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GLOBALITY IN SEMISIMPLE LIE GROUPS

by Karl-Hermann NEEB (*)

0. Introduction.

One of the most essential facts in the theory of Lie groups is that, given a Lie group G, there is a one-to-one correspondence between the analytic subgroups of G and the Lie subalgebra of $\mathbf{L}(G)$, the Lie algebra of G. We are interested in the corresponding situation in the Lie theory of semigroups.

Semigroups in connection with Lie groups became increasingly important in recent years in such contexts as representation theory (Ol'shanskiĭ [O1], [O2], Howe [Fo]), harmonic analysis (Faraut [Fa1], [Fa2]) and system theory (Kupka [HiLPy]). For further references see [HiHoL] and [HLP89].

Firstly one has to look for a suitable class of subsemigroups of Lie groups generalizing the analytic subgroups. These are the subsemigroups S of a Lie group G for which the group $G(S) \stackrel{\text{def}}{=} \langle S \cup S^{-1} \rangle$ generated by S is an analytic subgroup of G. We call this subsemigroups *preanalytic.* As is described in detail in [HiHoL] V, it is possible to define a *tangent wedge* for such semigroups S of G by

$$\mathbf{L}(S) = \{x \in \mathbf{L}(G) : \exp(\mathbb{R}^+ x) \subseteq \mathrm{cl}_{G(S)} S\}$$

where the closure has to be understood with respect to the Lie group topology of $G(S)$. This generalizes the notion of a tangent algebra of an

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analytic subgroup. Let us call a closed convex cone W in a finite dimensional vector space L a *wedge* and $H(W) \stackrel{\text{def}}{=} W \cap (-W)$ the *edge of the wedge*, *i.e.*, the largest vector space contained in W. The suitable generalization of the Lie subalgebras of $\mathbf{L}(G)$ are the *Lie wedges*. These are the wedges $W \subseteq \mathbf{L}(G)$ with the additional property that

$$e^{\operatorname{ad} h} W = W \quad \text{for all} \quad h \in H(W).$$

Notice that the Lie wedges which are vector spaces are exactly the subalgebras of $\mathbf{L}(G)$. This definition is justified by the fact that for every preanalytic subsemigroup S of a Lie group G the set $\mathbf{L}(S)$ is a Lie wedge ([HiHoL] V.1.6). It is also true that, given a Lie wedge $W \subseteq \mathbf{L}(G)$, we find a local subsemigroup $U \subseteq G$ having W as its tangent wedge in some local sense, but the circle is a simple example of a Lie group such that $\mathbf{L}(G) = \mathbb{R}$ contains a Lie wedge $W = \mathbb{R}^+$ which is not the tangent wedge of a subsemigroup of G. This shows that the correspondence between the subsemigroups of the Lie group G and the Lie wedges in $\mathbf{L}(G)$ is far from being surjective as is true in the group case where every subalgebra is the tangent object of a subgroup. We call the Lie wedges $W \subseteq \mathbf{L}(G)$ which occur as tangent wedges of subsemigroups of G *global in* G. If $W \neq \mathbf{L}(G)$ is global in G, we clearly have $S \stackrel{\text{def}}{=} \langle \exp W \rangle \neq G$. These Lie wedges are said to be *not controllable* in G. This definition has a control theoretic interpretation : if W is interpreted as the set of controls, then $S = \langle \exp W \rangle$ is the set of points in the state space G attainable by the system whose trajectories are obtained by piecewise constant steering functions.

To avoid technical difficulties in our formulations and proofs we often restrict our attention to subsemigroups $S \subseteq G$ for which $G(S) = G$ and Lie wedges $W \subseteq \mathbf{L}(G)$ which are *Lie generating in* $\mathbf{L}(G)$, *i.e.*, $\mathbf{L}(G)$ is the smallest subalgebra containing W. One knows from [HiHoL] that this is no loss in generality but it guarantees that all semigroups $S = \langle \exp W \rangle$ have dense interior and the same interior as $\overline{S}$ ([HiHoL] V.1.16).

We trivialize the tangent bundle of G with the mapping

$$\Psi : L \times G \to T(G), (x, g) \mapsto d\lambda_g(\mathbf{1})x.$$

If V is a finite dimensional vector space and $f : G \to V$ a differentiable function we define $f' : G \to \operatorname{Hom}(L, V)$ by

$$\langle f'(g), x \rangle = \langle df(g), d\lambda_g(\mathbf{1})x \rangle \quad \text{for all} \quad x \in L.$$

For a wedge W of a finite dimensional vector space L we define the dual $W^* \stackrel{\text{def}}{=} \{\omega \in \widehat{L} : \langle \omega, x \rangle \geqslant 0 \text{ for all } x \in W\}$. This set is always a wedge in

$\widehat{L}$. We also set algint $W = \operatorname{int}_{W-W} W$. According to [HiHoL] I.2.2, we find that

$$\operatorname{algint} W^* = \{\omega \in \widehat{L} : \langle \omega, x\rangle > 0 \quad \text{for all} \quad x \in W \setminus H(W)\}.$$

Our main tool will be the concept of *W-positive functions*. These are the real functions f on G which are contained in the set

$$\operatorname{Pos}(W) \overset{\text{def}}{=} \{f \in C^\infty(G) : f'(x) \in W^* \quad \text{for all} \quad x \in G\}.$$

A principal resul in [N1] II.12, states that W is global in G if and only if Pos(W) contains a function f with

$$f'(g) \in \operatorname{algint} W^* \quad \text{for all} \quad g \in G.$$

Furthermore it is shown in [N1] II.13, that Pos(W) contains a non-constant function if and only if W is not controllable in G.

Using these results, we give a characterization of those Lie wedges W in $L = \mathrm{sl}(2, \mathbb{R})^n$ which are invariant under the maximal torus of the adjoint group and which are controllable in the associated simply connected Lie group $G = \mathrm{Sl}, \mathbb{R})^{n\sim}$ (Theorem 1.3). The rest of Section 1 is dedicated to a more detailed analysis of this situation. In Section 2 we develop some algebraic tools concerning real root decompositions with respect to compactly embedded Cartan algebras and invariant cones in semisimple Lie algebras. To every invariant cone W in the semisimple Lie algebra L we associate the bigger Lie wedge $V \overset{\text{def}}{=} W + K'_H$ where K_H is a maximal compactly embedded subalgebra of L. An inspection of the orthogonal projection along K'_H yields some useful information about the intersections of V with $\mathrm{sl}(2, \mathbb{R})^m$ subalgebras of L (Lemma 2.23). In Section 3 this allows us to reduce the controllability problem for invariant cones in semisimple Lie groups to the controllability problem for Lie wedges in $\mathrm{sl}(2, \mathbb{R})^m$ which are invariant under a maximal torus of the adjoint group. Combined with the results from Section 1, we get a characterization of the invariant cones in a semisimple Lie algebra L which are controllable in the associated simply connected Lie group G (Theorem 3.5). If L is simple, much more is known. We even get a characterization of those $e^{\operatorname{ad} K_H}$-invariant wedges $W \subseteq L$ with $H(W) = K'_H$ which are *global in* G (Theorem 3.7). We conclude with a criterion for globality in the non-simply connected case.

1. Globality in $\mathrm{Sl}(2,\mathbb{R})^{n\sim}$.

In this section we consider the simply connected group $G = \mathrm{Sl}(2,\mathbb{R})^{n\sim}$ and its Lie algebra $L = \mathbf{L}(G) = \mathrm{sl}(2,\mathbb{R})^n$. We use the following notations for the elements of $\mathrm{sl}(2,\mathbb{R})$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These matrices satisfy the relations

$$[U,B] = 2A, \quad [U,A] = -2B, \quad \text{and} \quad [A,B] = 2U. \tag{1}$$

We denote the elements of the ideal $L_i = \{0\}^{i-1} \times \mathrm{sl}(2,\mathbb{R}) \times \{0\}^{n-i}$ with a subscript i and write $T \stackrel{\mathrm{def}}{=} \mathrm{span}\,\{U_i + A_i, B_i : i = 1,\dots,n\}$ for the Borel subalgebra of L, $N \stackrel{\mathrm{def}}{=} \mathrm{span}\,\{U_i + A_i : i = 1,\dots,n\}$ for its commutator algebra, $K \stackrel{\mathrm{def}}{=} \mathrm{span}\,\{U_i : i = 1,\dots,n\}$ for the maximal compactly embedded subalgebra and $P \stackrel{\mathrm{def}}{=} \mathrm{span}\,\{A_i, B_i : i = 1,\dots,n\}$. Then $L = K + T$ is an Iwasawa decomposition and $L = P + K$ a Cartan decomposition of L. We identify L with its dual $\widehat{L}$ using the non-degenerate symmetric bilinear form $-k$, where k is the normalized Cartan Killing form with

$$k(X) \stackrel{\mathrm{def}}{=} k(X,X) = \sum_{i=1}^{n} a_i^2 + b_i^2 - x_i^2 \quad \text{for} \quad X = \sum_{i=1}^{n} a_i A_i + b_i B_i + x_i U_i.$$

LEMMA 1.1. — *Let $g(z) = z(1-e^{-z})^{-1}$ for $z \in \mathbb{C} \setminus (2\pi i\mathbb{Z} \setminus \{0\})$ and $a \in \mathbb{R}$. Then the linear operator $g(\mathrm{ad}\, aA)$ on $\mathrm{sl}(2,\mathbb{R})$ is well defined ($\mathrm{ad} A$ has only real eigenvalues) and may be expressed as*

$$g(\mathrm{ad}\ aA) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{a}{\tanh a} & a \\ 0 & a & \dfrac{a}{\tanh a} \end{pmatrix}$$

with respect to the base (A,B,U) of $\mathrm{sl}(2,\mathbb{R})$.

Proof. — From (1) we get

$$[A, B+U] = 2(B+U), \quad [A, B-U] = -2(B-U) \quad \text{and} \quad [A,A] = 0.$$

Another simple computation shows that

$$\frac{1}{2}(g(2a)+g(-2a)) = \frac{a}{\tanh a} \quad \text{and} \quad \frac{1}{2}(g(2a)-g(-2a)) = a.$$

We conclude that $g(\operatorname{ad} aA)A = A$,

$$\begin{aligned} g(\operatorname{ad} aA)B &= \frac{1}{2}(g(\operatorname{ad} aA)(B+U) + g(\operatorname{ad} aA)(B-U)) \\ &= \frac{1}{2}(g(2a)(B+U) + g(-2a)(B-U)) = \frac{a}{\tanh a}B + aU, \end{aligned}$$

and

$$\begin{aligned} g(\operatorname{ad} aA)U &= \frac{1}{2}(g(\operatorname{ad} aA)(B+U) + g(\operatorname{ad} aA)(B-U)) \\ &= \frac{1}{2}(g(2a)(B+U) + g(-2a)(B-U)) = aB + \frac{a}{\tanh a}U. \end{aligned}$$

□

For later reference we record the following simple fact :

LEMMA 1.2. — *If the Lie generating wedge W in the Lie algebra $L(G)$ of the Lie group G is invariant under the differential $d\gamma(\mathbf{1})$ of the automorphism γ of G, then the Lie wedge $V \stackrel{\text{def}}{=} \mathbf{L}(\overline{\langle \exp W\rangle})$ is invariant under $d\gamma(\mathbf{1})$.*

Proof. — For $v \in V$ we have

$$\begin{aligned} \exp(\mathbf{R}^+ d\gamma(\mathbf{1})v) &= \gamma(\exp(\mathbf{R}^+ v)) \subseteq \gamma(\overline{\langle \exp W\rangle}) \\ &\subseteq \overline{\gamma(\langle \exp W\rangle)} = \overline{\langle \exp d\gamma(\mathbf{1})W\rangle} = \overline{\langle \exp W\rangle}. \end{aligned}$$

This shows that $d\gamma(\mathbf{1})v \in V$ and that V is invariant under $d\gamma(\mathbf{1})$. □

THEOREM 1.3. — *Let $W \subseteq L$ be a Lie wedge with non-empty interior which is invariant under $e^{\operatorname{ad} K}$. Then $W$is not controllable in G if and only if*

$$W^* \cap N \neq \{0\}.$$

Proof. — "$\Rightarrow$" : We assume that W is not controllable in G. Then

$$V \stackrel{\text{def}}{=} \mathbf{L}(\langle \exp W\rangle) = \mathbf{L}(\overline{\langle \exp W\rangle}) \neq L$$

is a global Lie generating Lie wedge which contains W ([Nel] II.13).

Therefore V has inner points. We see with Lemma 1.2 that $e^{\operatorname{ad} K} V = V$, so it suffices to prove that $V^* \cap N \neq \{0\}$ because $V^* \subseteq W^*$. According to Theorem II.12 and Proposition III.4 in [Ne1] we find $f \in \operatorname{Pos}(V)$ with $f'(g) \in \operatorname{algint} V^*$ for all $g \in G$ and $f \circ I_{\exp k} = f$ for all $k \in K$. We define $\widetilde{f} : L \to \mathbb{R}$ by $\widetilde{f} \stackrel{\text{def}}{=} f \circ \exp_G$. Using the formula for the differential of the exponential function from [He] p.105, we see that

$$\begin{aligned} f'(\exp p) &= df(\exp p) d\lambda_{\exp p}(\mathbf{1}) = df(\exp p) d\exp(p) g(\operatorname{ad} p) \\ &= d\widetilde{f}(p) g(\operatorname{ad} p) \in \operatorname{algint} V^* \quad \text{for all} \quad p \in P. \end{aligned}$$

The operator $g(\operatorname{ad} p)$ is well defined for every $p \in P$ because $\operatorname{ad} p$ has only real eigenvalues for $p \in P$. Furthermore we have $\widetilde{f} \circ e^{\operatorname{ad} k} = \widetilde{f}$ for all $k \in K$. For $p = \sum_{i=1}^n a_i A_i$ with $a_i \neq 0$, this leads to

$$0 = \frac{1}{a_i} \frac{d}{dt}\bigg|_{t=0} \widetilde{f}(e^{\frac{-t}{2} \operatorname{ad} U_i} p) = d\widetilde{f}(p)\Big[-\frac{1}{2} U_i, A_i\Big] = d\widetilde{f}(p) B_i = 0.$$

Hence we may represent $d\widetilde{f}(p)$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ as

$$d\widetilde{f}(p) = \sum_{i=1}^n \alpha_i(a) A_i + \beta_i(a) U_i.$$

With (2) we get

$$\begin{aligned} \omega(a) \stackrel{\text{def}}{=} d\widetilde{f}(p) g(\operatorname{ad} p) = \sum_{i=1}^n & \alpha_i(a) A_i \\ & + \beta_i(a)\Big(- a_i B_i + \frac{a_i}{\tanh a_i} U_i\Big) \in \operatorname{algint} V^* \end{aligned} \tag{3}$$

for all $a \in \mathbb{R}^n$. The averaging operator $p : L \to K = L_{\text{fix}}$ of the action of the torus group $e^{\operatorname{ad} K}$ on L agrees with the orthogonal projection onto K along P. Using the assumption that $\operatorname{int} W = \emptyset$, we find that $\emptyset \neq \pi(\operatorname{int} W) \subseteq \operatorname{int} W \cap K$ contains an element $U_0 \stackrel{\text{def}}{=} \sum_{i=1}^n \delta_i U_i$ with $\delta_i \neq 0$ for all $i = 1, \ldots, n$. Now (3) leads to

$$-k(\omega(a), U_0) = \sum_{i=1}^n \delta_i \beta_i(a) \frac{a_i}{\tanh a_i} > 0$$

for all $a \in \mathbb{R}^n$. Hence the element $\widetilde{\omega}(a) \stackrel{\text{def}}{=} \frac{1}{-k(\omega(a), U_0)} \omega(a)$ is contained in the compact base $C \overset{\text{def}}{=} \{\omega \in W^* : \omega(U_0) = 1\}$ of the pointed cone V^*. Setting $a_m \overset{\text{def}}{=} (m, m, \dots, m)$ we find a cluster point of the sequence $\widetilde{\omega}(a_m) \in C$, i.e., $\omega_0 = \lim\limits_{k\to\infty} \widetilde{\omega}(a_{m_k})$. We claim that $\langle \omega_0, U_i - B_i\rangle = 0$ for $i = 1, \dots, n$. If $\beta_i(a_{m_k}) = 0$ for almost all $k \in \mathbb{N}$, this is clear. Therefore we may assume that $\beta_i(a_{m_k}) \neq 0$ for all $k \in \mathbb{N}$. Now we have

$$\frac{\langle \widetilde{\omega}(a_{m_k}), B_i\rangle}{\langle \widetilde{\omega}(a_{m_k}), U_i\rangle} = \frac{\langle \omega(a_{m_k}), B_i\rangle}{\langle \omega(a_{m_k}), U_i\rangle} = \tanh(m_k)$$

which tends to 1 for $k \to \infty$. Combining both cases completes the proof of our claim. Using this information, we represent ω_0 as $\sum\limits_{i=1}^{n} \alpha_i A_i + \beta_i(U_i - B_i)$. Application of a suitable element $\gamma \in e^{\operatorname{ad} K}$ leads to

$$\gamma(\omega_0) = \sum_{i=1}^{n} \operatorname{sgn}(\beta_i)\sqrt{\alpha_i^2 + \beta_i^2}\, A_i + \beta_i U_i \in C \subseteq V^*$$

because U_0 and therefore C is invariant under $e^{\operatorname{ad} K}$. The element $\sum\limits_{i=1}^{n} \beta_i(A_i + U_i)$ lies on the line segment between $\gamma(\omega_0)$ and $\pi(\gamma(\omega_0)) = \sum\limits_{i=1}^{n} \beta_i U_i$, hence is contained in $C \cap N \subseteq V^* \cap N \setminus \{0\}$.

"$\Leftarrow$" : Let $\omega = \sum\limits_{i=1}^{n} \beta_i(A_i + U_i) \in W^* \cap N \setminus \{0\}$. We know from [He] p.270, that $G = \langle \exp T\rangle \exp(K)$ is a product decomposition in the sense of [N1] IV.7. This means that the mapping $\langle \exp T\rangle \times \exp(K) \to G$, $(x, y) \mapsto xy$ is a diffeomorphism. We know, in addition, that $\omega \in N = [T, T]^\perp = ([T, T] \oplus [K, K])^\perp$ which allows us to apply Proposition IV.11 in [N1] to find a function $f \in C^\infty(G)$ such that $f'(\mathbf{1}) = \omega \neq 0$ and $f'(g) \in \operatorname{Ad}(\exp K)\widehat{\ }\omega = (e^{\operatorname{ad} K})\widehat{\ }\omega \subseteq W^*$. Using again Corollary II.13 in [N1] we have proved that W is not controllable in G. □

Remark 1.4. — One should notice that for $n > 1$ there are Lie generating $e^{\operatorname{ad} K}$-invariant wedges in L without inner points. To see this, let

$$C \overset{\text{def}}{=} \left\{ \sum_{i=1}^{n} a_i A_i + b_i B_i : a_i^2 + b_i^2 \leqslant 1 \quad \text{for} \quad i = 1, \dots, n \right\}$$

$$\text{and} \quad W \overset{\text{def}}{=} \mathbb{R}^+(U_1 + C).$$

Then $e^{\operatorname{ad} K}C = C$. Hence W is $e^{\operatorname{ad} K}$-invariant, pointed, and contained in the hyperplane $U_2^{\perp}$. Every subalgebra containing W must contain L_1 and therefore C. Then it contains also $P_i + [P_i, P_i] = L_i$ and agrees with L. Consequently W is Lie generating. □

COROLLARY 1.5. — *Let $n = 1$, $G = \mathrm{Sl}(2,\mathbb{R})^{\sim}$, $L = \mathrm{sl}(2,\mathbb{R})$ and $W_s \overset{\text{def}}{=} \{aA + bB + xU : x \geqslant 0, a^2 + b^2 \leqslant s^2x^2\}$. Then the $e^{\operatorname{ad} K}$-invariant wedge W_s is global in G if and only if $s \leqslant 1$.*

Proof. — We have $N = T' = \mathbb{R}(A + U)$ and $W_s^* = W_{\frac{1}{s}}$. Therefore $W_s^* \cap N \neq \{0\}$ if and only if $U + A \in W_s^*$, which is equivalent to $s \leqslant 1$. Using Theorem 1.3, we see that W_s is controllable in G if and only if $s > 1$. If W_s is global, then it is not controllable and if W_s is not controllable, then it is contained in a global W_s' (Lemma I.2) which shows that $s \leqslant s' \leqslant 1$. □

Remark 1.6. — If we compare the proof of Theorem 1.3 with the proof of Proposition II.7 in [N2], it is remarkable that we did not use any explicit parametrization of G to prove Theorem 1.3. The difficulties arise from the great variety of $e^{\operatorname{ad} K}$-invariant cones for $n > 1$. For $n = 1$ the proof is much easier. From $W = W_s$ and $\omega(a) \in W_s^* = W_{\frac{1}{s}}$ for all $a \in \mathbb{R}$, we may conclude that

$$\alpha(a)^2 + \beta(a)^2 a^2 \leqslant \frac{1}{s^2}\ \beta(a)^2\ \frac{a^2}{\tanh^2(a)} \neq 0.$$

This shows that $s^2 \leqslant \dfrac{1}{\tanh^2(a)}$ for all $a \in \mathbb{R}$, hence $s \leqslant 1$. □

LEMMA 1.7. — *Let $F \subseteq L$ be a subalgebra with $[F, K] \subseteq F$ and $I \overset{\text{def}}{=} \{i : F \cap P_i \neq \{0\}\}$. Then*

$$F = (F \cap K) + \sum_{i \in I} L_i.$$

Proof. — The Lie algebra L is a K-module under the adjoint action and $L = K \oplus \bigoplus_{i=1}^{n} P_i$ is the decomposition into isotypical components. Consequently F decomposes as

$$F = (F \cap K) \oplus \bigoplus_{i=1}^{n} P_i \cap F = (F \cap K) \oplus \bigoplus_{i \in I} P_i \cap F.$$

The K-modules P_i are simple, hence $P_i \subseteq F$ for $i \in I$. But F is also a subalgebra which leads to $L_i = [P_i, P_i] + P_i \subseteq F$ for $i \in I$. $\square$

COROLLARY 1.8. — *Let $W \subseteq L = \mathrm{sl}(2, \mathbb{R})^n$ be an $e^{\mathrm{ad}\, K}$-invariant Lie wedge containing W_1^n. Then W is not controllable in $G = \mathrm{Sl}(2, \mathbb{R})^{n\sim}$ iff there exist numbers $\beta_i \in \mathbb{R}^+$ such that*

$$\omega \stackrel{\text{def}}{=} \sum_{i=1}^{n} \beta_i (A_i + U_i) \in W^* \setminus \{0\}. \tag{4}$$

If (4) is fulfilled, we have

$$H(\mathbf{L}(\langle \exp W \rangle)) \subseteq \omega^\perp \cap K \oplus \sum_{\beta_j = 0} L_j.$$

Proof. — Let W be controllable in G. Then, using Theorem 1.3, we find an element $\omega = \sum_{i=1}^{n} \beta_i (A_i + U_i) \in W^* \setminus \{0\}$. But $U_i \in W_1^n \subseteq W$ and therefore

$$\langle \omega, U_i \rangle = -k(\beta_i U_i, U_i) = \beta_i \geqslant 0.$$

The other implication is trivial from Theorem 1.3. Let us assume that (4) is fulfilled. Then $V \stackrel{\text{def}}{=} \mathbf{L}(\langle \exp W \rangle)$ is an $e^{\mathrm{ad}\, K}$-invariant global Lie wedge in L (Lemma 1.2) with $W \subset V$ and $H(V)$ is a subalgebra of L with $[K, H(V)] \subseteq H(V)$. The function $f \in \mathrm{Pos}(V) \subseteq \mathrm{Pos}(W)$ constructed in the proof of Theorem 1.3 satisfies $f'(\mathbf{1}) = \omega$ and therefore $\omega \in V^* \subseteq W^*$. Especially we find that $H(V) \subseteq \omega^\perp$. Let $I = \{i : H(V) \cap P_i \neq \{0\}\}$. It follows from Lemma 1.7 that $i \notin I$ for $\beta_i \neq 0$. Hence

$$H(V) \subseteq (H(V) \cap K) \oplus \sum_{\beta_j = 0} L_j \subseteq (K \cap \omega^\perp) \oplus \sum_{\beta_j = 0} L_j.$$

$\square$

PROPOSITION 1.9. — *Let $W \subseteq L = \mathrm{sl}(2, \mathbb{R})^n$ be an $e^{\mathrm{ad}\, K}$-invariant Lie wedge containing W_1^n with $H(W) = \omega^\perp \cap K$ for $\omega = \sum_{i=1}^{n} \lambda_i (U_i + A_i)$ and $\lambda_i > 0$. Then W is global in $G = \mathrm{Sl}(2, \mathbb{R})^{n\sim}$ if and only if $\omega \in W^*$.*

Proof. — "$\Rightarrow$" : Let W be global in G. Then W is not controllable in G and Corollary 1.8 provides $\beta_i \in \mathbb{R}^+$ with $\widetilde{\omega} \stackrel{\text{def}}{=} \sum_{i=1}^{n} \beta_i (U_i + A_i) \in W^* \setminus \{0\}$.

Then $\widetilde{\omega}$ vanishes on $H(W) = \omega^\perp \cap K$. Hence $\widetilde{\omega} \mid_K$ is a positive scalar multiple of $\omega \mid_K$. Consequently $\beta_i = \mu\lambda_i$ for $i = 1, \ldots, n$ and $\mu > 0$.

"$\Leftarrow$" : Assume that $\omega \in W^*$. We apply Corollary 1.8 to see that

$$H(\mathbf{L}(\langle \exp W \rangle)) \subseteq (\omega^\perp \cap K) \oplus \sum_{\lambda_i = 0} L_i = \omega^\perp \cap K = H(W).$$

This proves that W is global ([N1] III.1). □

LEMMA 1.10. — *Let $W \subseteq L$ be a Lie wedge and $F \subseteq L$ a subalgebra with*

$$e^{\mathrm{ad}\, F} W = W \quad \text{and} \quad W \cap F \subseteq H(W).$$

Then $V \overset{\mathrm{def}}{=} W + F$ is a Lie wedge with $H(V) = H(W) + F$ and $V^* = W^* \cap F^\perp$.

Proof. — Firstly we observe that $W \cap F = -W \cap F = H(W) \cap F$ is a vector space. Then [HiHoL] I.2.32, implies that V is closed and therefore a wedge. Clearly $H(W) + F$ is contained in the edge $H(V)$ of V. If for $v = w + f$ with $f \in F$ and $w \in W$ the element $-v$ is also in V, hence $-w - f = w' + f'$ with $w' \in W$ and $f' \in F$. Consequently $w + w' \in F \cap W \subseteq H(W)$ which proves that even $w \in H(W)$ because $H(W)$ is the unit group of the additive semigroup W, hence $H(V) = H(W) + F$. For $f \in F$ and $h \in H(W)$ we get $[h, f] \in H(W)$ because $e^{t \mathrm{ad}\, f} H(W) = H(W)$ for all $t \in \mathbb{R}$. Therefore

$$e^{\mathrm{ad}\, f} V = e^{\mathrm{ad}\, f} W + e^{\mathrm{ad}\, f} F = W + F = V \quad \text{and} \quad e^{\mathrm{ad}\, h} V = e^{\mathrm{ad}\, h} W + e^{\mathrm{ad}\, h} F = V.$$

We conclude that $e^{\mathrm{ad}\, H(V)} V = V$ and therefore V is a Lie wedge. That $V^* = W^* \cap F^\perp$ is clear because a linear functional is non-negative on V if and only if it is non-negative on W and vanishes on F. □

COROLLARY 1.11. — *Let $W \subseteq L = \mathrm{sl}(2, \mathbb{R})^n$ be a pointed $e^{\mathrm{ad}\, K}$-invariant Lie wedge containing W_1^n with $\omega = \sum_{i=1}^n \lambda_i (U_i + A_i) \in W^*$, $\lambda_i > 0$ and $\omega^\perp \cap K \cap W = \{0\}$. Then W is global in G.*

Proof. — Let $E = \omega^\perp \cap K$. Then $V = W + E$ is a Lie wedge because $E \cap W = \{0\}$ (Lemma 1.10). It is also invariant under $e^{\mathrm{ad}\, K}$ because this is true for W and E separately. It follows from Proposition 1.9 that V is global in G because $\omega \in V^* = W^* \cap E^\perp$. The fact that $W \cap H(V) = W \cap E = \{0\}$ allows us to apply [N1] III.1, to complete the proof. □

We conclude this section with some facts about the subsemigroups $S = \langle \exp W \rangle$ of G where W is an $e^{\mathrm{ad}\, K}$-invariant wedge in L.

PROPOSITION 1.12. — *Let $S \subseteq G = \mathrm{Sl}(2,\mathbb{R})^{n\sim}$ be a subsemigroup which is invariant under all inner automorphisms I_k with $k \in K^\diamond = \langle \exp K \rangle$ and $q : G = K^\diamond \exp(P) \to K^\diamond$ the projection onto $K^\diamond$. Then*

$$\{q(s)^2 : s \in S\} \subseteq S \cap K^\diamond \subseteq q(S).$$

Proof. — It is clear that $S \cap K^\diamond \subseteq q(S)$ because q fixes the elements of $K^\diamond$. Let $s = k\exp(p) \in S$ with $p \in P$ and $k \in K^\diamond$. Then $\mathrm{Ad}(k')p = -p$ for $k' = \exp\Big(\frac{\pi}{2}\sum_{i=1}^{n} U_i\Big)$ as can be easily seen from a direct computation using (1). This leads to

$$\begin{aligned} q(s^2) &= kk = k\exp(p)\exp(-p)k = skI_{k^{-1}}\big(\exp(-p)\big) \\ &= skI_{k^{-1}} \circ I_{k'}\big(\exp(p)\big) = sI_{k^{-1}k'}\big(k\exp(p)\big) = sI_{k^{-1}k'(s)} \in SS = S, \end{aligned}$$

which completes the proof. □

If the conditions of Corollary 1.11 are not satisfied, we get more information about the semigroup S :

PROPOSITION 1.13. *Let $W \subseteq L = \mathrm{sl}(2,\mathbb{R})^n$ be an $e^{\mathrm{ad}\, K}$-invariant Lie wedge containing W_1^n with $\omega = \sum_{i=1}^{n} \lambda_i(U_i + A_i) \notin W^*$, $\lambda_i > 0$ and $\omega^\perp \cap K \cap W = \{0\}$. Suppose that $z_1, \ldots, z_{n-1} \in \omega^\perp \cap K$ is a base of this vector space. Then*

$$G = \exp(\mathbb{Z}z_1 \oplus \ldots \oplus \mathbb{Z}z_{n-1})S \quad \textit{for} \quad S = \langle \exp W \rangle$$

and $K^\diamond = \exp(\mathbb{Z}z_1 \oplus \ldots \oplus \mathbb{Z}z_{n-1})(S \cap K^\diamond)$.

Proof. — To get a contradiction, we assume that $G \neq \widetilde{S} \stackrel{\mathrm{def}}{=} \exp(\mathbb{Z}z_1 \oplus \ldots \oplus \mathbb{Z}z_{n-1})S$. The semigroup S has dense interior ([HiHoL] V.1.10), and therefore $\overline{\widetilde{S}} \neq G$ ([HiHoL] V.5.14, V.5.16). Then [N1] 1.5 provides a non-constant function

$$f \in \mathrm{Mon}(\widetilde{S}) = \{f \in C^\infty(G) : f(gs) \geqslant f(g) \text{ for all } g \in G, s \in \widetilde{S}\} \subseteq \mathrm{Pos}(W).$$

Let $Z \stackrel{\mathrm{def}}{=} \exp(\mathbb{Z}z_1 \oplus \ldots \oplus \mathbb{Z}z_{n-1})$ and $K_1 \stackrel{\mathrm{def}}{=} \exp(\omega^\perp \cap K)/Z$ the associated torus group. The function f is constant on the cosets gZ, hence it factors to a function $\widetilde{f} : G/Z \to \mathbf{R}$. The group $K^\diamond$ acts by right multiplication on G/Z and Z acts trivially. So the torus K_1 acts on G/Z. If m is normalized Haar measure on K_1 and $\pi : G \to G/Z$ the quotient homorphism, then the function

$$h : g \mapsto \int_{K_1} \widetilde{f}(\pi(g) \cdot k) dm(k) = \int_{K_1} f(gk) dm(k)$$

is smooth and satisfies the conditions

1) $h(g_1) > h(g_2)$ if $f(g_1) > f(g_2)$,

2) $h(gs) \geqslant h(g)$ for $g \in G$, $s \in \widetilde{S}$, and

3) $h(gk_1) = h(g)$ for all $k \in K_1$.

This shows that $h \in \operatorname{Pos}(V)$ for the Lie wedge $V = W + \omega^\perp \cap K$ (Lemma 1.10). The fact that h is not constant implies that V is not controllable in G ([N1], II.13). Now Corollary 1.8 provides real numbers $\beta_i \in \mathbf{R}^+$ such that $\widetilde{\omega} = \sum_{i=1}^n \beta_i (U_i + A_i) \in V^* \setminus \{0\}$. We conclude that $\sum_{i=1}^n \beta_i U_i \in V^* \cap K = \mathbf{R}^+ \Big(\sum_{i=1}^n \lambda_i U_i \Big)$. This proves that $\omega \in \mathbf{R}^+ \widetilde{\omega} \subseteq V^* \subseteq W^*$, a contradiction. □

2. Invariant cones in semisimple Lie algebras.

In this section we denote with L a real semisimple Lie algebra and with $L = \bigoplus_{i=1}^n L_i$ its decomposition into simple constituents such that L_i is non-compact for $i \leqslant m$ and compact for $i > m$. We assume that L contains a pointed Lie generating wedge W which is invariant under the adjoint action, an *invariant cone* for short. We fix a compactly embedded Cartan algebra H of L ([HiHoL] III.2.14) and denote the unique maximal compactly embedded subalgebra containing H with K_H and its center with Z_K ([HiHoL] A.2.40). Then $H = \bigoplus_{i=1}^n H_i$, where $H_i \overset{\text{def}}{=} H \cap L_i$ is a compactly embedded Cartan algebra in L_i and $K_H = \bigoplus_{i=1}^n K_{H_i}$ with $K_{H_i} = L_i$ for $i > m$ because L_i is compact for $i > m$. We need the real root decomposition of L with respect to H :

THEOREM 2.1 ([HiHoL] III.6.5). — *Let L be a finite dimensional Lie algebra with compactly embedded Cartan algebra H and Λ be the set of roots of $L_{\mathbb{C}}$ with respect to $H_{\mathbb{C}}$. These are all purely imaginery on H. We set*

$$\Omega \overset{\text{def}}{=} \{-i\lambda|_H : \lambda \in \Lambda\} \text{ and } L^\omega = L^{-\omega} \overset{\text{def}}{=} L \cap (L_{\mathbb{C}}^{\lambda} \oplus L_{\mathbb{C}}^{-\lambda}) \quad \text{for } \omega = -i\lambda|_H.$$

Any choice of a closed halfspace E in $\widehat{H}$ whose boundary meets the finite set Ω only in $\{0\}$ allows us to represent Ω as $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ = E \cap \Omega$ and $\Omega^- = -\Omega^+$. We shall call Ω^+ a set of positive roots. For each choice of a set of positive roots there is a unique complex structure $I : H^+ \to H^+$ with $I^2 = -\mathrm{id}_{H^+}$ and a direct decomposition of L into isotypic H-submodules under the adjoint action

$$L = H \oplus H^+, \quad H^+ = \bigoplus_{0 \neq \omega \in \Omega^+} L^\omega, \tag{5}$$

where the action of H is described by

$$[h, x] = \omega(h) I x \quad \text{for all} \quad h \in H, x \in L^\omega. \tag{6}$$

The complexification of L^ω is $L_{\mathbb{C}}^{\lambda} \oplus L_{\mathbb{C}}^{-\lambda}$, where λ is the unique complex extension of $i\omega$. We have

$$[L^\omega, L^{\omega'}] \subseteq L^{\omega+\omega'} + L^{\omega-\omega'} \tag{7}$$

and if q is any invariant symmetric bilinear form on $L \times L$, then

$$q(x, Ix) = 0 \quad \text{and} \quad q(x) = q(Ix) \quad \text{for all} \quad x \in L^\omega. \tag{8}$$

Proof. — In view of [HiHoL] III.6.5/8, it only remains to show (7). This follows easily :

$$\begin{aligned}
[L^\omega, L^{\omega'}] &= [L \cap (L_{\mathbb{C}}^{\lambda} \oplus L_{\mathbb{C}}^{-\lambda}), L \cap (L_{\mathbb{C}}^{\lambda'} \oplus L_{\mathbb{C}}^{-\lambda'})] \\
&\subseteq L \cap [L_{\mathbb{C}}^{\lambda} \oplus L_{\mathbb{C}}^{-\lambda}, L_{\mathbb{C}}^{\lambda'} \oplus L_{\mathbb{C}}^{-\lambda'}] \\
&\subseteq L \cap (L_{\mathbb{C}}^{\lambda+\lambda'} \oplus L_{\mathbb{C}}^{-\lambda-\lambda'} \oplus L_{\mathbb{C}}^{\lambda-\lambda'} \oplus L_{\mathbb{C}}^{\lambda'-\lambda}) \\
&= L \cap (L^{\omega+\omega'} \oplus i L^{\omega+\omega'} \oplus L^{\omega-\omega'} \oplus i L^{\omega-\omega'}) \\
&= L^{\omega+\omega'} \oplus L^{\omega-\omega'}.
\end{aligned}$$

□

DEFINITION 2.2. — *A root* $\omega \in \Omega$ *is said to be* compact *if* $L^\omega \subseteq K_H$. *The set of compact roots is denoted with* Ω_K^+. *We write* Ω_P^+ *for the set of non-compact roots and*

$$P_H \stackrel{\text{def}}{=} \bigoplus_{\omega \in \Omega_P^+} L^\omega.$$

□

According to [HiHoL] III.6.38, we get a Cartan decomposition $L = K_H \oplus P_H$ of L and

$$K_H = \bigoplus_{\omega \in \Omega_K^+} L^\omega = H \oplus \bigoplus_{0 \neq \omega \in \Omega_K^+} L^\omega.$$

We get a disjoint decomposition

$$\Omega_P^+ = \bigcup_{i=1}^m \Omega_P^+(i)$$

such that

$$\{\omega^i|_{H_i} : \omega^i \in \Omega_P^+(i)\}$$

are the non-compact roots of L_i with respect to H_i ([HiHoL] III.9). For $x \in H^+$ we set $Q(x) \stackrel{\text{def}}{=} [Ix, x]$ and for $\omega \in \Omega_P^+$ we choose an element $x_\omega \in L^\omega$ such that $\omega\big(Q(x_\omega)\big) = 1$ and for $\omega \in \Omega_K^+$ an element $x_\omega \in L^\omega$ with $\omega\big(Q(x_\omega)\big) = -1$. The following lemma shows that this is possible.

LEMMA 2.3. — *Let* L *be a semisimple Lie algebra with the compactly embedded Cartan algebra* H, $\omega \in \Omega^+$ *and* $x \in L^\omega \backslash \{0\}$, *then* $\omega\big(Q(x_\omega)\big) \neq 0$.

(i) $\langle x \rangle = \mathbb{R}x \oplus \mathbb{R}Ix \oplus \mathbb{R}Q(x) \cong \mathrm{sl}(2,\mathbb{R})$ *iff* $\omega\big(Q(x_\omega)\big) > 0$ *iff* $\omega \in \Omega_P^+$ *and*

(ii) $\langle x \rangle = \mathbb{R}_x \oplus \mathbb{R}Ix \oplus \mathbb{R}Q(x) \cong \mathrm{su}(2)$ *iff* $\omega\big(Q(x_\omega)\big) < 0$ *iff* $\omega \in \Omega_K^+$.

Proof. — Using [Hu] p.37 and Theorem 2.1, we see that the complexification $\langle x \rangle_{\mathbb{C}}$ of $\langle x \rangle$ is isomorphic to $\mathrm{sl}(2,\mathbb{C})$. Consequently $\langle x \rangle$ is a real form of $\mathrm{sl}(2,\mathbb{C})$ and therefore a simple Lie algebra. An application of [HiHoL] III.6.12, shows that $\omega\big(Q(x)\big) \neq 0$ and that $\langle x \rangle \cong \mathrm{sl}(2,\mathbb{R})$ if $\omega\big(Q(x)\big) > 0$ and $\langle x \rangle \cong \mathrm{su}(2)$ otherwise. The rest follows from [HiHoL], III.6.16. □

LEMMA 2.4. — *For* $i = 1, \ldots, m$ *we have* $\dim Z_K \cap L_i = 1$ *and the* K_H*-module* $P_H \cap L_i$ *is irreducible. We may choose* Ω^+ *and* $z \in Z_K$ *such that* $\omega(z) = 1$ *for all* $\omega \in \Omega_P^+$.

Proof. — It follows from [HiHoL] p.249, that, for $i \leqslant m$, the ideal L_i is hermitean, i.e., $Z(K_{H_i}) \neq \{0\}$. Consequently it is an irreducible symmetric Lie algebra ([He] pp. 377, 379) and the K_{H_i}-module $P_H \cap L_i$ is irreducible. It follows from ([He], p.382) that $\dim Z(K_{H_i}) = \dim Z_K \cap L_i = 1$ and that $Z_K \cap L_i$ acts on $P_H \cap L_i$ as scalar multiples of the identity (Schur's Lemma). Hence $|\omega(z_i)| = c \neq 0$ for a $z_i \in Z_K \cap L_i$ and all $\omega \in \Omega_P^+(i)$. Therefore we may choose Ω^+ such that $\omega(z_i) = 1$ for all $\omega \in \Omega_P^+(i)$ and $i = 1, \ldots, m$ (cf. [HiHoL] III.6.37). Then $z = \sum_{i=1}^{m} z_i$ is the desired element. □

In the following we denote the Cartan-Killing form of L with B and set $\Lambda^+ = \{\lambda \in \Lambda : -i\lambda|_H \in \Omega^+\}$.

LEMMA 2.5. — *For $\lambda \in \Lambda^+$ we choose $t_\lambda \in H_{\mathbb{C}}$ such that $\lambda(h) = B(t_\lambda, h)$ for all $h \in H$ and define $(\lambda, \lambda') \stackrel{\text{def}}{=} B(t_\lambda, t_{\lambda'})$ for $\lambda, \lambda' \in \Lambda$. If $\omega = -i\lambda|_H$ and $\omega' = -i\lambda'|_H$ we get*

$$(9)\quad t_\lambda = -i\frac{Q(x_\omega)}{B(x_\omega)}, (\lambda, \lambda') = \frac{\omega\big(Q(x_{\omega'})\big)}{B(x_{\omega'})} \text{ and } \omega(h)B(x_\omega) = -B(Q(x_\omega), h).$$

Proof. — The last formula follows immediately from

$$\begin{aligned}\omega(h) \cdot B(x_\omega) &= B(x_\omega, \omega(h)x_\omega) = B(x_\omega, [h - Ix_\omega]) \\ &= B(x_\omega, [Ix_\omega, h]) = B([x_\omega, Ix_\omega], h) = -B(Q(x_\omega), h).\end{aligned}$$

Consequently we have the relation

$$\lambda(h) = i\omega(h) = -i\ \frac{B(Q(x_\omega), h)}{B(x_\omega)}$$

for $\lambda \in \Lambda^+$ and $h \in H$. This proves the formula for t_λ. The scalar product of two complex roots λ, λ' can be computed as follows :

$$(\lambda, \lambda') = B(t_\lambda, t_{\lambda'}) = -\frac{B\big(Q(x_\omega), Q(x_{\omega'})\big)}{B(x_\omega)B(x_{\omega'})} = \frac{\omega\big(Q(x_\omega)\big)}{B(x_{\omega'})}.$$

□

LEMMA 2.6. — *Let $\omega, \omega' \in \Omega^+$, $\omega \neq \pm\, \omega'$, $\omega = -i\lambda|_H$, $\omega' = -i\lambda'|_H$, $h_\lambda = 2t_\lambda/\lambda(t_\lambda)$ and $h_{\lambda'} = 2t_{\lambda'}/\lambda'(t_{\lambda'})$. Suppose that p and q are the greatest integers such that $\lambda + p\lambda'$ and $\lambda - q\lambda'$ respectively $\omega + p\omega'$ and $\omega - q\omega'$ are roots. Then*

$$(10)\qquad \omega\big(Q(x_{\omega'})\big) = \frac{q-p}{2} = \frac{1}{2}\lambda(h'_\lambda) = \frac{(\lambda, \lambda')}{(\lambda', \lambda')} \quad \textit{for} \quad \omega' \in \Omega_P^+$$

and

$$\omega\big(Q(x_{\omega'})\big) = \frac{p-q}{2} = -\frac{1}{2}\lambda(h'_\lambda) = -\frac{(\lambda,\lambda')}{(\lambda',\lambda')} \quad \text{for} \quad \omega' \in \Omega_K^+. \tag{11}$$

Proof. — According to [Hu] p.39 we have $q - p = \lambda(h_{\lambda'})$ for $h_{\lambda'} = 2t_{\lambda'}/\lambda'(t_{\lambda'})$ [Hu] p.37. Consequently

$$\begin{aligned}\frac{q-p}{2} &= \lambda(t_{\lambda'})/\lambda'(t_{\lambda'}) = \lambda\left(-\frac{iQ(x_{\omega'})}{B(x_{\omega'})}\right)/\lambda'\left(-\frac{iQ(x_{\omega'})}{B(x_{\omega'})}\right)\\ &= \omega\big(Q(x_{\omega'})\big)/\omega'\big(Q(x_{\omega'})\big) = \pm\omega\big(Q(x_{\omega'})\big)\end{aligned}$$

where the minus sign is valid if and only if ω' is compact (Lemma 2.3). □

THEOREM 2.7. — *Suppose that L is simple, $\mu_1, \ldots, \mu_l$ is a basis of the root system Λ^+ and $\omega_j = -i\mu_j|_H$. Then the following assertions hold :*

1) *There is a unique non-compact base root, we may assume ω_1, and every positive root ω may be written as*

$$\omega = \omega_1 + \sum_{i=2}^{l} \eta_i\omega_i \quad \text{if} \quad \omega \in \Omega_P^+ \quad \text{and} \quad \omega = \sum_{i=2}^{l} \eta_i\omega_i \quad \text{if} \quad \omega \in \Omega_K^+.$$

2) *The sum of two positive non-compact roots is never a root.*

3) *There is a system $\Pi \overset{\text{def}}{=} \{\nu_1, \ldots, \nu_k\}$ of roots in Ω_P^+ such that*

- a) *Two roots $\nu \neq \nu' \in \Pi$ are strongly orthogonal, i.e., $\nu \pm \nu' \notin \Omega$.*
- b) *$\nu_1 = \omega_1$ is the non-compact base-root.*
- c) *ν_i is a minimal non-compact positive root which is strongly orthogonal to $\nu_1, \ldots, \nu_{i-1}$.*
- d) *$D \overset{\text{def}}{=} \operatorname{span}\{Ix_\nu : \nu \in \Pi\}$ is a maximal abelian subspace of P_H.*

4) *Let $\widetilde{H} \overset{\text{def}}{=} \operatorname{span}\{Q(x_\nu) : \nu \in \Pi\}$. For a compact root $\omega \in \Omega_K^+$ there are three mutually exclusive possibilities :*

- a) *ω is strongly orthogonal to all $\nu \in \Pi$, i.e., $\omega|_{\widetilde{H}} = 0$.*
- b) *There exists one $\nu \in \Pi$ such that $\omega\big(Q(x_\mu)\big) = -\frac{1}{2}\delta_{\nu\mu}$ for $\mu \in \Pi$, $\omega + \nu$ is a root and ω is strongly orthogonal to all other roots in Π.*

c) *There exist two roots* $\nu \neq \nu' \in \Pi$ *such that* $\omega + \nu$ *and* $\omega - \nu'$ *are roots,* $\omega(Q(x_\mu)) = \frac{1}{2}(\delta_{\nu'\mu} - \delta_{\nu\mu})$ *for* $\mu \in \Pi$, *and* ω *is strongly orthogonal to all other roots in* Π.

5) *For a non-compact root* $\omega \in \Omega_P^+$ *there are three mutually exclusive possibilities :*

a) $\omega \in \Pi$.

b) *There exists one* $\nu \in \Pi$ *such that* $\omega(Q(x_\mu)) = \frac{1}{2}\delta_{\nu\mu}$ *for* $\mu \in \Pi$, $\nu - \omega$ *is a root and* ω *is strongly orthogonal to all other roots in* Π.

c) *There exist two roots* $\nu \neq \nu' \in \Pi$ *such that* $\nu - \omega$ *and* $\nu' - \omega$ *are roots,* $\omega(Q(x_\mu)) = \frac{1}{2}(\delta_{\nu'\mu} + \delta_{\nu\mu})$ *for* $\mu \in \Pi$, *and* ω *is strongly orthogonal to all other roots in* Π.

6) *Let* $\alpha \in \Omega$ *and* $\nu \in \Pi$. *Then* $\nu \pm \alpha$ *are not both roots.*

Proof. — 1) See [M] p.359 and [HarC1] p. 761.

2) This follows immediately from 1).

3) This is proved in [HarC2] pp.581-583.

4) This follows from [HarC2] p.586, [M] p.359 and Lemma 2.5.

5) See [HarC2] pp.587-588 and Lemma 2.5.

6) See [HarC2] p.585. □

Remark 2.8. — Theorem 2.7.2)-6) clearly may be generalized to the semisimple case $L = \sum_{i=1}^{n} L_i$. Then a basis of the root system Λ contains exactly m non-compact roots, one for each non-compact ideal L_i, $i = 1, \ldots, m$. □

COROLLARY 2.9. *The following assertions hold for* $\omega \in \Omega_P^+$:

1) *For* $\omega' \in \Omega_P^+$ *wa have*

$$\omega(Q(x_{\omega'})) = \frac{q}{2} \geqslant 0,$$

where q *is the greatest integer such that* $\omega - q\omega'$ *is a root,*

2) $\omega\big(Q(x_\nu)\big) \in \left\{0, \frac{1}{2}\right\}$ *for all* $\nu \in \Pi$ *and* $\omega \notin \Pi$, *and*

3) $\omega\Big(\sum\limits_{\nu\in\Pi} Q(x_\nu)\Big) \in \left\{\frac{1}{2}, 1\right\}$.

Proof. — 1) This follows from Lemma 2.6 and Theorem 2.7.2.

2), 3) These are consequences of Theorem 2.7.5. □

PROPOSITION 2.10. — *For* $\nu, \mu \in \Pi$ *we have*

(i) $\nu\big(Q(x_\mu)\big) = \delta_{\nu\mu}$,

(ii) $[Ix_\nu, x_\mu] = Q(x_\nu) \cdot \delta_{\nu\mu}$, *and*

(iii) $[Ix_\nu, Q(x_\mu)] = x_\nu \cdot \delta_{\nu\mu}$.

Proof. — For $\nu \neq \mu \in \Pi$ neither $\nu + \mu$ nor $\nu - \mu$ is a root. Therefore $\nu\big(Q(x_\mu)\big) \cdot Ix_\nu = [Q(x_\mu), x_\nu] = \big[[Ix_\mu, x_\mu], x_\nu\big] = \big[[Ix_\mu, x_\nu], x_\mu\big] + \big[Ix_\mu, [x_\mu, x_\nu]\big] = 0$ because $[L^\nu, L^\mu] = 0$ (Theorem 2.1). This gives i) because the elements x_ν were chosen such that $\nu(\big(Q(x_\nu)\big) = 1$. The second assertion follows from the definition of $Q(x_\nu)$ and $[L^\nu, L^\mu] = \{0\}$ for $\nu \neq \mu$. With $[Ix_\nu, Q(x_\mu)] = -\nu\big(Q(x_\mu)\big)IIx_\nu = x_\nu \cdot \delta_{\nu\mu}$ we prove iii). □

COROLLARY 2.11. — *Let* $k = \mathrm{card}(\Pi)$ *and* $S \stackrel{\mathrm{def}}{=} \bigoplus\limits_{\nu\in\Pi} \langle x_\nu \rangle$. *Then* $S \cong \mathrm{sl}(2, \mathbb{R})^k$ *with*

$$U_\nu = 2Q(x_\nu), \quad A_\nu = 2Ix_\nu \quad \textit{and} \quad B_\nu = 2x_\nu.$$

Proof. — With Lemma 2.3 we see that $\langle x_\nu \rangle = \mathbb{R}Ix_\nu \oplus \mathbb{R}x_\nu \oplus \mathbb{R}Q(x_\nu) \cong \mathrm{sl}(2, \mathbb{R})$, because $\nu\big(Q(x_\nu)\big) = 1$ for every $\nu \in \Pi$ and from Proposition 2.10 that S is a direct sum of these ideals. In addition we have

$$\begin{aligned}
&[U_\nu, B_\nu] = [2Q(x_\nu), 2x_\nu] = 4Ix_\nu = 2A_\nu,\\
&[U_\nu, A_\nu] = [2Q(x_\nu), 2Ix_\nu] = -4x_\nu = -2B_\nu, \quad \text{and}\\
&[A_\nu, B_\nu] = [2Ix_\nu, 2x_\nu] = 4Q(x_\nu) = 2U_\nu.
\end{aligned}$$

This completes the proof. □

For a subalgebra A of the Lie algebra L we define

$$\mathrm{Inn}_L(A) = \langle e^{\mathrm{ad}\,A}\rangle \quad \text{and} \quad \mathrm{INN}_L(A) = \overline{\mathrm{Inn}_L(A)}.$$

We usually omit the subscript if no confusion is possible. For every pointed generating invariant cone $W \subseteq L$ we have, according to [HiHoL] III.2.15, that

$$\text{(12)} \qquad \mathrm{int}\, W = (\mathrm{Inn}\, L)\mathrm{algint}(H \cap W).$$

If $N(H) \overset{\mathrm{def}}{=} \{g \in \mathrm{INN}\, L : I_g(\mathrm{INN}_L H) = \mathrm{INN}_L H\}$ is the normalizer of the maximal torus $\mathrm{INN}_L H$ in $\mathrm{INN} L$, then $N(H) \subseteq \mathrm{INN}_L K_H$ and the quotient group $N(H)/\mathrm{INN}_L H$ is finite ([HiHoL], III.5.6). We call it the *Weyl group* $\mathcal{W}(H,L) = \mathcal{W}$ of L. For $\nu \in \mathcal{W}$ with $\nu = n(\mathrm{INN}_L H)$, $n \in N(H)$ and $h \in H$ we have

$$\text{(13)} \qquad \nu \cdot h = n(h) \quad \text{and} \quad e^{\mathrm{ad}\,\nu\cdot h} = n \circ e^{\mathrm{ad}\,h} \circ n^{-1}.$$

It can be shown ([S] p.151) that the Weyl group agrees with the group W_K of automorphisms of H generated by the reflections on the hyperplanes $\ker \omega$ for $\omega \in \Omega_K^+$.

From now on we identify the duals of L and H with L and H respectively using the non-degenerate symmetric bilinear form $-B$ where B is the Cartan Killing form of L. Then $-B$ is positive definite on H ([HiHoL] III.6.8). Consequently we set for a cone $W \subseteq L$ and a cone $C \subseteq H$:

$$W^* \overset{\mathrm{def}}{=} \{x \in L : B(x,y) \leqslant 0 \quad \text{for all} \quad y \in W\}$$

and

$$C^\star \overset{\mathrm{def}}{=} \{h \in H : B(h,c) \leqslant 0 \quad \text{for all} \quad c \in C\}.$$

The next Lemma was already proved in [Pa] and [O1] for simple Lie algebras but the same proof works in the general case.

LEMMA 2.12. — *Let $W \subseteq L$ be a pointed generating invariant cone, then*

$$(W \cap H)^\star = W^* \cap H.$$

Proof. — That $W^* \cap H \subseteq (W \cap H)^\star$ is clear. Let $p : L \to H$ be the orthogonal projection. Then p is the averaging operator for the action of the compact group $\mathrm{INN}H$. Consequently $p(W) = W \cap H$. If $h \in (W \cap H)^\star$ and $x \in W$, we get

$$B(x,h) = B\big(p(x),h\big) \leqslant 0$$

because $p(x) - x$ is orthogonal to H. □

DEFINITION 2.13. — *The following two cones play the role of a minimal and a maximal trace of an invariant cone in H :*

$$C_{\min} \stackrel{\text{def}}{=} \sum_{\nu \in \Omega_P^+} \mathbb{R}^+ Q(x_\nu) \quad \text{and}$$

$$C_{\max} \stackrel{\text{def}}{=} C_{\min}^\star = \{h \in H : \omega(h) \geqslant 0 \quad \text{for all} \quad \omega \in \Omega_p^+\}.$$

That $C_{\min}^\star = \{h \in H : \omega(h) \geqslant 0 \text{ for all } \omega \in \Omega_p^+\}$ follows from Lemma 2.5. One knows from [HiHoL] III.9.15, that these two cones are invariant under the Weyl group, that $C_{\min}$ is generating and that $C_{\max}$ is pointed because L is semisimple. The fact that $C_{\min} \subseteq C_{\max}$ (Corollary 2.9) implies that both are pointed and generating.

THEOREM 2.14 (Classification of Invariant Cones). — *Let $W \subseteq L$ be a pointed generating invariant cone. Then there exists a choice of positive roots $\Omega^+ \subseteq \Omega$ such that the cone $C \stackrel{\text{def}}{=} W \cap H$ is invariant under the Weyl group and satisfies*

$$C_{\min} \subseteq C \subseteq C_{\max}. \tag{14}$$

Conversely, if $C \subseteq H$ is a pointed generating cone which is invariant under the Weyl group and satisfies (14), then there exists a pointed generating invariant cone $W \subseteq L$ with $W \cap H = C$.

Proof. — Let $C = W \cap H$ for a pointed generating cone $W \subseteq L$. We know from ([HiHoL] III.9.18) that C is invariant under the Weyl group and $\omega(C)Q(x_\omega) \subseteq C$ for every non-compact root ω because the set $\omega(C)Q(x_\omega)$ does not depend on the choice of positive roots. If one replaces ω by $-\omega$, one has to change the complex structure $I|_{L_\omega}$ which leads to a replacement of $Q(x_\omega)$ by $-Q(x_\omega)$. Therefore $\omega(C) \neq \mathbb{R}$ for all $\omega \in \Omega_P^+$. Hence $C^\star$ contains either ω or $-\omega$. If $E \subseteq \widehat{H}$ is a half space which contains C, satisfies $C \cap \partial E = \{0\}$ and no compact root lies on its boundary, we set $\Omega^+ \stackrel{\text{def}}{=} \Omega \cap E$. Then $\omega(C) = \mathbb{R}^+$ for all $\omega \in \Omega_P^+$. This implies that $C \subseteq C_{\max}$ and $Q(x_\omega) \in C$. This leads to $C_{\min} \subseteq C$ and completes the proof of the first part. Conversely (14) implies that $\omega(C) = \mathbf{R}^+$ for every $\omega \in \Omega_P^+$, hence

$$\omega(C)Q(x_\omega) = \mathbf{R}^+ Q(x_\omega) \subseteq C_{\min} \subseteq C.$$

Now a further application of [HiHoL] III.9.18, completes the proof. □

We say that a root $\omega \in \Omega_P^+$ is *long* if the associated complex root $\lambda \in \Lambda^+$ with $\omega = -i\lambda|_H$ is long.

PROPOSITION 2.15. — *Suppose that L is simple. Then the following assertions hold :*

1) *All roots $\nu \in \Pi$ are long.*

2) *If $\omega \in \Omega_P^+$ is short, then there are two long roots $\nu, \nu' \in \Omega_P^+$ such that $Q(x_\omega) = Q(x_\nu) + Q(x_{\nu'})$.*

3) *All long roots in Ω_P^+ are conjugate under the Weyl group $\mathcal{W}$.*

Proof. — 1) See [Pa], p.219.

2) Let $\omega \in \Omega_P^+$ be a short root and $\omega = -i\beta|_H$ for $\beta \in \Lambda^+$. With [Pa] p.219, we find two long roots λ, $\lambda' \in \Lambda^+$ with $\beta = \frac{1}{2}(\lambda + \lambda')$. For $h \in H$, $\nu = -i\lambda|_H$, and $\nu' = -i\lambda'|_H$ we have

$$\begin{aligned} B(h, Q(x_\omega)) &= -\omega(h)B(x_\omega) = \frac{i}{2}(\lambda + \lambda')(h)B(x_\omega) \\ &= -\frac{B(x_\omega)}{2}\,(\nu(h) + \nu'(h)) = \frac{B(x_\omega)}{2B(x_\nu)}B(h, Q(x_\nu) + Q(x_{\nu'})), \end{aligned}$$

because ν and ν' are long. One checks easily that Λ cannot be of type G_2, hence $(\lambda, \lambda) = 2(\beta, \beta)$ ([Hu] pp.58,59). This implies with Lemma 2.5 that

$$B(x_\nu) = B(x_{\nu'}) = \frac{1}{2}\,B(x_\omega) = \frac{1}{(\lambda, \lambda)} = \frac{1}{(\lambda', \lambda')} = \frac{1}{2(\beta, \beta)}.$$

Therefore $Q(x_\omega) = Q(x_\nu) + Q(x_{\nu'})$.

3) We choose the minimal root $\nu_1 \in \Pi$ and assume that not all long roots $\omega \in \Omega_P^+$ are conjugate to ν_1. Let $\omega \in \Omega_P^+$ be minimal with this property. Firstly we assume that there exists $\nu \in \Pi$ with $\omega - \nu \in \Omega_P^+ \setminus \{0\}$. Then ν is long and lower than ω, consequently it is conjugate to ν_1. The fact that ω is long implies with Lemma 2.5 and Theorem 2.7.c) that

$$\frac{1}{2} = \omega(Q(x_\nu)) = \nu(Q(x_\omega)).$$

Now Lemma 2.6 and Theorem 2.7 show that $\nu+2(\omega-\nu)=2\omega-\nu$ is no root and the reflection generated by $\omega-\nu$ interchanges ν and ω, a contradiction to the fact that ω is not conjugate to ν_1. Consequently we may assume that $\omega-\nu \notin \Omega_P^+ \setminus \{0\}$ for all $\nu \in \Pi$. We choose $\nu_i \in \Pi$ minimal with the property that $\nu_i-\omega \in \Omega_K^+$ (Theorem 2.7.5). Then ω is strongly orthogonal to all roots $\nu_j \in \Pi$ with $j<i$. If $i=1$, we get $\nu_1=\omega$ (Theorem 2.7.1) which contradicts our hypothesis. If $i>1$, we get $\omega=\nu_i$ (Theorem 2.7.3.c) and we find a baseroot $\alpha \in \Omega_K^+$ with $\omega-\alpha \in \Omega_P^+$, hence $\omega+\alpha \notin \Omega_P^+$ (Theorem 2.7). The other end $\omega-p\alpha$ of the α-string through ω is positive (Theorem 2.7.1) and conjugate to ω under the reflection s_α on $\ker\alpha$ which is contained in $\mathcal{W}=\mathcal{W}_K$, a final contradiction. □

COROLLARY 2.16. — *We have*

$$C_{\min}=\sum_{w\in\mathcal{W}}\mathbb{R}^+w\big(Q(x_\nu)\big)\quad \textit{for every long root}\quad \nu\in\Omega_P^+.$$

Proof. — This follows from the definition of $C_{\min}$ and Proposition 2.15. □

We know from Theorem 2.14 and Definition 2.13 that, if L is simple, there are invariant cones $W_{\min}$ and $W_{\max}$ in L with $W_{\min}\cap H=C_{\min}$ and $W_{\max}\cap H=C_{\max}$. Now let $L=\bigoplus_{i=1}^n L_i$ be semisimple. The fact that the cones $C_{\min}$ and $C_{\max}$ are adapted to the decomposition of L implies that there are invariant wedges

$$W_{\max}=\sum_{i=1}^m W_{\max,i}+\sum_{i=m+1}^n L_i \quad\text{and}\quad W_{\min}=\sum_{i=1}^m W_{\min,i}$$

where the cones with subscript i are the mininal and maximal cones in the non-compact simple Lie algebras L_i for $i\leqslant m$.

With respect to the maximal abelian subalgebra $D\overset{\text{def}}{=}\operatorname{span}\{Ix_\nu : \nu\in\Pi\}\subseteq P_H$ we get a real root of decomposition of L :

$$L=D\oplus\sum_{\rho\in\Sigma_D\setminus\{0\}}L_D^\rho\oplus\big(K(H)\cap L_D^0\big),$$

where Σ_D denotes the set of linear functionals on D such that

$$L_D^\rho=\{x\in L:[a,x]=\rho(a)x\quad\text{for all}\quad a\in D\}\neq\{0\}.$$

We define the functionals $\rho_\nu \in \widehat{D}$ by $\rho_\nu(Ix_\mu) = \delta_{\nu\mu}$ for all $\mu \in \Pi$ and choose a positive system $\Sigma_D^+ \subseteq \Sigma_D$ which contains functionals. Then

$$T \stackrel{\text{def}}{=} D \oplus \sum_{\rho \in \Sigma_D^+} L_D^\rho$$

is a Borel algebra of L.

DEFINITION 2.17. — *We fix the notations for the following special elements in L :*

$$b \stackrel{\text{def}}{=} \sum_{\nu\in\Pi} x_\nu, \quad u \stackrel{\text{def}}{=} z + b \quad \text{and} \quad l \stackrel{\text{def}}{=} z - Q(b) = z - \sum_{\nu\in\Pi} Q(x_\nu)$$

where $z \in Z_K$ is chosen such that $\omega(z) = 1$ for all $\omega \in \Omega_P^+$. We write the corresponding elements in the ideals L_i with a subscript i. □

LEMMA 2.18. — *For $\nu \in \Pi$ and $x_\nu \in L^\nu$ the following assertions hold :*

1) $x_\nu + Q(x_\nu) \in L_D^{\rho\nu}$, $x_\nu - Q(x_\nu) \in L_D^{-\rho\nu}$, *and*

2) $l = z - \sum_{\nu\in\Pi} Q(x_\nu) \in L_D^0 \cap H$.

Proof. — 1) follows immediately from Proposition 2.10 and 2) from

$$[Ix_\mu, l] = [Ix_\mu, z] - \sum_{\nu\in\Pi} \delta_{\nu\mu} x_\nu = -\mu(z) I I x_\mu - x_\mu = 0.$$

□

LEMMA 2.19. — *If L is simple, we have*

$$u^\perp = K'_H \oplus T = \{x \in L : B(x,u) = 0\}.$$

Proof (See also [V] p.10). — The space P_H is orthogonal to K_H, therefore $B(z,[k,k']) = B([z,k],k') = 0$ which leads to

$$B(u, K'_H) = B(z, K'_H) = \{0\}.$$

For $\rho + \rho' \neq 0$ the associated root of spaces are orthogonal ([S] p.151). Therefore

$$\begin{aligned} B(u,T) &= B\Big(l + \sum_{\nu\in\Pi}(Q(x_\nu) + x_\nu), T\Big) \\ &\subseteq B\Big(L_D^0 \cap K_H + \sum_{\rho\in\Sigma_D^+} L_D^\rho, D + \sum_{\rho'\in\Sigma_D^+} L_D^{\rho'}\Big) \subseteq B(K_H, D) = \{0\}. \end{aligned}$$

Consequently the hyperplane $K'_H \oplus T$ agrees with $u^\perp$. □

DEFINITION 2.20. — *The following cone plays an essential role for the globality of invariant cones in simple Lie groups (Section 3 and [O2], p.311). We define the cone C_0 by its dual*

$$C_0^\star \stackrel{\text{def}}{=} C_{\min} + \sum_{w\in\mathcal{W}} \mathbb{R}^+ w(l) = \operatorname{conv}(C_{\min} + \mathcal{W}\mathbb{R}^+ l).$$

PROPOSITION 2.21. — *Let L be simple. Then the following assertions hold :*

1) $l \in C_0^\star \subseteq C_{\max}$ *and* $l \in S^\perp$ *for* $S = \bigoplus_{\nu\in\Pi} \langle x_\nu \rangle$.

2) $Q(x_\nu) \pm x_\nu \in W_{\min}$ *for all* $\nu \in \Omega_P^+$.

3) *In L exists an invariant pointed generating cone W_0 with $W_0 \cap H = C_0$.*

4) $u \in W_0^* \subseteq W_{\max}$.

Proof. — 1) (Cf. [V] p.9). Corollary 2.9.3) implies for every $\omega \in \Omega_P^+$ that

$$\omega(l) = \omega(z) - \omega\big(Q(b)\big) = 1 - \omega\big(Q(b)\big) \geqslant 0.$$

Therefore $l \in C_{\max}$ and $C_0^* \subseteq C_{\max}$ because $C_{\max}$ is invariant under the Weyl group (Definition 2.13). For $\omega \in \Pi$ we have $\omega(l) = 0$ (Proposition 2.10), hence $B\big(l, Q(x_\omega)\big) = 0$ (Lemma 2.5). This proves that

$$l \in S^\perp \cap H = \bigcap_{\omega\in\Pi} \ker\omega.$$

2) With Proposition 2.10 we see that $[Ix_\nu, Q(x_\nu)] = x_\nu$. For $t \in \mathbb{R}$ this leads to

$$e^{\pm \operatorname{ad} tIx_\nu} Q(x_\nu) = \cosh(t)Q(x_\nu) \pm \sinh(t)x_\nu \in W_{\min}.$$

The closedness of $W_{\min}$ now shows that

$$\lim_{t\to\infty} \frac{1}{\sinh(t)}\, e^{\pm t \operatorname{ad} Ix_\nu} Q(x_\nu) = Q(x_\nu) \pm x_\nu \in W_{\min}.$$

3) This follows from 1) and Theorem 2.14.

4) With 1) and 2) we see that

$$u = l + Q(b) + b \in W_0^* + W_{\min} \subseteq W_0^* \subseteq W_{\max}.$$

□

LEMMA 2.22. — *Let $q : L \to P(H) \oplus Z_K$ be the orthogonal projection with* $\ker q = K'_H$, *then we have, for $h \in H_i$, that*

$$(15) \qquad q(h) = \frac{B(h, z_i)}{B(z_i)}\, z_i \quad \text{and} \quad q\big(Qx_\nu)\big) = -\frac{B(x_\nu)}{B(z_i)}\, z_i.$$

If $x = \sum_{\nu \in \Pi} t_\nu x_\nu$, *we get :*

$$(16) \quad e^{\operatorname{ad} x} h = h + \sum_{\nu \in \Pi_i} \nu(h)\Big((\operatorname{cosh} t_\nu - 1)Q(x_\nu) - \operatorname{sinh} t_\nu I x_\nu\Big) \quad \text{and}$$

$$(17) \qquad q(e^{\operatorname{ad} x} h)$$
$$= -\frac{z_i}{B(z_i)}\Big(\sum_{\nu \in \Pi_i} \nu(h) B(x_\nu)(\cosh(t_\nu) - 1) B(h, z_i)\Big) - \sum_{\nu \in \Pi_i} \nu(h) \operatorname{sinh}(t_\nu) I x_\nu.$$

Proof. — (15) : The first part follows directly from

$$B\Big(h - \frac{B(h, z_i)}{B(z_i)}\, z_i, z_i\Big) = 0 \quad \text{and} \quad \ker q \cap H_i = z_i^\perp \cap H_i = K'_H \cap H_i.$$

In view of Lemma 2.5, for $\nu \in \Pi$ we have

$$q\big(Q(x_\nu)\big) = \frac{B\big(Q(x_\nu), z_i\big)}{B(z_i)}\, z_i = -\frac{\nu(z_i) B(x_\nu)}{B(z_i)}\, z_i = -\frac{B(x_\nu)}{B(z_i)}\, z_i.$$

(16) : The subspace $ID = \sum_{\nu \in \Pi} \mathbb{R} x_\nu \subseteq P_H$ is abelian. By successive application of $e^{\operatorname{ad} t_{\nu_i} x_{\nu_i}}$ for $\nu_i \in \Pi$ we find with [HiHoL] III.7.8, that

$$\begin{aligned} e^{\operatorname{ad} x} h &= e^{\operatorname{ad} t_{\nu_1} x_{\nu=1}} \cdot \ldots \cdot e^{\operatorname{ad} t_{\nu_{k_i}} x_{\nu_{k_i}}} h \\ &= h + \sum_{\nu \in \Pi_i} \nu(h)\Big((\cosh t_i - 1)Q(x_\nu) - \operatorname{sinht} t_\nu I x_\nu\Big). \end{aligned}$$

(17) : This follows by combining 1) and 2). □

LEMMA 2.23. — *Let $W \subseteq L$ be a pointed generating invariant cone containing $W_{\min}$,*

$$U_i \stackrel{\text{def}}{=} 2 \sum_{\nu \in \Pi_i} Q(x_\nu), \quad A_i \stackrel{\text{def}}{=} 2 \sum_{\nu \in \Pi_i} I x_\nu, \quad B_i \stackrel{\text{def}}{=} 2 \sum_{\nu \in \Pi_i} x_\nu,$$

$E \overset{\text{def}}{=} \operatorname{span}\{U_i, A_i, B_i : i = 1, \dots, m\}$ and $V \overset{\text{def}}{=} (W + K'_H) \cap E$. Suppose that there exist numbers $\widetilde{\delta}_i \in \mathbb{R}^+$ with

$$\sum_{i=1}^{m} \widetilde{\delta}_i (U_i + A_i) \in V^*,$$

then

$$\sum_{i=1}^{m} \delta_i l_i \in W^*$$

for $\delta_i = \dfrac{2\widetilde{\delta}_i}{B(B_i)}$. The subalgebra E is isomorphic to $\mathrm{sl}(2)^m$.

Proof. — Using Corollary 2.11 we see that $E \cong \mathrm{sl}(2)^m$. We set $I \overset{\text{def}}{=} \{i : \widetilde{\delta}_i \neq 0\}$, $U = \sum_{i=1}^{m} U_i$, $A = \sum_{i=1}^{m} A_i$, $B = \sum_{i=1}^{m} B_i$ and

$$\delta_i = \frac{2\widetilde{\delta}_i}{B(B_i)} = \frac{\widetilde{\delta}_i}{2 \sum_{\nu \in \Pi_i} B(x_\nu)}.$$

To get a contradiction, we assume that $\sum_{i=1}^{m} \delta_i l_i \notin W^*$. Firstly we find an element $h \in W$ with $B(h, \sum_{i=1}^{m} \delta_i l_i) > 0$. For $\nu \in \Pi$ we have $B\big(l_i, Q(x_\nu)\big) = 0$ for $i = 1, \dots, n$ (Lemma 2.21.1) and $Q(x_\nu) \in W_{\min} \subseteq W$. Adding scalar multiples of the $Q(x_\nu)$ to h and rescaling, we may assume that $\omega(h) = 1$ for all $\omega \in \Pi$. The invariance of W implies that $e^{\operatorname{ad} tB} h \in W$ for all $t \in \mathbb{R}$. Using the previous lemma, we see that the following element is contained in $q(W)$:

$$\begin{aligned}q(e^{t\,\mathrm{ad}\,B}h) &= -\sum_{i=1}^{m}\frac{z_i}{B(z_i)}\Big(\sum_{\nu\in\Pi_i}B(x_\nu)(\cosh(2t)-1)-B(h,z_i)\Big)\\ &\quad-\sum_{\nu\in\Pi}\sinh(2t)Ix_\nu\\ &= -\sum_{i=1}^{m}\frac{z_i}{B(z_i)}\Big(\sum_{\nu\in\Pi_i}B(x_\nu)\cosh(2t)+\sum_{\nu\in\Pi_i}B\big(Q(x_\nu),h\big)\\ &\quad-B(h,z_i)\Big)-\frac{\sinh(2t)}{2}A\\ &= -\sum_{i=1}^{m}\frac{z_i}{B(z_i)}\Big(\frac{1}{4}B(B_i)\cosh(2t)-B(h,l_i)\Big)-\frac{\sinh(2t)}{2}A.\end{aligned}$$

We have, according to Lemma 2.22, that

$$q(U_i)=2\sum_{\nu\in\Pi_i}q\big(Q(x_\nu)\big)=-2\sum_{\nu\in\Pi_i}\frac{B(x_\nu)}{B(z_i)}z_i=-\frac{1}{2}\frac{B(B_i)}{B(z_i)}z_i.$$

The fact that $q(e^{t\,\mathrm{ad}\,B}h)\in W+K'_H$ implies that the following element is contained in $V=(W+K'_H)\cap E$ because its image under q agrees with $q(e^{t\,\mathrm{ad}\,B}h)$.

$$\begin{aligned}X &\stackrel{\mathrm{def}}{=}\sum_{i=1}^{m}\frac{2}{B(B_i)}U_i\Big(\frac{1}{4}B(B_i)\cosh(2t)-B(h,l_i)\Big)-\frac{\sinh(2t)}{2}A\\ &=\frac{\cosh(2t)}{2}U-\frac{\sinh(2t)}{2}A-\sum_{i=1}^{m}\frac{2B(h,l_i)}{B(B_i)}U_i.\end{aligned}$$

According to our assumption $Y\stackrel{\mathrm{def}}{=}\sum_{i=1}^{m}\widetilde{\delta}_i(U_i+A_i)\in V^*$. This shows that there exists $\lambda>0$ such that

$$\begin{aligned}\lambda B(X,Y) &= -\frac{1}{2}\sum_{i=1}^{m}\widetilde{\delta}_i\big(\cosh(2t)+\sinh(2t)\big)+\sum_{i=1}^{m}\frac{2\widetilde{\delta}_iB(h,l_i)}{B(B_i)}\\ &= -\frac{1}{2}\sum_{i=1}^{m}\widetilde{\delta}_ie^{2t}+B\Big(h,\sum_{i=1}^{m}\delta_il_i\Big)\leqslant 0\end{aligned}$$

for all $y\in\mathbb{R}$. For $t\to-\infty$ this leads to

$$B\Big(h,\sum_{i=1}^{m}\delta_il_i\Big)\leqslant 0,$$

a contradiction. □

PROPOSITION 2.24. — *Let W be a pointed generating invariant cone which contains $W_{\min}$ in the semisimple Lie algebra L. Then the following assertions are equivalent :*

1) $W^* \cap (K'_H \oplus T)^\perp \neq \{0\}$.

2) *There exists* $\widetilde{\delta_i} \geqslant 0$, $i = 1, \ldots, m$ *such that* $\sum_{i=1}^m \widetilde{\delta_i} u_i \in W^* \setminus \{0\}$ *and* $\widetilde{\delta_i} \neq 0$ *for* $i \in I \subseteq \{1, \ldots, m\}$.

3) *There exist* $\delta_i \geqslant 0$, $i = 1, \ldots, m$ *such that* $\sum_{i=1}^m \delta_i l_i \in W^* \cap H \setminus \{0\}$ *and* $\delta_i \neq 0$ *for* $i \in I \subseteq \{1, \ldots, m\}$.

Proof. — 1) $\Rightarrow$ 2) : From Lemma 2.19 we get $(K'_H \oplus T)^\perp = \sum_{i=1}^m \mathbf{R} u_i$ because $\mathbf{R} u_i = (K'_{H_i} \oplus T_i)^\perp \cap L_i$. Therefore every non-zero element $\omega \in W^* \cap (K'_H \oplus T)^\perp$ may be written as

$$\omega = \sum_{i=1}^m \widetilde{\delta_i} u_i.$$

From $z_i \in W_{\min} \subseteq W$ for $i = 1, \ldots, m$ and $B(u_i, z_i) < 0$ we find that

$$\langle \omega, z_i \rangle = -\widetilde{\delta_i} B(u_i, z_i) = -\widetilde{\delta_i} B(z_i) \geqslant 0,$$

which proves that $\widetilde{\delta_i} \geqslant 0$.

2) $\Rightarrow$ 3) : Let $\widetilde{V} \stackrel{\text{def}}{=} W + K'_H$, then $\widetilde{V}$ is a Inn K_H-invariant Lie wedge in L with $H(\widetilde{V}) = K'_H$ (Lemma 1.10) because $W \cap K'_H = \{0\}$ ([HiHoL] III.5.16). Therefore

$$\sum_{i=1}^m \widetilde{\delta_i} u_i \in W^* \cap K_H'^{\perp} = \widetilde{V}^*.$$

We set $V \stackrel{\text{def}}{=} \widetilde{V} \cap E$ with E as in Lemma 2.23. The elements l_i are orthogonal to E (Proposition 2.21), hence

$$2 \sum_{i=1}^m \widetilde{\delta_i}(u_i - l_i) = \sum_{i=1}^m \widetilde{\delta_i}(U_i + B_i) \in (\widetilde{V}^* + E^\perp) \cap E \subseteq V^*.$$

From the invariance of V under $\mathrm{Inn}(\sum_i \mathbb{R}U_i)$ we get that $\sum_{i=1}^m \widetilde{\delta}_i(U_i + A_i) \in V^*$ and with Lemma 2.23 that $\sum_{i=1}^m \delta_i l_i \in W^*$ for

$$\delta_i = \frac{2\widetilde{\delta}_i}{B(B_i)}.$$

3) $\Rightarrow$ 2) : With the notations from above we find with Lemma 2.21 that

$$\begin{aligned}\sum_i \delta_i u_i &= \sum_i \delta_i \Big(l_i + \sum_{\nu\in\Pi_i} (Q(x_\nu) + x_\nu)\Big) \\ &\in \sum_i \delta_i l_i + W_{\min} \subseteq W^* + W_{\min} \subseteq W^*.\end{aligned}$$

□

3. Globality of invariant cones in semisimple Lie groups.

PROPOSITION 3.1. — *Let G be a Lie group, $K \subseteq G$ a compact subgroup, $L = \mathbf{L}(G)$, $W \subseteq L$ a pointed generating $\mathbf{Ad}(K)$-invariant cone with $\mathbf{L}(K) \cap W = \{0\}$ and $V \stackrel{\text{def}}{=} W + \mathbf{L}(K)$. Then the following assertions hold :*

1) *W is global in G if and only if V is global in G.*

2) *W is controllable in G if and only if V is controllable in G.*

Proof. — 1) This follows from Proposition III.5 in [N1].

2) If W is controllable in G, it is clear that V is controllable in G, too. Let us assume that W is not controllable in G, i.e., $S \stackrel{\text{def}}{=} \langle \exp W \rangle \neq G$. Then $\widetilde{V} \stackrel{\text{def}}{=} \mathbf{L}(S) \neq L$ is a wedge which is global in G (Lemma 1.2). Applying [N1], III.5, we see that $\widetilde{V} + \mathbf{L}(K)$ is global in G. So we are done if we can show that $\widetilde{V} + \mathbf{L}(K) \neq L$. If this is false, we have $H\big(\widetilde{V} + \mathbf{L}(K)\big) = H(\widetilde{V}) + \mathbf{L}(K) = L$. This leads to

$$\widetilde{V} \cap \mathbf{L}(K) \not\subseteq H(\widetilde{V}),$$

a contradiction to [N1] III.5. □

Remark 3.2. — If $K \stackrel{\text{def}}{=} \langle \exp K'_H \rangle$ is the subgroup of G with Lie algebra K'_H, then K is a compact semisimple group and Proposition 3.1 is applicable with $V = W + K'_H$ because $W \cap K'_H = \{0\}$ ([HiHoL] III.5.16). □

COROLLARY 3.3. — *Let L be a semisimple Lie algebra, $W \subseteq L$ an invariant generating cone, L_I the sum of the non-compact ideals and L_{II} the sum of the compact ideals. The simply connected Lie group G with $\mathbf{L}(G) = L$ is a direct product $G = G_I \times G_{II}$ with $\mathbf{L}(G_I) = L_I$, $\mathbf{L}(G_{II}) = L_{II}$ and the Lie wedge W is global [controllable] in G if and only if the projection $dp_I(\mathbf{1})W = W \cap L_I$ is global [controllable] in G_I, where p_I is the projection $G \to G_I$ onto the first factor.*

Proof. — We want to apply Proposition 3.1 with $K = G_{II}$ and $V = W + L_{II}$. It follows from Lemma III.8 in [N1] that V is global in G if and only if $dp_I(\mathbf{1})V = dp_I(\mathbf{1})W$ is global in G_I. The Lie algebra L is a G_{II}-module under the adjoint action with $L_{\text{fix}} = L_I$ and $dp_I(\mathbf{1}) : L \to L_I$ is the averaging operator. Therefore $dp_I(\mathbf{1})W = W \cap L_I$. It remains to show that V is controllable in G if and only if $V \cap L_I = dp_I(\mathbf{1})V$ is controllable in G_I, but this follows directly from

$$\langle \exp V \rangle = \langle \exp\big((V \cap L_I) + L_{II}\big)\rangle = \langle \exp(V \cap L_I)\rangle G_{II}.$$

This semigroup agrees with G if and only if $\langle \exp(V \cap L_I)\rangle$ agrees with G_I.

□

PROPOSITION 3.4. — *Let G be a semisimple simply connected Lie group with $\mathbf{L}(G) = L$. Suppose that $W \subseteq L$ is an $e^{\operatorname{ad} K_H}$-invariant wedge and $W^* \cap (K'_H \oplus T)^\perp \neq \{0\}$. Then W is not controllable in G. If $\omega \in W^* \cap (K'_H \oplus T)^\perp$ with $\omega(L_j) \neq \{0\}$, then $L_j \not\subseteq H(\mathbf{L}(S))$, where $S = \langle \exp W \rangle$.*

Proof (Cf. [V] p.7, where the same idea is used in the simple case for invariant cones). — Let $\omega \in W^* \cap (K'_H \oplus T)^\perp \setminus \{0\}$, $T^\diamond = \langle \exp T \rangle$ and $K^\diamond_H = \langle \exp K_H \rangle$. According to [He] p.270, the mapping $T^\diamond \times K^\diamond \to G$, $(t,k) \mapsto tk$ is a diffeomorphism. Therefore $T^\diamond$ and $K^\diamond$ are simply connected and we may apply Lemmas IV.10 and IV.11 in [N1] to find a function $f \in C^\infty(G)$ with $f'(tk) = \omega \circ \operatorname{Ad}(k)$ for all $t \in T^\diamond$ and $k \in K^\diamond$. The function f is non-constant and W-positive because W is invariant under $e^{\operatorname{ad} K_H}$ and $\omega = f'(\mathbf{1}) \in W^*$. The last assertion follows from the fact that f is constant on the unit group $H(S)$ of S and $\omega(L_j) \neq \{0\}$ implies that $df(\mathbf{1})$ does not vanishes on L_j. □

THEOREM 3.5 (Controllability Theorem). — *Let W be a pointed generating invariant wedge with $W_{\min} \subseteq W$ in the semisimple Lie algebra L. Then the following are equivalent :*

1) *W is not controllable in the simply connected Lie group G with* $\mathbf{L}(G) = L$.

2) $W^* \cap (K'_H \oplus T)^\perp \neq \{0\}$.

Proof. — 1) $\Rightarrow$ 2) : According to Remark 3.2 the controllability of W is equivalent to the controllability of $\widetilde{V} = W + K'_H$ in G. We use the notations of Lemma 2.23 to show that $V \stackrel{\text{def}}{=} \widetilde{V} \cap E$ is not controllable in $E^\diamond \stackrel{\text{def}}{=} \langle \exp E \rangle$. We assume that this is false and set $S \stackrel{\text{def}}{=} \langle \exp \widetilde{V} \rangle$. The invariance of $\widetilde{V}$ under $e^{\operatorname{ad} K_H}$ implies that $\mathbf{L}(S)$ is an $e^{\operatorname{ad} K_H}$-invariant wedge containing E because $\langle \exp \widetilde{V} \cap E \rangle = E^\diamond \subseteq S$. We show that $F \stackrel{\text{def}}{=} H(\mathbf{L}(S)) = L$. The simplicity of the ideals L_i implies the irreducibility of the K_H-module $P_H \cap L_i$ (Lemma 2.4), but the K_H-module F contains $B_i \in P_H \cap L_i$, hence $P_H \subseteq F$. The fact that the edge of the Lie wedge $\mathbf{L}(S)$ is a subalgebra shows that

$$[P_H, P_H] = \sum_{i=1}^{m} K_H \cap L_i \subseteq F$$

and therefore

$$\sum_{i=1}^{m} L_i + K'_H = \sum_{i=1}^{m} L_i + \sum_{i=m+1}^{n} L_i = L \subseteq F.$$

This proves that V is not controllable in $E^\diamond$. From

$$K'_H \cap E \subseteq \sum_{i=1}^{m} K'_H \cap \mathbb{R}U_i = \{0\}$$

we see that V is a pointed, $e^{\operatorname{ad} \sum_{i=1}^{m} \mathbb{R}U_i}$-invariant wedge in E which contains all elements $U_i + A_i$ (Proposition 2.21) and therefore the product W_1^n of the Cartan Killing cones in the $\mathrm{sl}(2,\mathbb{R})$-subalgebras $\operatorname{span}\{U_i, A_i, B_i\}$. Now Corollary 1.8 provides $\beta_i \in \mathbb{R}^+$ with $\sum_{i=1}^{m} \beta_i(U_i + A_i) \in V^* \setminus \{0\}$ and an application of Lemma 2.23 and Proposition 2.24 completes the proof.

2) $\Rightarrow$ 1) : This follows from Proposition 3.4. □

Theorem 3.5 is our main result on controllability in semisimple Lie groups. Now we shall see that much more can be said about simple Lie groups. See also [O2], where it is proved that an invariant cone W in a simple Lie algebra L with $W_{\min} \subseteq W$ is global in the associated simply connected group if and only if W is contained in W_0 (see Corollary 3.8).

LEMMA 3.6. — *Suppose that L is simple and $l \neq 0$, then $Q(x_\nu)+x_\nu \in T''$ for all $\nu \in \Pi$.*

Proof. — We claim that $\frac{1}{2}\rho_\nu \in \Sigma_D^+$ for every $\nu \in \Pi$. Since $l \neq 0$, l is not contained in $H(C_{\max}) = \{0\}$ and, according to Proposition 2.21.1 and Theorem 2.7.5. we find $\omega \in \Omega_P^+$ with $\omega(l) = 1 - \omega(b) > 0$. Now Corollary 2.9 implies that there is exactly one $\nu \in \Pi$ with $\omega\big(Q(x_\mu)\big) = \frac{1}{2}\delta_{\nu\mu}$ for $\mu \in \Pi$. It follows from [M] p.362, that we find for every $\nu' \in \Pi$ an $\omega' \in \Omega_p^+$ with $\omega'\big(Q(x_\mu)\big) = \frac{1}{2}\delta_{\nu'\mu}$ for $\mu \in \Pi$. Using Lemma III.9.5 and III.9.7 in [HiHoL] we see that we find an inner automorphism α of $L_{\mathbb{C}}$ with $\alpha^{-1}(Ix_\nu) = -iQ(x_\nu)$ and the function $\lambda \mapsto \lambda \circ \alpha^{-1}|_D : \Lambda \to \Sigma_D$ is surjective. Taking $\lambda \in \Lambda^+$ such that $\omega = -i\lambda|_H$, we see that

$$\frac{1}{2}\rho_\nu(Ix_\mu) = \frac{1}{2}\delta_{\nu\mu} = \omega\big(Q(x_\nu)\big) = \lambda \circ \alpha^{-1}(Ix_\mu)$$

and therefore $\frac{1}{2}\rho_\nu \in \Sigma_D$. Hence $\frac{1}{2}\rho_\nu \in \Sigma_D^+$. We know from Lemma 2.18 that $Q(x_\nu) + x_\nu \in L_D^{\rho_\nu}$, so it is enough to show that

$$T'' = \sum_{\rho \in \Sigma_D^+ + \Sigma_D^+} L_D^\rho.$$

It is clear that $T' = \sum_{\rho \in \Sigma_D^+} L_D^\rho$. We consider the complexification $L_{\mathbb{C}}$ of L. According to [He] p.530, we may extend $D_{\mathbb{C}}$ to a Cartan algebra $\widetilde{H}_{\mathbb{C}}$ of $L_{\mathbb{C}}$. Let $\widetilde{\Lambda}$ be the system of roots of $L_{\mathbb{C}}$ with respect to $\widetilde{H}_{\mathbb{C}}$. For $\rho \in \Sigma_D$ we have

$$L_D^\rho = (L_D^\rho)_{\mathbb{C}} \cap L = \Big(\bigoplus_{\lambda \in \Lambda_\rho} L_{\mathbb{C}}^\lambda\Big) \cap L$$

for $\Lambda_\rho = \{\lambda \in \widetilde{\Lambda} : \lambda|_D = \rho\}$. It remains to show that $L_{\mathbb{C}}^\lambda \subseteq T''_{\mathbb{C}}$ for all $\lambda \in \Lambda_{\rho_\nu}$. We choose $\lambda \in \Lambda_{\frac{1}{2}\rho_\nu}$ and a lexicographic ordering on $\widetilde{\Lambda}$ such that $\lambda \geqslant 0$ whenever $\lambda \circ \alpha^{-1}|_D \in \Sigma_D^+$ and $\lambda - 2\lambda_\nu \geqslant 0$. Then λ cannot be a base root with respect to $\widetilde{\Lambda}$. Therefore there exist roots λ_1, λ_2 with $\lambda = \lambda_1 + \lambda_2$. From [Hu] p.39, we get that

$$[L_{\mathbb{C}}^{\lambda_1}, L_{\mathbb{C}}^{\lambda_2}] \subseteq [L_D^{\lambda_1|_D}, L_D^{\lambda_2|_D}]_{\mathbb{C}} \subseteq T''_{\mathbb{C}}.$$

□

THEOREM 3.7 (Globability in Simple Lie Groups). — *Let G be a simply connected simple Lie group and $V \subseteq L = \mathbf{L}(G)$ be an $e^{\operatorname{ad} K_H}$-invariant Lie generating Lie wedge with $H(V) = K'_H$ which contains z. Then the following are equivalent :*

1) *V is global in G.*

2) *V is not controllable in G.*

3) *For all subalgebras $E \cong \mathrm{sl}(2)^k$ of L, the wedge $V \cap E$ is global in $E^\diamond = \langle \exp E \rangle$.*

4) $u = z + \sum_{\nu \in \Pi} x_\nu \in V^*$.

5) $\operatorname{int} V \cap T = \varnothing$.

6) $\operatorname{int} V \cap T' = \varnothing$.

If $l \neq 0$, these assertions are equivalent to

7) $\operatorname{int} V \cap T'' = \varnothing$.

8) *For all nilpotent subalgebras $N \subseteq L$, the wedge $V \cap N$ is global in $N^\diamond = \exp N$.*

Proof. — 1) $\Leftrightarrow$ 2) : If V is global in G, then it is not controllable in G. Conservely we assume that V is not controllable in G and set $S \overset{\text{def}}{=} \langle \exp V \rangle$. Hence $\mathbf{L}(S) \neq L$ and we have to show that the assumption $\mathbf{L}(S) \neq V$ leads to a contradiction. Then $F \overset{\text{def}}{=} H(\mathbf{L}(S)) \neq H(V) = K'_H$ ([N1] III.1) and F is an K_H-submodule of L (Lemma 1.2). Consequently

$$F = F \cap P_H \oplus K'_H \oplus F \cap Z_K.$$

If $F \cap P_H \neq \{0\}$, the irreducibility of P_H (Lemma 2.4) implies that $P_H \subseteq F$ and therefore $L = P_H + [P_H, P_H] = P_H + K_H \subseteq F$ because F is a subalgebra, a contradiction to $F \neq L$. Hence $F \cap P_H = \{0\}$. Now $F \neq K'_H$ yields $F = K_H$. The Lie wedge $\mathbf{L}(S)$ is Lie generating and cannot be contained in K_H. Consequently $\mathbf{L}(S) \cap P_H$ must be an $e^{\operatorname{ad} K_H}$-invariant pointed cone. This is impossible since $-\mathrm{id}_{P_H} = (\operatorname{ad} z)^2|_{P_H}$ would imply that $\mathbf{L}(S) \cap P_H = -\mathbf{L}(S) \cap P_H$.

1) $\Rightarrow$ 3) : If S a closed subsemigroup of the simply connected Lie group G with $\mathbf{L}(S) = V$, then $S \cap E^\diamond$ is a closed subsemigroup of $E^\diamond$ with

$$\mathbf{L}(S \cap E^\diamond) = \{x \in E : \exp(\mathbb{R}^+ x) \subseteq S \cap E^\diamond\} = \mathrm{L}(S) \cap E = V \cap E.$$

3) $\Rightarrow$ 4) : We know from Lemma 2.19 that $K'_H \oplus T = u^\perp$. Therefore we have to show that $u \in V^*$ because $B(u,z) = B(z) < 0$. If this is not true, we find an element $z+p$ in the compact base $C = \{p+z : p+z \in V, p \in P_H\}$ of the pointed cone $\widetilde{V} \overset{\text{def}}{=} V \cap (Z_K \oplus P_H)$ with $\delta \overset{\text{def}}{=} B(u, p+z) > 0$ because $V = \widetilde{V} + K'_H$ and $K'_H \subseteq u^\perp$. We set $M \overset{\text{def}}{=} \max\{B(p) : p+z \in C\}$. The set of elements $\widetilde{b} = \sum_{\nu \in \Pi} \alpha_\nu x_\nu$, whose centralizer in P_H is ID, is dense in ID because these are exactly the elements in ID for which the one-parameter group $e^{i\mathbb{R}\,\mathrm{ad}\,\widetilde{b}}$ is dense in the torus $e^{i\,\mathrm{ad}\,D} \subseteq \mathrm{Aut}(L_\mathbb{C})$ ([He] pp.247, 248, [BD] p.38). Consequently we find such an $\widetilde{b}$ with

$$B(b - \widetilde{b}) \leqslant \frac{\delta^2}{9M}.$$

In the compact set $\mathrm{I\small{NN}}(K_H)(p+z)$ there exists an element $\gamma_0(p+z)$ such that the function $\gamma \mapsto B\big(z + \widetilde{b}, \gamma(p+z)\big)$ attains its maximum at γ_0. For $k \in K_H$ this implies for $p_1 = \gamma_0(p)$ that

$$0 = \frac{d}{dt}\bigg|_{t=0} B(e^{t\,\mathrm{ad}\,k} p_1 + z, z + \widetilde{b}) = B([k, p_1, \widetilde{b}]) = B(k, [p_1, \widetilde{b}]).$$

But B is negative definite on K_H ([He] p.184) and $[p_1, \widetilde{b}] \in K_H$ which leads to $[p_1, \widetilde{b}] = \{0\}$ and therefore $p_1 \in ID$ because the centralizer of $\widetilde{b}$ in P_H is ID. Now we get

$$\begin{aligned} B(p_1 + z, u) &\geqslant B(p_1 + z, z + \widetilde{b}) - |B(p_1 + z, b - \widetilde{b})| \\ &\geqslant B(p_1 + z, z + \widetilde{b}) - \sqrt{|B(p_1 + z)|}\sqrt{B(b - \widetilde{b})} \\ &\geqslant B(p_1 + z, z + \widetilde{b}) - \sqrt{M}\sqrt{\frac{\delta^2}{9M}} \geqslant B(p_1 + z, z + \widetilde{b}) - \frac{\delta}{3} \\ &\geqslant B(p + z, z + \widetilde{b}) - \frac{\delta}{3} \geqslant B(p + z, u) - \frac{2\delta}{3} = \frac{\delta}{3} > 0. \end{aligned}$$

This shows that $p_1 + z \in (ID + z) \cap V$ and

$$B(p_1 + z, u) = B\Big(p_1, \sum_{\nu \in \Pi} x_\nu\Big) + B(z) > 0. \tag{18}$$

We choose the subalgebra $E \stackrel{\text{def}}{=} \operatorname{span}\{x_\nu, Ix_\nu, Q(x_\nu) : \nu \in \Pi\} \cong \mathrm{sl}(2,\mathbf{R})^k$ with $U_\nu = 2Q(x_\nu)$, $A_\nu = 2Ix_\nu$ and $B_\nu = 2x_\nu$. According to our assumption, $V \cap E$ is global in $E^\diamond$. To apply Theorem 1.3 we have to show that V has inner points with respect to E. As for $H(\mathbf{L}(S))$ in 1) $\Leftrightarrow$ 2) we see that $V - V = L$ because V is a Lie generating and invariant under $e^{\operatorname{ad} K_H}$, hence V has inner points and must contain z in its interior because $Z_K = L_{\text{fix}}$ for the action of the compact group $\mathrm{INN} K_H$. The orthogonal projection $q : L \to Z_K \oplus P_H$ maps the element $Q(x_\nu) \in E$ onto a positive multiple of z (Lemma 2.22), therefore $Q(x_\nu) \in \operatorname{int} V \cap E \subseteq \operatorname{int}_E(V \cap E)$ because $z \in \operatorname{int} V$. Using Theorem 1.3 we find an element $\alpha = \sum_{\nu\in\Pi} \beta_\nu \big(Q(x_\nu) + x_\nu\big) \in (V \cap E)^*$. All differences $Q(x_\nu) - Q(x_\mu)$ are contained in $z^\perp \cap H \subseteq K'_H$ and $B\big(\alpha, Q(x_\nu) - Q(x_\mu)\big) = (\beta_\nu - \beta_\mu) B\big(Q(x_\nu) - Q(x_\mu)\big) = 0$, hence $\beta_\nu = \beta > 0$ for all $\nu \in \Pi$ and $\alpha \stackrel{\text{def}}{=} \beta \sum_{\nu\in\Pi} x_\nu + Q(x_\nu) \in (V \cap E)^*$. In $q^{-1}(p_1 + z) \subseteq V$ we find, according to Lemma 2.22, the element $p_1 - \dfrac{B(z)}{B(x_\nu)} Q(x_\nu) \in V \cap E$. Applying α yields

$$\frac{1}{\beta} B\left(p_1 - \frac{B(z)}{B(x_\nu)} Q(x_\nu), \alpha\right) = B(p_1, b) - \frac{B\big(Q(x_\nu)\big)}{B(x_\nu)} B(z)$$
$$= B(p_1, b) + B(z) \leqslant 0,$$

a contradiction to $B(p_1, b) + B(z) = B(p_1 + z, u) > 0$.

4) $\Leftrightarrow$ 5) : According to the separation theorems of Hahn Banach, the existence of a functional $\omega \in V^* \cap T^\perp$ is equivalent to $\operatorname{int} V \cap T = \varnothing$. But

$$V^* \cap T^\perp = V^* \cap (K'_H)^\perp \cap T^\perp = V^* \cap (K'_H \oplus T)^\perp = V^* \cap \mathbf{R}u.$$

A glance at Lemma 2.19 completes the proof.

5) $\Rightarrow$ 6) $\Rightarrow$ 7) : This is trivial.

6), 7) $\Rightarrow$ 4) : From 6) or 7) (if $l \not\equiv 0$) and Lemma 3.6 we see that

$$F \cap \operatorname{int} V = \varnothing, \quad \text{where} \quad F \stackrel{\text{def}}{=} \operatorname{span}\{Q(x_\nu) + x_\nu : \nu \in \Pi\}.$$

Let $q : L \to P_H \oplus Z_K$ be the orthogonal projection. To get a contradiction to $\operatorname{int} V \cap F = \varnothing$, it is enough to show that $\operatorname{int} V \cap q(F) \neq \varnothing$. According to Lemma 2.22 we have

$$q\big(Q(x_\nu) + x_\nu\big) = x_\nu - \frac{B(x_\nu)}{B(z)} Z \quad \text{for all} \quad \nu \in \Pi$$

and with Lemma 2.5 we find that

$$B\Big(u, q(Q(x_\nu) + x_\nu)\Big) = B\big(u, Q(x_\nu) + x_\nu\big) = -B(x_\nu) + B(x_\nu) = 0.$$

Therefore $q(F) \subseteq u^\perp \cap (ID \oplus Z_K)$. The element $q\big(Q(x_\nu) + x_\nu\big)$ are linearly independent, therefore

$$q(F) = u^\perp \cap (ID \oplus Z_K)$$

because the dimensions are equal (Lemma 2.19). We assume that 4) is false. Then we find $p_1 \in P_H$ with $p_1 + z \in \operatorname{int} V$ such that $B(p_1 + z, u) > 0$ (see 3) $\Rightarrow$ 4)). Using the invariance of V under $\text{Inn}K_H$, we see that $-p_1 + z = e^{\pi \operatorname{ad} z}(p_1 + z) \in V$, too. But $B(u, p_1 + z) = B(z) + B(b, p_1) > 0$ and

$$B(-p_1 + z, u) = B(z) - B(p_1, b) < B(z) + B(z) = 2B(z) < 0.$$

Consequently the line segment from $p_1 + z$ to $-p_1 + z$, which lies completely in the interior of V, intersects $u^\perp \cap (ID \oplus Z_K) = q(F)$.

4) $\Rightarrow$ 2) : This is a consequence of Proposition 3.4.

1) $\Rightarrow$ 8) : This is the same proof as in 1) $\Rightarrow$ 3).

8) $\Rightarrow$ 7) : The subalgebra $N \stackrel{\text{def}}{=} T'$ of L is nilpotent. We assume that $\operatorname{int} V \cap T'' = \operatorname{int} V \cap N' \neq \emptyset$. Then the Lie wedge $\widetilde{V} \stackrel{\text{def}}{=} V \cap N$ is global in the simply connected covering group $\widetilde{N}^\diamond$ of $N^\diamond$ ([N1] III.7) and $\operatorname{int}_N \widetilde{V} \cap N' \neq \emptyset$. We show that this is impossible. Firstly we observe that $\widetilde{V}$ has non-empty interior and therefore is Lie generating in N and the semigroup $S \stackrel{\text{def}}{=} \langle \exp_{\widetilde{N}^\diamond} \widetilde{V} \rangle$ has inner points. Using [HiHoL] V.5.40, we see that $\widetilde{V}$ is contained in a half space $F \subseteq N$ such that ∂F is a subalgebra of codimension 1. A look at the classification theorem in [Ho] p.638 shows that ∂F has to be an ideal of N which contains the commutator algebra. This is a contradiction to $\operatorname{int} \widetilde{V} \subseteq \operatorname{int} F$ and $N' \subseteq \partial F$. □

Corollary 3.8. — *Let L be simple, $W \subseteq L$ an invariant pointed generating cone containing $W_{\min}$ and G the simply connected group with $\mathbf{L}(G) = L$. Then the following are equivalent :*

1) *W is global in G.*

2) *W is not controllable in G.*

3) *$W \subseteq W_0$, i.e., $u \in W^*$.*

There are invariant cones in W which are not global in the simply connected group G with $\mathbf{L}(G) = L$ if and only if $l \neq 0$.

Proof. — 1) $\Rightarrow$ 2) : This is clear from the definitions because $W \neq L$.

2) $\Rightarrow$ 3) : It follows from Remark 3.2 and Theorem 3.7 that $u \in W^*$. Proposition 2.24 implies that $l \in (W \cap H)^\star = W^* \cap H$. Therefore $C_0^\star \subseteq (W \cap H)^\star$ and $W \cap H \subseteq C_0$. Now Theorem 2.14 yields $W \subseteq W_0$.

3) $\Rightarrow$ 1) : It is clear that $l \in W_0^* \cap H \subseteq W^*$. Now Proposition 2.24 together with Remark 3.2 and Theorem 3.6 imply that $u \in W^* \cap K_H^\perp = (V + K'_H)^*$ and that W is global in G.

We prove the last assertion. If $l = 0$, we have $W_0 = W_{\max}$ and every invariant cone containing $W_{\min}$ is contained in $W_{\max}$. If $l \neq 0$, we set $h \overset{\text{def}}{=} 2Q(b) - z$. Then $h \in C_{\max}$ (Corollary 2.9) and $B(l, h) = -B(l, z) > 0$. Therefore the $\mathcal{W}$-invariant cone

$$C \overset{\text{def}}{=} C_{\min} + \sum_{w \in \mathcal{W}} \mathbb{R}^+ w(h)$$

is the trace of an invariant cone W in L which contains $W_{\min}$ but which is not contained in W_0 and therefore not global in G. □

Remark 3.9. — The simple Lie algebras containing invariant cones are the following ([Pa]) :

1) $\mathrm{su}(p,q), p \geqslant q \geqslant 1$, $l = 0 \Leftrightarrow p = q$.

2) $\mathrm{so}(p,2), p \geqslant 3$, $l = 0$.

3) $\mathrm{sp}(n,\mathbb{R}), n \geqslant 3$, $l = 0$.

4) $\mathrm{so}^*(2n), n \geqslant 3$, $l = 0 \Leftrightarrow n \in 2\mathbb{Z}$.

5) $e_{6(-14)}$, $l \neq 0$.

6) $e_{7(-25)}$, $l = 0$. □

With Lemma 3.10 to Remark 3.13 we demonstrate how Theorem 3.6 can be used in special cases to prove the globability of certain Lie wedges.

LEMMA 3.10. — *For an* $\mathrm{INN}K_H$*-invariant Lie wedge* V *in the semisimple Lie algebra* L *with* $H(V) = K'_H$ *we have*

$$q(V) = V \cap (P_H \oplus Z_K) = \mathrm{Inn}\, K_H\big(V \cap D \oplus Z_K)\big),$$

where $q : L \to P_H \oplus Z_K$ *is the orthogonal projection.*

Proof. — It is clear that

$$q(V) = (V + K'_H) \cap (P_H \oplus Z_K) = V \cap (P_H \oplus Z_K)$$

because $H(V) = K'_H = \ker q$. The invariance of $q(V)$ under $\mathrm{Inn} K_H$ shows that the set on the right is contained in $q(V)$. Conversely, if $v = \widetilde{z}+p \in q(V)$ with $\widetilde{z} \in Z_K$ and $p \in P_H$, we find with [He] p.247, an element $\gamma \in \mathrm{Inn}\, K_H$ with $\gamma(p) \in D$ and therefore $v \in \gamma^{-1}\big(V \cap (D \oplus Z_K)\big)$. □

Example 3.11. — We consider the special case $L = \mathrm{su}(p,1)$. These are exactly the simple Lie algebras containing non-degenerate invariant cones with $D = 1$ and $\Pi = \{\omega_1\}$ for an $\omega_1 \in \Omega_P^+$. The set of all $\mathrm{Inn} K_H$-invariant Lie wedges V with $H(V) = K_H$ is a one-parameter family. For $r > 0$ we define V_r by (Lemma 3.10)

$$V_r \cap (D + z) = \{z + \lambda I x_1 : B(\lambda I x_1) \leqslant -r^2 B(z)\}.$$

Then

$$q(V_r) = \{\mu z + p : B(p) \leqslant -r^2 B(\mu z), \mu \geqslant 0\}.$$

Then $u = z + x_1$ and therefore

$$\begin{aligned} u \in V_r^* &\Leftrightarrow B(z + x_1, z + \lambda x_1) \leqslant 0 \quad \text{for all } z + \lambda x_1 \in V_r \\ &\Leftrightarrow B(z) + \lambda B(x_1) \leqslant 0 \text{ for all } \lambda \text{ with } B(z)r^2 + \lambda^2 B(x_1) \leqslant 0 \\ &\Leftrightarrow B(z) + \sqrt{\frac{-B(z)}{B(x_1)}}\, r B(x_1) \leqslant 0 \\ &\Leftrightarrow r^2 \leqslant \frac{-B(z)}{B(x_1)}. \end{aligned}$$

As explicit inspection of $\mathrm{su}(p,1)$ shows, we have

$$B(z) = -2p \quad \text{and} \quad B(x_{\omega_1}) = p + 1.$$

We set $r_0 \overset{\text{def}}{=} \sqrt{\frac{-B(z)}{B(x_1)}} = \sqrt{\frac{2p}{p+1}}$. Then we see with Theorem 3.6 that V_r is global in the associated simply connected group $\mathrm{SU}(p,1)^\sim$ if and only if $r \leqslant r_0$.

For $p = 1$ we have $L = \mathrm{su}(1,1) = \mathrm{sl}(2,\mathbb{R})$, $r_0 = 1$ and V_r agrees with the cones W_r (see Corollary 1.5). Therefore Corollary 1.5 is a special case of the result above. □

PROPOSITION 3.12. — *Let L be simple and $V_0 = \{x \in P_H \oplus Z_k : B(x) \leqslant 0, B(x,z) \leqslant 0\} + K'_H$. Then V_0 is an* InnK_H*-invariant Lie wedge with $H(V_0) = K'_H$ and $q(V_0)$ is the Lorentzian cone in $P_H \oplus Z_K$, which contains z and which is associated to the* InnK_H*-invariant Lorentzian form $B|_{P_H \oplus Z_K}$. The wedge V_0 is global in the simply connected group G with $\mathbf{L}(G) = L$.*

Proof. — It only remains to prove the last statement. Using Theorem 3.6, we have to show that $u \in V_0^*$. Let $x = p + z \in q(V_0)$ with $p \in P_H$. Our assumption implies that $B(p+z) = B(p) + B(z) \leqslant 0$ and $B(b) \leqslant -B(z)$ follows from

$$-B(z) = -B(l) - B\big(Q(b)\big) = -B(l) + B(b).$$

Therefore

$$\begin{aligned} B(x,u) &= B(z) + B(b,p) \leqslant B(z) + \sqrt{B(b)}\sqrt{B(p)} \\ &\leqslant B(z) + \sqrt{-B(z)}\sqrt{-B(z)} = B(z) - B(z) \leqslant 0. \end{aligned}$$

Hence $u \in V_0^*$. □

Remark 3.13.— Among the Lie wedges $V \subseteq L$ with $H(V) = K'_H$ and $z \in V$ which are global in the simply connected group G with Lie algebra L, there is a maximal one :

$$V_{\text{glob}} = \bigcap_{\gamma \in \text{Inn} K(H)} \gamma u^*,$$

where u^* is the half space $\{x \in L : B(x,u) \leqslant 0\}$. With $u^\perp \supseteq K'(H)$ we see that $K'(H) \subseteq H(V_{\text{glob}})$ and from $B(\gamma z, u) = B(z,u) < 0$ it follows that $z \in \operatorname{int} V_{\text{glob}}$, hence $H(V_{\text{glob}}) = K'_H$. We may apply Theorem 3.7 to see that V_{glob} is global in G. If $V \subseteq L$ is an $e^{\operatorname{ad} K_H}$-invariant Lie wedge with $H(V) = K'_H$, it follows from Theorem 3.7 that $V \subseteq u^*$, therefore $V \subseteq V_{\text{glob}}$. In general the wedge V_0 from Proposition 3.12 is different from V_{glob}. For $L = \mathrm{su}\,(p,1)$ (Example 3.11), we have

$$V_{\text{glob}} = V_{r_0} \quad \text{and} \quad V_0 = V_1$$

with $r_0 > 1$ for $p > 1$ (Example 3.11). □

We conclude this section, and also this paper, with a result on the non-simply connected case (Theorem 3.16). We need two lemmas to prepare the proof.

LEMMA 3.14. — *Let $\widetilde{G}$ be the simply connected Lie group with $\mathbf{L}(G) = L$ and $Z^\diamond \stackrel{\text{def}}{=} \langle \exp Z_K \rangle$. As in Section 2 we choose $z_i \in Z_K \cap L_i$ such that $\omega(z_i) = 1$ for all $\omega \in \Omega^+_{p,i} \subseteq \Omega^+_i$. Then*

$$Z^\diamond \cap Z(G) = \exp\big(2\pi(\mathbb{Z}z_1 \oplus \ldots \mathbb{Z}z_m)\big).$$

Proof. — For $a = \sum_{i=1}^{m} \lambda_i z_i$ we have

$$\mathrm{Ad}(\exp a) = e^{\mathrm{ad}\, a} = e^{\lambda_1 \mathrm{ad}\, z_1} e^{\lambda_2 \mathrm{ad}\, z_2} \ldots e^{\lambda_m \mathrm{ad} z_m}$$

and this element is the identity on $L = \sum_{i=1}^{n} L_i$ if and only if $\lambda_i \in 2\pi\mathbb{Z}$ because $\mathrm{Spec}(\mathrm{ad}\, z_i) = \{0, i\}$. The lemma follows from $Z(G) \cap Z^\diamond = \ker(\mathrm{Ad}) \cap Z^\diamond$. □

LEMMA 3.15. — *Let $W = (\mathbb{R}^+)^n \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n$ a discrete subgroup with $W \cap D = \{0\}$. Then*

$$W \cap \operatorname{span} D = \{0\}.$$

Proof. — We give a proof with induction over decreasing codimension $n = \operatorname{rank} D$ of the grid D. For $n = \operatorname{rank} D$ there is nothing to show because only $D = \{0\}$ is possible. We assume that the lemma holds for $n - \operatorname{rank} D \leqslant m - 1$ and that $n - \operatorname{rank} D = m$. Suppose, in addition, that we find an element $v \neq 0$ in $W \cap \operatorname{span} D$ and choose the dimension n as small as possible such that the lemma does not hold. We consider the sup-norm $\|x\| = \max_{i=1\ldots n} |x_i|$ on $\mathbb{R}^n$. Let $D = \bigoplus_{j=1}^{k} \mathbb{Z} d_j$ and $M \stackrel{\text{def}}{=} \max_i \|d_i\|$. If $v \in \operatorname{int} W$, there exists a $\lambda > 0$ such that $\min_i(\lambda v_i) > kM$. If $\lambda v = \sum_{j=1}^{k} \lambda_j d_j$, we get $\|\lambda v - \sum_{j=1}^{k} [\lambda_j] d_j\| \leqslant kM$ and $d = \sum_{j=1}^{k} [\lambda_j] d_j \in D$. Hence $d \in D \cap \operatorname{int} W$, a contradiction. Therefore we may assume that $v \in \partial W$ and w.l.o.g. that $v_n = 0$. Let $\pi : \mathbb{R}^n \to \{0\}^{n-1} \times \mathbb{R} \subseteq \mathbb{R}^n$ be the orthogonal projection. There are two cases to consider.

Case 1 : The group D is contained in $\ker \pi$. Then $D \cap (\mathbb{R}^+)^{n-1} = \{0\}$ and $n - 1 - \operatorname{rank} D = m - 1$, a contradiction to the induction hypothesis.

Case 2 : The group D is not contained in ker π. Then, according to the Rank-Theorem for finitely generated abelian groups,

$$\operatorname{rank}\pi(D) + \operatorname{rank}(\ker\pi \cap D) = \operatorname{rank} D$$

and therefore

$$\operatorname{rank}(D \cap \ker\pi) = \operatorname{rank} D - 1,$$

because $\pi(D)$ is a non-trivial subgroup of $\mathbb{Z}$ and has rank 1. Now we find linearly independant elements $d'_1, \ldots, d'_n \in D$ with $d'_1, \ldots, d'_{n-1} \subseteq \ker\pi$ and $d'_n \notin \ker\pi$. Consequently the d'_n-coefficient of v in the representation with respect to the basis of span$D$ vanishes. Thus we have reduced the dimension of the problem :

$$D' = \bigoplus_{i=1}^{n-1} \mathbb{Z}d'_i, \quad D' \cap (\mathbb{R}^+)^{n-1} = \{0\}, \quad \text{and} \quad 0 \neq v \in \operatorname{span} D' \cap (\mathbb{R}^+)^{n-1}.$$

This is a contradiction to the minimality of n which completes the proof for $n - \operatorname{rank} D = m$.

THEOREM 3.16. — *Suppose that the wedges $W_i \subseteq L_i$ are invariant pointed generating and that $W \overset{\text{def}}{=} \bigoplus_{i=1}^{n} W_i$ is global in the simply connected Lie group $\widetilde{G}$ with $\mathbf{L}(G) = L$ and contains $W_{\min}$. Let $Z^\diamond \overset{\text{def}}{=} \exp Z_K$, $D \subseteq Z^\diamond \cap Z(G)$ be a discrete central subgroup and $D' \overset{\text{def}}{=} \exp^{-1}(D) \cap Z_K$. Then W is global in $G \overset{\text{def}}{=} \widetilde{G}/D$ iff*

$$D' \cap \sum_i \mathbb{R}^+ z_i = \{0\}. \tag{19}$$

Proof. — "$\Leftarrow$" : If W is global in G and $d \in (\Sigma\mathbb{R}^+ z_i) \cap D'$, then $\exp(d) \in \exp W$ because $\Sigma\mathbb{R}^+ z_i \subseteq W_{\min} \subseteq W$. According to [N1] II.12, we find a function $f \in C^\infty(G)$ such that $f'(g) \in \operatorname{int} W^*$ for all $g \in G$. Let $\widetilde{f} = f \circ \pi$, where $\pi : \widetilde{G} \to G$ is the canonical projection. Then clearly $\widetilde{f}'(g) \in \operatorname{int} W^*$ for all $g \in \widetilde{G}$ and therefore

$$\widetilde{f}(\mathbf{1}) = f(\mathbf{1}) = \widetilde{f}(\exp d) = \widetilde{f}(\mathbf{1}) + \int_0^1 \langle \widetilde{f}'(\exp(td)), d\rangle dt.$$

This proves that $\langle \widetilde{f}'(\mathbf{1}), d\rangle = 0$ and therefore $d = 0$.

"$\Rightarrow$" : Let $D' \cap \Sigma\mathbb{R}^+ z_i = \{0\}$. According to Lemma 3.14 we have $D' \subseteq \bigoplus_i \mathbb{Z}2\pi z_i$, then Lemma 3.15 shows that span$D' \cap \Sigma\mathbb{R}^+ z_i = \{0\}$ and we find a linear functional $\omega \in \widehat{Z_K}$ vanishing on D' and taking positive values on all elements z_i. If $\omega = \sum_{i=1}^{m} \delta_i z_i$ and $\delta_i > 0$ for $i = 1, \ldots, m$ we set $\widetilde{\omega} = \sum_{i=1}^{m} \delta_i u_i$ and $V \stackrel{\text{def}}{=} W + K'_H + \omega^\perp \cap Z_K$. Using our hypothesis, Corollary 3.8 and Lemma 2.19, we have

$$\widetilde{\omega} \in \sum_{i=1}^{m} W_i^* = W^* \quad \text{and} \quad \widetilde{\omega} \in W^* \cap H(V)^\perp = V^*.$$

Let $T \stackrel{\text{def}}{=} \langle \exp_{\widetilde{G}} V \rangle$. Then Proposition 3.4 applies and shows that V is not controllable in $\widetilde{G}$ and $L_i \not\subseteq H(\mathbf{L}(S))$ for $i = 1, \ldots, m$ because all numbers δ_i are positive. The fact that W is generating in L shows that $n = m$, *i.e.*, there are no compact ideals in L. Hence $\mathbf{L}(T)$ contains no ideal of L. Let $p : \widetilde{G} \to G = \widetilde{G}/D$ be the covering homomorphism. From

$$D = \ker p \subseteq \exp D' \subseteq \exp V \subseteq T$$

we conclude that $\mathbf{L}(T)$ is global in G ([N1] III.8). Setting $S \stackrel{\text{def}}{=} \langle \exp_G W \rangle$ we have

$$\mathbf{L}(S) \subseteq \mathbf{L}(\langle \exp_G \mathbf{L}(T) \rangle) = \mathbf{L}(T).$$

Consequently $\mathbf{L}(S)$ is an invariant wedge in L (Lemma 1.2) which contains no ideal of L. Hence $\mathbf{L}(S)$ is pointed. Now Proposition 3.1 in [N1] shows that W is global in G. □

Corollary 3.17. — *Let G be a simple Lie group, $L = \mathbf{L}(G)$ and $W \subseteq L$ be an invariant pointed, generating cone which is global in G and contains $W_{\min}$. Then $\pi_1(G)$ is finite.*

Proof. — Set $D = \pi_1(G) \subseteq \widetilde{G}$ where $\widetilde{G}$ is the universal covering group of G. Then W is global in $\widetilde{G}$ ([N1] III.7). If D is infinite, then $D \cap \exp_{\widetilde{G}}(Z_K) \neq \{1\}$ because $D \subseteq Z(\exp_{\widetilde{G}} K_H) = Z(\exp_{\widetilde{G}} K'_H) \exp_{\widetilde{G}}(Z_K)$ and $Z(\exp_{\widetilde{G}} K'_H)$ is finite. Consequently $D' \stackrel{\text{def}}{=} \exp^{-1}(D) \cap Z_K \neq \{0\}$. Now $Z_K = \mathbb{R}z$ together with Theorem 3.16 shows that W is not global in G. □

Open problems : Two of the most interesting open questions related to this subjects are :

1) Given an invariant wedge W in the semisimple Lie algebra L, when is W global in the associated simply connected group?

2) Classify the maximal global Lie wedges $W \neq \mathbf{L}(G)$ in the Lie algebra of a simply connected simple Lie group G (see [N2] for $G = \mathrm{Sl}(2,\mathbb{R})^{\sim}$).

BIBLIOGRAPHY

[BD] T. BRÖCKER and T. Tom DIECK, Representations of Compact Lie Groups, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985.

[Fa1] J. FARAUT, Algèbres de Volterra et Transformation de Laplace Sphérique sur certains espaces symétriques ordonnés, Symposia Math., 29(1989), 183-196.

[Fa2] J. FARAUT, Algèbres de Jordan et Cônes symétriques, Notes d'un cours de l'Ecole d'Eté CIMPA-Universités de Poitiers, 22 Août-16 Septembre 1988.

[Fo] G.B. FOLLAND, Harmonic Analysis in Phase Space, Princeton University Press, Princeton, New Jersey, 1989.

[HawE] S.W. HAWKING and G.F.R. ELLIS, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge, 1973.

[HarC1] HARISH-CHANDRA, Representations of semi-simple Lie groups, IV, Amer. J. Math., 77(1955), 743-777.

[HarC2] HARISH-CHANDRA, Representations of semi-simple Lie groups, VI, Amer. J. Math., 78(1956), 564-628.

[He] S. HELGASON, Differential Geometry, Lie Groups, and Symmetric Spaces, Acad. Press. London, 1978.

[HiHoL] J. HILGERT, K.H. HOFMANN, J.D. LAWSON, Lie groups, Convex cones, and Semigroups, Oxford University Press, 1989.

[HiLPy] K.H. HOFMANN, J.D. LAWSON, J.S. PYM, Eds., The Analytic and Topological Theory of Semigroups – Trends and Developments, to appear.

[Ho] K.H. HOFMANN, Lie algebras with subalgebras with codimension one, Illinois J. Math., 9(1965), 636-643.

[Hu] J.E. HUMPHREYS, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1972.

[L] J.D. LAWSON, Maximal subsemigroups of Lie groups that are total, Proceedings of the Edinburgh Math. Soc., 87(1987), 497-501.

[M] C.C. MOORE, Compactifications of symmetric spaces, II, The Cartan domains, Amer. J. Math., 86(1964), 358-378.

[N1] K.-H. NEEB, The duality between subsemigroups of Lie groups and monotone functions, Transactions of the Amer. Math Soc., to appear.

[N2] K.-H. NEEB, Semigroups in the Universal Covering Group of $SL(2)$, Semigroup Forum, to appear.

[O1] G.I. OL'SHANSKIĬ, Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series, Funct. Anal. and Appl., 15(1982), 275-285.

[O2] G.I. OL'SHANSKIĬ, Invariant orderings in simple Lie groups. The solution to E.B. Vinberg's problem, Funct. Anal. and Appl., 16(1982), 311-313.

[Pa] S. PANEITZ, Determination of invariant convex cones in simple Lie algebras, Arkiv för Mat., 21(1984), 217-228.

[S] K. SPINDLER, Invariante Kegel in Liealgebren, Mitt. aus dem mathematischen Sem. Gießen, Heft 188, 1988.

[V] E.B. VINBERG, Invariant cones and orderings in Lie groups, Funct. Anal. and Appl., 14(1980), 1-13.

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