathscr{P}\!ic_X$, is the jacobian of X over S. We also let $\mathscr{P}^0_{X,Y}$ denote the inverse image of $\mathscr{P}\!ic^0_{X/S}$ in $\mathscr{P}_{X,Y}$ under the natural map.

Suppose Z is a closed subscheme of X étale and surjective over S. This implies that $\mathscr{I}_Z$ is invertible. Let $\rho : R \to X$ denote a closed subscheme of X such that $\mathscr{I}_R$ is locally a power of $\mathscr{I}_Z$. Let $R(-1)$ denote the closed subscheme of X such that $\mathscr{I}_R \otimes \mathscr{I}_Z^{-1} = \mathscr{I}_{R(-1)}$. For an S scheme T, ∇_{triv} will denote the connection : exterior differentiation on the structure sheaf of T. Then,

$\mathscr{P}^{\#}_{X,R}(S')$ will denote the group of isomorphism classes of triples $(\mathscr{L},\nabla,\alpha)$ where $\mathscr{L}$ is an invertible sheaf on $X_{S'}$, ∇ is a connection on $\mathscr{L}$ over S' and $\alpha : \mathcal{O}_{R_{S'}} \to \rho^*\mathscr{L}$ is an isomorphism such that

$$(\rho^*\nabla)(\alpha(1)) \equiv 0 \bmod \mathscr{I}_{R(-1)}\Omega^1_{R_{S'}/S'} \otimes \rho^*(\mathscr{L})$$

where here $1 \in \mathcal{O}_{R_{S'}}$ and $\rho^*\nabla : \rho^*\mathscr{L} \to \Omega^1_{R_{S'}/S'} \otimes \rho^*(\mathscr{L})$ is the pullback of ∇ to R.

$\mathscr{U}^{\#}_{X,R}(S')$ will denote the subgroup of $\mathscr{P}^{\#}_{X,R}(S')$ consisting of isomorphism classes of triples $(\mathscr{L},\nabla,\alpha)$ where $\mathscr{L}$ is isomorphic to $\mathcal{O}_{X_{S'}}$.

$\mathscr{V}^{\#}_{X,R}(S')$ will denote the subgroup of $\mathscr{U}^{\#}_{X,R}(S')$ consisting of isomorphism classes of triples $(\mathscr{L},\nabla,\alpha)$ where $(\mathscr{L},\nabla)$ isomorphic to $(\mathcal{O}_{X_{S'}}, \nabla_{\text{triv}})$.

$\mathrm{Pic}^{\#}_X(S')$ will denote the group of isomorphism classes of pairs $(\mathscr{L},\nabla)$ where $\mathscr{L}$ is an invertible sheaf on X, ∇ is a connection on $\mathscr{L}$ over S.

$\mathscr{W}_X = \pi_*(\Omega^1_X)$ so that $\mathscr{W}_X(S')$ is canonically isomorphic to the subgroup of $\mathrm{Pic}^{\#}_X(S')$ consisting of isomorphism classes of pairs $(\mathscr{L},\nabla)$ where $\mathscr{L}$ is isomorphic to $\mathcal{O}_{X_{S'}}$.

Remark. – We note that $\mathscr{I}_{R(-1)}\Omega^1_{R_{S'}/S'} = 0$ if $\mathscr{I}_R \supseteq (\mathscr{I}_Z)^{n+1}$ and $(n+1)!$ is invertible on S and so in this case if $(\mathscr{L},\nabla,\alpha)$ represents an element of $\mathscr{P}^{\#}_{X,R}(S')$, α is a horizontal isomorphism from $(\mathcal{O}_{R_{S'}}, \nabla_{\text{triv}})$ onto $\rho^*(\mathscr{L},\nabla)$.

The functors $\mathscr{P}^{\#}_{X,R}(S')$, $\mathscr{U}^{\#}_{X,R}(S')$, $\mathscr{V}^{\#}_{X,R}(S')$ and $\mathscr{W}_X$ are sheaves on the *fppf* site over S whereas $\mathrm{Pic}^{\#}_X$ is only a presheaf. We let $\mathscr{P}\!ic^{\#}_X$ denote the associated sheaf. Then $\mathscr{P}^{\#}_{X,R}$ and $\mathscr{P}\!ic^{\#}_X$ map to $\mathscr{P}\!ic_X$ and we let $(\mathscr{P}^{\#}_{X,R})^0$ and $(\mathscr{P}\!ic^{\#}_X)^0$ denote the respective inverse images of $\mathscr{P}\!ic^0_X$. Then $(\mathscr{P}\!ic^{\#}_X)^0$ is representable by the universal vectorial extension of the jacobian of X over S (see [MaMe]. We note that $\mathscr{V}^{\#}_{X,R}(S') \subseteq \mathscr{U}^{\#}_{X,R}(S') \subseteq (\mathscr{P}^{\#}_{X,R}(S'))^0$. Subsequently, we will not distinguish between a representable sheaf and the scheme which represents it.

Suppose now that Y is as above and $\mathcal{I}_Z \subseteq \mathcal{I}_Y \subseteq \mathcal{I}_R$. We have a natural commutative diagram of sheaves

$$(1)\qquad \begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{W}_X & \leftarrow & \mathcal{U}^{\#}_{X,R} & \rightarrow & \mathcal{U}_Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}ic^{\#}_X & \leftarrow & \mathcal{P}^{\#}_{X,R} & \rightarrow & \mathcal{P}_{X,Y} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}ic_X & \leftarrow & \mathcal{P}ic_X & \rightarrow & \mathcal{P}ic_X \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

in which the columns are exact.

As is well known, $\mathcal{P}ic_X(S')$ is naturally isomorphic to $\mathbf{R}^1_{\pi_*}(\mathcal{O}^*_X)(S')$ and $\mathcal{P}_{X,Y}(S')$ is naturally isomorphic to $\mathbf{R}^1_{\pi_*}(1+\mathcal{I}_Y)(S')$ whereas $\mathcal{P}ic^{\#}_X(S')$ is naturally isomorphic to $\mathbf{R}^1_{\pi_*}(\Omega^*_X)(S')$ where Ω^*_X is the log-de Rham complex $\mathcal{O}^*_X \to \Omega^1_X$ with boundary $f \mapsto d\log f$. Let $\mathcal{C}_R$ denote the subcomplex of Ω^*_X, $(1+\mathcal{I}_R) \to \mathcal{I}_{R(-1)}\Omega^1_X$ (that this is a subcomplex requires our hypothesis that $\mathcal{I}_R$ is locally a power of $\mathcal{I}_Z$). Then

LEMMA 2.1. – *The group* $\mathbf{R}^1_{\pi_*}(\mathcal{C}_R)(S')$ *is naturally isomorphic to* $\mathcal{P}^{\#}_{X,R}(S')$.

Proof. – Let $R' = R_{S'}$. Let $(\{\omega_U\},\{f_{U,V}\})$ be a one-hypercocycle of the complex $\mathcal{C}_R$ over S' with respect to some open cover of $X_{S'}$. Let $\mathcal{L}$ denote the sheaf on $X_{S'}$ with sections s_U on U such that $\mathcal{L}(U) = \mathcal{O}(U)s_U$ and on $U \cap V s_U = f_{U,V}s_V$. Let ∇ denote the connection on $\mathcal{L}$ determined by $\nabla s_U = \omega_U \otimes s_U$. Finally, there exists a unique isomorphism $\alpha : \mathcal{O}_{R'} \to \mathcal{L} \otimes \mathcal{O}_{R'}$ such that $\alpha(1)_{(U\cap R')} = (s_U)_{R'}$. Then the triple $(\mathcal{L},\nabla,\alpha)$ determines an element of $\mathcal{P}^{\#}_{X,R}(S')$. It is easy to check that this induces an isomorphism from $\mathbf{R}^1_{\pi_*}(\mathcal{C}_{R'})(S')$ onto $\mathcal{P}^{\#}_{X,R}(S')$. □

Remark. – We are grateful to the referee for suggesting that we express our functors in hypercohomological terms as well as extracting Lemma 2.2 from our previous exposition. The referee also proposed the use of the complex $\mathcal{O}^*_X \to \mathcal{F}$ where $\mathcal{F}$ is the subsheaf of $\Omega^1_X \oplus \mathcal{O}^*_R$ whose sections are pairs (ω,f) such that $\omega \equiv d\log f \bmod \mathcal{I}_{R(-1)}\Omega^1_R$ as well as the cone of the map of complexes $\Omega^*_X \to (\mathcal{O}^*_R \to \mathcal{I}_{R(-1)}\Omega^1_R)$.

However, one can easily show that both of these complexes are quasi-isomorphic to $\mathscr{C}_R$ which suffices for our purposes.

It is easy to see that the commutative diagram (1) may be obtained from morphisms and spectral sequences associated to the relevant complexes.

LEMMA 2.2. — *If $n!$ is invertible on S, the following sequence of sheaves is exact :*

$$0 \to \mathcal{O}_Z^* \to \mathcal{O}_R^* \xrightarrow{d\log} \Omega_R^1/\mathscr{I}_{R(-1)}\Omega_R^1 \to 0,$$

in particular, the complexes $(\mathcal{O}_Z^* \to 0)$ *and* $(\mathcal{O}_R^* \xrightarrow{d\log} \Omega_R^1/\mathscr{I}_{R(-1)}\Omega_R^1)$ *are quasi-isomorphic.*

Proof. — Since $\mathscr{I}_Z$ is locally principal it suffices to prove the following lemma.

LEMMA 2.3. — *Suppose B is an A algebra and $T \in B$ such that* Spec (B) *is smooth over* Spec (A) *of relative dimension one and* Spec $(B/(T))$ *is étale over A. Suppose $n!$ is invertible in A. Then*

(i) *The subring of $C =: B/(T^{n+1})$ consisting of elements b such that $d_{C/A} b \equiv 0 \bmod T^n\Omega_{C/A}^1$ maps isomorphically onto $B/(T)$ via the natural map.*

(ii) *If $\nabla : C \to \Omega_{C/A}^1$ is a connection, there exists a unique element r of C such that $r \equiv 1 \bmod (T)$ and $\nabla r \equiv 0 \bmod T^n\Omega_{C/A}^1$.*

Proof. — Let $C_i = B/(T^i)$. Since Spec C_1 is étale over Spec A, there exists a unique section from C_1 into C_{n+1} of $C_{n+1} \to C_1$ (see [G] Exp. III Cor. 3.2). Using the fact that B is smooth over A, we deduce that $C_1[x]/(x^{n+1})$ is isomorphic to C_{n+1} (via the map which takes x to T) (see [G] Exposé II Cor. 4.16) and we identify these two rings.

Suppose $g(x) = \sum_{i=0}^{n} r_i x^i \in C_{n+1}$ where $r_i \in C_1$ is such that $d_{C_{n+1}/A}\, g(x) = 0$. Then $r_i = 0$ for $1 \leqslant i \leqslant n$ since $n!$ is invertible in A and hence in C_1. This proves (i).

Suppose the hypotheses of (ii) and let $\nabla 1 = \omega \in \Omega_{C_{n+1}/A}^1 = C_{n+1}\, dx$. Since $n!$ is invertible we can find a polynomial $f(x)$ such that $f(0) = 0$ and $df = -\omega$ modulo $T^n\Omega^1_{C_{n+1}/A}$. Now let

$$r(x) = 1 + \sum_{i=1}^{n} f(x)^i/i\,!.$$

Then $r \in C^*_{n+1}$ and $\nabla r = dr + r\omega \equiv rdf + r\omega \equiv 0$ modulo $x^n\Omega^1_{C_{n+1}/A}$. Hence, r satisfies the requirements of (ii) of the lemma. If r' is another solution, then

$$d_{C_{n+1}/A} \log(r'/r) \equiv 0 \text{ modulo } x^n\Omega^1_{C_{n+1}/A}.$$

It follows that $d_{C_{n+1}/A}(r'/r) \equiv 0$ modulo $x^n\Omega^1_{C_{n+1}/A}$. If we represent r'/r as a polynomial of degree n in x, we see that, by part (i), this implies $r'/r \in C_1$. However, $r'/r \equiv 1$ modulo (x) and hence $r' = r$. $\square$

Now $\mathscr{V}^{\#}_{X,R}(S')$ is isomorphic to the subgroup of $\mathscr{U}_R(S')$ consisting of elements represented by sections s of $\pi_*(\mathscr{O}_{R_{S'}})^*$ such that the image of $d_{R/S}(S)$ in $\mathscr{I}_{R(-1)}\Omega^1_{R_{S'}/S'}$ equals zero. We have a natural map from R to Z using the unique infinitesimal lifting property of étale morphisms (see Cor. 2 of Exp. III [G]). Because $\Omega^1_{Z/S} = 0$, this map induces a map from $\mathscr{U}_Z(S')$ into $\mathscr{V}^{\#}_{X,R}(S')$. Also, $\mathscr{V}^{\#}_{X,R}(S')$ is the kernel of the map from $\mathscr{P}^{\#}_{X,R}(S')$ to $\mathscr{P}\!\mathit{ic}^{\#}_X(S')$.

Suppose from now on that n is a positive integer such that $\mathscr{I}_R \supseteq (\mathscr{I}_Z)^{n+1}$.

Taking hypercohomology of the exact sequence

$$0 \to \mathscr{C}_R \to \Omega^*_X \to (\mathscr{O}^*_R \xrightarrow{d\log} \Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R)$$

and applying Lemma 2.2 we obtain

Proposition 2.4. — *Suppose $n\,!$ is invertible on S. Then the sequence*

$$0 \to \mathscr{U}_Z \to \mathscr{P}^{\#}_{X,R} \to \mathscr{P}\!\mathit{ic}^{\#}_X \to 0$$

is exact and $\mathscr{V}^{\#}_{X,R}$ is isomorphic to $\mathscr{U}_Z$.

Corollary 2.4.1. — *If $n\,!$ is invertible, the sequence*

$$0 \to \mathscr{V}^{\#}_{X,R} \to \mathscr{U}^{\#}_{X,R} \to \mathscr{W}_X \to 0$$

is exact and splits canonically.

Proof. – The exactness follows from Proposition 2.4. The splitting follows from the fact that the composition $\mathscr{V}^{\#}_{X,R} \to \mathscr{U}^{\#}_{X,R} \to \mathscr{U}_Z$ is an isomorphism where $\mathscr{U}^{\#}_{X,R} \to \mathscr{U}_Z$ is the natural map. □

Remark. – When $n!$ is invertible, it follows that $\mathscr{P}^{\#}_{X,R}$ is the fiber product of $\mathscr{P}ic^{\#}_X$ and $\mathscr{P}_{X,Z}$ over $\mathscr{P}ic_X$ and so $(\mathscr{P}^{\#}_{X,R})^0$ is representable and depends only on Z.

There is an exact sequence of sheaves on the $fppf$ site over S,

$$0 \to \pi_*((1+\mathscr{I}_Z)/(1+\mathscr{I}_Y)) \to \mathscr{U}_Y \to \mathscr{U}_Z \to 0.$$

This sequence canonically splits because Z is étale over S. Moreover, if $n!$ is invertible the sheaf of groups $(1+\mathscr{I}_Z)/(1+\mathscr{I}_Y)$ is isomorphic to $\mathscr{I}_Z/\mathscr{I}_Y$ via the map

$$1 + a \to - \sum_{k=1}^{n} (-a)^k/k \bmod \mathscr{I}_Y$$

for a section a of $\mathscr{I}_Z$. Hence, in this case, $\mathscr{U}_Y/\mathscr{U}_Z = \pi_*((1+\mathscr{I}_Z)/(1+\mathscr{I}_Y))$ is naturally isomorphic to the vector group $V(\pi_*(\mathscr{I}_Z/\mathscr{I}_Y))$ and so $\mathscr{P}^0_{X,Y}/\mathscr{U}_Z$ is a vectorial extension of $\mathscr{P}ic^0_X$.

THEOREM 2.5. – *Suppose $n!$ is invertible on S. Then the canonical map of vectorial extensions from $(\mathscr{P}ic^{\#}_X)^0$ to $\mathscr{P}^0_{X,Y}/\mathscr{U}_Z$ is equal to the composition of the inverse of the isomorphism $(\mathscr{P}^{\#}_{X,R})^0/\mathscr{V}^{\#}_{X,R} \to (\mathscr{P}ic^{\#}_X)^0$ described in Proposition 2.4 and the natural map $(\mathscr{P}^{\#}_{X,R})^0/\mathscr{V}^{\#}_{X,R} \to \mathscr{P}^0_{X,Y}/\mathscr{U}_Z$.*

Proof. – It follows from the surjectivity of the map from $(\mathscr{P}ic^{\#}_X)^0$ to $\mathscr{P}ic^0_X$ (see [MaMe]) that the homomorphism from $(\mathscr{P}^{\#}_{X,R})^0/\mathscr{V}^{\#}_{X,R}$ to $\mathscr{P}^0_{X,Y}/\mathscr{U}_Z$ is the pushout of the homomorphism from $\mathscr{U}^{\#}_{X,R})^0/\mathscr{V}^{\#}_{X,R}$ to $\mathscr{U}_Y/\mathscr{U}_Z$ which is a map of vector groups. On the other hand the isomorphism $(\mathscr{P}ic^{\#}_X)^0 \to (\mathscr{P}^{\#}_{X,R})^0/\mathscr{V}^{\#}_{X,R}$ is the pushout of the homomorphism from $\mathscr{W}_X$ to $\mathscr{U}_R/\mathscr{U}_Z$ which makes the following diagram commute.

$$\begin{tikzcd} \mathscr{W}_X \arrow[rr] \arrow[dr] & & \pi_*(\Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R) \\ & \mathscr{U}_R/\mathscr{U}_Z \arrow[ur, "d\log"'] & \end{tikzcd} \tag{2}$$

Since $d\log$ and $\mathscr{W}_X \to \pi_*(\Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R)$ are homomorphisms of vector groups and $d\log$ is an injection by Lemma 2.2, the theorem follows from the universal property of $(\mathscr{P}ic^{\#}_X)(S'))^0$. □

3. Non-special divisors of degree g.

Let notation be as in the previous section, so that, in particular, R is a closed subscheme of X containing a closed subscheme Z of X étale over S such that $\mathscr{I}_R$ is locally a power of $\mathscr{I}_z$ and n is a positive integer such that $\mathscr{I}_R \supseteq (\mathscr{I}_Z)^{n+1}$. Then, $\pi_*(\Omega^1_X)$, $\pi_*(\Omega^1_X/\mathscr{I}_{R(-1)}\Omega^1_R)$ and $\pi_*(\mathscr{O}_R)$ are locally free on S, rank $(\pi_*(\Omega^1_X)) = g$ and rank $(\pi_*(\Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R)) = \text{rank}\,(\pi_*(\mathscr{O}_{R(-1)}))$. (See the proof of Lemma 2.3.) Let w denote the natural map from $\pi_*(\Omega^1_X)$ to $\pi_*(\Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R)$.

THEOREM 3.1. — *Suppose that w is an isomorphism and $n\,!$ is invertible. Then the map from $(\mathscr{P}ic^{\#}_X)^0$ to $\mathscr{P}^0_{X,R}/\mathscr{U}_Z$ is an isomorphism.*

Proof. — To prove this map is an isomorphism, we only need to show that the corresponding map from $\mathscr{W}_X$ to $\mathscr{U}_R/\mathscr{U}_Z$ is an isomorphism using the commutativity of (1) as $(\mathscr{P}ic^{\#}_X)^0$ surjects onto $\mathscr{P}ic^0_X$. But this follows immediately from the commutative diagram (2) as the map from $\mathscr{U}_R/\mathscr{U}_Z$ to $\pi_*(\Omega^1_R/\mathscr{I}_{R(-1)}\Omega^1_R)$ is an isomorphism by Lemma 2.2. □

Remark. — If S is the spectrum of a field, the condition that w is an isomorphism is equivalent to the divisor $R(-1)$ being non-special of degree g.

Suppose that S' is an S scheme and let $X' = X_{S'}$, $R' = R_{S'}$, $Z' = Z_{S'}$, etc. Let $x \in U(S')$ be an S' valued point of $U =: X - R = X - Z$. Let x also denote the corresponding divisor on X' and $\mathscr{L}_x$ denote the invertible sheaf on X' corresponding to x. Then there exists an isomorphism α_x from $\mathscr{O}_{X'}$ to $\mathscr{L}_x$ on $X' - x$ which is unique up to multiplication by a global section of $\mathscr{O}^*_{S'}$. Since $x \in (X-R)(S')$, $\rho' : R' \to X'$ factors through $U_{S'}$. Hence,

$$\rho'^*\alpha_x : \mathscr{O}_{R'} \to \rho'^*\mathscr{L}_x$$

is an isomorphism well-defined up to multiplication by a global section of $\mathscr{O}^*_{S'}$ and so the isomorphism class of $(\mathscr{L}_x, \rho'^*\alpha_x)$ is a well defined element of $\mathscr{P}_{X,R}(S')$. Moreover, this map from $\text{Mor}_S(*,U)$ to $\mathscr{P}_{X,R}$ is a natural transformation of functors and hence a morphism j_R from $X - R$ into $\mathscr{P}_{X,R}$. In fact, j_R may also be described more simply as the point represented by $(\mathscr{L}_y, \rho'^*\alpha_y)$ of $\mathscr{P}_{X,R}(U)$ where y is the inclusion of U into X.

Let $\mathscr{I}nv(\Omega^1_{\mathscr{P}_{X,R}})$ denote the sheaf of invariant differentials on $\mathscr{P}_{X,R}$ over S. Our hypotheses imply that $\mathscr{I}_R$ is an invertible sheaf. Let $\mathscr{I}_R^{-1}$ denote its inverse and $\Omega^1_X(R) = \Omega^1_X \otimes \mathscr{I}_R^{-1}$.

THEOREM 3.2. – *The sheaf $j_R{}^*\mathscr{I}nv(\Omega^1_{\mathscr{P}_{X,R}})$ equals the sheaf $\pi_*(\Omega^1_X(R))$ regarding both as subsheaves of $\pi_*(\Omega^1_U)$.*

Proof. – Both $\mathscr{I}nv(\Omega^1_{\mathscr{P}_{X,R}})$ and $\pi_*(\Omega^1_X(R)$ are locally free sheaves and hence commute with arbitrary base change. Moreover since both $\pi_*(\Omega^1_X(R))$ and $j_R{}^*\mathscr{I}nv(\Omega^1_{\mathscr{P}_{X,R}})$ have finite rank, they may be naturally identified with subsheaves of $\pi_*(\Omega^1_X(mZ))$ for some positive integer m. Hence the result, in general, will follow from the result in the case in which S is the spectrum of a field. When S is the spectrum of an algebraically closed field this is Prop. 5, Chapt. V, No. 10 of [S]. The result for an arbitrary field may be deduced from this either by Galois descent or by checking that all the arguments in [S] carry over. □

Suppose that y is an S valued point of U. Then the morphism $j_{R,y} =: j_R - j_R(y)$ is a morphism from U into $\mathscr{P}^0_{X,R}$. We deduce immediately from the previous theorem that

COROLLARY 3.2.1. – *The sheaf $j_{R,y}{}^*\mathscr{I}nv(\Omega^1_{\mathscr{P}^0_{X,R}})$ equals the sheaf $\pi_*(\Omega^1_X(R))$ regarding both as subsheaves of $\pi_*(\Omega^1_U)$.*

In fact, if $x \in U(S')$, $j_{R,y}(x)$ is represented by the pair $(\mathscr{L}_{x,y}, \rho'^*(\alpha_x \otimes \alpha_{y'}^{-1}))$ where $\mathscr{L}_{x,y}$ is the invertible sheaf $\mathscr{L}_x \otimes (\mathscr{L}_{y'}^{-1})$ on X'. We note that $\alpha_x \otimes \alpha_{y'}^{-1}$ is an isomorphism from $\mathcal{O}_{X'}$ onto $\mathscr{L}_x \otimes (\mathscr{L}_{y'}^{-1})$ on $X' - \{x, y'\}$ well defined up to multiplication by a global section of $\mathcal{O}^*_{S'}$.

Suppose now w is an isomorphism, $n\,!$ is invertible and $y \in U(S)$. Let $y' = y_{S'}$. Let $v_{R,y}$ denote the morphism from U to $(\mathscr{P}ic^\#_X)^0$ equal to the composition of $j_{R,y}$, the projection from $\mathscr{P}^0_{X,R}$ to $\mathscr{P}^0_{X,R}/\mathscr{U}_z$ and the isomorphism from $\mathscr{P}^0_{X,R}/\mathscr{U}_z$ to $(\mathscr{P}ic^\#_X)^0$ of Theorem 3.1. Then it follows from the definitions that

PROPOSITION 3.3. – *If $x \in U(S')$, $v_{R,y}(x) \in (\mathscr{P}ic^\#_X)^0(S')$ is represented by the pair $(\mathscr{L}_{x,y}, \nabla_{R,x,y})$ where $\mathscr{L}_{x,y}$ is the sheaf on $U_{S'}$ defined above and $\nabla_{R,x,y}$ is the unique connection on $\mathscr{L}_{x,y}$ such that*

$$\tau_{R'}{}^*\nabla_{R,x,y}(\alpha_x \otimes \alpha_{y'}^{-1}(1)) \equiv 0 \bmod \mathscr{I}_{R(-1)}\Omega^1_{R'/S'} \otimes \mathscr{L}_{x,y}.$$

Suppose T is a Zariski open of S and W is a Zariski open of X_T. If $\omega \in \Omega^1_X(W)$ we say ω is a differential of the second kind on W if there exists a Zariski open covering $\mathscr{C}$ of X_T and elements $\omega_U \in \Omega^1_X(U)$ and $f_{U,V} \in \mathcal{O}_{U \cap V}$ for any U and V in $\mathscr{C}$ such that $\omega_U - \omega_V = d_{X/S} f_{U,V}$ on $U \cap V$ and such that $W \in \mathscr{C}$ and $\omega_W = \omega$. In short, we say ω is of the second kind on W, if there exists a one-hypercocycle $(\{\omega_U\}, \{f_{U,V}\})$ for Ω^1_X above T with respect to a cover $\mathscr{C}$ such that $W \in \mathscr{C}$ and $\omega_W = \omega$. (In this context, we think of second kind as locally holomorphic plus exact.) Then we have,

PROPOSITION 3.4. – *The sheaf* $v_{R,y}{}^*\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,R}/\mathscr{U}_Z})$ *equals the subsheaf* $\mathscr{S}$ *of* $\pi_*(\Omega^1_X(R))$ *whose sections are differentials of the second kind.*

Proof. – Let P denote the S-scheme $\mathscr{P}^0_{X,R}/\mathscr{U}_Z$. The proof of [AD] Theorem 1.2.2 (*i*), may be adapted to show that $v_{R,y}{}^*\mathscr{I}nv(\Omega^1_P) \subseteq \mathscr{S}$.

Let $f: \mathscr{P}^0_{X,R} \to P$ and $g: \mathscr{P}^0_{X,R} \to \mathscr{P}^0_{X,Z}$ denote the natural maps. Then

$$\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,R}}) = f^*\,\mathscr{I}nv\,(\Omega^1_P) + g^*\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,Z}})$$

and so by Theorem 3.2

$$\pi_*(\Omega^1_X(R)) = v_{R,y}{}^*\mathscr{I}nv\,(\Omega^1_P) + j_{Z,y}{}^*\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,Z}})$$

since $v_{R,y} = j_{R,y} \circ f$ and $j_{Z,y} = j_{R,y} \circ g$. Hence

$$\mathscr{S} = v_{R,y}{}^*\mathscr{I}nv\,(\Omega^1_P) + \mathscr{S} \cap (j_{Z,y}{}^*\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,Z}})).$$

Theorem 3.2 also implies that

$$j_{Z,y}{}^*\mathscr{I}nv\,(\Omega^1_{\mathscr{P}^0_{X,Z}}) = \pi_*(\Omega^1_X(Z)).$$

Since $\mathscr{S} \cap \pi_*(\Omega^1_X(Z)) = \pi_*(\Omega^1_X) \subseteq v_{R,y}{}^*\mathscr{I}nv(\Omega^1_P)$, the proposition follows. □

4. Over an algebraically closed field again.

In this section we will deduce the propositions stated in Section 1 from the results of the preceding two sections.

Let notation be as in Section 1. In particular, X is a curve over an algebraically closed field K and D is an effective divisor on X.

Considering D as a closed subscheme of X, let $\rho : D \to X$ denote the corresponding closed immersion. Suppose (E,η) is a differential pair on X of the third kind. Let $U = X - |E|$ and V be a Zariski open containing $|E|$, $g \in K(X)^*$ such that $(g)|_V = E$. Let $(\mathscr{L},\nabla)$ be the invertible sheaf with connection defined by the data: s_U generates $\mathscr{L}(U)$, s_V generates $\mathscr{L}(V)$, $s_U = gs_V$, $\nabla s_U = \eta \otimes s_U$ and $\nabla s_V = (\eta - d\log g) \otimes s_V$. Then (E,S) corresponds to the point of $G(K) = \mathscr{P}^0_{X/K}(K)$ represented by $(\mathscr{L},\nabla)$. Now suppose $f \in K(X)^*$ such that $(\eta - df/f)_0 \geqslant D(-1)$, then

$$\nabla(f^{-1}s_U) \equiv 0 \text{ modulo } \mathscr{I}_{D(-1)}\Omega^1_{D/K} \otimes \mathscr{L}.$$

Hence, the image of (E,η) in $G_D(K)$ is represented by the pair $(\mathscr{L},\alpha)$ where α is the isomorphism from $\mathscr{O}_D$ to $\mathscr{L}_D$ which sends $1 \in \mathscr{O}_D(D \cap U)$ to $\rho^*(f^{-1}s_U)$. Since the pair $(\mathscr{L},\alpha)$ corresponds to the class of the divisor $E - (f)$, this together with Theorem 2.3 proves Theorem 1.1.

Proposition 1.2 follows from Theorem 3.1. Finally, let $(P-Q,\omega_{D,P,Q})$ be the differential pair of the third kind defined after the statement of Proposition 1.2 above. This pair corresponds to a pair $(\mathscr{L},\nabla)$ defined by data s_U, s_V and g as above. In particular,

$$\rho^*(\nabla s_U) = \rho^*(\omega_{D,P,Q}) \otimes \rho^*(s_U) \equiv 0 \text{ modulo } \mathscr{I}_{D(-1)}\Omega^1_{D/K} \otimes \mathscr{L}.$$

Hence, $(\mathscr{L},\nabla) = v_{D,Q}(P)$. This combined with Proposition 3.4 yields Proposition 1.3.

5. Differentials and extensions of the Jacobian by $\mathbf{G}_m$.

Suppose the characteristic of K equals zero. Let ω be a differential of the third kind on X. Then, as we know, ω modulo logarithmic differentials corresponds to a point in $G(K)$. We also know, [C] Theorem 2.2.1, that points in $G(K)$ correspond to isomorphism classes of pairs (E,ν) where, E is an extension of J by $\mathbf{G}_m$ and ν is an invariant differential on E whose pullback to $\mathbf{G}_m$ is dT/T. We call such differentials on E, *normal invariant differentials*. The following result makes the relationship of ω so (E,ν) more explicit.

Let $\alpha : X \to A$ be an Albanese morphism.

PROPOSITION 5.1. — *There is a pair* (E,ν), *unique up to isomorphism, consisting of an extension* E *of* J *by* $\mathbf{G}_m$ *and a normal invariant differential* ν *on* E *with the property that there exists a morphism* $w : X - |D| \to E$ *such that* $\pi \circ w = \alpha$ *and* $w^*\nu = \omega$. *The point on* G *corresponding to* ω *is equal to that corresponding to* (E,ν).

Proof. — Suppose $(\mathscr{L},\nabla)$ is a pair consisting of a connection ∇ on an invertible sheaf $\mathscr{L}$ on X. Then $(\mathscr{L},\nabla)$ is the pullback via α of a pair, unique up to isomorphism, $(\mathscr{L}^*,\nabla^*)$ consisting of an integrable connection ∇^* on an invertible sheaf $\mathscr{L}^*$ on J (char $K=0$). Let (T,ξ) denote the pair consisting of a $\mathbf{G}_m$ torseur T on X and one-form ξ invariant under the action of $\mathbf{G}_m$ with residue one corresponding to $(\mathscr{L},\nabla)$ via [C1] Proposition 0.2.1, and if (E,ν) denotes the $\mathbf{G}_m$-torseur with normal invariant differential corresponding to $(\mathscr{L}^*,\nabla^*)$ via [C1] Theorem 2.2.1, then (T,ξ) is naturally isomorphic to the fiber product of X and (E,ν) over J. In other words, if $g : T \to E$ denotes the natural morphism then we have a Cartesian square

$$\begin{array}{ccc} T & \to & E \\ \downarrow & & \downarrow \\ X & \to & J \end{array}$$

and $g^*\nu = \xi$,

Suppose U is a Zariski open of X. It follows that the map $u \to g \circ w$ gives a one-to-one correspondence between morphisms from $U \to E$ such that $\pi \circ w = \alpha$ and sections $u : U \to T$. In particular, if $U = X - |\mathrm{Res}\,(\omega)|$ there exists a morphism $w : U \to E$ such that $\pi \circ w = \alpha$ and $w^*\nu = \omega$ iff there exists a section $u : U \to T$ such that $u^*\xi = \omega$.

The isomorphism class of the pair $(\mathscr{L},\nabla)$, consisting of an invertible sheaf on X together with a connection, corresponds to ω modulo logarithmic differentials iff there exists a generator s of $\mathscr{L}(U)$ such that $\nabla s = \omega \otimes s$. Moreover, using the characterization (0.2) of [C1], such an $(\mathscr{L},\nabla)$ corresponds to the $\mathbf{G}_m$-torseur (T,ξ) on X iff there exists a section $u : U \to T$ such that $u^*\xi = \omega$. We conclude that the point of $G(K)$ represented by ω is also represented by the pair (E,ν) iff there exists a morphism $w : U \to E$ such that $\pi \circ w = \alpha$ and $w^*\nu = \omega$. □

BIBLIOGRAPHY

[C1] R. COLEMAN, The Universal Vectorial Bi-extension and p-adic Heights, to appear in Inventiones.

[C2] R. COLEMAN, Duality for the de Rham Cohomology of Abelian Schemes, to appear.

[CG] R. COLEMAN, and B. GROSS, p-adic Heights on Curves, Advances in Math., 17 (1989), 73-81.

[C-C] C. CONTOU-CARRÈRE, La jacobienne généralisée d'une courbe relative, C.R. Acad. Sci., Paris, t. 289 (279), 203-206.

[G] A. GROTHENDIECK, Revêtement Étale et Groupe Fondamental (SGA I) SLN 224 (1971).

[MaMe] B. MAZUR and W. MESSING, Universal Extensions and One Dimensional Crystalline Cohomology, Springer Lecture Notes, 370, 1974.

[MaT] B. MAZUR and J. TATE, Canonical Height Pairings via Bi-extensions, Arithmetic and Geometry, Vol. I, Birkhauser, (1983), 195-237.

[O] H. ONSIPER, Rational Maps and Albanese Schemes, Thesis, University of California at Berkeley, (1984).

[S] J.-P. SERRE, Groupes Algébriques et Corps de Classes, Hermann, 1959.

Manuscrit reçu le 25 août 1990.

Robert F. COLEMAN,
Department of Mathematics
University of California
Berkeley CA 94720 (USA).