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MASLOV INDICES ON THE METAPLECTIC GROUP $Mp(n)$

by Maurice de GOSSON

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0. Introduction and results.

The symplectic group $Sp(n)$ has for every $q = 2, 3, \ldots, +\infty$ a unique (up to an isomorphism) covering group $Sp_q(n)$ of order q. It is known since André Weil [W], that the double cover $Sp_2(n)$ has a unitary representation in $L^2(\mathbb{R}^n)$, the metaplectic group $Mp(n)$. That group plays a very special role in the theory of asymptotic solutions to partial differential equations (see [L], [GS]).

Jean Leray has shown in his treatise "Lagrangian analysis and quantum mechanics" ([L]) that $Mp(n)$ is generated by a class of unitary operators in $L^2(\mathbb{R}^n)$ appearing as generalized Fourier transforms and associated to quadratic forms of the type

$$A(x,x') = \frac{1}{2}\langle Px, x\rangle - \langle Lx, x'\rangle + \frac{1}{2}\langle Qx', x'\rangle \tag{1}$$

P, L, Q being real matrices of order n, $P = {}^tP$, $Q = {}^tQ$, $\det(L) \neq 0$. A choice of $\arg\det(L) = m\pi$, $m \in \mathbb{Z}$, being made, J. Leray calls the class modulo 4 of the integer m the "Maslov index" of the unitary operator associated with the pair (A, m).

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We are going to show in this paper that J. Leray's Maslov index (or rather a slight variant of that index) can be extended to the whole of $Mp(n)$, and that the properties of this extended index yields very precise informations on the algebraic and topological structure of the metaplectic group $Mp(n)$.

This paper is structured as follows :

– in §1 we recall the main definitions and results in J. Leray's theory and prove a relation between the Maslov indices of products of unitary operators associated with quadratic forms given by (1) (proposition 4).

– in §2 we define and study the properties of the extended Maslov index.

Our main result is theorem 1 :

THEOREM.

1) *The Maslov index μ_0 is the only function $Mp(n) \to \mathbb{Z}/8\mathbb{Z}$ such that :*

(2) *the mapping $(S, \ell) \mapsto \mu_0(S) - \operatorname{sign}(s\ell_0, \ell_0, \ell)$ is locally constant on* $\{(S, \ell) \in Mp(n) \times \Lambda(n)\,;\, s\ell_0 \cap \ell = \ell \cap \ell_0 = \{0\}\}$ [$\Lambda(n)$ being the Lagrangian Grassmannian, $\ell_0 = \{0\} \times \mathbb{R}^n$, s the projection on the symplectic group $Sp(n)$ of $S \in Mp(n)$, and sign the class modulo 8 of the signature of a triple of elements of $\Lambda(n)$].

(3) $\mu_0(SS') = \mu_0(S) + \mu_0(S') + \operatorname{sign}(\ell_0, s\ell_0, ss'\ell_0)$ *for all S, S' in $Mp(n)$.*

2) *The Maslov index μ_0 has furthermore the following properties :*

$$\mu_0(S^{-1}) = -\mu_0(S)\ ;\ \mu_0(I) = \dot{0}\ . \tag{4}$$

$$\mu_0(-S) = \mu_0(S) + \dot{4} \tag{5}$$

($\dot{k}$ being the class modulo 8 of $k \in \mathbb{Z}$).

(6) $\mu_0(S)$ *and* $n - \dim(s\ell_0 \cap \ell_0)$ *have same images in $\mathbb{Z}/2\mathbb{Z}$.*

The key to the proof of that theorem is proposition 4, §1 which makes possible the definition of the Maslov index μ_0, together with the properties, due to Masaki Kashiwara, of the signature of a triple of elements of $\Lambda(n)$.

The Maslov index thus characterized is associated with a particular element ℓ_0 of $\Lambda(n)$;

– in §3 we show that it is actually possible to associate a "Maslov index" μ_ℓ to any choice of an element $\ell \in \Lambda(n)$ by the formula :

$$\mu_\ell(S) = \mu_0(S_0^{-1} S S_0)\ ,\ S \in Mp(n) \tag{7}$$

where $S_0 \in Mp(n)$ projects onto $s_0 \in Sp(n)$ such that $\ell = s_0\ell_0$; in fact that Maslov index μ_ℓ *only depends on* ℓ, and we have, if $(\ell, \ell') \in \Lambda(n) \times \Lambda(n)$:

$$\mu_\ell(S) - \mu_{\ell'}(S) = \operatorname{sign}(s\ell, \ell, \ell') - \operatorname{sign}(s\ell, s\ell', \ell') \tag{8}$$

(proposition 1), hence in particular :

$$\mu_\ell(S) = \mu_0(S) + \operatorname{sign}(s\ell, \ell, \ell_0) - \operatorname{sign}(s\ell, s\ell_0, \ell_0) \tag{9}$$

since $\mu_0 = \mu_{\ell_0}$.

Those results are used to show that there exists, for every $\ell \in \Lambda(n)$, a bijection H_ℓ of $Mp(n)$ onto a subset $[Sp(n) \times \mathbb{Z}/8\mathbb{Z}]_\ell$ of the cartesian product $Sp(n) \times \mathbb{Z}/8\mathbb{Z}$, which is characterized by theorem 1; from this follows (corollary 1) :

COROLLARY.

1) *The set* $[Sp(n) \times \mathbb{Z}/8\mathbb{Z}]_\ell$ *can be equipped with a structure of topological group for which* H_ℓ *is an isomorphism*

$$H_\ell\ :\ Mp(n) \xrightarrow{\approx} [Sp(n) \times \mathbb{Z}/8\mathbb{Z}]_\ell\ . \tag{10}$$

2) *The composition law of that group is given by the formula :*

$$(s\dot\mu)(s', \dot\mu') = (ss', \dot\mu + \dot\mu' + \operatorname{sign}(\ell, s\ell, ss'\ell)) \tag{11}$$

where the dots on μ *and* μ' *indicate classes modulo 8.*

Remark. — We have shown in [G2] that the universal covering group $Sp_\infty(n)$ of the symplectic group $Sp(n)$ can be identified with a subset $[Sp(n) \times \mathbb{Z}]_\ell$ of $Sp(n) \times \mathbb{Z}$, equipped with an adequate topology and algebraic structure; the definition of $[Sp(n) \times \mathbb{Z}]_\ell$ is similar to that of $[Sp(n) \times \mathbb{Z}/8\mathbb{Z}]_\ell$, the group $(\mathbb{Z}/8\mathbb{Z}, +)$ being replaced everywhere by the group $(\mathbb{Z}, +)$. The construction in [G2] was based upon the definition in [G1] of a Maslov index on the universal covering space $\Lambda_\infty(n)$ of the lagrangian Grassmannian $\Lambda(n)$. It is possible to show, using the results in [G1] and [G2] that the Maslov index on $Mp(n)$ defined in this paper is related to the Maslov index defined in [G1] by :

$$\mu_\ell(S) = \text{ class of } \mu_\ell(s_\infty\ell_{0,\infty}, \ell_{0,\infty}), \text{ modulo } 8 \tag{12}$$

where

$$S \in Mp(n) \text{ and } s_\infty \in Sp_\infty(n) \text{ have same projection on } Sp(n); \tag{13}$$

(14) $\quad \ell_{0,\infty}$ is any element of $\Lambda_\infty(n)$ with projection $\ell_0 \in \Lambda(n)$.

These facts, and their consequences, will be developed in a forthcoming paper; see also [G3] for the applications to $q-$symplectic geometry.

Notations.

For $n \geq 1$, $V = \mathbb{R}^n \times \mathbb{R}^n$ is equipped with its usual real vector space structure. The usual scalar product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$: if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$. The standard symplectic form on V is defined by $\omega(z, z') = \langle x', y \rangle - \langle x, y' \rangle$ if $z = (x, y)$ and $z' = (x', y')$ are in V. The symplectic group $Sp(n)$ is the subgroup of the special linear group $S\ell(2n, \mathbb{R})$ consisting of all automorphisms $s : V \to V$ such that $\omega(sz, sz') = \omega(z, z')$ for all $(z, z') \in V \times V$. The lagrangian Grassmannian $\Lambda(n)$ is the set of $n-$dimensional linear subspaces of V on which ω vanishes identically.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing smooth functions $\mathbb{R}^n \to \mathbb{C}$; its dual is $\mathcal{S}'(\mathbb{R}^n)$, the Schwartz space of tempered distributions on $\mathbb{R}^n$. The Fourier transform $\mathcal{F}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by $\mathcal{F}f(x) = (2\pi)^{-n/2} \int \exp(-i\langle x, x' \rangle) f(x') dx'$. It is a unitary operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, whose inverse is given by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-n/2} \int \exp(i\langle x, x' \rangle f(x') dx' \ .$$

1. Quadratic Fourier transforms and the metaplectic group.

In this section we review and complement some of J. Leray's results on the metaplectic group ([L], ch. I).

Let $\mathcal{A}$ be the set of all pairs $\widetilde{A} = (A, m)$ where :

1) A is a quadratic form $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ whose value at (x, x') is given by :

$$A(x, x') = \frac{1}{2}\langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2}\langle Qx', x' \rangle \tag{1.1}$$

where P, L, Q are real $n \times n$ matrices, P and Q being symmetric and L invertible. Throughout this paper we will use the shorthand notation $A = (P, L, Q)$ to denote quadratic forms of the type (1.1).

2) $m \in \mathbb{Z}$ is associated to a choice of $\arg\det(L)$ by the formula :

$$m\pi \equiv \arg\det(L), \text{ modulo } 2\pi . \tag{1.2}$$

To each $\widetilde{A} \in \mathcal{A}$, [L] (ibid., §1,2) associates an automorphism of $\mathcal{S}(\mathbb{R}^n)$ given by :

$$S_{\widetilde{A}}f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \Delta(A) \int e^{iA(x,x')} f(x')dx' \tag{1.3}$$

where :

$$\Delta(A) = i^m \sqrt{|\det(L)|} \text{ if } \widetilde{A} = (A, m), \text{ and } i^x = \exp(ix\pi/2). \tag{1.4}$$

It is easily checked that the $S_{\widetilde{A}}$ extend into automorphisms of $\mathcal{S}'(\mathbb{R}^n)$, whose restrictions to $L^2(\mathbb{R}^n)$ are unitary. In fact :

Proposition 1. — *$S_{\widetilde{A}} = i^m V_{-P} M_L F V_{-Q}$ where*

a) $V_{-R}f(x) = \exp(i\langle Rx, x\rangle/2)f(x)$

b) $M_L f(x) = \sqrt{|\det(L)|}f(Lx)$

c) $Ff(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \int e^{-i\langle x,x'\rangle} f(x')dx'$,

and it is indeed clear that the operators defined in a), b), c) all are unitary.

Corollary 1. *$S_{\widetilde{A}}^{-1} = S_{\widetilde{A}^*}$, where $\widetilde{A}^* = (A^*, m^*)$ with $A^*(x, x') = -A(x', x)$ (that is : $A^* = (-Q, {}^{-t}L, -P)$ and $m^* \equiv n - m$, mod 4.*

Proof. — Immediate in view of the obvious relations $V_{-R}^{-1} = V_R$, $F^{-1}M_L^{-1} = i^n M_{-{}^tL} F$.

Let $\mathcal{U}(L^2(\mathbb{R}^n))$ be the group of all unitary automorphisms of $L^2(\mathbb{R}^n)$ and denote by $Mp(n)$ the subgroup of $\mathcal{U}(L^2(\mathbb{R}^n))$ generated by the set $\{S_{\widetilde{A}} \,;\, \widetilde{A} \in \mathcal{A}\}$. That group is called the metaplectic group, and we have :

Theorem.

1) *The metaplectic group $Mp(n)$ is a covering group of order 2 of $Sp(n)$, the projection map $\pi : Mp(n) \to Sp(n)$ being characterized by :*

$$(x, y) = \pi(S_{\widetilde{A}})(x', y') \iff \begin{cases} y = \operatorname{grad}_x A(x, x') , \\ y' = -\operatorname{grad}_{x'} A(x, x') . \end{cases} \tag{1.5}$$

2) *Each $S \in Mp(n)$ is the product of two elements of* $\{S_{\widetilde{A}}\,;\ \widetilde{A} \in \mathcal{A}\}$;

3) $S_{\widetilde{A}} = S_{\widetilde{A}'}$ *if and only if* $A = A'$ *and* $m \equiv m'$, mod 4.

Proof. — For 1) : [L], ch. I, §1,2, theorem 2,1) and formula (1.11), §1,1. For 2) : ibid., §1,2, lemma 2.5. For 3) : ibid., §2,8, formula 8.6. Setting $v_{-R} = \pi(V_{-R})$, $m_L = \pi(M_L)$, $J = \pi(F)$, (1.5) in theorem 1 immediately yields :

PROPOSITION 2. — (a) $v_{-R} = \begin{pmatrix} I & 0 \\ R & I \end{pmatrix}$, (b) $m_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & {}^tL \end{pmatrix}$, (c) $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$,

where we are using the standard block-matrix notation (I (resp. 0) being the $n \times n$ identity (resp. 0) matrix).

PROPOSITION 3. — *Let $S \in Mp(n)$, $\ell_0 = \{0\} \times \mathbb{R}^n$. There exists $\widetilde{A} \in \mathcal{A}$ such that $S = S_{\widetilde{A}}$ if and only if $s\ell_0 \cap \ell_0 = \{0\}$.*

Proof. — See [L], ch. I, §2,4, Remark 4.1, p.38 (J. Leray uses the notation X^* instead of ℓ_0, and denotes by Σ_{sp} the subset of $Sp(n)$ whose elements are not projections of the $S_{\widetilde{A}}$).

Let R be a real symmetric $n \times n$ matrix; we denote by $\mathrm{Inert}(R)$ the number of negative eigenvalues of R, and call that integer the index of inertia of R; the signature of R will be denoted $\mathrm{Sign}(R)$; obviously $\mathrm{Sign}(R) + 2\,\mathrm{Inert}(R)$ is the rank of R; in our calculations we will actually only need $\mathrm{sign}(R)$, the class modulo 8 of $\mathrm{Sign}(R)$.

Let $(\widetilde{A}, \widetilde{A}', \widetilde{A}'') \in \mathcal{A}^3$ with $\widetilde{A} = (A, m)$; $\widetilde{A}' = (A', m')$, $\widetilde{A}'' = (A'', m'')$, where $A = (P, L, Q)$, $A' = (P', L', Q')$, $A'' = (P'', L'', Q'')$.

THEOREM 2.

1) *There exists $\widetilde{A}'' \in \mathcal{A}$ such that $S_{\widetilde{A}} S_{\widetilde{A}'} S_{\widetilde{A}''} = I$ (the identity) if and only if $P' + Q$ is invertible; if $S_{\widetilde{A}} S_{\widetilde{A}'} S_{\widetilde{A}''} = I$, then :*

$$(1.6) \qquad m - m'^{*} + m'' \equiv \mathrm{Inert}(P' + Q) \equiv \mathrm{Inert}(P'' + Q) \equiv \mathrm{Inert}(P''' + Q'), \ \mathrm{mod}\, 4\ ;$$

2) *The mapping*

$$(1.7) \qquad \{S_{\widetilde{A}}\,;\ \widetilde{A} \in \mathcal{A}\} \ni S_{\widetilde{A}} \mapsto \mathrm{class}(m) \in \mathbb{Z}/4\mathbb{Z}$$

is locally constant.

Proof. — For 1) : [L], ch. I, §1,2, lemma 2.3; remark (2.2), p. 14 and its proof, p. 15. For 2); ibid., theorem 2,3), using definition 1.3, p. 7–8.

We are going to supplement theorem 2. We will need the following technical result :

LEMMA 1. — *Let $R = (\lambda_{ij}\delta_{ij})_{1\leq i,j\leq n}$ be a real diagonal matrix with $\lambda_i = \lambda_{ii} \neq 0$ for $1 \leq i \leq j \leq n$, $\lambda_i = 0$ for $i > j$. Assume there are p (resp. q) positive (resp. negative) λ_i. Set $x_{(j)} = (x_1,\dots,x_j)$, $x_{(n-j)} = (x_{j+1},\dots,x_n)$, $dx_{(n-j)} = dx_{j+1}\cdots dx_n$, and let $f \in \mathcal{S}(\mathbb{R}^n)$. For $k \in \mathbb{R}$, we define :*

$$I(k) = \int \exp\left[\frac{ik}{2}\langle Rx, x\rangle\right] f(x)dx \ . \tag{1.8}$$

There exists a constant $C(R) > 0$ such that :

$$I(k) \sim C(R)k^{-j/2}(e^{i\frac{\pi}{4}})^{p-q} \int_{\mathbb{R}^{n-m}} f(0_{(j)}, x_{(n-j)})dx_{(n-j)} \tag{1.9}$$

when $k \to +\infty$, $j < n$;

$$I(k) \sim C(R)k^{-n/2}(e^{i\frac{\pi}{4}})^{p-q} f(0) \tag{1.9'}$$

when $k \to +\infty$, $j = n$.

Proof. — Recall the classical formula

$$\int_{\mathbb{R}} \exp\left[\pm \frac{ik}{2}x^2\right] g(x)dx \sim \left(\frac{2\pi}{k}\right)^{1/2} e^{\pm i\frac{\pi}{4}} g(0) \tag{1.10}$$

when $k \to +\infty$, which is valid for every $g \in C^2(\mathbb{R})$ which is bounded and with bounded second derivatives (see for instance [GS], p. 5). Suppose $j = n$; then (1.9′) immediately follows from (1.10) with $C(R) = (2\pi)^{n/2}|\det(R)|^{-1/2}$. If $j < n$, write :

$$I(k) \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \exp\left[\frac{ik}{2}\langle Rx, x\rangle\right] f(x)dx_{(j)} \right) dx_{(n-j)}$$

and apply (1.10) to the integral in $x_{(j)}$.

Let R be a real symmetric matrix of rank $j \leq n$, and let p (resp. q) be the number of positive (resp. negative) eigenvalues of R; thus $p + q = j$.

With the notations of §0, we have :

PROPOSITION 4. — *If $S_{\widetilde{A}}S_{\widetilde{A}'} = S_{\widetilde{A}''}S_{\widetilde{A}'''}$, where $\widetilde{A} = (A, m)$, $\widetilde{A}' = (A', m')$ and so on, then $P' + Q$ and $P''' + Q''$ have the same rank and :*

$$m + m' + \frac{1}{2}\operatorname{Sign}(P' + Q) \equiv m'' + m''' + \frac{1}{2}\operatorname{Sign}(P''' + Q''), \ \bmod 4 \ .$$

Proof. — Set $R = P' + Q$, $S = P''' + Q''$; in view of proposition 1 we have :

$$S_{\widetilde{A}} S_{\widetilde{A'}} = i^{m+m'} V_{-P} M_L F V_{-R} M_{L'} F V_{-Q'} \ , \tag{1.11}$$

$$S_{\widetilde{A''}} S_{\widetilde{A'''}} = i^{m''+m'''} V_{-P''} M_{L''} F V_{-S} M_{L'''} F V_{-Q'''} \ . \tag{1.11'}$$

Noting the obvious relations :

$$M_L V_{-R} = V_{{}^tLRL} M_L \ ; \quad F M_L = M_{{}^tL^{-1}} F \tag{1.12}$$

it is indeed no restriction to assume $L' = L''' = I$; let H (resp. K) be an orthogonal matrix such that $R' = HR\,{}^tH$ (resp. $S' = KS\,{}^tK$) is diagonal; we may even choose H and K such that $R' = (\lambda_{ij}\delta_{ij})_{1\le i,j\le n}$, $\lambda_{ij} \ne 0$ for $1 \le i \le j = \operatorname{rank}(R)$, $\lambda_{ii} = 0$ otherwise (resp. $S' = (\mu_{ij}\delta_{ij})_{1\le i,j\le n}$, $\mu_{ii} \ne 0$ for $1 \le i \le j' = \operatorname{rank}(S)$, $\mu_{ii} = 0$ otherwise).

Since we have, again in view of the relations (1.12) :

$$\begin{aligned} S_{\widetilde{A}} S_{\widetilde{A'}} &= i^{m+m'} V_{-P} M_L F V_{-R} F V_{-Q'} \\ &= i^{m+m'} V_{-P} M_L F V_{-{}^tHR'H} M_H M_{{}^tH} F V_{-Q'} \\ &= i^{m+m'} V_{-P} M_L F M_H V_{-R'} M_{{}^tH} F V_{-Q'} \\ &= i^{m+m'} V_{-P} M_{LH} F V_{-R'} M_{{}^tH} F V_{-Q'} \end{aligned}$$

and, similarly :

$$S_{\widetilde{A''}} S_{\widetilde{A'''}} = i^{m''+m'''} V_{-P} M_{L''K} F V_{-S'} M_{{}^tK} F V_{-Q'''} \ ,$$

we can finally assume that $S_{\widetilde{A}} S_{\widetilde{A'}}$ and $S_{\widetilde{A''}} S_{\widetilde{A'''}}$ are given by the expressions (1.11), (1.11'), L' and L''' being orthogonal, and R and S diagonal, with $R = (\lambda_{ij})_{1\le i,j\le n}$, $\lambda_{ii} \ne 0$ for $1 \le i \le j$, $\lambda_{ii} = 0$ otherwise, and $S = (\mu_{ij})_{1\le i,j\le n}$, $\mu_{ii} \ne 0$ for $1 \le i \le j'$, $\mu_{ii} = 0$ otherwise. Choose now a function $f \in \mathcal{S}(\mathbb{R}^n)$ which is radially symmetric ($f(x) = \exp(-|x|^2)$ is for instance a perfect choice), and set $f_k(x) = f(kx)$, $k \in \mathbb{R}$.

We have :

$$S_{\widetilde{A}} S_{\widetilde{A'}} f_k(0) = i^{m+m'-n} \int \exp\left[\frac{i}{2}\langle Rx', x'\rangle\right] g_k(x')dx'$$

with

$$g_k(x') = (2\pi)^{-n}\sqrt{|\det(L)|} \int \exp\left[-i\langle L'x', x''\rangle + \frac{1}{2}\langle Q''x'', x''\rangle\right] f(kx'')dx'' \ .$$

Performing the substitutions $kx'' \to x''$, $x'/k \to x'$ we get :

$$S_{\widetilde{A}} S_{\widetilde{A'}} f_k(0) = i^{m+m'-n} \int \exp\left[\frac{ik}{2}\langle Rx', x'\rangle\right] h_k(x')dx'' \tag{1.13}$$

with :

$$h_k(x') = (2\pi)^{-n}\sqrt{|\det(L)|}\int \exp\left[-i\langle L'x', x''\rangle + \frac{i}{2k^2}\langle Q''x'', x''\rangle\right] f(x'')dx'' ,$$

that is, since L' is orthogonal and f radial :

$$h_k(x') = (2\pi)^{-n}\sqrt{|\det(L)|}\int \exp\left[-i\langle x', x''\rangle + \frac{i}{2k^2}\langle {}^tLQ''Lx'', x''\rangle\right] f(x'')dx'',$$

from which readily follows that :

(1.14)

$$S_{\widetilde{A}}S_{\widetilde{A}'}f_k(0) - i^{m+m'-n}\int \exp\left[\frac{ik}{2}\langle Rx', x'\rangle\right] g(x')dx' = O(k^{-2-n/2}) ,$$

when $k \to +\infty$, the function g being given by :

$$g(x') = (2\pi i)^{n/2}\sqrt{|\det(L)|}F[f](x').$$

Applying lemma 1 to the integral in (1.14) we get :

$$(1.15) \qquad S_{\widetilde{A}}S_{\widetilde{A}'}f_k(0) \sim C(R,f) i^{m(A)+m(A')-\frac{n}{2}}(e^{i\frac{\pi}{4}})^{\mathrm{Sign}(R)}k^{-j/2}$$

when $k \to +\infty$, $C(R,f)$ being given by :

$$(1.16) \qquad C(R,f) = C(R)\sqrt{|\det(L)|}\int_{\mathbf{R}^{n-m}} F[f](0_{(j)}, x_{(n-j)})dx_{(n-j)} ,$$

$C(R)$ being a positive constant (the integral in (1.16) being replaced by $F[f](0)$ if $j = n$).

Performing a similar calculation for $S_{\widetilde{A}''}S_{\widetilde{A}'''}f_k(0)$, we get as well :

$$(1.16') \qquad S_{\widetilde{A}''}S_{\widetilde{A}'''}f_k(0) \sim C'(S,f) i^{m''+m'''-\frac{n}{2}}(e^{i\frac{\pi}{4}})^{\mathrm{Sign}(S)}k^{-j'/2}$$

with :

$$(1.17) \qquad C'(S,f) = C(S)\sqrt{|\det(L'')|}\int_{\mathbf{R}^{n-j'}} F[f](0_{(j')}, x_{(n-j')})dx_{(n-j')}$$

and $C(S) > 0$. Now $S_{\widetilde{A}}S_{\widetilde{A}'} = S_{\widetilde{A}''}S_{\widetilde{A}'''}$ implies that the right hand sides of (1.16) and (1.16′) must be of the same order of magnitude as $k \to +\infty$, hence we must have $j = j'$, and $C(R,f)$, $C'(S,f)$ must have same sign, hence :

$$(1.18) \qquad i^{m+m'-\frac{n}{2}}(e^{i\frac{\pi}{4}})^{\mathrm{Sign}(R)} = i^{m''+m'''-\frac{n}{2}}(e^{i\frac{\pi}{4}})^{\mathrm{Sign}(S)} ;$$

this is proposition 4.

2. The Maslov index on $Mp(n)$

Let $\mathbb{Z}_8 = \mathbb{Z}/8\mathbb{Z}$; if $x \in \mathbb{Z}$ we denote by $\dot{x} \in \mathbb{Z}_8$ the class of x modulo 8. Proposition 4, §1, makes the following possible :

DEFINITION.

1) *Let $\widetilde{A} \in \mathcal{A}$. We call Maslov index of $S_{\widetilde{A}} \in Mp(n)$, and denote by $\mu_0(S_{\widetilde{A}})$ the class of $2m - n$ modulo 8 :*

$$\mu_0(S_{\widetilde{A}}) = \dot{2m} - \dot{n} \in \mathbb{Z}_8 \ . \tag{2.1}$$

2) *Let $S \in Mp(n)$, and $(\widetilde{A}, \widetilde{A}') \in \mathcal{A}^2$, $A = (P, L, Q)$, $A' = (P', L', Q')$ be such that $S = S_{\widetilde{A}} S_{\widetilde{A}'}$ (cf. theorem 1, 2),§1). We call Maslov index of S, and denote by $\mu_0(S)$ the class modulo 8 of $2(m + m' - n) + \mathrm{Sign}(P' + Q)$:*

$$\mu_0(S) = \mu_0(S_{\widetilde{A}}) + \mu_0(S_{\widetilde{A}'}) + \mathrm{sign}(P' + Q) \ . \tag{2.2}$$

It is indeed clear in view of theorem 1, 3), §1 that (2.1) makes sense. Suppose now $S = S_{\widetilde{A}} S_{\widetilde{A}'} = S_{\widetilde{A}''} S_{\widetilde{A}'''}$; then in view of proposition 4, §1, we have : $2m + 2m' + \mathrm{Sign}(P' + Q) \equiv 2m'' + 2m''' + \mathrm{Sign}(P''' + Q'')$, mod 8, hence definition (2.2) is independent of the choice of the pair $(S_{\widetilde{A}}, S_{\widetilde{A}'})$ such that $S = S_{\widetilde{A}} S_{\widetilde{A}'}$.

Before we study the properties of that Maslov index we need to recall the definition and the properties of the signature of a triple of lagrangian planes, due to M. Kashiwara (unpublished). For proofs and details we refer to [LV], part 1, pp. 39–45, or to [G3].

Let $(\ell, \ell', \ell'') \in \Lambda^3(n) = \Lambda(n) \times \Lambda(n) \times \Lambda(n)$ be a triple of lagrangian planes, and consider the quadratic form Q on $\ell \times \ell' \times \ell''$ given by :

$$Q(z, z', z'') = \omega(z, z') + \omega(z', z'') + \omega(z'', z) \ . \tag{2.3}$$

The symmetric matrix of Q has p (resp. q) positive (resp. negative) eigenvalues. The integer $p - q$ (*i.e.* the signature of the quadratic form Q) is called the signature of the triple (ℓ, ℓ', ℓ''), and denoted $\mathrm{Sign}(\ell, \ell', \ell'')$ ([LV] uses the notation $\tau(\ell, \ell', \ell'')$). That signature has the following properties :

(2.4) Sign *is antisymmetric* :

$$\mathrm{Sign}(\ell, \ell', \ell'') = -\,\mathrm{Sign}(\ell, \ell', \ell'') = -\,\mathrm{Sign}(\ell, \ell'', \ell') \ .$$

(2.5) Sign *is invariant by* $Sp(n)$:

$$\mathrm{Sign}(s\ell, s\ell', s\ell'') = \mathrm{Sign}(\ell, \ell', \ell''), \ \forall s \in Sp(n) \ .$$

(2.6) Sign *is a* $\mathbb{Z}$*−valued cocycle :*

$$\mathrm{Sign}(\ell,\ell',\ell'') - \mathrm{Sign}(\ell,\ell',\ell''') + \mathrm{Sign}(\ell,\ell'',\ell''') - \mathrm{Sign}(\ell,\ell'',\ell''') = 0 .$$

(2.7) Sign *is locally constant on the set* $\{(\ell,\ell',\ell'') \in \Lambda^3(n);\ \ell\cap\ell' = \ell'\cap\ell'' = \ell''\cap\ell = \{0\}\}$.

(2.8) $\mathrm{Sign}(\ell,\ell',\ell'') \equiv n + \dim(\ell\cap\ell') + \dim(\ell'\cap\ell'') + \dim(\ell''\cap\ell)$, mod 2.

When ℓ and ℓ'' are transverse (*i.e.* $\ell\cap\ell'' = \{0\}$, which is equivalent to $V = \ell\oplus\ell'$), $\mathrm{Sign}(\ell,\ell',\ell'')$ can be expressed as the signature of a quadratic form of rank n :

(2.9) *When* $\ell\cap\ell' = \{0\}$, $\mathrm{Sign}(\ell,\ell',\ell'')$ *is the signature of the quadratic form* R *on* ℓ' *defined by :*

$$R(z') = \omega(Pz',z') = \omega(z',P''z') = \omega(Pz',P''z')$$

where P *(resp.* P''*) is the projection operator of* V *onto* ℓ *along* ℓ'' *(resp. onto* ℓ'' *along* ℓ*).*

The properties of the signature, together with the results of §1, allow us to prove the following result, *where we have set* $\mathrm{sign}(\ell,\ell',\ell'')$ = *class of* $\mathrm{Sign}(\ell,\ell',\ell'')$ *modulo 8* :

THEOREM 1.

1) *The Maslov index* μ_0 *is the only function* $Mp(n) \to \mathbb{Z}_8$ *having the two following properties :*

(2.10) *The mapping* $(S,\ell) \mapsto \mu_0(S) - \mathrm{sign}(s\ell_0,\ell_0,\ell)$ *is locally constant on* $\{(S,\ell) \in Mp(n)\times\Lambda(n);\ s\ell_0\cap\ell = \ell\cap\ell_0 = \{0\}\}$;

(2.11) $\mu_0(SS') = \mu_0(S) + \mu_0(S') + \mathrm{sign}(\ell_0,s\ell_0,ss'\ell_0)$ *for every pair* $(S,S') \in Mp(n)\times Mp(n)$;

2) *The Maslov index* μ_0 *has the following properties :*

(2.12) $$\mu_0(S^{-1}) = \mu_0(S)\ ;\ \mu_0(I) = \dot{0} .$$

(2.13) $$\mu_0(-S) = \mu_0(S) + \dot{4} .$$

(2.14) $\mu_0(S)$ *and* $n - \dim(s\ell_0\cap\ell_0)$ *have the same images in* $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Proof. — Let us first show the uniqueness of a function $Mp(n) \to \mathbb{Z}_8$ satisfying (2.10) and (2.11). Noting that property (2.10) implies that $S \mapsto \mu_0(S)$ is locally constant on $\{S \in Mp(n)\ ;\ s\ell_0\cap\ell_0 = \{0\}\}$ that is, in view of proposition 3, on $\{S_{\widetilde{A}}\ ;\ \widetilde{A} \in \mathcal{A}\}$, it is sufficient to show that every function :

$$\nu : \{S_{\widetilde{A}}\ ;\ \widetilde{A} \in \mathcal{A}\} \longrightarrow \mathbb{Z}_8$$

such that :

(2.10′) ν is locally constant on $\{S_{\widetilde{A}}, \widetilde{A} \in \mathcal{A}\}$,

(2.11′) $\nu(SS') = \nu(S) + \nu(S')$ for every pair $(S, S') \in Mp(n) \times Mp(n)$,

is identically zero.

Let $S \in Mp(n)$; in view of theorem 1, 2), §1, we can find $(\widetilde{A}, \widetilde{A}') \in \mathcal{A}^2$ such that $S = S_{\widetilde{A}} S_{\widetilde{A}'}$. In view of (2.11′) we then have :

$$\nu(S) = \nu(S_{\widetilde{A}}) + \nu(S_{\widetilde{A}'})\ ;$$

hence ν is locally constant on $Mp(n)$ in view of (2.10′); but $Mp(n)$ is connected (theorem 1, 1); §1) hence ν is constant on $Mp(n)$; choosing $S = S' = I$ in (2.11′) the value of that constant is zero.

To prove (2.10) and (2.11) we need :

LEMMA 1. — *Let $S = S_{\widetilde{A}} S_{\widetilde{A}'} \in Mp(n)$; then :*

$$\mu_0(S) = \mu_0(S_{\widetilde{A}}) + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s_A\ell_0, s_A s_{A'}\ell_0) \tag{2.15}$$

where $s_A = \pi(S_{\widetilde{A}})$, $s_{A'} = \pi(S_{\widetilde{A}'})$.

Proof of the lemma. — Let $A = (P, L, Q)$, and $A' = (P', L', Q')$. In view of definition (2.2) of the Maslov index, it is of course sufficient to prove that :

$$\operatorname{Sign}(P' + Q) = \operatorname{Sign}(\ell_0, s_A\ell_0, s_A s_{A'}\ell_0)\ ,$$

or, which amounts to the same, in view of (2.4), (2.5) :

$$\operatorname{Sign}(P' + Q) = -\operatorname{Sign}(\ell_0, s_A^{-1}\ell_0, s_{A'}\ell_0)\ . \tag{2.16}$$

Now, s_A^{-1} and $s_{A'}$ are in fact characterized by theorem 1, 1), §1, (formula (1.5)), and the corollary to proposition 1, §1 :

$$(x, y) = s_A^{-1}(x', y') \Longleftrightarrow \begin{cases} y = -Qx + Lx' \\ y' = -{}^tLx + Px' \end{cases} \tag{2.17}$$

$$(x, y) = s_{A'}(x', y') \Longleftrightarrow \begin{cases} y = -P'x - {}^tL'x' \\ y' = L'x + Q'x' \end{cases} \tag{2.18}$$

hence :

$$\begin{cases} s_A^{-1}\ell_0 = \{(x, -Qx)\ ;\ x \in \mathbb{R}^n\}\ , \\ s_{A'}\ell_0 = \{(x, P'x)\ ;\ x \in \mathbb{R}^n\}\ . \end{cases} \tag{2.19}$$

Since $s_{A'}\ell_0 \cap \ell_0 = \{0\}$ in view of proposition 3, §1, $\operatorname{sign}(\ell_0, s_A^{-1}\ell_0, s_{A'}\ell_0)$ is, in view of (2.9), the signature of the quadratic form $x \mapsto -\langle (P'+Q)x, x\rangle$ hence (2.16) and the lemma.

Proof of (2.10). — Let $S = S_{\tilde{A}} S_{\tilde{A}'} \in Mp(n)$ and $\ell \in \Lambda(n)$. Using the lemma we get :

$$\begin{aligned}\mu_0(S) - &\operatorname{sign}(s\ell_0, \ell_0, \ell)\\ &= \mu_0(S_{\tilde{A}}) + \mu_0(S_{\tilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s_A s_{A'}\ell_0) - \operatorname{sign}(s\ell_0, \ell_0, \ell)\\ &= \mu(S_{\tilde{A}}) + \mu(S_{\tilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s\ell_0) - \operatorname{sign}(s\ell_0, \ell_0, \ell)\end{aligned}$$

that is, in view of the cocycle property (2.6) of the signature :

$$\begin{aligned}\mu_0(S) - &\operatorname{sign}(s\ell_0, \ell_0, \ell)\\ &= \mu(S_{\tilde{A}}) + \mu(S_{\tilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, \ell) + \operatorname{sign}(s_A\ell_0, s\ell_0, \ell).\end{aligned}$$

Modifying slightly A (and A') if necessary, we can always assume that $s_A\ell_0 \cap \ell = s_A\ell_0 \cap \ell_0 = s_A\ell_0 \cap s\ell_0 = \{0\}$, hence (2.10) in view of property (2.7).

Proof of (2.11). — We are first going to show that (2.11) holds for $S' = S_{\tilde{A}'}$, *i.e.* :

$$\mu_0(SS_{\tilde{A}}) = \mu_0(S) + \mu_0(S_{\tilde{A}}) + \operatorname{sign}(\ell_0, s\ell_0, ss_A\ell_0)\ . \tag{2.20}$$

For that purpose, define a function : $f_{\tilde{A}} : Mp(n) \to \mathbb{Z}_8$ by the relation

$$f_{\tilde{A}}(S) = \mu_0(SS_{\tilde{A}}) - \mu_0(S) - \operatorname{sign}(\ell_0, s\ell_0, ss_A\ell_0)\ , \tag{2.21}$$

and let $\ell \in \Lambda(n)$. Noting that in view of the cocycle property (2.6) of the signature we have :

$$\operatorname{sign}(\ell_0, s\ell_0), ss_A\ell_0) = \operatorname{sign}(\ell_0, s\ell_0, \ell) - \operatorname{sign}(\ell_0, ss_A\ell_0, \ell) + \operatorname{sign}(s\ell_0, ss_A\ell_0, \ell)$$

hence, using the antisymmetric property (2.4) :

$$f_{\tilde{A}}(S) = h_\ell(SS_{\tilde{A}}) - h_\ell(S) - \operatorname{sign}(s\ell_0, ss_A\ell_0, \ell) \tag{2.22}$$

where :

$$h_\ell(S) = \mu_0(S) - \operatorname{sign}(s\ell_0, \ell_0, \ell)\ . \tag{2.23}$$

Choosing ℓ such that :

$$s\ell_0 \cap \ell = ss_A\ell_0 \cap \ell = \ell_0 \cap \ell = \{0\}\ ,$$

$g_{\tilde{A},\ell}$ and h_ℓ are constant in a neighborhood of S in view of (2.10); since on the other hand : $s\ell_0 \cap ss_A\ell_0 = \ell_0 \cap s_A\ell_0 = \{0\}$ in view of proposition 3, §1, $S \mapsto \operatorname{sign}(s\ell_0, ss_A\ell_0, \ell)$ is also constant in a neighborhood of S in view of (2.7), hence $f_{\widetilde{A}}$ is locally constant on $Mp(n)$ and hence constant since $Mp(n)$ is connected. Choosing $S = S_{\widetilde{A}'}$, lemma 1 hereabove implies :

$$f_{\widetilde{A}}(S_{\widetilde{A}'}) = \mu_0(S_{\widetilde{A}'}S_{\widetilde{A}}) - \mu_0(S_{\widetilde{A}'}) - \operatorname{sign}(\ell_0, s\ell_0, s_{A'}s_A\ell_0) = \mu_0(S_{\widetilde{A}}) ;$$

hence $f_{\widetilde{A}}(S) = \mu_0(S_{\widetilde{A}})$ for all $S \in Mp(n)$, which proves (2.20).

To prove (2.11) in the general case, set $S' = S_{\widetilde{A}}S_{\widetilde{A}'}$; then by (2.20) :

$$\begin{aligned}\mu_0(SS') &= \mu_0(SS_{\widetilde{A}}) + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, ss_A\ell_0, ss'\ell_0)\\ &= \mu_0(S) + \mu_0(S_{\widetilde{A}}) + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, ss_A\ell_0, ss'\ell_0)\\ &\qquad + \operatorname{sign}(\ell_0, s\ell_0, ss_A\ell_0) .\end{aligned}$$

Now, using (2.4), (2.5) and (2.6) we have :

$$\begin{aligned}\operatorname{sign}(\ell_0, ss_A\ell_0, ss'\ell_0) &= \operatorname{sign}(\ell_0, s\ell_0, ss_A\ell_0)\\ &= \operatorname{sign}(\ell_0, ss_A\ell_0, ss'\ell_0) - \operatorname{sign}(\ell_0, ss_A\ell_0, s\ell_0)\\ &= \operatorname{sign}(ss_A\ell_0, ss'\ell_0, s\ell_0) - \operatorname{sign}(\ell_0, ss'\ell_0, s\ell_0)\\ &= \operatorname{sign}(\ell_0, s_A\ell_0, s'\ell_0) + \operatorname{sign}(\ell_0, s\ell_0, ss'\ell_0) ;\end{aligned}$$

hence, noting that $s' = s_As_{A'}$:

$$\begin{aligned}\mu_0(SS') = \mu_0(S) + (\mu_0(S_{\widetilde{A}}) + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s_As_{A'}\ell_0))\\ + \operatorname{sign}(\ell_0, s\ell_0, ss'\ell_0) ,\end{aligned}$$

that is :

$$\mu_0(SS') = \mu_0(S) + \mu_0(S') + \operatorname{sign}(\ell_0, s\ell_0, ss'\ell_0)$$

which is (2.11).

Proof of (2.12). — In view of (2.11) and (2.4) we have :

$$\begin{aligned}\mu_0(I) &= \mu_0(I) + \mu_0(I) + \operatorname{sign}(\ell_0, \ell_0, \ell_0)\\ &= 2\mu_0(I)\end{aligned}$$

hence $(2.12)_2$; $(2.12)_1$ follows writing

$$\mu_0(I) = \mu_0(SS^{-1}) = \mu_0(S) + \mu_0(S^{-1}) + \operatorname{sign}(\ell_0, s\ell_0, \ell_0) .$$

Proof of (2.13). — Let $S = S_{\widetilde{A}}S_{\widetilde{A}'} \in Mp(n)$, with $\widetilde{A} = (A, m)$. Define $\widetilde{A}'' \in \mathcal{A}$ by $\widetilde{A}'' = (A, m + 2)$; then $-S = S_{\widetilde{A}''}S_{\widetilde{A}'}$. In view of definition (2.1) we have :

$$\mu_0(S_{\widetilde{A}''}) = \dot{2}(\dot{m} + \dot{2}) - \dot{n} = \mu_0(S_{\widetilde{A}}) + \dot{4} ,$$

hence, since $S_{\widetilde{A}}$ and $S_{\widetilde{A}''}$ have same projection s_A on $Sp(n)$:

$$\begin{aligned}\mu_0(-S) &= \mu_0(S_{\widetilde{A}''}) + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s_As_{A'}\ell_0)\\ &= \mu_0(S_{\widetilde{A}}) + \dot{4} + \mu_0(S_{\widetilde{A}'}) + \operatorname{sign}(\ell_0, s_A\ell_0, s_As_{A'}\ell_0)\\ &= \mu_0(S) + \dot{4} .\end{aligned}$$

Proof of (2.14). — Let $S = S_{\tilde{A}} S_{\tilde{A}'} \in Mp(n)$; we have, by definition (2.1) :

$$\mu_0(S_{\tilde{A}}) + \mu_0(S_{\tilde{A}'}) = \dot{2}(\dot{m} + \dot{m}' - \dot{n}) \ .$$

On the other hand, by (2.8)

$$\begin{aligned} &\operatorname{sign}(\ell_0, s_A\ell_0, s_A s_{A'}\ell_0) \\ &\quad \equiv n + \dim(\ell_0 \cap s_A\ell_0) + \dim(s_A\ell_0 \cap s_A s_{A'}\ell_0) + \dim(s_A s_{A'}\ell_0 \cap \ell_0), \ \operatorname{mod} 2 \\ &\quad \equiv n + \dim(\ell_0 \cap s_A\ell_0) + \dim(\ell_0 \cap s_{A'}\ell_0) + \dim(s\ell_0 \cap \ell_0), \ \operatorname{mod} 2 \ , \end{aligned}$$

hence the result by (2.11) and proposition 3, §1.

Remark. — Theorem 1, 1) implies theorem 2, 1), §1; but we have not used the latter to establish our results.

3. A relation between $Mp(n)$ and $Sp(n) \times \mathbb{Z}_8$.

Recall that the symplectic group $Sp(n)$ acts transitively on the lagrangian Grassmannian $\Lambda(n)$.

Let $\ell \in \Lambda(n)$ and $s_0 \in Sp(n)$ be such that $\ell = s_0\ell_0$. There exist exactly two elements S_0 and $-S_0$ in $Mp(n)$ projecting onto s_0. Consider the function

$$Mp(n) \ni S \longmapsto \mu_0(S_0^{-1} S S_0) \in \mathbb{Z}_8 \ . \tag{3.1}$$

It is clear that $S_0^{-1}SS_0 = (-S_0)^{-1}S(-S_0)$, hence the function (3.1) does not depend on the choice of the element of $Mp(n)$ projecting onto s_0. Suppose now s'_0 is another element of $Sp(n)$ such that $\ell = s'_0\ell_0$; then $s'_0 = s_0 h$ where $h \in Sp(n)$ is such that $h\ell_0 = \ell_0$. Let H be an element of $Mp(n)$ projecting onto h.

We have, in view of formula (2.11) in theorem 1, 1), §2, and property (2.6) of Sign :

$$\begin{aligned} &\mu_0(H^{-1}S_0^{-1}SS_0H) \\ &\quad = \mu_0(H^{-1}) + \mu_0(S_0^{-1}SS_0) + \mu_0(H) + \operatorname{sign}(\ell_0, h^{-1}\ell_0, h^{-1}s_0^{-1}ss_0h) \\ &\qquad\qquad + \operatorname{sign}(\ell_0, s_0^{-1}ss_0\ell, s_0^{-1}ss_0h\ell_0) \\ &\quad = -\mu_0(H) + \mu_0(S_0^{-1}SS_0) + \mu_0(H) + \operatorname{sign}(h\ell_0, \ell_0, s_0^{-1}ss_0h\ell_0) \\ &\qquad\qquad + \operatorname{sign}(\ell_0, s_0^{-1}ss_0\ell, s_0^{-1}ss_0h\ell_0) \\ &\quad = \mu_0(S_0^{-1}SS_0) + \operatorname{sign}(\ell_0, \ell_0, s_0^{-1}ss_0\ell_0) + \operatorname{sign}(\ell_0, s_0^{-1}ss_0\ell, s_0^{-1}ss_0\ell_0) \\ &\quad = \mu_0(S_0^{-1}SS_0) \ ; \end{aligned}$$

hence the function (3.1) only depends on $\ell \in \Lambda(n)$. That makes possible the following definition and notation :

DEFINITION. — *Let $\ell \in \Lambda(n)$ and $s_0 \in Sp(n)$ be such that $\ell = s_0\ell_0$. Let $S_0 \in Mp(n)$ have projection s_0. The function $\mu_\ell : Mp(n) \to \mathbb{Z}_8$ defined by $\mu_\ell(S) = \mu_0(S_0^{-1}SS_0)$ is called a Maslov index on $Mp(n)$; $\mu_\ell(S)$ depends only on $S \in Mp(n)$ and $\ell \in \Lambda(n)$.*

Remark. — It is obvious that $\mu_{\ell_0} = \mu_0$.

The Maslov index μ_ℓ has properties which are very similar to those of μ_0 :

THEOREM 1.

1) *The Maslov index μ_ℓ is the only function $Mp(n) \to \mathbb{Z}_8$ having the two following properties :*

(3.2) *the mapping $(S,\ell,\ell') \mapsto \mu_\ell(S) - \mathrm{Sign}(s\ell,\ell,\ell')$ is locally constant on $\{(S,\ell,\ell')\ ;\ s\ell\cap\ell' = \ell\cap\ell' = \{0\}\}$, [hence $(S,\ell) \mapsto \mu_\ell(S)$ is locally constant on the set $\{(S,\ell)\ ;\ s\ell\cap\ell = \{0\}\}$];*

$$\mu_\ell(SS') = \mu_\ell(S) + \mu_\ell(S') + \mathrm{sign}(\ell, s\ell, ss'\ell) \ . \tag{3.3}$$

2) *The function μ_ℓ has moreover the following properties :*

$$\mu_\ell(S^{-1}) = -\mu_\ell(S)\ ;\ \mu_\ell(I) = \dot{0}\ ; \tag{3.4}$$

$$\mu_\ell(-S) = \mu_\ell(S) + \dot{4}\ ; \tag{3.5}$$

(3.6) *$\mu_\ell(S)$ and $n - \dim(s\ell\cap\ell)$ have same image in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.*

Proof. — In view of the definition of the function μ_ℓ we have :

$$\mu_\ell(S) - \mathrm{sign}(s\ell,\ell,\ell') = \mu_0(S_0^{-1}SS_0) - \mathrm{sign}(ss_0\ell_0, s_0\ell_0, \ell')$$

that is, by the $Sp(n)$–invariance property (2.5) of the signature :

$$\mu_\ell(S) - \mathrm{sign}(s\ell,\ell,\ell') = \mu_0(S_0^{-1}SS_0) - \mathrm{sign}(s_0^{-1}ss_0\ell_0, \ell_0, s_0^{-1}\ell') \ .$$

Now, the right-hand side of that equality is locally constant, in view of (2.10) in theorem 1, §2, for :

$$s_0^{-1}ss_0\ell_0 \cap s_0^{-1}\ell' = \ell_0 \cap s_0^{-1}\ell' = \{0\}\ ,$$

that is, for :

$$s\ell\cap\ell' = \ell\cap\ell' = \{0\}$$

hence property (3.2).

To prove (3.3), note that :

$$\begin{aligned}\mu_\ell(SS') &= \mu_0((S_0^{-1}SS_0)(S_0^{-1}S'S_0)) \\ &= \mu_0(S_0^{-1}SS_0) + \mu_0(S_0^{-1}S'S_0) + \operatorname{sign}(\ell_0, s_0^{-1}ss_0\ell_0, s_0^{-1}ss's_0\ell_0) \ ,\end{aligned}$$

hence, again by (2.5), noting that $\ell = s_0\ell_0$:

$$\mu_\ell(SS') = \mu_\ell(S) + \mu_\ell(S') + \operatorname{sign}(\ell, s\ell, ss'\ell)$$

which is (3.3).

The uniqueness of a function f_ℓ satisfying (3.2), (3.3) is equivalent to the uniqueness of a function $g_0(S) = f_\ell(S_0SS_0^{-1})$ satisfying properties (2.10), (2.11) in theorem 1, 1), §2, but the uniqueness of that function has been established in that theorem.

The proofs of properties (3.4), (3.5), (3.6) readily follow from the proof of the corresponding properties (2.12), (2.13), (2.4) for μ_0 in theorem 1, 2), §2; the details are left to the reader.

The following result gives an explicit formula for the change of ℓ :

PROPOSITION 1. — *Let $(\ell, \ell') \in \Lambda(n) \times \Lambda(n)$; then*

$$\mu_\ell(S) - \mu_{\ell'}(S) = \operatorname{sign}(s\ell, \ell, \ell') - \operatorname{sign}(s\ell, s\ell', \ell') \tag{3.7}$$

for all $S \in Mp(n)$.

Proof. — Set $\ell = s_0\ell_0$, $\ell' = s_0'\ell_0$, and let $S_1 \in Mp(n)$ have projection $s_1 = s_0 s_0'^{-1} \in Sp(n)$; then $\ell = s_1\ell'$ and we can write, in view of (3.3), noting that $\mu_\ell(S) = \mu_{\ell'}(S_1^{-1}SS_1)$:

$$\begin{aligned}\mu_\ell(S) &= \mu_{\ell'}(S_1^{-1}) + \mu_{\ell'}(SS_1) + \operatorname{sign}(\ell', s_1^{-1}\ell', s_1^{-1}ss_1\ell') \\ &= \mu_{\ell'}(S_1^{-1}) + \mu_{\ell'}(S) + \mu_{\ell'}(S_1) + \operatorname{sign}(\ell', s\ell', ss_1\ell') \\ &\qquad + \operatorname{sign}(\ell', s_1^{-1}\ell', s_1^{-1}ss_1\ell')\end{aligned}$$

hence, by (3.4), (2.4) and (2.5) :

$$\mu_\ell(S) = \mu_{\ell'}(S) + \operatorname{sign}(s\ell, \ell, \ell') - \operatorname{sign}(s\ell, s\ell', \ell')$$

that is (3.7).

In view of (3.5) in theorem 1, we have $\mu_\ell((-1)^kS) = \mu_\ell(S) + \dot{4}k$ for $k \in \mathbb{Z}$ and $S \in Mp(n)$, hence the images of $\mu_\ell(S)$ and $\mu_\ell(-S)$ in $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ are equal, and only depend on the projection $s \in Sp(n)$ of $\pm S$ and $\ell \in \Lambda(n)$; we will denote that image $\mu_\ell[s]$ (we could call the function : $Sp(n) \ni s \longmapsto \mu_\ell[s] \in \mathbb{Z}_4$ "Maslov index on $Sp(n)$".)

THEOREM 2.

1) *For every* $\ell \in \Lambda(n)$ *the mapping :*

(3.8) $Mp(n) \ni S \longmapsto (s, \mu_\ell(S)) \in Sp(n) \times \mathbb{Z}_8$ *is a bijection :*

(3.9) $H_\ell : Mp(n) \longmapsto (Sp(n) \times \mathbb{Z}_8)_\ell = \{(s, \dot{\mu}) \ ; \ \mu \in \mu_\ell[s]\}$ *(here* $\dot{\mu} \in \mathbb{Z}_8$ *is the class modulo 8 of* $\mu \in \mathbb{Z}$*).*

2) *The restriction of that bijection :*

$$(3.10) \qquad \{S \in Mp(n) \ ; \ s\ell \cap \ell = \{0\} \to \{(s, \dot{\mu}) \ ; \ s\ell \cap \ell = \{0\}, \mu \in \mu_\ell[s]\}$$

is a homeomorphism when $\mathbb{Z}_8$ *comes equipped with the discrete topology.*

Proof. — The mapping H_ℓ is surjective in view of the definition of $\mu_\ell[s]$. It is also injective. Suppose in fact $(s, \mu_\ell(S)) = (s', \mu_\ell(S'))$; then $s = s'$ and $S = (-1)^k S'$ for some $k \in \mathbb{Z}$, hence $\mu_\ell(S) + \mu_\ell(S') + \dot{4k}$ in view of (3.5) in theorem 1, hence $k \equiv 0, \bmod 2$, and $S = S'$. The proof of 2) is straightforward in view of (3.2) in theorem 1 : the function $S \mapsto \mu_\ell(S)$ is locally constant on the set $\{S \in Mp(n) \ ; \ s\ell \cap \ell = \{0\}\}$, hence continuous when $\mathbb{Z}_8$ is equipped with the discrete topology.

COROLLARY 1. — *The set of all homeomorphisms* $\widetilde{H}_\ell$ *defined in (3.10) is a system of local charts of* $Mp(n)$ *whose transition functions are given by :*

$$H_\ell \circ H_{\ell'^{-1}} : (s, \mu_{\ell'}(S)) \longmapsto (s, \mu_\ell(S))$$

with $\mu_\ell(S) = \mu_{\ell'}(S) + \operatorname{sign}(s\ell, \ell, \ell') - \operatorname{sign}(s\ell, s\ell', \ell')$.

Proof. — Immediate in view of formula (3.7) in proposition 1.

COROLLARY 2.

1) *For every* $\ell \in \Lambda(n)$, *the set* $(Sp(n) \times \mathbb{Z}_8)_\ell$ *defined by (3.9) can be equipped with a structure of topological group* $[Sp(n) \times \mathbb{Z}_8]_\ell$ *for which* H_ℓ *is an isomorphism* $Mp(n) \mapsto [Sp(n) \times \mathbb{Z}_8]_\ell$.

2) *The composition law of that group* $[Sp(n) \times \mathbb{Z}_8]_\ell$ *is given by :*

$$(3.11) \qquad (s, \dot{\mu}) \cdot (s', \dot{\mu}') = (ss', \dot{\mu} + \dot{\mu}' + \operatorname{sign}(\ell, s\ell, ss'\ell))$$

where $\mu \in \mu_\ell[s]$, $\mu' \in \mu_\ell[s']$ *and hence* $\mu + \mu' + \operatorname{sign}(\ell, s\ell, ss'\ell) \in \mu_\ell[ss']$.

Proof. — Immediate in view of theorem 1, (3.3), theorem 2, (3.9) and corollary 1.

Remark. — Let $\ell' \in \Lambda(n)$. It is straightforward to show, using the properties (2.4) through (2.8) of the signature, and (3.7), that the mapping :

$$\psi_{\ell,\ell'} : [Sp(n) \times \mathbb{Z}_8]_\ell \longrightarrow [Sp(n) \times \mathbb{Z}_8]_{\ell'} \tag{3.12}$$

defined by :

$$\psi_{\ell,\ell'} \ : \ (s, \dot{\mu}) \longmapsto (s, \dot{\mu} + \operatorname{sign}(\ell, s\ell, s\ell') + \operatorname{sign}(\ell, s\ell', \ell')) \tag{3.13}$$

is an isomorphism of topological groups.

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