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OPEN BOOK STRUCTURES AND UNICITY OF MINIMAL SUBMANIFOLDS

by R. HARDT(*) and H. ROSENBERG(**)

0. Introduction.

Suppose Γ is a compact codimension 2 submanifold of a compact orientable smooth Riemannian manifold N. In this note, we consider some ways that a foliation of $N - \Gamma$ by minimal hypersurfaces controls the uniqueness of minimal hypersurfaces in $M - \Gamma$. Our discussion is motivated by the examples of certain disks and annuli in $\mathbb{S}^3$. These are described below in §1.2, 1.3, 1.4 and 2.2. We are also reminded of the beautiful theorem of Shiffman [Sh] concerning minimal annuli in $\mathbb{R}^3$ bounded by two convex curves (respectively, circles) that lie in parallel planes. His conclusion is that all intermediate parallel planar sections are also convex curves (respectively, circles).

Our argument in §1 combines the maximum principle and the Hopf boundary point lemma with various topological conditions. Somewhat analogous discussions occur (without the topological conditions) in [HS], pp. 478–479, and (without boundary considerations) in [So], Lemma 1. In §2 we examine, in an analytic 3–manifold, the intersection between a fixed immersed minimal surface and an open book structure whose leaves are minimal surfaces and whose binding consists of analytic curves bounding

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the fixed minimal surface. The behavior away from the binding has been studied by J. Hass [H] §2. F. Morgan [M] gives conditions under which area-minimizing embeddings form open book structures. Some open problems suggested by our work are given in §3.

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1. Embedded minimal hypersurfaces and open books.

1.1. *Open books.* — Suppose N is a compact connected $n+1$ dimensional orientable smooth Riemannian manifold, and Γ is a compact $n-1$ dimensional smooth submanifold of N. A fibration $\mathcal{L}$ of $N-\Gamma$ into smooth hypersurfaces is called an *open book structure* for N with *binding* Γ if Γ is covered by open sets U for which there is a smooth diffeomorphism mapping U onto $(\Gamma \cap \mathsf{U}) \times \{z \in \mathbb{C} : |z| < 1\}$ that sends $\{L \cap \mathsf{U} : L \in \mathcal{L}\}$ onto the product foliation

$$\{(\Gamma \cap \mathsf{U}) \times \{\lambda v : 0 < \lambda < 1\} : v \in \mathbb{S}^1\}.$$

It follows that each leaf L of $\mathcal{L}$ is the interior of a compact manifold with boundary. Moreover, the orientability of N implies that the foliation $\mathcal{L}$ is transversally orientable.

1.2. THEOREM. — *Suppose that the closure of each leaf of $\mathcal{L}$ is an embedded minimal hypersurface with boundary Γ and $\varphi : M \to N$ is a minimal immersion of a compact connected orientable n dimensional manifold with boundary such that $\varphi^{-1}(\Gamma) = \partial M$. If either*

(1) *M is simply connected, or*

(2) *$H_n(N, \mathbb{R}) = 0$ and φ is an embedding,*

then M must be the closure of a leaf of $\mathcal{L}$.

Proof. — For each $L \in \mathcal{L}$, $\operatorname{Clos} L = L \cup \Gamma$. Moreover, each point a of $N - \Gamma$ has a neighborhood in N whose intersection with the leaf through a is a single embedded disk. Thus by the connectedness of $N - \Gamma$, the quotient space $(N-\Gamma)/\mathcal{L}$ has naturally the structure of a smooth circle with the quotient map being a smooth submersion. Let ω be the pull-back under the quotient map of an orienting 1 form of this circle. Then $d\omega = 0$ in $N - \Gamma$, and hence $d(\varphi^*\omega) = 0$ in $M - \Gamma$. Moreover, by the maximality of the rank of φ along ∂M, $\varphi^*\omega$ extends up to the boundary (as in the argument of [HS] p. 478) to a smooth 1 form on all of M.

In case M is simply connected, the closed form $\varphi^*\omega$ is exact, that is, $\varphi^*\omega = d\theta$ for some real-valued smooth function θ on M. Using θ, we can choose a point $a \in M$ where θ attains a maximum. Near a, one leaf of $\mathcal{L}$ will lie entirely on one side of M, with tangential contact at a. Then, (as in [HS] p. 479), the maximum principle, in case $a \in M - \partial M$, or the Hopf boundary point lemma [GT] 3.2, in case $a \in \partial M$, implies that θ is identically equal to $\theta(a)$. Thus, $M - \Gamma$ is completely contained in a single leaf L of $\mathcal{L}$. Being open, as well as closed, relative to L, $M - \Gamma$ actually equals L.

To complete the proof we will show that $\varphi^*\omega$ is also exact for non-simply connected M provided that φ is embedded and that $H_n(N, \mathbb{R}) = 0$. For this we assume, for notational convenience, that φ is the inclusion map and observe that it now suffices to verify that

$$\int_\beta \omega = 0 \quad \text{for any smooth closed curve } \beta \text{ in } M - \Gamma.$$

Since M is orientable and embedded, $\int_\beta \omega = \int_\gamma \omega$ for some smooth closed curve γ in $N - M$; here, γ may be found by lifting β a small positive distance off of M in a normal direction, obtained from the orientability of M. We may also choose γ to be transverse to all but finitely many leaves of $\mathcal{L}$. Noting that γ does not intersect M, we find, after orienting M, that the total intersection numbers of the corresponding oriented chains satisfy

$$\beta \sharp M = \gamma \sharp M = 0.$$

Choose a leaf L which intersects γ transversely and orient it so that $\partial L = \partial M$. Since $H_n(N, \mathbb{R}) = 0$, $L - M = \partial E$ for some $n+1$ chain E in N, and so

$$\gamma \sharp L = (\gamma \sharp M) + (\gamma \sharp \partial E) = 0 + 0.$$

Now observe that

$$\int_\beta \omega = \int_\gamma \omega = \gamma \sharp L = 0. \qquad \square$$

1.3. *Embedded Minimal Surfaces in* $\mathbb{S}^3$. — In

$$\mathbb{S}^3 = \{(y, z) \in \mathbb{C}^2 : |y|^2 + |z|^2 = 1\},$$

$$C_1 = \mathbb{S}^3 \cap \{y = 0\} \quad \text{and} \quad C_2 = \mathbb{S}^3 \cap \{z = 0\}$$

are great circles that are linked and constant spherical distance $\pi/2$ apart. For $\theta \in [0, 2\pi)$

$$D_\theta = \mathbb{S}^3 \cap \{\arg(y) = \theta\}$$

is a (totally geodesic) embedded minimal disk with boundary C_1. Applying Theorem 1.2 with $N = \mathbb{S}^3$, $\Gamma = C_1$, and $\mathcal{L} = \{D_\theta : \theta \in [0, 2\pi)\}$, we find (as in [HS] p. 441) that :

Any connected embedded orientable minimal surface in $\mathbb{S}^3$ that has boundary C_1 must be one of the disks D_θ for some $\theta \in [0, 2\pi)$.

Also, for $\theta \in [0, 2\pi)$,

$$A_\theta = \mathbb{S}^3 \cap \{\arg(y\bar{z}) = \theta\}$$

is an embedded minimal annulus with boundary $C_1 \cup C_2$. Reflection (as in [La]) of A_θ about either boundary component gives the Clifford torus

$$\overline{A_\theta \cup A_{\theta+\pi}} = A_\theta \cup A_{\theta+\pi} \cup C_1 \cup C_2 \quad \text{for each } \theta \in [0, \pi).$$

Applying Theorem 1.2 with $N = \mathbb{S}^3$, $\Gamma = C_1 \cup C_2$, and $\mathcal{L} = \{A_\theta : \theta \in [0, 2\pi)\}$, we similarly find that :

Any connected embedded orientable minimal surface in $\mathbb{S}^3$ that has boundary $C_1 \cup C_2$ must be one of the annuli A_θ for some $\theta \in [0, 2\pi)$.

Recalling that the proof of Theorem 1.1 required only that the closed one form $\varphi^*\omega$ on M be exact we may replace the assumption on the ambient manifold N by an assumption on M.

2. Immersed minimal surfaces and open books.

In this section we obtain, for $n = 2$, some results allowing minimal surfaces that are a priori only immersed and open book structures whose leaves may have higher boundary multiplicity. We assume that N is a compact orientable real analytic Riemannian 3−manifold, that Γ is a finite disjoint union of analytic Jordan curves in N, and that $\mathcal{L}$ is a transversally orientable open book structure on N with binding Γ and with leaves that are embedded minimal surfaces.

As in §1, for any point $a \in N - \Gamma$, the leaf through a intersects some neighborhood of a in a single disk. It also implies that for any point $b \in \Gamma$ each leaf intersects some neighborhood of b in a finite number of disjoint half disks.

We will now consider a smooth immersion φ from a connected compact orientable bordered surface M into N for which $\varphi^{-1}(\Gamma) = \partial M$. Such a φ induces a 2 chain $[\varphi]$ $(= \varphi_* M)$ in N.

Suppose U is a tubular neighborhood of Γ in N. For almost all positive numbers $\varepsilon < \text{dist}(\Gamma, \partial \mathsf{U})$, the slice

$$[\varphi]_\varepsilon = \partial([\varphi] \llcorner \{x : \text{dist}(x, \Gamma) < \varepsilon\})$$

is a finite sum of oriented smooth curves. Here the 2 chain $[\varphi] \llcorner A$ denotes the restriction of $[\varphi]$ to A in the sense that

$$([\varphi] \llcorner A)(\alpha) = [\varphi](\alpha \cdot \text{char}.fcn(A))$$

for any differential 2 form α on N. For almost all $0 < \delta < \varepsilon$, the 1 chains $[\varphi]_\delta$ and $[\varphi]_\varepsilon$ are homologous in $\mathsf{U} - \Gamma$ because

$$[\varphi]_\varepsilon - [\varphi]_\delta = \partial([\varphi] \llcorner \{x : \delta \leq \text{dist}(x, \Gamma) < \varepsilon\}).$$

More generally, we say any two such 2 chains $[\varphi]$ and $[\psi]$ are *link homologous at* Γ if $[\varphi]_\varepsilon$ and $[\psi]_\varepsilon$ are homologous in $\mathsf{U} - \Gamma$ for some positive number $\varepsilon < \text{dist}(\Gamma, \partial U)$. Note that, by orienting N and Γ, the link homology class at Γ may be described by associating, to each component of Γ, a pair of integers, one being a boundary multiplicity and the other a linking multiplicity (see *e.g.* §2.2 below) corresponding to generators of the homology of a torus link of the component.

2.1. THEOREM. — *Suppose M is an annulus and $N, \Gamma, \mathcal{L}$ and $\varphi : M \to N$ are as above with φ a minimal immersion that maps distinct components of ∂M onto distinct components of Γ. If φ is link homologous at Γ to an oriented leaf of $\mathcal{L}$ then $\varphi(M) - \Gamma$ is a leaf of $\mathcal{L}$.*

Proof. — Let γ be a loop on M that generates the fundamental group and lies arbitrarily close to a component of ∂M.

Then the link homology condition implies that, in N, the corresponding 1−chain $\varphi * \gamma$ is homologous to a 1−chain supported in a single leaf L of $\mathcal{L}$. Thus, in the notation of the proof of 1.4,

$$\int_\gamma \varphi * \omega = \int_{\varphi * \gamma} \omega = 0,$$

which implies as before, that $\varphi(M - \partial M)$ lies entirely in a single leaf L of $\mathcal{L}$. The image $\varphi(M - \partial M)$ is open in L because φ is an immersion and closed in L because $\varphi^{-1}(\Gamma) = \partial M$. Thus $\varphi(M) - \Gamma = L$. □

2.2. *Immersed disks in* $\mathbb{S}^3$. — Recall the notation of 1.2.

Any immersed minimal disk $\varphi : D \to \mathbb{S}^3$ *with* $\varphi^{-1}(C_1) = \partial D$ *must be* D_θ *or a multiple covering of* D_θ *for some* $\theta \in [0, 2\pi)$.

In fact, if $\varphi : D \to \mathbb{S}^3$ were such an immersion, then the closed form $\varphi^* dy$, (which is, as in the proof of 2.1, smooth up to ∂D), would be exact because $H^1(D, \mathbb{R}) = 0$. Then, as in the proof of 2.1, the image of φ would have to lie in some leaf D_θ.

2.3. *Some minimal annuli in* $\mathbb{S}^3$. — Next we consider, for each $\theta \in [0, 2\pi)$ and each pair, p, q of nonzero integers with $|p|$, $|q|$ relatively prime, the set

$$A_\theta^{p,q} = \mathbb{S}^3 \cap \{\arg(y^p \bar{z}^q) = \theta\}.$$

As in [L] one can easily verify that $A_\theta^{p,q} - (C_1 \cup C_2)$ is an (open) embedded minimal annulus. Moreover, by suitably orienting $A_\theta^{p,q}$, C_1 and C_2, we obtain a 2−chain $A_\theta^{p,q}$ satisfying

$$\partial A_\theta^{p,q} = qC_1 + pC_2.$$

Any immersed minimal annulus $\varphi : A \to \mathbb{S}^3$, *with* $\varphi^{-1}(C_1 \cup C_2) = \partial A$ *and with*

$$\varphi|\{a \in A : 0 < \operatorname{dist}(a, \partial A) < \varepsilon\},$$

being injective for some $\varepsilon > 0$, *must be embedded with image equal to* $A_\theta^{p,q}$ *for some* θ, p, q.

To see this, we may choose a small positive $\delta < \varepsilon$ so that each torus

$$T_i = \{x \in \mathbb{S}^3 : \operatorname{dist}(x, C_i) = \delta\}$$

intersects $\varphi(A)$ transversely in a simple closed curve γ_i for $i = 1, 2$. We fix a usual pair of generators for the homologies of the T_i with the first of each pair corresponding to a shortest geodesic on T_i. We find that γ_1 is homotopic in T_1 to a (p, q) geodesic, while γ_2 is homotopic in T_2 to a (q, p) geodesic, for some nonzero integers p, q with $|p|$, $|q|$ relatively prime. It also follows that φ is link homologous at $C_1 \cup C_2$ to $\pm \widetilde{A}_\theta^{p,q}$ for any $\theta \in [0, 2\pi)$. Thus we may apply Theorem 2.1 with

$$M = A\,, \quad N = \mathbb{S}^3\,, \quad \Gamma = C_1 \cup C_2\,, \quad \text{and } \mathcal{L} = \{A_\theta^{p,q} : \theta \in [0, 2\pi)\}. \quad \square$$

3. Remarks and questions.

1. The sets $A_\alpha^{2,1} \cup C_2$ are embedded minimal Möbius bands with boundary C_1 (and with waist C_2). This shows the necessity of the orientability assumption in Theorem 1.1. Are these, along with the disks D_θ of §1.2, the only embedded minimal surfaces in $\mathbb{S}^3$ with boundary C_1?

2. The minimal annuli $\Lambda_\theta^{1,q}$ of §2.3 are embedded in $\mathbb{S}^2 \times \mathbb{S}^1$. This shows the necessity of the assumption in Theorem 1.1 on the vanishing of either $H^1(M, \mathbb{R})$ or $H_n(N, \mathbb{R})$. Are these the only embedded minimal surfaces in $\mathbb{S}^3$ with boundary $\Gamma_1 \cup \Gamma_2$?

3. Is the near-boundary injectivity assumption necessary in §2.2 and §2.3? In particular, is there an oriented immersed minimal annulus in $\mathbb{S}^3$ whose (2−chain) boundary is $2C_1 + 2C_2$ other than $2\widetilde{A}_\theta^{2,2}$?

4. Are there analogues of 2.2 and 2.3 with Γ being two great circles in $\mathbb{S}^3$ that are a constant distance apart, other than $\pi/2$?

5. What complex polynomials give rise to open book structures of the sphere?

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