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PLURISUBHARMONIC MARTINGALES AND BARRIERS IN COMPLEX QUASI-BANACH SPACES

by N. GHOUSSOUB and B. MAUREY

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0. Introduction.

Our main goal in this paper is to describe the geometrical structure on a complex quasi-Banach space X that is necessary and sufficient for the following analytical property to hold :

(*) *Every bounded* X-*valued analytic function on the open unit disc of the complex plane, has boundary limits almost surely.*

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This property was first studied by Bukhvalov and Danilevich [BD] in the context of Banach spaces. They observed that if « analytic » is replaced by « harmonic » in (*), then the property characterizes those Banach spaces possessing the so-called *« Radon-Nikodym Property »* (R.N.P). This latter class of spaces is now well-known to have a nice « convex geometrical structure » and that it contains all reflexive spaces as well as separable duals. (See for instance [DU].)

However, the « analytic case » is more general and concerns a larger class of spaces. Indeed, it is shown in [BD] that L^1 and more generally all Banach lattices not containing c_0 also verify (*). More recently, Haagerup and Pisier [HP] showed that the same holds for preduals of Von Neuman algebras while in [GMS] the same property is established for the predual of James-tree space. Another proof of this fact – based on the methods of this paper – is given in section 3. In [GLM], a geometric study of those Banach spaces verifying (*) is carried out. In that case, (*) is known to be equivalent to the *« Analytic Radon-Nikodym Property »* (A.R.N.P) : that is X-valued *analytic* measures of bounded variation are differentiable. In other words, these are the spaces where the vector-valued version of the brothers Riesz theorem holds ; (see Dowling [D]). The results in [GLM] show that – in a Banach space setting – there exists a « geometric theory » for the A.R.N.P, analogous to the R.N.P case where Phelps' theorem (see [DU]) gives the existence of strongly exposing linear functionals as opposed to the strongly exposing plurisubharmonic functions obtained in the « analytic case ».

However, this « analytic case » is not exclusively a locally convex problem as the spaces L^p, H^p and the Schatten classes C^p indicate for $0 < p < 1$ (see [A] and [K1]). Linear functionals are not relevant in this setting and all what is needed on a quasi-Banach space X to verify (*) is a « nice plurisubharmonic structure ». Already, Kalton had shown in [K2], the existence of a plurisubharmonic equivalent quasi-norm on X as a necessary condition. Since this is trivially not sufficient, we prove in section 4 that (*) holds if and only if, in addition to the plurisubharmonicity of the quasi-norm, all closed bounded subsets of X have *strong barrier points* i.e. points where plurisubharmonic functions strongly expose the set in question.

The proofs rely on martingale techniques, mostly the *analytic martingales* introduced by Davis *et al.* [DGT]. As shown by Edgar [E2], these are « reasonable discrete approximations » for the processes that are images of complex Brownian motion by X-valued analytic functions. We shall actually show that in this setting, all bounded X-valued plurisubharmonic martingales converge almost surely : a result obtained recently by Bu-Schachermayer [BS] in the Banach space case, via different methods. This leads naturally to an integral representation result in terms of Jensen boundary measures, which is the « complex » counterpart of the non-compact but convex Choquet theorem established by Edgar [E3].

Section 1 contains the basic definitions. In section 2, we discuss the various notions of plurisubharmonic envelopes for functions and sets. In section 3, we exhibit some properties of plurisubharmonic martingales and holomorphic mappings that are relevant for showing that certain spaces verify (*). Section 4 begins with the « complex structure » of the compact subsets of a quasi-Banach space X equipped with a plurisubharmonic quasi-norm. The – well known – topological methods used there are worth comparing to the martingale techniques employed in the second part of the section where the non-compact case is considered. The main result being the existence of barrier points in all closed bounded subsets of spaces verifying (*). It is actually shown that any bounded upper semi-continuous function on such a set has arbitrarily small plurisubharmonic perturbations that are strongly exposing. This is then used to prove the convergence of bounded X-valued PSH martingales. In section 5 we give a *« descriptive set-theoretical »* representation of the unit ball of X in a compactification that is compatible with the plurisubharmonic – but not the linear – structure of X. This representation yields – among other things – another proof of the convergence of PSH-martingales announced above. In an appendix, we include a general result about « embedding » such martingales in analytic functions, which might illuminate the connections between these concepts.

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1. Definitions and preliminaries.

Throughout this paper all vector spaces are assumed to be on the complex field. If X is a vector space, a map $x \to \|x\|$ from X into R^+ is called a *quasi-norm* if :

(i) $\|x\| > 0$ when $x \neq 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, for $\alpha \in \mathbf{C}$, $x \in X$

(iii) $\|x_1 + x_2\| \leqslant C(\|x_1\| + \|x_2\|)$ for all x_1, $x_2 \in X$. Here C is a constant larger or equal to one.

We call $\| \ \|$ a *p-norm* for $0 < p \leqslant 1$, if in addition it is p-subadditive, that is

(iv) $\|x_1 + x_2\|^p \leqslant \|x_1\|^p + \|x_2\|^p$, for x_1, $x_2 \in X$.

Note that the Aoki-Rolewicz theorem [KPR] asserts that every quasi-norm is equivalent to a p-norm for some $p (0 < p \leqslant 1)$. If $(X, \| \ \|)$ is complete we say it is a *quasi-Banach space* and if it has a p-subadditive quasi-norm we say it is a *p-Banach space.*

An upper-semi-continuous function $\varphi : X \to [-\infty, +\infty)$ is called *plurisubharmonic* if for every $x, y \in X$,

$$\varphi(x) \leqslant \int_0^{2\pi} \varphi(x + e^{i\theta} y) \frac{d\theta}{2\pi}.$$

We denote by $PSH(X)$ the space of all such functions. If the quasi-norm $\| \ \|$ on X is plurisubharmonic then X is called *PL-convex* by Davis, Garling and Tomczak-Jaegermann [DGT]. If X can be equivalently normed with a plurisubharmonic quasi-norm, then we shall follow Kalton [K1] and say that *X is A-convex.* (The terms *locally pseudo-convex* and *locally holomorphic* have been used by Peetre [P] and Alexandrov [A] respectively.)

A recent result of Kalton [K1] shows that an A-convex quasi-Banach space has an equivalent quasi-norm which is both plurisubharmonic and p-subadditive for some $0 < p \leqslant 1$. We shall say that such a *space is $(A-p)$ convex.* Note that L^p and $C^p (0 < p < 1)$ are $(A-p)$ convex (Etter [Et] and Kalton [K1]) while L^p/H^p is not A-convex (Alexandrov [A]). It is clear that all Banach spaces are $(A-1)$-convex since the norm is then convex and subadditive.

Let now $\Delta = \{z \in \mathbf{C}, |z| < 1\}$ be the open unit disc and denote by $\partial\Delta$ or $\mathbf{T}$ the unit circle $\{z \in \mathbf{C}; |z| = 1\}$. An *X-valued holomorphic (or analytic) function on* Δ will be a function of the form

$$f(z) = \sum_{n=0}^{\infty} x_n z^n \text{ where } |z| < 1, \ x_n \in X \quad \text{and} \quad \limsup_n \|x_n\|^{1/n} \leqslant 1 .$$

For $0 < p < \infty$, let $H^p(\Delta, X)$ be the space of holomorphic functions such that :

$$\|f\|_{H^p(X)} = \sup_{0 \leqslant r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p \, d\theta \right)^{1/p} < \infty .$$

For $p = \infty$, we let $H^\infty(\Delta, X)$ be the space of such functions verifying :

$$\|f\|_{H^\infty(X)} = \sup_{|z|<1} \|f(z)\| < \infty .$$

We shall also consider the Nevannlina class $N(\Delta, X)$ consisting of those holomorphic functions verifying :

$$\|f\|_{N(X)} = \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|f(re^{i\theta})\| \, d\theta < \infty .$$

It is clear that $H^\infty(X) \subset H^p(X) \subset H^q(X) \subset N(X)$ if $0 < q \leqslant p < +\infty$ and if $\| \ \|$ is plurisubharmonic. We say that f has *radial limit a.s.* if for almost all $\theta \in [0, 2\pi]$, $\lim_{r \uparrow 1} f(re^{i\theta})$ exists in X.

Let now $(W_t)_t$ be Brownian motion in $\mathbf{R}^2 = \mathbf{C}$ starting at 0. Let $r_k = 1 - 2^{-k}$ and define the stopping times

$$\tau_k = \inf\{t > 0; |W_t| \geqslant r_k\}, \qquad \tau_\infty = \inf\{t > 0; |W_t| \geqslant 1\}$$

so that $\tau_k \uparrow \tau_\infty$ and $\tau_\infty < \infty$ a.s. The close connection between the radial limits of f in $H^\infty(\Delta, X)$ and the convergence of the process $(f(W_{\tau_n}))_n$ is well known in the finite dimensional case [Du]. The same connections hold in infinite dimensional Banach spaces (Edgar [E2]). We shall recall these facts in Section 2 while dealing with quasi-Banach spaces.

We will also need the concept of *Analytic martingales* introduced by Davis, Garling and Tomczak [DGT]. This is a sequence of X-valued random variables $(F_n)_{n=0}^{\infty}$ defined on $\Omega = [0, 2\pi]^N$ of the form : $F_n(\underline{\theta}) = \sum_{k=1}^{n} f_k(\theta_1, \theta_2, \cdots, \theta_{k-1}) e^{i\theta_k}$ where $\underline{\theta} = (\theta_1, \theta_2, \cdots)$ belongs to Ω

and such that the coefficients $f_k : [0,2\pi]^{k-1} \to X$ are X-valued random variables.

In the case where X is a Banach space, Dowling [D] showed that functions in $H^p(\Delta,X)$ have radial limits a.s. if and only if X-valued analytic measures of bounded variation on $\partial\Delta$ are differentiable. (Recall that the *analytic measures* are those μ defined on the Borel σ-field of $\partial\Delta$, whose negative *Fourier coefficients*, $\hat{\mu}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\, d\mu(t)$, $n<0$ are zero.) Such spaces are then said to have the *Analytic Radon-Nikodym Property* (A.R.N.P). The connection with the above stems from the following correspondence: if μ is an analytic measure on $\partial\Delta$, then its « harmonic extension » to Δ defined by $\phi_\mu(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, d\mu(t)$ is analytic. Here $P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}$, $0 \leqslant r < 1$, $t \in \mathbf{R}$ is the *Poisson kernel.* We shall use the classical fact that if Brownian motion starts at a point $z_0 = re^{i\theta}$ in Δ, then the distribution of W_{τ_∞} on $\partial\Delta$ will have density $P_{z_0} = P_r$. (See [Du] p. 36.)

2. Plurisubharmonic envelopes and hulls.

Let $(X, \| \ \|)$ be a quasi-normed vector space and let $\varphi : X \to \mathbf{R} \cup \{-\infty\}$ be an upper semi-continuous function that is bounded above on bounded sets. Set $\varphi_0 = \varphi$ and define for each $n > 0$, the function :

$$\varphi_{n+1}(x) = \inf\left\{\int_0^{2\pi} \varphi_n(x + e^{i\theta}v)\,\frac{d\theta}{2\pi} \,;\, v \in X\right\}.$$

The sequence $(\varphi_n)_n$ decreases pointwise to a function $\hat{\varphi}$ that we shall call the *plurisubharmonic envelope* of φ since it is shown in [E1] that $\hat{\varphi}$ is the largest plurisubharmonic function less or equal to φ.

We are interested here in the case where the function φ is Hölder-continuous of order $p (0 < p \leqslant 1)$. We denote by $LIP_p(X)$ the set of functions φ on X verifying $|\varphi(x) - \varphi(y)| \leqslant K\|x-y\|^p$ for some $K > 0$ and for all x, y in X. The following class of functions will also be relevant to our study: denote by $UC(X)$ the class of all functions that are bounded and uniformly continuous on the bounded sets of X. We shall write $PSH_p(X)$ for $PSH(X) \cap LIP_p(X)$ and $PSH_{uc}(X)$ for $PSH(X) \cap UC(X)$. The cone of plurisubharmonic and continuous functions on X will be denoted by $PSH_c(X)$.

In the sequel, the space $PSH_p(X)$ will be equipped with the following norm: for $\varphi \in PSH_p(X)$, we let

$$\|\varphi\|_p = \max\{|\varphi(0)|, \sup\{|\varphi(x)-\varphi(y)|/\|x-y\|^p ; x \neq y\}\}.$$

We summarize in the following lemmas the properties of the envelope that will be needed later on:

LEMMA 2.1. – *Let φ be a function in $LIP_p(X)$ such that $\hat{\varphi}$ is not identically $-\infty$, then:*

a) *$\hat{\varphi}$ is the largest function in $PSH(X)$ that is smaller or equal to φ. Moreover $\hat{\varphi}$ belongs to $PSH_p(X)$.*

b) *For each n, $\varphi_n(x) = \inf E[\varphi(x+U_n)]$ where the infimum is taken over all analytic martingales $(U_k)_{k=0}^n$ with $U_0 = 0$ and whose coefficients are finitely valued in X.*

c) *For each $x \in X$, $\hat{\varphi}(x) = \inf \int_0^{2\pi} \varphi(P(e^{i\theta}))\frac{d\theta}{2\pi}$ where the infimum is taken over all polynomials $P: \mathbf{C} \to X$ such that $P(0) = x$.*

Proof. – a) Suppose $\psi \in PSH_p(X)$ and $\psi \leqslant \phi$, it is clear that $\psi \leqslant \varphi_n \leqslant \varphi$ for each n. On the other hand if x, y belong to X, we have for each $\varepsilon > 0$ a v in X so that:

$$\varphi_1(x) - \varphi_1(y) \leqslant \varphi_1(x) - \int_0^{2\pi} \varphi(y+e^{i\theta}v)\frac{d\theta}{2\pi} + \varepsilon.$$

It follows that:

$$\varphi_1(x) - \varphi_1(y) \leqslant \int_0^{2\pi} [\varphi(x+e^{i\theta}v) - \varphi(y+e^{i\theta}v)]\frac{d\theta}{2\pi} + \varepsilon \leqslant K\|x-y\|^p + \varepsilon,$$

where K is the Lipschitz constant of φ. An easy induction implies that $\hat{\varphi} \in PSH_p(X)$.

b) was proved in [E1] in the case where X is separable but φ was only upper semi-continuous. If $\varphi_n \in UC(X)$ for each n – in particular when $\varphi \in LIP_p(X)$ – the separability assumption is not needed as the following sketch shows: indeed, assume the formula in b) is true for $n-1$. Fix $x \in X$ and $\varepsilon > 0$. Choose $v \in X$ so that

$$\varphi_n(x) \leqslant \int_0^{2\pi} \varphi_{n-1}(x+e^{i\theta}v)\frac{d\theta}{2\pi} + \varepsilon/2.$$

Let F_1 have the uniform distribution on the circle $C = \{x + e^{i\theta}v\,; 0 \leqslant \theta \leqslant 2\pi]$. To get the formula for n, it is enough to choose measurably for each θ, an analytic martingale $(F_k)_{k=1}^n$ with $F_1(\theta) = x + e^{i\theta}v$ and $\varphi_{n-1}(x+e^{i\theta}v) \leqslant E[\varphi(F_n)] + \varepsilon/2$. The analytic martingale $(F_k)_{k=0}^n$ with $F_0 = x$ will clearly verify $\varphi_n(x) \leqslant E[\varphi(F_n)] + \varepsilon$.

To avoid the use of selection theorems which require the separability of the space, we can use the following observation to make the selection finite: suppose $\varphi_{n-1}(z) \leqslant E[\varphi(z+U)] + \varepsilon$. Choose $\rho > 0$ so that if $\|z-z'\| \leqslant \rho$ and $z' \in C$ then $|\varphi_{n-1}(z) - \varphi_{n-1}(z')| \leqslant \varepsilon$ and $\|\varphi(z) - \varphi(z')\| \leqslant \varepsilon$. We then get for such $z' \in C$,

$$\varphi_{n-1}(z') \leqslant \varphi_{n-1}(z) + \varepsilon \leqslant E[\varphi(z+U)] + 2\varepsilon \leqslant E[\varphi(z'+U)] + 3\varepsilon.$$

It follows that the same random variable U can be used to realize an approximation for both z and z' provided they are close enough. The details of the proof as well as the fact that the coefficients can be chosen to be « step functions » are left to the interested reader.

c) was proved by Kalton [K1] in the case $\varphi(x) = \|x\|^p$ for some p-norm $\|\ \|(0<p\leqslant 1)$. It also follows from b) and the correspondence between analytic martingales and analytic functions established in the Appendix. □

In many cases, we shall consider the envelopes of functions which are not in $LIP_p(X)$ but there p^{th}-power are, such as $\varphi(x) = \|x\|$ or if φ is the distance function to a given set, provided of course, the space is p-normed. Here are some properties of the envelopes of such functions:

LEMMA 2.2. – *Let φ be a positive function such that $\varphi^p \in LIP_p(X)$ for some $0 < p \leqslant 1$. Then for each $n \geqslant 0$, φ_n belongs to $UC(X)$ and $\hat{\varphi}$ belongs to $PSH_{uc}(X)$.*

Proof. – Write $\varphi = f^\alpha$ where $\alpha = 1/p \geqslant 1$ and $f \in LIP_p(X)$. Let x, $h \in X$ and consider a random variable U. Apply the following two elementary inequalities: for all a, b in $\mathbf{R}_+$ and $\alpha \geqslant 1$; $|a^\alpha - b^\alpha| \leqslant \alpha|a-b| \max(|a|^{\alpha-1}, |b|^{\alpha-1})$ and for all a, k and $\beta > 0$, $(a+k)^\beta \leqslant a^\beta + K_\beta(|k|+|k|^\beta)(1+a^\beta)$, to obtain:

$$E[\varphi(x+h+U) - \varphi(x+U)] \leqslant \alpha\|h\|^p E[f(x+U)^{\alpha-1} + K_{\alpha-1}(\|h\|^p + \|h\|^{p(\alpha-1)})(1+f(x+U)^{\alpha-1})].$$

By Hölder's inequality, we get:

$$(*)\quad E[\varphi(x+h+U)-\varphi(x+U)] \leqslant \alpha\|h\|^p[(E[\varphi(x+U))^{\frac{\alpha-1}{\alpha}} + K_{\alpha-1}(\|h\|^p+\|h\|^{p(\alpha-1)})(1+(E[\varphi(x+U)])^{\frac{\alpha-1}{\alpha}})].$$

Suppose now we have for some $x \in X$ and $\varepsilon > 0$, a vector v so that $\varphi_1(x) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \varphi(x+e^{i\theta}v) + \varepsilon$. By applying (*) to the uniform distribution on the circle we obtain

$$\varphi_1(x+h)-\varphi_1(x) \leqslant \alpha\|h\|^p[\varphi_1(x)^{\frac{\alpha-1}{\alpha}} + K_{\alpha-1}(\|h\|^p+\|h\|^{p(\alpha-1)})(1+\varphi_1(x)^{\frac{\alpha-1}{\alpha}})].$$

This coupled with the fact that $0 \leqslant \varphi_1 \leqslant \varphi$ implies that $\varphi_1 \in UC(X)$. An immediate induction gives that $\varphi_n \in UC(X)$ for all n and that $\hat{\varphi} \in PSH_{uc}(X)$.

Remark 2.3. – a) The above proof actually uses the easy fact that if φ is positive and verifies for all x, $h \in X$, $\varphi(x+h) - \varphi(x) \leqslant \omega(\|h\|)[1+\varphi(x)]$, where ω is any modulus of continuity, then the same will hold for $\hat{\varphi}$.

b) Note that the proof of Lemma 2.1b and the result in Lemma 2.2 give that if $\varphi \geqslant 0$ and $\varphi^p \in LIP_p(X)$ for some $0 < p \leqslant 1$, then $\hat{\varphi}(x) = \inf E[\varphi(F_n)]$ where the infimum is taken over all analytic martingales $(F_n)_n$ starting at x. □

Let now C be a subset of X and denote by $LIP_p^1(C)$ the class of all normalized p-Hölder-continuous functions on C, that is $\varphi \in LIP_p^1(C)$ if $|\varphi(x)-\varphi(y)| \leqslant \|x-y\|^p$ for all x, $y \in C$. When $C = X$ we denote by $PSH_p^1(X)$ the set $PSH(X) \cap LIP_p^1(X)$. The next lemma deals with "maximal" extensions of elements in $LIP_p^1(C)$ to functions that are p-Hölder-continuous on the whole space X.

LEMMA 2.4. – *Suppose X is a p-normed space for some $p(0<p\leqslant 1)$. For φ in $LIP_p^1(C)$, define $\tilde{\varphi}$ by $\tilde{\varphi}(x) = \inf\{\varphi(y)+\|y-x\|^p\,;\, y \in C\}$ for each x in X. Then:*

a) *$\tilde{\varphi} = \varphi$ on C and $\tilde{\varphi} \in LIP_p^1(X)$.*

b) *If $\psi \in LIP_p^1(X)$ and $\psi \leqslant \varphi$ on C, then $\psi \leqslant \tilde{\varphi}$ on X.*

c) *If φ has an extension in $PSH_p^1(X)$, then $\bar{\varphi} = \hat{\tilde{\varphi}}$ is the largest extension of φ in $PSH_p^1(X)$.*

Proof. — a) and b) follow immediately from the p-subadditivity of the norm. For c) note that if $\psi \in LIP_p^1(X)$ and $\psi = \varphi$ on C, then $\psi \leqslant \tilde{\varphi}$ on X by b) and if $\psi \in PSH_p^1(X)$ then $\psi \leqslant \hat{\tilde{\varphi}} \leqslant \tilde{\varphi}$ on X by Lemma 2.1. It follows that $\varphi = \hat{\tilde{\varphi}}$ on C. □

Denote now by $PSH_p^1(C)$ the set of all functions on C admitting extensions in $PSH_p^1(X)$. If we equip these spaces with the distance of uniform convergence on C (resp. X) it is easy to see that the map $\varphi \to \bar{\varphi}$ defines an isometric embedding from $PSH_p^1(C)$ to $PSH_p^1(X)$. In other words, for any φ_1, φ_2 in $PSH_p^1(C)$ we have:

$$\sup_{x \in X} |(\bar{\varphi}_1 - \bar{\varphi}_2)(x)| = \sup_{c \in C} |(\varphi_1 - \varphi_2)(c)| = d(\varphi_1, \varphi_2).$$

It follows that $(PSH_p^1(C), d)$ is a complete metric space.

Finally we define $\overline{PSH_p(C)}$ to be the closure of $\bigcup_{n=1}^{\infty} nPSH_p^1(C)$ for the metric d, i.e. those functions on C which can be approximated uniformly on C by functions which are restrictions on C of functions in $PSH_p(X)$. It is clear that $\overline{PSH_p(C)}$ is a closed convex cone.

We now discuss the notions of plurisubharmonic hull of a subset A of X. For that denote by d_A the distance function to A that is $d_A(x) = \inf\{\|y - x\|; y \in A\}$. We shall say that the set $\hat{A} = \{x \in X; \hat{d}_A(x) = 0\}$ *is the plurisubharmonic hull of* A. One can easily verify the following observations:

Proposition 2.5. — *Let A be a subset of a quasi-normed space X. Then*

(a) $A \subset \hat{A}$.

(b) *A vector x belongs to $\hat{A}$ if and only if for each $\varepsilon > 0$, there exists an analytic martingale $(F_k)_{k=0}^n$ with $F_0 = x$ and $E[d_A(F_n)] < \varepsilon$.*

(c) *The plurisubharmonic hull is not altered by an equivalent renorming of X.*

On the other hand, for any subset $\mathscr{H} \subset PSH(X)$, we can define the *$\mathscr{H}$-hull of A* to be:

$$\hat{A}_{\mathscr{H}} = \{x \in X; \varphi(x) \leqslant \sup_A \varphi \text{ for all } \varphi \in \mathscr{H}\}.$$

The cases we are concerned with are $\mathscr{H} = PSH_{uc}(X)$ and $\mathscr{H} = PSH_p(X)$. It is clear that if $\mathscr{H}_1 \subset \mathscr{H}_2$ then $\hat{A}_{\mathscr{H}_2} \subset \hat{A}_{\mathscr{H}_1}$.

PROPOSITION 2.6. — *Let A be a subset of a p-normed quasi-Banach space $X (0<p\leqslant 1)$. For a vector x in X, the following conditions are equivalent :*

(i) $x \in \hat{A}_{PSH_p}$.

(ii) $\hat{d}_A^p(x) = 0$.

(iii) *For each $\varepsilon > 0$, there exists an analytic martingale $(F_k)_{k=0}^n$ with $F_0 = x$ and $E[d_A^p(F_n)] \leqslant \varepsilon$.*

(iv) *For each $\varepsilon > 0$, there exists a polynomial $P : \mathbf{C} \to X$ such that $P(0) = x$ and $E[d_A^p(P(W_{\tau_\infty}))] \leqslant \varepsilon$.*

Moreover, if A is compact the above are then equivalent to

(v) *There exists a Radon probability measure μ supported on A such that $\varphi(x) \leqslant \int \varphi \, d\mu$ for all φ in $PSH_p(X)$.*

Proof. — i) $\Rightarrow$ ii) $x \in \hat{A}_{PSH_p}$ implies that there is no $\varphi \in PSH_p(X)$ so that $\varphi \leqslant 0$ on A and $\varphi(x) > 0$. This means that there is no $\varphi \in PSH_p(X)$ with $\varphi \leqslant d_A^p$ and $\varphi(x) > 0$. Lemma 2.1.a then implies that $0 \leqslant \hat{d}_A^p(x) \leqslant 0$.

ii) $\Rightarrow$ i) If $\varphi \leqslant 0$ on A and $\varphi \in PSH_p(X)$, then $\varphi \leqslant d_A^p$ on X by Lemma 2.4.b. Hence $\varphi \leqslant \hat{d}_A^p$ on X and $\varphi(x) \leqslant 0$.

(ii), (iii) and (iv) are readily equivalent in view of Lemma 2.1. Also note that (v) always implies (i).

To prove that (i) $\Rightarrow$ (v) consider the set

$$\mathscr{U} = \{u \in C(A)\,;\, \exists \varphi \in PSH_p(X),\ \varphi(x)=0 \text{ and } \varphi \leqslant u \text{ on } A\},$$

where $C(A)$ is the space of continuous functions on A. Note that $\mathscr{U}$ is a convex cone that contains $C_+(A)$. Moreover, the hypothesis i) implies that the constant function -1 does not belong to the closure of $\mathscr{U}$. By Hahn-Banach theorem, if A is compact there exists a Radon measure μ on A such that $\int u \, d\mu \geqslant 0$ for all $u \in \mathscr{U}$ and $-\mu(A) < 0$. Since $\mathscr{U} \supset C_+(A)$, μ is positive and since $\mu(A) \neq 0$, $\nu = \mu(A)^{-1}\mu$ is a Radon probability measure on A. Moreover, if $\varphi \in PSH_p(X)$, $\varphi - \varphi(x)$

restricted to A belongs to $\mathscr{U}$, hence $\int (\varphi - \varphi(x))\, d\nu \geqslant 0$ and $\varphi(x) \leqslant \int \varphi\, d\nu$.

□

The following proposition clarifies the relations between the various types of plurisubharmonic hulls.

PROPOSITION 2.7. — *Let C be a bounded subset of an A-convex quasi-Banach space X and let x be an element in X. The following conditions are then equivalent :*

(i) $x \in \hat{C}$.

(ii) $x \in \bigcap_{0<p\leqslant 1} \hat{C}_{PSH_p}$.

(iii) *There exists* $q(0<q\leqslant 1)$ *such that* X *is* $(A-q)$ *convex and* $x \in \hat{C}_{PSH_q}$.

(iv) $x \in \hat{C}_{PSH_{uc}}$.

Proof. — (i) ⇒ (ii) If $0 \leqslant \varphi \leqslant d_C^p$, $\varphi \in PSH(X)$ and $0 < p \leqslant 1$, we have $0 \leqslant \varphi^{1/p} \leqslant d_C$ and $\varphi^{1/p} \in PSH(X)$. It follows that $\hat{d}_C^p \leqslant (\hat{d}_C)^p$ and $\hat{C} \subset \hat{C}_{PSH_p}$ for all $0 < p \leqslant 1$.

(ii) ⇒ (iii) is immediate in view of the result of Kalton [K1] mentioned in section 1 about the existence of an equivalent q-norm that is also plurisubharmonic.

(iii) ⇒ (iv) Assume $\|\ \|^q$ is plurisubharmonic and subadditive. Without loss of generality we can suppose that $0 \in C$ and $C \subset \{z \in X ; \|z\| \leqslant 1\} = B$. We shall first show the following
claim : If $\|z\|^q \geqslant 2$ then $d_C^q(z) \leqslant 2\hat{d}_B^q(z)$.

Indeed, first note that for each z in B we have $\|y\|^q \leqslant \|y-z\|^q + \|z\|^q \leqslant \|y-z\|^q + 1$. Hence $\|y\|^q \leqslant d_B^q(y) + 1$ and since $y \to \|y\|^q$ is plurisubharmonic we have $\|y\|^q \leqslant \hat{d}_B^q(y) + 1$ for all y in X. On the other hand $\hat{d}_B^q(y) \leqslant d_B^q(y) \leqslant d_C^q(y) \leqslant \|y\|^q$ for all y, hence if z is such that $\|z\|^q \geqslant 2$, we get : $d_C^q(z) \leqslant \|z\|^q \leqslant \|z\|^q + (\|z\|^q - 2) = 2(\|z\|^q - 1) \leqslant 2\hat{d}_B^q(z)$ and the claim is proved.

Suppose now $\hat{d}_C^q(x) = 0$. For each $\varepsilon > 0$ there exists by Prop. 2.6 (iv) an X-valued polynomial P on $\mathbf{C}$ such that $P(0) = x$ and

$E[d_C^q \circ P(W_{\tau_\infty})] \leqslant \varepsilon$. Let τ be the stopping time defined by

$$\tau(\omega) = \begin{cases} +\infty & \text{if } \sup_t \|F_t(\omega)\|^q < 2 \\ \inf\{t; \|F_t(w)\|^q \geqslant 2\} & \text{otherwise} \end{cases}$$

where (F_t) is the martingale $(P(W_{t \wedge \tau_\infty}))_t$.

Let $\Omega_\tau = \{\tau < \infty\}$ and note that on Ω_τ, $\|F_\tau\|^q = 2$ since the martingale $(F_t)_t$ is continuous. Hence $d_C^q(F_\tau) \leqslant 2\hat{d}_B^q(F_\tau)$ by the claim.

Since $\hat{d}_B^q$ is plurisubharmonic we get on Ω_τ:

$$\frac{1}{2} d_C^q(F_\tau) \leqslant \hat{d}_B^q(F_\tau) \leqslant E[\hat{d}_B^q(F_\infty) | F_\tau] \leqslant E[d_C^q(F_\infty) | F_\tau].$$

Since $F_\tau = F_\infty$ outside Ω_τ we obtain:

$$E[d_C^q(F_\tau)] \leqslant 2E[d_C^q(F_\infty)] \leqslant 2\varepsilon \qquad \text{and} \qquad P[d_C^q(F_\tau) > \sqrt{\varepsilon}] \leqslant 2\sqrt{\varepsilon}.$$

Suppose now φ is in $PSH_{uc}(X)$ such that $\varphi \leqslant 0$ on C. There exists $M > 0$ so that $\|y\|^q \leqslant 2 \Rightarrow |\varphi(0) - \varphi(y)| \leqslant M$ hence $\varphi(F_\tau) \leqslant \varphi(F_\tau) - \varphi(0) \leqslant M$.

Fix now $\eta > 0$. Since $\varphi \in UC(X)$, we can find $\varepsilon > 0$ so that $\|x - y\|^p \leqslant 2\sqrt{\varepsilon}$ implies $|\varphi(x) - \varphi(y)| \leqslant \eta$. For each ω, find $Z(\omega) \in C$ so that $\|F_\tau(\omega) - Z(\omega)\|^p \leqslant d_C^p(F_\tau(\omega)) + \sqrt{\varepsilon}$. It follows that on the set $\{\omega ; d_C^p(F_\tau(\omega)) \leqslant \sqrt{\varepsilon}\}$ we have $\varphi(F_\tau) \leqslant \varphi(F_\tau) - \varphi(Z) \leqslant \eta$ since $\|F_\tau - Z\|^p \leqslant 2\sqrt{\varepsilon}$. We finally obtain $E[\varphi(F_\tau)] \leqslant \eta + 2\sqrt{\varepsilon}M$ and since $\varphi \in PSH(X)$, $\varphi(x) \leqslant E[\varphi(F_\tau)] \leqslant \eta + 2\sqrt{\varepsilon}M$. Hence $\varphi(x) \leqslant 0$ and $x \in \hat{C}_{PSH_{uc}}$.

(iv) $\Rightarrow$ (i) It is enough to notice that d_C^p is in $LIP_p(X)$ for some $p (0 < p \leqslant 1)$, hence $\hat{d}_C \in PSH_{uc}(X)$ by Lemma 2.2.

We shall say that a subset C of X is *PSH-convex* if $C = \hat{C}$. On the other hand, say that x is a *Jensen barycenter* of a probability measure μ if $\varphi(x) \leqslant \int \varphi \, d\mu$ for all $\phi \in PSH_{uc}(X)$. In this case μ is said to be a *Jensen measure representing* x. We then say that C is *J-convex* if the barycenter of any Jensen probability measure supported on C, belongs to C. The following proposition follows immediately from the above.

PROPOSITION 2.8. — *Let C be a closed bounded subset of an A-convex quasi-Banach space X. The following conditions are equivalent :*

(i) *C is PSH-convex.*

(ii) *For any $p(0<p\leqslant 1)$ such that X is p-normed, there exists a family $(\varphi_\alpha)_{\alpha\in I}$ of functions in $PSH_p(X)$ such that $C = \bigcap_{\alpha\in I}\{x\in X\,;\,\varphi_\alpha(x)\leqslant 0\}$.*

Moreover, any of the above conditions implies

(iii) *C is J-convex.*

If C is compact then (iii) $\Rightarrow$ (i).

3. Analytic functions, *PSH*-martingales and Holomorphic injections.

Let (Ω,Σ,P) be a probability space and let $(\Sigma_n)_n$ be an increasing sequence of sub-σ-fields of Σ. Suppose X is a quasi-normed space, and let $(F_n)_n$ be a sequence of X-valued p-integrable (for some $p>0$) random variables such that F_n is Σ_n-measurable for each n. We shall say that $(F_n,\Sigma_n)_n$ is a *PSH-martingale* if for every $\varphi\in PSH_{uc}(X)$, $\varphi\geqslant 0$, the sequence $(\varphi(F_n))_n$ is a real-valued submartingale: that is $\int_A \varphi(F_n)\,dP \leqslant \int_A \varphi(F_{n+1})\,dP$ for every n and any $A\in\Sigma_n$.

It is clear that an analytic martingale is a PSH-martingale. Moreover, $(f(W_{\tau_n}))_n$ is a PSH-martingale whenever $f\in H^\infty(\Delta,X)$. Note that PSH-martingales are martingales when X is a Banach space since continuous linear functionals are Lipschitz harmonic functions on X and separate the points of X. Note that in the quasi-Banach case, it is not possible in general to define vector-valued conditional expectation operators and hence martingales.

We shall say that $(F_n)_n$ *is a closed PSH-martingale* if there exists a p-integrable $(p>0)$ X-valued random variable F such that for every $A\in\bigcup_n \Sigma_n$, we have: $\lim_n \int_A \varphi(F_n)\,dP = \int_A \varphi(F)\,dP$ for every φ in $PSH_{uc}(X)$.

We shall need the following :

LEMMA 3.1. — *Let X be a separable A-convex quasi-Banach space. Then there exists a countable family $(\varphi_m)_m$ in $PSH_p(X)$ for some*

$0 < p \leqslant 1$ *with the following property : whenever* $((y_n)_n, y)$ *is a sequence in* X *that verifies* $\lim_n \varphi_m(y_n) = \varphi_m(y)$ *for each* m, *then* $\lim_n \|y_n - y\| = 0$.

Proof. – Let $\| \ \|$ be an equivalent plurisubharmonic p-norm on $X(0<p\leqslant 1)$. Let $(x_m)_m$ be a dense sequence in X with $x_0 = 0$. It is clear that the sequence of functions $(\varphi_m)_m$ defined by $\varphi_m(x) = \|x - y_m\|^p$ for $x \in X$ will do the job.

We can deduce the following :

PROPOSITION 3.2. – *Let* X *be an* A*-convex quasi-Banach space. Then every closed* PSH*-martingale converges a.s.*

Proof. – Let $\{F_n, F\}$ be an X-valued uniformly bounded closed PSH-martingale. Since it is almost surely separably valued, we can assume that X is separable. Apply Lemma 3.1 to find suitable plurisubharmonic functions $(\varphi_m)_m$. Note that for each m, $\varphi_m(F_n) \to \varphi_m(F)$ a.s. outside a set Ω_m of measure 0. We then get $\lim_n \|F_n - F\| = 0$ outside $\Omega^0 = \bigcup_m \Omega_m$ which is also of measure zero.

PROPOSITION 3.3. – *Let* X *be an* A*-convex quasi-Banach space. Then a function* f *in* $H^\infty(\Delta, X)$ *has radial limits a.s. if and only if* $(f(W_{\tau_n}))_n$ *is a closed* PSH*-martingale.*

Proof. – Assume $g(e^{i\theta}) = \lim_{r\uparrow 1} f(re^{i\theta})$ exists for almost all θ. For any $\varphi \in PSH_{uc}(X)$, $\varphi \circ f$ is a subharmonic function on Δ, hence by the one-dimensional complex case (see [Du] p. 105]) $(\varphi \circ f(W_{\tau_n}))_n$ converges a.s. to $\varphi \circ g(W_{\tau_\infty})$ hence $(f(W_{\tau_n}))_n$ is a closed PSH-martingale.

For the converse assume $(f(W_{\tau_n}))_n$ converges a.s. to an X-valued random variable F. Again from the one-dimensional case we get that for any $\varphi \in PSH_c(X)$, $\varphi \circ f(W_{\tau_n})$ converges a.s. to $k(W_{\tau_\infty})$ where k denotes the boundary limit of $\varphi \circ f$. Hence $\varphi \circ F = k(W_{\tau_\infty})$ a.s. By applying this to the sequence $(\varphi_m)_m$ given by Lemma 3.1, we obtain that F is measurable for the σ-field generated by W_{τ_∞}, hence there exists a measurable $g : \mathbf{T} \to X$ such that $F = g \circ W_{\tau_\infty}$. Moreover, $\lim_{r\uparrow 1} \varphi_m \circ f(re^{i\theta}) = \varphi_m \circ g(e^{i\theta})$ for almost all θ and all m, hence by Lemma 3.1, $\lim_{r\uparrow 1} f(re^{i\theta}) = g(e^{i\theta})$ a.s.

Another way to construct the candidate for the boundary limit g of f is to argue that for almost all $\theta \in \mathbf{T}$, $(f(W^{\theta}_{\tau_n}))_n$ also converges when $n \to \infty$, where $(W^{\theta}_t)_t$ is Brownian motion starting at 0 and conditioned to exit Δ at $e^{i\theta}$ ([Du] p. 94). It is then enough to set $g(e^{i\theta}) = \lim_{n\to\infty} f(W^{\theta}_{\tau_n})$. □

The following proposition is known in the Banach space setting ([D], [E2]). We shall only indicate how the proofs can be adapted to the quasi-normed case, just to emphasize the role of A-convexity.

PROPOSITION 3.4. — *Let X be a quasi-Banach space. The following conditions are then equivalent :*

1) *Every function f in $H^{\infty}(\Delta,X)$ has radial limits a.s.*
2) *Every function f in $N(\Delta,X)$ has radial limits a.s.*
3) *X is (A-p) convex for some $p(0<p\leqslant 1)$ and all X-valued L_p-bounded analytic martingales converge a.s.*

Proof. — 1) ⇒ 2) By a result of Kalton [K2], 1) implies that X is A-convex. That is there exists an equivalent quasi-norm $\| \ \|$ on X such that $\log^+ \| \ \|$ is plurisubharmonic on X. Suppose now $f \in N(\Delta,X)$. Since $\log^+ \|f\|$ is now subharmonic on Δ and

$$\sup_{0\leqslant r<1} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \|f(re^{i\theta})\|\, d\theta < \infty ,$$

we can use a standard technique to find $g \in H^{\infty}(\Delta,X)$ and $h \in H^{\infty}(\Delta,\mathbf{C})$ such that $f = g/h$. Actually, it is enough to find a positive harmonic function $u:\ \Delta \to \mathbf{R}^+$ such that $\log^+ \|f\| \leqslant u$, to take its harmonic conjugate v and then to let $g(z) = e^{-(u(z)+iv(z))}f(z)$ while $h(z) = e^{-(u(z)+iv(z))}$. (See for instance [Ko]). 2) then follows immediately.

2) ⇒ 3) was proved in the Banach space case by Edgar [E2]. It also follows immediately from the « embedding » of analytic martingales into analytic functions established in the Appendix.

3) ⇒ 1) can be obtained from the geometrical characterizations of section 4. Edgar [E2] proved it in the Banach space case by showing that analytic images of Brownian motion are approximable in the L^2-norm by analytic martingales. We sketch the proof of this fact for completeness.

LEMMA 3.5. — *Let X be $(A-p)$-convex Banach space for some $p(0<p\leqslant 1)$, and let $f \in H^{\infty}(\Delta,X)$ and $\varepsilon > 0$. There exists then an analytic* *martingale* $(F_n)_{n=0}^{\infty}$ *with* $F_0 = f(0)$ *and integers* $n_1 < n_2 \cdots < n_k < \cdots$ *so that* $E[\|F_{n_k} - f(W_{\tau_k})\|^p] < \varepsilon$.

Sketch of proof. – Modulo a standard induction, it is enough to show the following claim : If f is X-valued and analytic in a neighborhood of $\bar{\Delta}$, then for any $\varepsilon > 0$ and any $z_0 \in \Delta$, there exists an analytic martingale $(F_k)_{k=0}^{n}$ with $F_0 = f(z_0)$ and a random variable H distributed on $\partial\Delta$ with density P_{z_0} such that $E(\|F_n - f(H)\|^p) < \varepsilon$.

To do that, it is enough to notice that $\hat{d}_A^p(f(z_0), z_0) = 0$ where A is the set $A = \{(f(e^{i\theta}), e^{i\theta}) ; \theta \in [0, 2\pi]\}$ in $X \oplus \mathbf{C}$ and where the plurisubharmonic envelope of the function d_A^p is taken in the open set $X \oplus \Delta$. By Proposition 2.7, there exists for any $\varepsilon' > 0$, an analytic martingale $(Y_k)_{k=0}^{n}$ valued in $X \oplus \Delta$ such that $E[d_A^p(Y_n)] < \varepsilon'$ and $Y_0 = (f(z_0), z_0)$. Write $Y_n = F_n + G_n$ where $F_n \in X$ and $G_n \in \Delta$ are both analytic martingales. Let now H be Brownian motion stopped on $\partial\Delta$ after having started at G_n. It is clear that $E[|H - G_n|^2] \leqslant 2E[1 - |G_n|]$ and if $M > 0$ is such that $\|f(z_1) - f(z_2)\| \leqslant M|z_1 - z_2|$ for z_1, z_2 in $\bar{\Delta}$ we get $E[\|f(H) - f(G_n)\|^2] \leqslant 2M^2\varepsilon'$. This coupled with the estimates $E[\operatorname{dist}(F_n, f(\Delta))^p] < \varepsilon'$ and $E[\operatorname{dist}(G_n, \partial\Delta)^p] < \varepsilon'$ easily gives the above claim. □

It is clear that the properties discussed in Proposition 3.4 are stable under isomorphic linear embeddings. We shall show in the sequel the stability of such properties under much weaker types of injections. We then give few examples where this simple method is applicable to prove that various spaces have A.R.N.P.

Let us say that a subset C of a quasi-normed space X is a *PSH_δ-set* if $\bar{C} \setminus C = \bigcup_n F_n$ where each F_n is closed and PSH-convex. By Proposition 2.8, this implies the existence of a family $\{\varphi_{\alpha,n}\}$ of functions in $PSH_{uc}(X)$ such that $C = \bigcap_n \bigcup_{\alpha \in I} \{x \in \bar{C} ; \varphi_{\alpha,n}(x) > 0\}$.

We shall say that C is a *strict PSH_δ-set in* $\bar{C}$ if there exists a sequence $(\alpha_n)_n$ of reals so that for each n, $\sup_\alpha \varphi_{\alpha,n} \geqslant \alpha_n > 0$ on C while $\sup_C \varphi_{\alpha,n} \leqslant 1$ for all α and n. In other words, $\bar{C} \setminus C$ can be written as a countable union of closed sets $(F_n)_n$ such that for any $x \in C$ and any $n \in \mathbf{N}$, there exists $\varphi \in PSH_{uc}(X)$ with $\sup_C \varphi \leqslant 1$ such that $\varphi(x) > \alpha_n$ while $\varphi \leqslant 0$ on F_n.

Say that a one-to-one (not necessarily linear) map $S : X \to Y$ is a

PSH_δ-injection (resp. a *semi-embedding*) if the image by S of the unit ball of X is a bounded strict PSH_δ (resp. bounded closed) set in Y. We shall only consider here the cases where X is a linear or a holomorphic map. More general types of maps will be considered in section 5. Facts about the holomorphic ones can be found in the book [C].

PROPOSITION 3.6. — *Let X be a separable Banach space and let S be a holomorphic PSH_δ-injection from X into a quasi-Banach space Y. Let f be in $H^\infty(\Delta, X)$ and suppose that Sf has radial limits a.s. in Y, then f itself has radial limits a.s. in X.*

Proof. — Suppose $S \circ f(re^{i\theta})$ converges for almost all θ when $r \uparrow 1$ and denote by $g(e^{i\theta})$ its limit. Note that $S \circ f \in H^\infty(\Delta, Y)$ since S is holomorphic [C, p. 202]. Assume without loss of generality that $f(\Delta) \subset \mathrm{Ball}\,(X)$, we shall first show that $g(\partial\Delta) \subset S(\mathrm{Ball}\,(X))$ a.s.

To do that write $\overline{S(\mathrm{Ball}\,(X))} \backslash S(\mathrm{Ball}\,(X)) = \bigcup_n F_n$ where each F_n is PSH-convex. Suppose there exists n so that the set $A = \{\theta \in \partial\Delta\,; g(e^{i\theta}) \in F_n\}$ has strictly positive Lebesgue measure. It follows that for every $\varepsilon > 0$, there exists $z_0 \in \Delta$ such that Brownian motion starting at z_0 exits Δ at A with probability larger than $1 - \varepsilon$; that is $P_{z_0}(A) > 1 - \varepsilon$. Since $S(\mathrm{Ball}\,(X))$ is a strict PSH_δ-set, we can find φ in $PSH_{uc}(Y)$ such that $\varphi(S \circ f(z_0)) > \alpha_n$ and $\varphi \leqslant 0$ on F_n while $\sup \varphi(S(\mathrm{Ball}\,(X))) \leqslant 1$. Since $\varphi \circ S \circ f$ is subharmonic we have:

$$0 < \alpha_n < \varphi(S \circ f(z_0)) \leqslant \int \varphi(g)\, dP_{z_0}$$
$$= \int_A \varphi(g)\, dP_{z_0} + \int_{A^c} \varphi(g)\, dP_{z_0} \leqslant P_{z_0}(A^c) \leqslant \varepsilon .$$

If we choose $\varepsilon < \alpha_n$ we get a contradiction. Hence $m(A) = 0$ and $g(\partial\Delta) \subset S(\mathrm{Ball}\,(X))$ a.s.

Consider now $\tilde{f} = S^{-1} \circ g : \partial\Delta \to X$ and notice that it is measurable in view of Lusin's theorem. Moreover, the PSH-martingale $(f(W_{\tau_n}), \tilde{f}(W_{\tau_\infty}))$ is clearly closed, hence f has radial limits a.s by Proposition 3.3.

COROLLARY 3.7. — *Let X be a separable Banach space. Suppose there exists a holomorphic PSH_δ-injection from X into a Banach space Y with the A.R.N.P, then X also verifies the A.R.N.P.*

Remark 3.8. — A typical example where Corollary 3.7 is applicable is when X is a Banach lattice not containing c_0. In this case there exists a « linear semi-embedding » from X into L^1 and the latter has the A.R.N.P. (See [GR] or [D].) □

We shall now present an example of a different nature where a certain holomorphic and non-linear map arises in a natural way. It is in the context of the predual of James-tree space J_*T defined below. We note that in [GMS] it is shown that J_*T has the A.R.N.P via a construction of *a linear PSH_δ-injection* from J_*T into Hilbert space. In the following, we shall establish the existence of *a holomorphic semi-embedding from J_*T* into a separable dual Banach space. This will also imply that J_*T has the A.R.N.P in view of Corollary 3.7.

First we construct the appropriate separable dual. Let $T = \bigcup_{n=0}^{\infty} \{0,1\}^n$ be the usual diadic tree and let Γ be the set of infinite branches γ of T. Denote by J the James space of complex-valued sequences $x = (x_n)_n$ such that $\|x\|_J = \sup\left(\sum_{k=0}^{K}\left|\sum_{i \in I_k} x_i\right|^2\right)^{1/2} < +\infty$, where the supremum is taken over all families $(I_k)_{k=0}^{K}$ of mutually disjoint segments in $\mathbf{N}$.

Denote by Y the Banach space of complex-valued functions $x = (x_t)_{t \in T}$ on the tree T, verifying

$$\|x\|_Y = \sup_{\gamma \in \Gamma} \|(x_{\gamma(n)})_{n=0}^{\infty}\|_J < +\infty .$$

Let Y_0 be the closed subspace of Y generated by the finitely supported vectors $x = (x_t)_t$ in Y verifying $\sum_{t \in \gamma} x_t = 0$ for all $\gamma \in \Gamma$.

PROPOSITION 3.9. — *The dual of Y_0 is isomorphic to the closed subspace Z of Y^* generated by the coefficient functionals $(e_t^*)_t$. In particular, Y_0^* is a separable dual.*

Proof. — Let $q^* : Y^* \to Y_0^*$ be the quotient map. We show first that for any $x^* \in Z \subset Y^*$ we have $\|q^*(x^*)\| \leqslant \|x^*\| \leqslant 2\|q^*(x^*)\|$. Indeed, the first inequality is evident. For the second, assume without loss that x^* is finitely supported, that is : $x^* = \sum_{t \in T} x_t^* e_t^*$ with $x_t^* = 0$ when $|t| > n$. Consider $x = (x_t)_t \in Y$ with $\|x\| \leqslant 1$ and define a vector

x' by $x'_t = x_t$ if $|t| \leqslant n$, $x'_t = 0$ if $|t| > n+1$ and $x'_t = -\sum_{s<t} x_s$ if $|t| = n+1$. It is clear that $x^*(x) = x^*(x')$, $x' \in Y_0$ and $\|x'\| \leqslant 2$. It follows that $\|x^*\| \leqslant 2\|q^*(x^*)\|$.

It remains to show that $q^*(Z) = Y_0^*$. For that, let $\alpha^* \in Y_0^*$. For each $t \in T$ and $n > |t|$ define $f_{t,n} = e_t - \sum_{\substack{s \\ s>t, |s|=n}} e_s$. We have that $f_{t,n} \in Y_0$ and $\|f_{t,n}\| = \sqrt{2}$. Note that for each strictly increasing sequence $(n_k)_{k=0}^{\infty}$ with $n_0 > |t|$, the sequence $(f_{t,n_{2k+1}} - f_{t,n_{2k}})_k = (g_k)_k$ is (in the Y-norm) equivalent to the canonical basis of ℓ_2. It follows that $\alpha^*(g_k) \to 0$ and $\alpha_t = \lim_n \alpha^*(f_{t,n})$ exists for each t.

It is easy to see that the partial sums $\sum_{|t| \leqslant n} \alpha_t e_t^*$ are bounded and that they converge weak* to an element y^* in Y^* such that $q^*(y^*) = \alpha^*$. We now prove the convergence in norm that insures that $y^* \in Z$.

Indeed if not, there is $\delta > 0$ such that for all m, there exist $n \geqslant m$ with $\|w^* \sum_{\substack{t \in T \\ |t| \geqslant n}} \alpha_t e_t^*\|_{Y_0^*} > \delta$. We can then construct a strictly increasing sequence $(n_k)_k$ of integers, a bounded sequence $(y_k)_k$ in Y_0 such that for all k,

(i) $y_{k,t} = 0$ if $|t| \notin [n_k, n_{k+1}[$

and

(ii) $\left\langle \sum_{n_k \leqslant |t| < n_{k+1}} \alpha_t e_t^*, y_k \right\rangle > \delta$.

But (i) gives that $(y_k)_k$ is dominated by the canonical basis of ℓ_2 which implies that $\lim_k \alpha^*(y_k) = 0$. On the other hand (ii) gives that $\alpha^*(y_k) > \delta$ for all k. A contradiction which completes the proof of Proposition 3.9. □

We now recall the definition of JT. It is the space of functions $x: T \to \mathbf{C}$ such that $\|x\|_{JT} = \sup \left(\sum_{i=1}^{m} \left| \sum_{t \in S_i} x_t \right|^2 \right)^{1/2} < \infty$ where the sup is taken over all families $S_1, \ldots, S_m$ of disjoint segments in T. Denote by J_*T the subspace of JT^* spanned by the coefficient functionals $(e_t^*)_{t \in T}$. It is well known [LS] that $(J_*T)^* = JT$. A *molecule* of JT^* is an element of the form $m = \sum_{j=1}^{n} \lambda_j a_j$ where $\lambda_j \in \mathbf{C}$, $\sum_{j=1}^{n} |\lambda_j|^2 \leqslant 1$ and the a_j's are the indicators of mutually disjoint (possibly infinite) segments. It is easy to verify that the molecules form a norming subset of the unit ball of JT^*. It is also shown in [SSW] that the convex combinations of the molecules form a norm dense subset of the unit ball of JT^*. We have the following.

LEMMA 3.10. – *For any couple of molecules m, n in JT^* we have* $\|m \cdot n\|_{Y*} \leqslant 2$.

Proof. – Let $m = \sum_i \lambda_i a_i$ and $n = \sum_j \mu_j b_j$ be two molecules. Let s_i (resp. t_j) be the origin of the segment a_i (resp. b_j). We define a_i^* and b_j^* in the following way: If there exists an index j so that $s_i \in b_j$ and $s_i < t_j$ we set $a_i^* = b_j$. If not we let $a_i^* = 0$. If there exists an index i so that $t_j \in a_i$ we set $b_j^* = a_i$. If not we let $b_j^* = 0$. Define now

$$a'_i = a_i \cdot a_i^* \qquad a''_i = a_i - a'_i$$
$$b'_j = b_j \cdot b_j^* \qquad b''_j = b_j - b'_j$$

and note that a'_i and a''_i are disjoint as well as b'_j and b''_j. Less straightforward but easily verifiable is the fact that $a'_i \cdot b'_j = 0$ and $a''_i \cdot b''_j = 0$ for all (i,j). The details are left to the reader.

Let now $I(j) = \{i; b_j \cdot a'_i \neq 0\}$ and $J(i) = \{j; a_i b'_j \neq 0\}$. The sets $(I(j))_j$ (resp. $J(i)_i$) are mutually disjoint. Set

$$\mu_j^* = \Big(\sum_{i \in I(j)} |\lambda_i|^2\Big)^{1/2} \qquad \text{and} \qquad \lambda_i^* = \Big(\sum_{j \in J(i)} |\mu_j^2|\Big)^{1/2}.$$

We have

$$\sum_j |\mu_j^*|^2 \leqslant 1 \qquad \text{and} \qquad \sum_i |\lambda_i^*|^2 \leqslant 1.$$

Let $x_i = \sum_j \mu_j a_i b'_j$ and $y_j = \sum_i \lambda_i b_j a'_i$. Note that the non-zero terms $(a_i b'_j)_j$ are mutually disjoint segments that are contained in the segment a_i and are affected with coefficients $(\mu_j)_j$ such that $\sum_j \mu_j^2 \leqslant (\lambda_i^*)^2$. It follows – from the structure of J – that $\|x_i\|_{Y*} \leqslant \lambda_i^*$.

Similarly we have $\|y_j\|_{Y*} \leqslant \mu_j^*$.

We finally get :

$$m \cdot n = \sum_{i,j} \lambda_i \mu_j (a_i' + a_i'')(b_j' + b_j'') = \sum_{i,j} \lambda_i \mu_j a_i'' b_j' + \sum_{i,j} \lambda_i \mu_j a_i' b_j''$$
$$= \sum_i \lambda_i x_i + \sum_i \mu_j y_j .$$

But $\left\| \sum_i \lambda_i x_i \right\|_{Y*} \leqslant \sum_i \lambda_i \lambda_i^* \leqslant \left(\sum_i |\lambda_i|^2 \right)^{1/2} \left(\sum_i |\lambda_i^*|^2 \right)^{1/2} \leqslant 1$. The same is valid for $\sum_j \mu_i y_i$ hence $\|mn\|_{Y*} \leqslant 2$ and the Lemma is proved. □

Now we can deduce the following.

PROPOSITION 3.11. — *There exists a holomorphic semi-embedding from $J_* T$ into Y_0^*.*

Proof. — Let S be the map that associates to an element $x = (x_t)_t$ in JT^* its square $Sx = (x_t^2)_t$. The above lemma shows that S maps each molecule in JT^* to an element in Y^*. Suppose now $(\theta_\alpha)_\alpha$ and $(\eta_\beta)_\beta$ are positive coefficients such that $\sum_\alpha \theta_\alpha \leqslant 1$ and $\sum_\beta \eta_\beta \leqslant \eta \leqslant 1$. Let $(m_\alpha)_\alpha$ and $(n_\beta)_\beta$ be two families of molecules. We have :

$$\Delta = \left(\sum_\alpha \theta_\alpha m_\alpha + \sum_\beta \eta_\beta n_\beta \right)^2 - \left(\sum_\alpha \theta_\alpha m_\alpha \right)^2 = 2 \left(\sum_\alpha \theta_\alpha m_\alpha \right) \left(\sum_\beta \eta_\beta n_\beta \right)$$
$$+ \left(\sum_\beta \eta_\beta n_\beta \right)^2 = 2 \sum_{\alpha,\beta} \theta_\alpha \eta_\beta (m_\alpha \cdot n_\beta) + \sum_{\beta,\beta'} \eta_\beta \eta_{\beta'} (n_\beta . n_{\beta'}) .$$

From Lemma 3.10 we have :

$$\|\Delta\|_{Y^*} \leqslant 2 \left[2 \sum_{\alpha,\beta} \theta_\alpha \eta_\beta + \sum_{\beta'} \eta_\beta \eta_{\beta'} \right] \leqslant 2\,[2\eta + \eta^2] \leqslant 6\eta .$$

It follows that $S : JT^* \to Y^*$ maps bounded sets in JT^* into bounded ones in Y^* and that it is uniformly continuous on bounded subsets of JT^*. It is also clear that $S(B(JT^*))$ is norm closed in Y^*. If we restrict S to $J_* T$ we see that the range falls in Z (or Y_0^*). By the results in [LS], the elements of $J_* T$ are those in JT^* that go to zero on the branches. This implies that $S(B(J_* T)) = S(B(JT^*)) \cap Z$ and hence it is also closed.

To remedy the fact that S is not one-to-one, it is enough to take any linear one-to-one map $R: J_*T \to Y_0^*$ and to notice that the map $(S,R): J_*T \to Y_0^* \oplus Y_0^*$ is still a holomorphic semi-embedding. Another way to do it is to take for R the cube operator $(x_t)_t \to (x_t^3)_t$ which also maps J_*T into Y_0^*.

Remark 3.12. – Say that a map $S: X \to Y$ is an *H_δ-embedding* if $\overline{S(Ball(X))}\backslash S(Ball(X)) = \bigcup_n F_n$ where each F_n is closed and convex. By taking any dense range compact operator $T: \ell_2 \to Y_0$ and by composing T^* with the map obtained in Proposition 3.11, we obtain a *holomorphic H_δ-embedding from J_*T into ℓ_2*. It is worth noting that one cannot find a linear H_δ-embedding from J_*T into ℓ_2 nor a linear semi-embedding from J_*T into a separable dual [GM1]. However, as noted above, it is shown in [GMS] that there exists a *linear PSH_δ-injection from J_*T into ℓ_2*. For more general type of injections that are compatible with the A.N.R.P, we refer to section 5.

4. Plurisubharmonic denting points and barriers.

We start by defining various types of geometrically distinguished points. Let C be a bounded subset of a quasi-Banach space X and let x be a point in C. We shall say that:

(a) x is a *complex extreme point of* C if there is no non-zero vector $y \in X$ with $\{x+re^{i\theta}y; 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\} \subset C$.

(b) x is a *Jensen boundary point of* C if the Dirac measure δ_x is the only Jensen Radon probability measure on C with barycenter x.

(c) x is a *plurisubharmonic denting point of* C if for each $\varepsilon > 0$, $x \notin \hat{A}_\varepsilon$ where $A_\varepsilon = C\backslash B(x,\varepsilon)$ and $B(x,\varepsilon)$ is the ball centered at x with radius ε.

(d) x is a *barrier* (resp. *strong barrier*) of C if there exists a plurisubharmonic function φ in $PSH_{uc}(X)$ that exposes (resp. strongly exposes) C at x: that is:

(i) $\varphi(x) = \sup_C \varphi$

(ii) $\varphi(c) < \varphi(x)$ for each c in C (resp. $\sup_{A_\varepsilon} \varphi < \varphi(x)$ for each $\varepsilon > 0$).

Such a function φ will be called a *plurisubharmonic barrier* (resp. *strong plurisubharmonic barrier*) for C.

Note that any (real) extreme point is a complex extreme point. On the other hand, any point in the unit sphere of L^1 is complex extreme for the unit ball (Theorem 2.8 [C]) while none of these points is extreme.

It is clear that a barrier point is a Jensen boundary point which in turn is a complex extreme point. Moreover, every strong barrier point is a plurisubharmonic denting point.

We shall start by dealing with the case of a compact subset K of X. The proofs in the following proposition are modelled on standard techniques. We include sketches for completeness.

PROPOSITION 4.1. — *Let K be a compact subset of an $(A-p)$ convex quasi-Banach space X where $0 < p \leqslant 1$. Then :*

(a) *Every function in $PSH_c(K)$ attains its maximum on K at a Jensen boundary point of K.*

(b) *Every Jensen boundary point of K is exposed by a function in $\overline{PSH_p(K)}$.*

(c) *K is contained in the plurisubharmonic hull of its barrier points.*

Proof. — The proof of (a) can be modelled on a standard proof of the Krein-Milman theorem. Say that a closed subset F of a compact space M is a *J-face in M* if any Jensen Radon probability measure on M is supported on F whenever its barycenter belongs to F. The following observations are easy to check :

(i) A singleton $\{x\}$ is a J-face in K if and only if x is a Jensen boundary point in K.

(ii) For any $\varphi \in PSH_c(K)$ and any compact $M \subset X$, the set $B_\varphi(M) = \{x \in M; \varphi(x) = \sup_M \varphi\}$ is a J-face in M.

(iii) The family of J-faces of a given compact set is an inductive family once ordered by inclusion.

Let now $\varphi \in PSH_c(K)$. By Zorn's lemma, there exists a minimal J-face F_0 of K contained in $B_\varphi(K)$. If F_0 is not a singleton, we use the A-convexity assumption to find $\psi \in PSH_p(K)$ that separates two distinct points of F_0. The set $B_\psi(F_0)$ contradicts then the minimality of F_0. This clearly establishes claim (a).

For (b) we shall use a technique developed by E. Bishop (see [Gam]). Let x_0 be a Jensen boundary point of K and let h be a function in $LIP_p(X)$ that attains its maximum on K uniquely at x_0. We shall prove the existence of a function $\varphi \in \overline{PSH_p(K)}$ such that $\varphi \leqslant h$ on K and $\varphi(x_0) = h(x_0)$. This will clearly establish (b).

Fix real numbers b and c so that $b < \min_K h < \max_K h < c$. For any compact subset $E \subset K$ not containing x_0, we have $x_0 \notin \hat{E}$ since x_0 is a Jensen boundary point. We can use Proposition 2.6 to find a function u in $PSH_p(X)$ such that: $u \leqslant c$, $u(x_0) = h(x_0)$ and $u \leqslant b$ on E.

Indeed, for $\varepsilon > 0$, let g be a continuous function on K such that $g(x_0) = h(x_0)$, $g \leqslant b - \varepsilon$ on E and $g \leqslant h(x_0)$ on K. We claim that $g(x_0) = \sup\{\varphi(x_0); \varphi \in PSH_p(X), \varphi \leqslant g \text{ on } K\}$ since if not the function $g - g(x_0)$ does not belong to the closure of the cone $\mathcal{U} = \{f \in C(K); \exists \varphi \in PSH_p(X), \varphi(x_0) = 0 \text{ and } \varphi \leqslant f \text{ on } K\}$. By the Hahn-Banach theorem, there exists a non-zero measure μ on K, positive on $\mathcal{U}$ while $\int (g - g(x_0))\, d\mu < 0$. It follows that $\nu = \dfrac{\mu}{\mu(K)}$ is a Jensen probability measure on K with barycenter x_0 while it is different from δ_{x_0}. A contradiction that implies the above formula for $g(x_0)$. Now, we can find $\varphi \in PSH_p(X)$ with $\varphi \leqslant g$ on K while $\varphi(x_0) = g(x_0) - \varepsilon = h(x_0) - \varepsilon$. If ε is small enough, then $u = \varphi + \varepsilon$ will do the job.

Let now $\lambda (0 < \lambda < 1)$ so that $(1-\lambda)c + \lambda b - \min_K h < 0$ and choose a sequence $(\varepsilon_m)_m$ of positive reals decreasing to 0 in such a way that for every $m \geqslant 1$ we have:

$$(1-\lambda^m)\varepsilon_m + \lambda^m[(1-\lambda)c + \lambda b - \min_K h] < 0.$$

Define, by induction a sequence $(u_i)_i$ in $PSH_p(X)$ in the following way: u_0 is the constant function $h(x_0)$ and if $u_0, \dots, u_{m-1}$ have been chosen so that $u_j(x_0) = h(x_0) (0 \leqslant j \leqslant m-1)$, the compact set $E_m = \{x \in K; \max_{0 \leqslant j \leqslant m-1} u_j(x) \geqslant h(x) + \varepsilon_m\}$ does not contain x_0, hence by applying the above observation, there exists $u_m \in PSH_p(X)$ so that $u_m(x_0) = h(x_0)$, $u_m \leqslant c$ on K and $u_m \leqslant b$ on E_m.

We now show that the function $u = (1-\lambda) \sum_{j=0}^{\infty} \lambda^j u_j$ which is in $\overline{PSH_p(K)}$, verifies our claim. (If the u_j's are not uniformly bounded below, we replace them by $\max(-M,u_j) \in PSH_p(X)$ where M is a large enough constant.) Indeed $u(x_0) = h(x_0)$ and to prove that $u \leqslant h$ on K we first note that it is obviously the case if $\sup_j u_j \leqslant h$. Otherwise, let $m \geqslant 0$ be the first integer such that there exists $x \in E_{m+1}$ while $x \notin E_m$. Then $u_j(x) < h(x) + \varepsilon_m$ for $0 \leqslant j \leqslant m-1$ while $u_j(x) \leqslant b$ for $j > m$. It follows that:

$$\begin{aligned} u(x) &\leqslant (1-\lambda)[(h(x)+\varepsilon_m) \sum_{j=0}^{m-1} \lambda^j + \lambda^m c + b \sum_{j=m+1}^{\infty} \lambda^j] \\ &= (1-\lambda^m)(h(x)+\varepsilon_m) + (1-\lambda)\lambda^m c + \lambda^{m+1} b \\ &= h(x) + (1-\lambda^m)\varepsilon_m + \lambda^m[(1-\lambda)c+\lambda b - h(x)] < h(x). \end{aligned}$$

For c) we shall prove that the functions in $PSH_p^1(K)$ that are exposing for K are dense. By the Baire category theorem, it is enough to show that for each $\varepsilon > 0$, the set $O(\varepsilon) = \{\varphi \in PSH_p^1(K)\,; \exists \tau > 0, \operatorname{diam} S(K,\varphi,\tau) \leqslant \varepsilon\}$ is open and dense in $PSH_p^1(K)$. Here $S(K,\varphi,\tau)$ denotes the set $\{x \in K; \varphi(x) > \sup_K \varphi - \tau\}$.

Assume $K \subset Ball(X)$ and note first that $O(\varepsilon)$ is trivially open. To show it is dense, consider an open set Ω in $PSH_p^1(K)$ and cover K by a finite union of *balls* $B_k = B(x_k, \varepsilon/2^{3/p})$. Denote by $F_k = \{\varphi \in PSH_p^1(K)\,;\ \varphi$ attains its maximum on K at a point of $B_k\}$. Since $\Omega = \bigcup_k \Omega \cap F_k$, there exists – by Baire's category theorem – a k_0 such that $\Omega \cap F_{k_0}$ has a non-empty interior. Assume $B(\psi,\alpha) \subset \Omega \cap F_{k_0}$ for some $\psi \in PSH_p^1(K)$ and $\alpha > 0$. We shall show that $\psi \in \Omega \cap O(\varepsilon)$. Actually we claim that $S(K,\psi,\tau) \subset B(x_{k_0}, \varepsilon/2^{1/p})$ for a small enough $\tau > 0$.

Indeed if $y \in S(K,\psi,\tau)$ but $y \notin B(x_{k_0}, \varepsilon/2^{1/p})$, we consider the function g in $PSH_p^1(K)$ equal to $g(x) = (\|x-x_{k_0}\|^p - \varepsilon^p/4)^+$. Note that g vanishes on $B_{k_0}, g(y) > \varepsilon^p/4$ and $0 \leqslant g \leqslant 2$ on K. If $y \in S(K,\psi,\tau)$, we claim that $\psi + \frac{8\tau}{\varepsilon^p} g$ does not attain its maximum on B_{k_0}. Indeed

$$\sup_{B_{k_0}} \left(\psi + \frac{8\tau}{\varepsilon^p} g\right) = \sup_{B_{k_0}} \psi, \text{ while}$$

$$\sup_K \left(\psi + \frac{8\tau}{\varepsilon^p} g\right) \geqslant \left(\psi + \frac{8\tau}{\varepsilon^p} g\right)(y) > \sup_K \psi - \tau + 2\tau > \sup_K \psi = \sup_{B_{k_0}} \psi .$$

This contradicts the fact that $\left(1+\frac{8\tau}{\varepsilon^p}\right)^{-1}\left(\psi+\frac{8\tau}{\varepsilon^p}g\right)\in B(\psi,\alpha)\subset\Omega\cap F_{k_0}$ for a small enough $\tau>0$.

Remark 4.2. — The assumption of A-convexity is essential for the above results to hold. Indeed, Kalton constructed recently in [K4] a convex compact subset K of a quasi-Banach space such that any continuous plurisubharmonic function on K is necessarily constant. This means that K has no barriers.

The following is the main result of this section.

THEOREM 4.3. — *Let X be a quasi-Banach space. The following properties are then equivalent :*

(1) *Every function in $H^\infty(\Delta,X)$ has radial limit a.s.*

(2) *X is A-convex and every closed bounded subset of X is contained in the closed plurisubharmonic hull of its plurisubharmonic denting points.*

(3) *X is $(A-p)$ convex for some p $(0<p\leqslant 1)$, and for any closed bounded subset C of X, $PSH_p(X)$ contains a dense G_δ-set consisting of plurisubharmonic strong barriers for C.*

(4) *X is $(A-p)$ convex for some $p(0<p\leqslant 1)$, and for any closed bounded subset C of X and every bounded above upper semi-continuous function f on C, the set $\{\varphi\in PSH_p(X)\ ;f+\varphi$ strongly exposes $C\}$ is a dense G_δ in $PSH_p(X)$.*

(5) *X is $(A-p)$ convex for some $p(0<p\leqslant 1)$ and all X-valued, L_p bounded PSH-martingales converge a.s.*

Remarks. — The implication 1) $\Rightarrow$ 5) was proved recently (for Banach spaces) by Bu-Schachermayer in [BS] where they show that PSH-martingales can be appropriately approximated by analytic martingales. Our proof is less direct and goes first through the optimization principle (4) which is the « analytic analogue » of results of Bourgain [Bo] and Stegall [St] established in the context of the Radon-Nikodym theory, and where they show that the perturbations can then be chosen to be linear.

We shall deduce the above theorem from the following propositions. The key will be the following — slightly more technical — condition which is also equivalent to the above assertions.

(4 *bis*) *X is $(A-p)$ convex for some p $(0<p\leqslant 1)$ and for every application F from a set K into X such that $F(K)$ is separable and any real-valued function f on K verifying for some $\alpha\in\mathbf{R}$ and $\beta>0$, that $f(t)\leqslant\alpha-\beta\|F(t)\|^p$ for all $t\in K$, there exist for every $\varepsilon>0$, a $\tau>0$ and $\varphi\in PSH_p^1(X)$ such that $\rho=\sup\{f(t)+\varphi(F(t))\ ;\ t\in K\}<\infty$ and* $\operatorname{diam}\{F(t)\ ;\ t\in K$ *and* $f(t)+\varphi(F(t))>\rho-\tau\}<\varepsilon$.

First define a *plurisubharmonic slice of a set C* to be any non-empty subset of the form $S(C,\varphi)=C\cap\{x\in X;\varphi(x)>0\}$ where φ is a function in $PSH_{uc}(X)$.

It is then easy to see that a point x in C is PSH-denting if and only if it is contained in plurisubharmonic slices of C of arbitrarily small diameter.

We first prove the following :

Proposition 4.4. — *Let X be a p-Banach space for some $p(0<p\leqslant 1)$. Suppose for any $\varepsilon>0$ and every non-empty bounded subset $C\subset X$ not contained in $B(0,(\varepsilon/2)^{1/p})$, there exists a plurisubharmonic slice $S(C,\varphi)$ of C such that $S(C,\varphi)\cap B(0,(\varepsilon/2)^{1/p})=\emptyset$ and $\operatorname{diam}(S(C,\varphi))<\varepsilon^{1/p}$. Then any F in $H^\infty(\Delta,X)$ has radial limits a.s.*

Proof. — Let F be an analytic function from Δ into X such that $\|F(z)\|\leqslant 1$ for all $z\in\Delta$. To show it has radial limits a.s, it is enough to prove — modulo a standard exhaustion argument — the following :

(*) For every measurable subset $\Omega\subset\mathbf{T}$ with $m(\Omega)>0$ and any $\varepsilon>0$, there exists a measurable subset $\Omega'\subset\Omega$ with $m(\Omega')>0$ such that $\limsup_{r,r'\uparrow 1}\|F(re^{i\theta})-F(r'e^{i\theta})\|<\varepsilon^{1/p}$ for almost all θ in Ω'. (Here m is Lebesgue measure on $\mathbf{T}$.)

To prove (*), first choose an outer function H in $H^\infty(\Delta,\mathbf{C})$ such that $|H(z)|=1$ if $z\in\Omega$ and $|H(z)|=(\varepsilon/4)^{1/p}$ if $z\in\mathbf{T}\backslash\Omega$. This can be done by taking $H=\exp(\log k+ih)$ where k is equal to 1 on Ω and $(\varepsilon/4)^{1/p}$ on $\mathbf{T}\backslash\Omega$ and h is the Hilbert transform of $\log k$.

Let $C=HF(\Delta)\subset X$. We distinguish two cases :

(1) If $C=H\cdot F(\Delta)\subset B(0,(\varepsilon/2)^{1/p})$, then

$$\limsup_{r,r'\uparrow 1}\|H(re^{i\theta})\cdot F(re^{i\theta})-H(r'e^{i\theta})F(r'e^{i\theta})\|<\varepsilon^{1/p}$$

for all θ in $\mathbf{T}$ and the conclusion follows since $\lim_{r\uparrow 1} |H(re^{i\theta})| = 1$ if $\theta \in \Omega$.

(2) If $C = H\cdot F(\Delta)$ is not contained in $B(0,(\varepsilon/2)^{1/p})$, find a plurisubharmonic slice $S(\varphi,C)$ of C with $S(\varphi,C) \cap B(0,(\varepsilon/2)^{1/p}) = \emptyset$ and diam $(S(\varphi,C)) \leqslant \varepsilon^{1/p}$. The function $\ell = \varphi \circ (H\cdot F)$ is continuous subharmonic on Δ, hence it has radial limits for almost all $\theta \in \mathbf{T}$. Let $\tilde{\ell}(\theta) = \lim_{r\uparrow 1} \ell(re^{i\theta})$ and consider the set $\Omega' = \{\theta \in \mathbf{T}; \tilde{\ell}(\theta) > 0\}$. Let us show that Ω' verifies the claims in (*):

(i) $m(\Omega') > 0$: Indeed if $\tilde{\ell} \leqslant 0$ a.s. on $\mathbf{T}$, then since ℓ is bounded and $\ell(W_t)$ is a submartingale, $\ell(W_t) \leqslant E[\tilde{\ell}(W_\tau)|F_t] \leqslant 0$ where $\tau = \inf\{t; |W_t| \geqslant 1\}$. This contradicts the fact that $P[\ell(W_t) > 0] = P[W_t \in (H\cdot F)^{-1}(S(\varphi,\mathrm{C}))] > 0$.

(ii) $\Omega' \subset \Omega$: For that we shall prove that $\tilde{\ell} \leqslant 0$ on $\mathbf{T}\backslash\Omega$. Indeed if $\theta \in \mathbf{T}\backslash\Omega$, then $\limsup_{r\uparrow 1} \|H(re^{i\theta})F(re^{i\theta})\| < \varepsilon/2^{1/p}$. It follows that for r close enough to 1, we have $(H\cdot F)(re^{i\theta}) \in \mathrm{B}(0,\varepsilon/2^{1/p})$, hence $H\cdot F(re^{i\theta}) \notin S(\varphi,C)$ and $\ell(re^{i\theta}) = \varphi(H\cdot F(re^{i\theta})) \leqslant 0$. Since φ is continuous, we get that $\tilde{\ell}(\theta) \leqslant 0$ for all θ in $\mathbf{T}\backslash\Omega$.

(iii) If $\theta \in \Omega'$, then $\lim_{r\uparrow 1} \ell(re^{i\theta}) > 0$, hence for r close enough to 1, we have $H\cdot F(re^{i\theta}) \in S(\varphi,C)$. It follows that $\limsup_{r,r'\uparrow 1} \|H\cdot F(re^{i\theta}) - H\cdot F(r'e^{i\theta})\| < \varepsilon^{1/p}$. The rest follows again from the fact that if $\theta \in \Omega' \subset \Omega$, then $\lim_{r\uparrow 1} |H(re^{i\theta)}| = 1$. □

PROPOSITION 4.5. — *Let X be a p-Banach space for some $p (0<p\leqslant 1)$ and assume that all X-valued L^p-bounded analytic martingales converge a.s, then X verifies Property* (4 *bis*).

Proof. — Define for each $t \in K$ the following function on X,

$$\varepsilon_t(y) = \inf\{f(t) - f(u) + \|y - F(u)\|^p\,;\, u \in K \quad \text{and} \quad \|F(u) - F(t)\|^p > \varepsilon/2\}.$$

If Δ is a countable subset of K such that $F(\Delta)$ is dense in $F(K)$, it is clear that the above infimum can be restricted to the elements of Δ. Note also that the function ε_t is in $LIP_p(X)$ and is bounded below by the constant $f(t) - \alpha$, it then follows from Lemma 2.1 that $\hat{\varepsilon}_t$ is also finite and belongs to $PSH_p^1(\mathrm{X})$. To establish the above Proposition, it is enough to prove the following claim:

There exists t in Δ such that $\hat{\varepsilon}_t(F(t)) > 0$.

Indeed, in this case the set $A = \{s \in K; f(s)+\hat{\varepsilon}_t(F(s))>f(t)\}$ is non-empty since it contains t. Moreover, if $s \in K$ is such that $\|F(s)-F(t)\|^p > \varepsilon/2$, then by taking $u = s$ to get an upper bound for $\varepsilon_t(F(s))$ we obtain

$$\hat{\varepsilon}_t(F(s)) \leqslant \varepsilon_t(F(s)) \leqslant f(t)-f(s).$$

This means that $s \notin A$ and consequently diam $F(A) \leqslant \varepsilon^{1/p}$.

Back to the claim and assume it is not true: that is $\hat{\varepsilon}_t(F(t) \leqslant 0$ for all $t \in K$. Let $(\tau_n)_n$ be a sequence of positive reals so that $\tau = \sum_{n=1}^{\infty} \tau_n < \infty$. We shall construct two sequences of random variables $(T_n)_{n\geqslant 0}$ and $(U_n)_{n\geqslant 1}$ on $\mathbf{T}^N$ such that for each $n \in \mathbf{N}$

(i) T_n is Δ-valued and $\|F(T_{n+1})-F(T_n)\|^p > \varepsilon/2$.

(ii) U_n is the k_n – th variable of an X-valued analytic martingale starting at 0 and

$$\mathbf{E}[f(T_n)-f(T_{n+1})+\|F(T_n)+U_{n+1}-F(T_{n+1})\|^p] < \tau_{n+1}.$$

Start with any t_0 in Δ and set $T_0 = t_0$. Suppose T_j and U_j have been constructed for $j \leqslant n$. Since for every $\omega \in \mathbf{T}^n$ we have that $\hat{\varepsilon}_{T_n(\omega)}(T_n(\omega)) \leqslant 0$, we can use Lemma 2.1 to find U_{n+1} that is the k_{n+1} – th variable of an X-valued analytic martingale starting at 0 such that

$$\mathbf{E}[\varepsilon_{T_n(\omega)}(T_n(\omega)+U_{n+1})] < \tau_{n+1}.$$

Use now the definition of $\varepsilon_{T_n(\omega)}$ to find a Δ-valued random variable T_{n+1} such that (i) and (ii) hold.

Note that (ii) gives

$$\mathbf{E}[f(T_0)-f(T_{n+1})] \leqslant \tau$$

and hence that

$$\mathbf{E}[\beta\|F(T_{n+1})\|^p] \leqslant \alpha - \mathbf{E}[f(T_{n+1})] \leqslant \alpha + \tau - f(t_0).$$

On the other hand we have

$$(*) \qquad \mathbf{E}\left[\sum_{n=0}^{\infty} \|(F(T_n)-F(T_{n+1})+U_{n+1}\|^p\right] < \tau - f(T_0) + \alpha.$$

Hence

$$\mathbf{E}\|(F(T_{n+1})-F(T_0))-M_{n+1}\|^p \leqslant \tau - f(t_0) + \alpha$$

where $(M_n)_n$ is the subsequence of an analytic martingale defined by $M_0 = 0$ and $M_n = U_1 + \cdots + U_n$ for $n > 0$. Note that $(M_n)_n$ is clearly L^p-bounded and hence must converge a.s by the hypothesis. But (*) implies that the sequence $F(T_n)_n$ also converges a.s. This clearly contradicts (i) and the claim is therefore established.

We now deal with the problem of convergence of *PSH*-martingales. Consider first a measurable function f from a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ into a complete metric space (Z, d). We shall say that a point $\omega \in \Omega$ is *regular for* $(f, \mathscr{F})$ if for every $\varepsilon > 0$, there exists $B \in \mathscr{F}$ with $P(B) > 0$ such that for every $\omega' \in B$ we have $d(f(\omega), f(\omega')) \leqslant \varepsilon$. It is easy to see that if $(f_n)_n$ is a countable sequence of random variables and if (Z, d) is separable, then there is $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that every $\omega \in \Omega'$ is regular for each f_n.

PROPOSITION 4.6. — *Let X be an $(A-p)$ convex Banach space for some p $(0<p\leqslant 1)$ which verifies Property (4 bis). Then*

(i) *Every X-valued and L^p-bounded PSH-martingale converges a.s.*

(ii) *Every function in $H^p(\Delta, X)$ has radial limits a.s.*

Proof. — Let $(M_n)_n$ be an X-valued and L^p-bounded *PSH*-martingale. By a standard exhaustion argument, it is enough to prove the following :

claim : *For every measurable set $A \subset \Omega$ with $P(A) > 0$ and any $\varepsilon > 0$, there exists a measurable set $A' \subset A$ with $P(A') > 0$ such that for all $\omega \in A'$,*

$$\limsup_{m,n} \|M_n(\omega) - M_m(\omega)\|^p < \varepsilon .$$

Note first that $(\|M_n\|^{p/2})_n$ is an L^2-bounded real submartingale. It follows from Doob's inequality that $\sup_n \|M_n\|^p \in L^1$. The real submartingale convergence theorem gives the L^1 as well as the almost sure convergence of $(\|M_n\|^p)_n$ to a random variable that we denote by Z. Fix $A \subset \Omega$ and $\varepsilon > 0$ and let $\lambda > 0$ be such that $A_\lambda = A \cap \{Z \leqslant \lambda\}$ has non-zero measure. Set $D_\lambda = \Omega \backslash A_\lambda$, $h = -1_{D_\lambda}(Z+\lambda+1)$ and $h_n = \mathrm{E}[h; \mathscr{F}_n]$. By the above remark we can find a measurable set $\Omega' \subset \Omega$ of full measure such that $\forall \omega \in \Omega'$ we have $h(\omega) = \lim_n h_n(\omega)$, $Z(\omega) = \lim_n \|M_n(\omega)\|^p$ and ω is regular for the sequence $\{((h_n, M_n), \mathscr{F}_n)$; $n \geqslant 0\}$.

We want to apply Property (4 *bis*) to the set $K = \mathbf{N} \times \Omega'$ and the functions $F(n,\omega) = M_n(\omega)$ and $f(n,\omega) = h_n(\omega)$. For that let us check that we have the right hypothesis. First f is clearly bounded above by 0. On the other hand we have for all $(n,\omega) \in K$ that

$$f(n,\omega)) \leqslant \lambda - \|F(n,\omega)\|^p .$$

Indeed, if $(n,\omega) \in K$ and since ω is regular, there exists for every $\varepsilon > 0$, a set $C \in \mathscr{F}_n$, with $P(C) > 0$ such that

$$\begin{aligned}\|M_n(\omega)\|^p &\leqslant \frac{1}{P(C)} \int_C \|M_n\|^p \, dP + \varepsilon \leqslant \frac{1}{P(C)} \int_C Z dP + \varepsilon \\ &\leqslant \frac{1}{P(C)} \int_C (\lambda + 1_{D_\lambda}(Z + \lambda + 1)) dP + \varepsilon = \lambda - \frac{1}{P(C)} \int_C h_n \, dP + \varepsilon \\ &\leqslant \lambda - h_n(\omega) + 2\varepsilon .\end{aligned}$$

Apply now Property (4 *bis*) to obtain $\varphi \in PSH_p^1(X)$ such that

$$\rho = \sup \{ h_n(\omega) + \varphi(M_n(\omega)) \, ; (n,\omega) \in K \} < \infty$$

and a $\tau (0 < \tau < 1)$ such that $\operatorname{diam}(F(K_0)) \leqslant \varepsilon^{1/p}$ where

$$K_0 = \{ (n,\omega) \in K \, ; h_n(\omega) + \varphi(M_n(\omega)) > \rho - \tau \} .$$

The real submartingale $\varphi(M_n)_n$ converges a.s and in L^1 to a random variable ψ. Let

$$A' = \{ \omega \in \Omega'' \, ; (h + \psi)(\omega) > \rho - \tau \}$$

where Ω'' is the subset of Ω' on which $\varphi(M_n)_n$ converges to ψ. It is clear that if $\omega \in A'$ then $(n,\omega) \in K_0$ for n large enough which implies that

$$\limsup_{n,m} \|M_n(\omega) - M_m(\omega)\|^p < \varepsilon .$$

So it remains to show that $A' \subset A$ while having a non-zero measure.

For that, we first show that

$$\psi \leqslant Z + \lambda + \rho \quad \text{on the set } \Omega'' .$$

Indeed, for $\omega_0 \in A_\lambda \cap \Omega''$ we have $\lim_m h_m(\omega_0) = 0$ hence

$$\psi(\omega_0) = \lim_m (h_m(\omega_0) + \varphi(M_n(\omega_0))) \leqslant \rho .$$

It follows that for any $\omega \in \Omega''$ we have

$$\psi(\omega)-\psi(\omega_0)\leqslant \lim_n \|M_n(\omega)-M_n(\omega_0)\|^p \leqslant Z(\omega) + Z(\omega_0) \leqslant Z(\omega) + \lambda .$$

But this implies that $A' \subset A_\lambda$ since if $\omega \in \Omega''\backslash A_\lambda$ we have

$$\begin{aligned}(h+\psi)(\omega) &= \{\psi - 1_{D_\lambda}(Z+\lambda+1)\}(\omega)\\ &= \psi(\omega)-Z(\omega)-\lambda-1<\rho-\tau,\end{aligned}$$

hence $\omega \notin A'$. To show that the latter has a non zero measure, pick $(n_1,\omega_1) \in K_0$. Since ω_1 is regular for $\{(h_{n_1}, M_{n_1}), \mathscr{F}_{n_1}\}$ there exists $C \in \mathscr{F}_{n_1}$ with $P(C) > 0$ and $h_{n_1}(\omega) + \varphi(M_{n_1}(\omega)) > \rho - \tau$ for all $\omega \in C$. By the submartingale property, we get

$$\rho - \tau < \frac{1}{P(C)} \int_C (h_{n_1} + \varphi(M_{n_1}))\, dP \leqslant \frac{1}{P(C)} \int_C (h+\psi)\, dP .$$

This clearly implies that $P(A') > 0$ and claim (i) of the proposition is proved.

(ii) Let now F be a function in $H^p(\Delta, X)$. As above we shall prove that for any $\varepsilon > 0$, any measurable subset A of $\mathbf{T}$ with Lebesgue measure $m(A) > 0$, contains a set A' such that $m(A') > 0$ and

$$\limsup_{r,r'\uparrow 1} \|F(rt)-F(r't)\|^p < \varepsilon .$$

To do that, note first that the real-valued subharmonic function $\|F\|^p$ has radial limits almost surely and in L^1. Let $Z: \mathbf{T} \to \mathbf{R}$ be such a limit and choose $\lambda > 0$ such that the set $A_\lambda = A \cap \{Z \leqslant \lambda\}$ has non-zero measure. Set $B_\lambda = \mathbf{T}\backslash A_\lambda$, and let f be the harmonic extension of $-1_{B_\lambda}(Z+\lambda+1)$ to Δ. Again the function f admits radial limits in L^1 and almost surely. If $z \in \Delta$ we have

$$\begin{aligned}\|F(z)\|^p &\leqslant \int_{\mathbf{T}} Z(t)\, dP_z(t)\\ &\leqslant \int_{\mathbf{T}} \{\lambda + 1_{B_\lambda}(Z+\lambda+1)\}\, dP_z(t) = \lambda - f(z)\end{aligned}$$

hence $f(z) \leqslant \lambda - \|F(z)\|^p$ for all $z \in \Delta$, and we can therefore apply property (4 *bis*) to f, F, and $K = \Delta$, to obtain $\varphi \in PSH^1_p(X)$ such that

$$\rho = \sup\{f(z) + \varphi(F(z)); z \in \Delta\} < \infty$$

and a $\tau(0<\tau<1)$ such that $\operatorname{diam}(F(\Delta_0)) < \varepsilon^{1/p}$ where

$$\Delta_0 = \{z \in \Delta; f(z) + \varphi(F(z)) > \rho - \tau\}.$$

Since $\sup_{r<1} \varphi(F(rt)) \in L^1(\mathbf{T})$, the real-valued subharmonic function $\varphi \circ F$ converges almost surely and in L^1 to a limit $\psi \in L^1(\mathbf{T})$. Let

$$A' = \{t \in \mathbf{T}; (f+\psi)(t) > \rho - \tau\}.$$

It is clear that for almost all $t \in A'$, $rt \in \Delta_0$ if r is close enough to 1, and consequently

$$\limsup_{r,r'\uparrow 1} \|F(rt) - F(r't)\|^p \leqslant \varepsilon.$$

So it remains to show that $A' \subset A$ and that $m(A') > 0$.

For that, note first that if $t_0 \in A_\lambda$ we have $\lim_{r\uparrow 1} f(rt_0) = 0$, hence

$$\psi(t_0) = \lim_{r\uparrow 1} \{f(rt_0) + \varphi(F(rt_0))\} \leqslant \rho.$$

If $t \in \mathbf{T}$, then

$$\psi(t) - \psi(t_0) \leqslant \lim_{r\uparrow 1} \|F(rt) - F(rt_0)\|^p \leqslant Z(t) + Z(t_0) \leqslant Z(t) + \lambda$$

hence $\psi(t) \leqslant Z(t) + \lambda + \rho$. But this implies that $A' \subset A_\lambda$ since if $t \in B_\lambda = \mathbf{T} \backslash \mathbf{A}_\lambda$, then

$$(f+\psi)(t) = -(Z(t) + \lambda + 1) + \psi(t) \leqslant \rho - 1 < \rho - \tau$$

and hence $t \notin A'$. To show that the latter has a non-zero measure, pick $z_1 \in \Delta_0$ and note that

$$\rho - \tau < f(z_1) + \varphi(F(z_1)) \leqslant \int (f+\psi)\, dP_{z_1}.$$

This clearly implies that $m(A') > 0$ and claim (ii) of the Proposition is proved.

Now we can prove Theorem 4.3.

(1) $\Rightarrow$ (4 *bis*). By Proposition 3.4, there exists p $(0<p\leqslant 1)$ such that an equivalent quasi-norm is p-subadditive and for which all L^p-bounded, X-valued analytic martingales converge a.s. The rest follows from Proposition 4.5.

(4 *bis*) $\Rightarrow$ (5) and (1). This is Proposition 4.6.

(4 *bis*) $\Rightarrow$ (4). Let C be a closed bounded subset of X and let $f : C \to \mathbf{R}$ be a bounded above upper semi-continuous function. Apply (4 *bis*) to f, $K = C$ and $F(x) = x$ to obtain for each $\varepsilon > 0$ a $\varphi \in PSH_p^1(X)$ and $\tau > 0$ such that the set

$$S(C, f+\varphi, \tau) = \{x \in C; (f+\varphi)(x) > \sup_C (f+\varphi) - \tau\}$$

has a diameter less than ε. The rest of the claim will follow from a standard application of the Baire category theorem. (See for instance [Bo] or [GLM].)

(4) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) Let $\varphi \in PSH_p^1(X)$ such that $C \cap \{\varphi > 0\} \neq \emptyset$. Choose $\psi \in PSH_p^1(X)$ such that $\sup_{x \in C} |\psi(x) - \varphi(x)| < \varepsilon$ and ψ strongly exposes C at $x_0 \in C$. It is clear that when ε is chosen small enough, x_0 would belong to $C \cap \{\varphi > 0\}$.

(2) $\Rightarrow$ (1) Let $\| \ \|$ be a plurisubharmonic equivalent quasi-norm such that $\| \ \|^p$ is subadditive. If $C \nsubseteq B(0, \varepsilon)$, this means that $C \cap \{\varphi > \varepsilon\}$ is a non-empty PSH-slice where $\varphi(x) = \|x\|$ is clearly in $PSH_{uc}(X)$. Hence 2) implies the hypothesis of Proposition 4.4 which in turn implies 1).

We also note that (4) can be used to prove (4 *bis*) directly. Indeed, if K, f, F and ε are as in the hypothesis of (4 *bis*), consider the function g defined on $C = F(K)$ by

$$g(x) = \sup \{h(t); t \in K \text{ and } F(t) = x\}.$$

It is clearly bounded above on C and let $\tilde{g}$ be its « upper semi-continuous regularization » on $\bar{C}$, i.e. for every $x \in \bar{C}$,

$$\tilde{g}(x) = \limsup_{y \in C, y \to x} g(y).$$

Apply the optimization principle (4) to find $\varphi \in PSH_p^1(X)$ such that the set $C_0 = \{x \in \bar{C} : \tilde{g}(x) + \varphi(x) > 0\}$ is non-empty and $\operatorname{diam}(C_0) < \varepsilon$. The set $C_1 = \{x \in C : g(x) + \varphi(x) > 0\}$ is non-empty and is contained in C_0. Let $K_0 = \{t \in K : f(t) + \varphi(F(t)) > 0\}$. It is easy to verify that it is also non-empty, that its image under F is contained in C_0 and hence $\operatorname{diam}(F(K_0)) \leqslant \varepsilon$.

Remark 4.7. — a) Recently S. Bu [Bu] showed that the existence of a function in $H^\infty(\Delta, X)$ that does not have radial limits on a set of positive measure, actually implies the existence of a function in $H^\infty(\Delta, X)$

that has radial limits nowhere – i.e. there is $\eta > 0$ such that for almost all $\theta \in \mathbf{T}$, $\limsup_{r,r'\uparrow 1} \|f(re^{i\theta}) - f(r'e^{i\theta})\| > \eta$. This clearly implies that Theorem 4.3 holds if and only if every closed bounded subset of X has plurisubharmonic slices of arbitrarily small diameter : a hypothesis which is slightly weaker than assertion (2) of Theorem 4.3.

b) One may ask whether Theorem 4.3 implies the existence of *holomorphic slices* of arbitrarily small diameter for closed subsets of X: that is slices of the form $\{x \in C; \phi(x) > \sup_C \phi - \alpha\}$ where ϕ is the real part of a holomorphic function on X (i.e. a pluriharmonic function). The example of L^1 gives a negative answer. Indeed any holomorphic function on L^1 is necessarily weakly continuous since one can easily see that monomials $p(f_1, \ldots, f_n)$ on L^1 can be written as

$$p(f_1, f_2, \ldots, f_n) = \int\int \cdots \int f_1(x_1) f_2(x_2) \ldots f_n(x_n) \, K(x_1, x_2, \ldots, x_n) \, dx_1 dx_2, \ldots, dx_n$$

where $K(x_1, x_2, \ldots, x_n)$ is an L^∞-bounded kernel.

It follows that slices determined by holomorphic functions give rise to weak neighborhoods in L^1. Since the unit ball of L^1 has no points of weak to norm continuity, one cannot expect such slices to have arbitrarily norm small diameter. The case where slices can be determined by weakly continuous plurisubharmonic or pluriharmonic functions is studied in [GMS].

5. A plurisubharmonic renorming and an integral representation.

The following is the main result of this section

THEOREM 5.1. – *Let X be a separable quasi-Banach space. The following assertions are equivalent :*

1) *Every function in $H^\infty(\Delta, X)$ has radial limits a.s.*

2) *X is $(A-p)$ convex for some $0 < p \leqslant 1$ and there exists a uniformly bounded countable family $\{\ell_{n,i}; (n,i) \in \mathbf{N}^2\}$ in $PSH_p(X)$ such that every bounded sequence $(x_k)_k$ in X is convergent if and only if it verifies :*

a) *$(\ell_{n,i}(x_k))_k$ is a Cauchy sequence for each $(n,i) \in \mathbf{N}^2$.*

b) $\lim_k \sup_i \ell_{n,i}(x_k) = \sup_i \lim_k \ell_{n,i}(x_k)$ *for each $n \in \mathbf{N}$.*

We start by proving 1) $\Rightarrow$ 2) which can be seen as the « plurisubharmonic » analogue of the *Asymptotic-norming property* introduced by James-Ho [JH] : A property shown to be – for separable Banach spaces – equivalent to the Radon-Nikodym property in [GM2]. Let us say that *a subset C of X verifies the PSH-optimization principle* if for every proper (i.e. not identically $+\infty$), bounded below and lower semi-continuous function $f : C \to \mathbf{R} \cup \{+\infty\}$, the set $\{\varphi \in PSH_p(X) ; f-\varphi$ *strongly exposes C from below*$\}$ is a dense G_δ in $PSH_p(X)$. We shall say that a fonction g ρ-*strongly exposes C from below* for some $\rho > 0$ if there exists $\alpha > 0$ such that $C \cap \{g < \alpha\} \neq \emptyset$ and diam $(C \cap \{g < \alpha\}) \leqslant \rho$.

We denote by $\widetilde{PSH}_p(C)$, the set of functions on C that are supremum (on C) of families of functions in $PSH_p(X)$. In the sequel we shall assume that the set C is contained in the unit ball of X. The following lemma is borrowed from [GMS].

LEMMA 5.2. – *Let C be a separable closed bounded subset of X verifying the PSH-optimization principle, then there exists a separable closed convex subset $\mathscr{S}_0$ of $PSH_p(X)$ such that for every proper lower semi-continuous fonction $f : C \to [0, +\infty]$ and any $\varepsilon > 0$, there exists $\varphi \in \mathscr{S}_0$, $\|\varphi\|_p \leqslant \varepsilon$ and $f - \varphi$ strongly exposes C from below.*

Proof. – We proceed in two steps. First we establish the following :

Claim. – For a given $\varepsilon > 0$ and a separable subset $\mathscr{C} \subset PSH_p(X)$, there exists a sequence $(\varphi_n)_n$ in $PSH_p(X)$ such that for every ψ in $\mathscr{C}$ and any proper l.s.c. $f : C \to [0, +\infty]$, there is $n \in \mathbf{N}$ so that $\|\psi - \varphi_n\|_p < \varepsilon$ and $f - \varphi_n$ ε-strongly exposes C from below.

Indeed, first fix ψ in $\mathscr{C}$ and proceed with the following transfinite induction : Start with $\varphi_0 = \tilde{\varphi}_0 = 0$ and suppose that up to the ordinal α, functions $(\varphi_\beta)_{\beta<\alpha}$ in $PSH_p(X)$ and $(\tilde{\varphi}_\beta)_{\beta<\alpha}$ in $\widetilde{PSH}_p(C)$ have been chosen. We denote by $F_\beta = \mathrm{Epi}(\tilde{\varphi}_\beta) = \{(x,\lambda) \in C \times [0,\infty] ; \tilde{\varphi}_\beta(x) \leqslant \lambda\}$.

(i) If $\alpha = \beta + 1$ and $F_\beta \neq \emptyset$, use the hypothesis to find $\varphi_\alpha \in PSH_p(X)$ and $c_\alpha \in \mathbf{R}$ such that

$$\|\psi - \varphi_\alpha\|_p < \varepsilon/2 \qquad \text{and} \qquad S_\alpha = \{x \in C ; (\tilde{\varphi}_\beta - \varphi_\alpha)(x) < c_\alpha\}$$

non-empty while diam $(S_\alpha) < \varepsilon$. Then set $\tilde{\varphi}_\alpha = \tilde{\varphi}_\beta \vee (\varphi_\alpha + c_\alpha)$.

(ii) If α is a limit ordinal, we let $\varphi_\alpha = \psi$ and $\tilde{\varphi}_\alpha = \sup_{\beta<\alpha} \tilde{\varphi}_\beta$ on C.

Since $C \times [0, +\infty]$ is separable, and since $(F_\alpha)_\alpha$ is a decreasing family of closed subsets, there exists $\gamma < \Omega$ (the first uncountable ordinal) such that $F_\gamma = \emptyset$. This implies that $\tilde{\varphi}_\alpha \uparrow + \infty$.

Suppose now $f: C \to [0, +\infty]$ is proper and lower-semi-continuous and let $\alpha \leqslant \gamma$ be the first ordinal such that $\tilde{\varphi}_\alpha$ is not less or equal to f. That is, $\tilde{\varphi}_\beta \leqslant f$ for all $\beta < \alpha$ while $\tilde{\varphi}_\alpha \nleqslant f$. It follows that α is necessarily of the form $\beta + 1$ and $\varphi_\alpha + c_\alpha$ is not less or equal than f. Hence $S = \{x \in C ; (f - \varphi_\alpha)(x) < c_\alpha\}$ is a non-empty subset of S_α. In other words, $f - \varphi$ ε-exposes C from below, while $\|\psi - \varphi_\alpha\|_p < \varepsilon/2$.

To finish the proof of the claim, it is enough to do the construction for a dense sequence $(\psi^i)_{i=1}^{\infty}$ of $\mathscr{C}$, to obtain a countable set $\{\varphi_\alpha^i \in PSH_p(X) ; \alpha < \gamma_i, i \in \mathbf{N}\}$ that will do the job.

To establish the lemma, we proceed with the following induction: Set $\mathscr{C}_1 = \{0\}$ and apply the above claim to $\mathscr{C} = \mathscr{C}_1$ and $\varepsilon = 1$ to obtain a sequence $(\varphi_n^1)_n$ in $PSH_p(X)$. Then let $\mathscr{C}_2$ be the convex set generated by $(\varphi_n^1)_n$. More generally, assume $\mathscr{C}_1 \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_k$ have been defined. Apply the claim to $\mathscr{C} = \mathscr{C}_k$ and $\varepsilon = 1/k$ to obtain an appropriate sequence $(\varphi_n^k)_n$ in $PSH_p(X)$. We then let $\mathscr{C}_{k+1}$ be the convex set generated by $\mathscr{C}_k$ and $(\varphi_n^k)_n$. Finally set $\mathscr{S}_0 =$ the closure of $\bigcup_k \mathscr{C}_k$ in $PSH_p(X)$.

If now $f: C \to [0, +\infty]$ is proper and lower semi-continuous, we obtain that for every $\varepsilon > 0$, the set $O(\varepsilon) = \{\varphi \in \mathscr{S}_0 ; f - \varphi$ ε-strongly exposes C from below$\}$ is dense and open in $\mathscr{S}_0$. The rest follows from Baire's category theorem.

LEMMA 5.3. — *Let C be a separable closed bounded subset of X verifying the PSH-optimization principle, then there exists a separable convex cone $\mathscr{S} \subset PSH_p(X)$ such that if we denote by δ the evaluation map from C into the dual of $P = \mathscr{S} - \mathscr{S}$ equipped with $\| \ \|_p$, the following then holds :*

(i) *$\delta : C \to P^*$ is a p-isometry from C onto its image $\delta(C)$, that is $\|\delta_x - \delta_y\|_p^* = \|x - y\|^p$ for every x, y in C.*

(ii) *$\overline{\delta(C)}^* \setminus \delta(C) = \bigcup_n L_n$ where each L_n is a weak*-compact subset of P^* of the form :* $L_n = \left(\bigcap_{m \in \mathbf{N}} \{\varphi_{n,m} \leqslant 0\} \right) \cap \left(\bigcap_{i=0}^{N_n} \{-\psi_{n,i} \leqslant 0\} \right)$ *where $(\varphi_{n,m})_{m \in \mathbf{N}}$ (resp. $(\psi_{n,i})_{i=0}^{N_n}$) is a countable (resp. finite) family of elements in $PSH_p(X)$ with $\| \ \|_p$-norm less than one. Moreover,*

(iii) *There exists a sequence* $(\lambda_n)_n$ *of reals such that for each* $n \in N$ *we have for all* $x \in C$

$$h_n(x) = (\sup_{m \in \mathbf{N}} \varphi_{n,m}(x)) \vee (\sup_{0 \leqslant i \leqslant N_n} (-\psi_{n,i}(x))) \geqslant \lambda_n > 0 .$$

Remark. — In the above Lemma and the sequel we will be identifying functions on C with their extensions to the weak*-closure $\overline{\delta(C)}^{*}$ of $\delta(C)$ in P^*. No confusion can occur as long as we are dealing with functions in $\mathscr{S}$.

Proof. — Let $\mathscr{S}_0$ be the separable subset of $PSH_p(X)$ obtained in Lemma 5.2. We can assume without loss that it contains the constant functions. Let $(x_n)_n$ be a dense sequence in C and consider the countable family $\mathscr{S}_1$ of functions $(\varphi_n)_n$ in $PSH_p(X)$ defined by $\varphi_n(x) = \|x-x_n\|^p$ for each n. Let $\mathscr{S}$ be the separable convex subcone of $PSH_p(X)$ generated by $\mathscr{S}_0 \cup \mathscr{S}_1$ and let P be the vector space $\mathscr{S} - \mathscr{S}$ equipped with $\| \ \|_p$. The evaluation map $\delta : C \to P^*$ is defined for each $x \in C$ by $\delta_x(\varphi) = \varphi(x)$ for all $\varphi \in P$. It is clear that $\|\delta_x - \delta_y\|_p^* = \|x-y\|^p$ for all x, y in C and that $\delta(C)$ is a bounded closed subset of P^* which is identifiable to C. One can also easily see that $\delta(C)$ is a weak*-G_δ in its weak*-closure $K = \overline{\delta(C)}^{*}$ in P^*. This will also follow from the representation of $\delta(C)$ claimed in (ii) and that we shall establish in the following three steps :

Step (1). — We claim that for any closed subset $F \subset C$ and any $\varepsilon > 0$, there exists a non-empty slice $S = F \cap \{\varphi > 0\}$ where $\varphi \in \mathscr{S}_0$ such that $\operatorname{diam}(S) \leqslant \varepsilon$. Indeed, let $f\colon C \to [0,1]$ be defined by $f(x) = 0$ if $x \in F$ and $f(x) = 1$ if $x \in C \setminus F$. Since f is l.s.c, use Lemma 5.2 to find $\varphi \in \mathscr{S}_0$, $\|\varphi\|_p < 1/2$ such that $f - \varphi$ strongly exposes C from below at a point x_0. It is clear that $x_0 \in F$ and that $\lim_{\alpha \downarrow 0} \operatorname{diam} \{x \in F ; \varphi(x) > \varphi(x_0) - \alpha\} = 0$.

Step (2). — We now prove that $\delta(C) = \bigcap_k (K_k \cup O_k)$ where each K_k is w^*-compact in P^* and each O_k is a countable union of w^*-open sets in P^* of the form $\{\varphi > 0\}$ where $\varphi \in \mathscr{S}_0$.

Indeed, for each $\varepsilon > 0$, we define by transfinite induction a decreasing family of norm closed subsets (F_α) of C in the following manner :

(i) $F_0 = C$.

(ii) If $\alpha = \beta + 1$ and $F_\beta \neq \emptyset$, use step (1) to find $\varphi_\beta \in \mathscr{S}_0$ with $\|\varphi_\beta\|_p \leqslant 1$ such that $H_\beta \cap F_\beta$ is non-empty and has diameter less than ε, where $H_\beta = \{x \in X ; \varphi_\beta(x) > 0\}$. Then set $F_\alpha = F_\beta \backslash H_\beta$.

(iii) If α is a limit ordinal, let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$.

Since C is separable, there exists a countable ordinal γ_ε so that $F_{\gamma_\varepsilon} = \emptyset$. Let K_β be the w^*-closure of $\delta(F_\beta)$ in P^* and let $\tilde{H}_\beta = \{\mu \in P^* ; \varphi_\beta(\mu) > 0\}$ for each $\beta < \gamma_\varepsilon$. It is clear that:

$$\delta(C) \subseteq \bigcap_{\alpha<\gamma\varepsilon} \left(K_\alpha \cup \left(\bigcup_{\beta<\alpha} \tilde{H}_\beta\right)\right).$$

If now μ belongs to the right hand side, there is $\beta < \gamma_\varepsilon$ such that $\mu \in K_\beta \cap \tilde{H}_\beta$. Hence there is a sequence $(x_j)_j$ in $F_\beta \cap H_\beta$ such that $\mu = \text{weak*} - \lim_j \delta_{x_j}$. Since $\operatorname{diam}(F_\beta \cap H_\beta) < \varepsilon$ we get that $\operatorname{dist}_{P^*}(\mu,(\delta(C))) < \varepsilon^p$. It follows that if we repeat the construction for each $\varepsilon = 1/n$, we obtain from the fact that $\delta(C)$ is norm closed in P^*, that

$$\delta(C) = \bigcap_n \bigcap_{\alpha<\gamma_n} \left(K_{\alpha,n} \cup \bigcup_{\beta<\alpha} \tilde{H}_{\beta,n}\right).$$

This clearly gives the above claim.

Step (3). — After relabeling we can write $\delta(C) = \bigcap_k (K_k \cup O_k)$ where each O_k is of the form $\cup_n \tilde{H}_{k,n}$. Since each K_k is w^*-compact, we can write $P^* \backslash K_k$ as a countable union of sets of the form $V = \left(\bigcap_{i=0}^{L} \{-\psi'_i \leqslant 0\}\right) \cap \left(\bigcap_{j=L+1}^{M} \{\varphi'_j \leqslant 0\}\right)$ where ψ'_i and φ'_j belong to $\mathscr{S}$ and $\|\varphi'_j\|_p \leqslant 1$. On the other hand, each $P^* \backslash O_k$ is of the form $\bigcap_{\ell=0}^{\infty} \{\varphi'_{k,\ell} \leqslant 0\}$ where again $\varphi'_{k,\ell} \in \mathscr{S}$ for all (k,ℓ). An obvious relabeling gives now conclusion (ii) of the Lemma. That is $\overline{\delta(C)}^* \backslash \delta(C) = \bigcup_n L'_n$ where each L'_n is of the form $L'_n = \left(\bigcap_{m=0}^{\infty} \{\varphi'_{n,m} \leqslant 0\}\right) \cap \left(\bigcap_{i=0}^{N} \{-\psi'_{n,i} \leqslant 0\}\right)$ where $\varphi'_{n,m}$ and $\psi'_{n,i}$ belong to $\mathscr{S}$ and their $\| \ \|_p$-norm is less than one.

We shall now split each L'_n in such a way that conclusion (iii) holds true. This will be done in the next two steps. We need the following notation: for each weak*-compact subset $L \subset K = \overline{\delta(C)}^*$, we define the function $\tilde{\varphi}_L$ on K by $\tilde{\varphi}_L(x) = \sup\{\varphi(x) - \sup_L \varphi\ ;\ \varphi \in \mathcal{S}$ and $\|\varphi\|_p \leqslant 1\}$. Note that $\tilde{\varphi}_L$ is always non-negative.

Step (4). – We now show the following:

(*) *Let χ be a function on $\overline{\delta(C)}^*$ that is lower semi-continuous and bounded above by one. Let L be a weak*-compact subset of $\overline{\delta(C)}^*$ such that : $\chi \leqslant 0$ on L and $(\tilde{\varphi}_L \vee \chi)(x) > 0$ for each x in $\delta(C)$. There exists then $\lambda > 0$ and $\psi \in \mathcal{S} \subset PSH_p(X)$ with $\|\psi\|_p \leqslant 5$ such that $S = L \cap \{\psi > 0\} \neq \emptyset$ and $\tilde{\varphi}_L \vee \chi \vee (-\psi) \geqslant \lambda > 0$ on $\delta(C)$.*

Indeed, use Lemma 5.2 to find $h \in \mathcal{S}_0$ with $\|h\|_p \leqslant 1/4$ and an x_0 in C such that $(\tilde{\varphi}_L \vee \chi) - h$ attains its minimum on C at x_0. That is

$$\tilde{\varphi}_L \vee \chi(x) \geqslant (\tilde{\varphi}_L \vee \chi)(x_0) + h(x) - h(x_0) \qquad \text{for all} \qquad x \in C.$$

Let $\lambda = \frac{1}{2}(\tilde{\varphi}_L \vee \chi)(x_0)$ and $\psi = 4h - 4h(x_0) + 3\lambda$ which is in $\mathcal{S}$. Note that $\lambda > 0$ and $\|\psi\|_p \leqslant 5$.

Moreover, $\tilde{\varphi}_L \vee \chi \geqslant \lambda$ on $C \cap \{h > h(x_0) - \lambda\}$, while $-\psi \geqslant \lambda$ on $C \cap \{h \leqslant h(x_0) - \lambda\}$. Hence $\tilde{\varphi}_L \vee \chi \vee (-\psi) \geqslant \lambda$ on C.

On the other hand, note that if $y \in L \backslash S$, then $4h(y) \leqslant 4h(x_0) - 3\lambda$ hence:

$$\begin{aligned}(\tilde{\varphi}_{L\backslash S} \vee \chi)(x_0) \geqslant \tilde{\varphi}_{L\backslash S}(x_0) &\geqslant 4h(x_0) - \sup_{L\backslash S} 4h \\ &\geqslant 4h(x_0) - 4h(x_0) + 3\lambda = 3/2(\tilde{\varphi}_L \vee \chi(x_0)).\end{aligned}$$

This clearly implies that $S \neq \emptyset$ and the claim is proved.

Step (5). – Consider now a weak*-compact subset $L \subset \overline{\delta(C)}^*$ such that $L \cap \delta(C) = \emptyset$ and $L = \left(\bigcap_{k=0}^{\infty} \{\varphi'_k \leqslant 0\}\right) \cap \left(\bigcap_{i=0}^{M} \{-\psi'_i \leqslant 0\}\right)$ where φ'_k and ψ'_i belong to $\mathcal{S}$ for all (k,i). Define $\chi = \sup_{1 \leqslant i \leqslant N} (-\psi'_i)$ on $\overline{\delta(C)}^*$ and note that $\chi \leqslant 0$ on L while $(\tilde{\varphi} \vee \chi)(x) > 0$ for each x in C. Use step (4) – with an appropriate normalization – to find $\lambda_0 > 0$ and $\psi_0 \in \mathcal{S}$ with $\|\psi_0\| \leqslant 1$ such that $S_0 = L \cap \{\psi_0 > 0\} \neq \emptyset$ and $\varphi_L \vee \chi \vee (-\psi_0) \geqslant \lambda_0 > 0$ on C. Set $L_1 = L \backslash S_0$. By transfinite induction, we can define a decreasing family of weak*-compact subsets $(L_\alpha)_\alpha$ of L and a family $(\psi_\alpha)_\alpha$ of functions in $\mathscr{S} \subset PSH_p(X)$ in the following manner:

(i) If $\alpha = \beta + 1$ and $L_\beta = \emptyset$ apply step (4) to $\tilde{\varphi}_{L_\beta} \vee \chi$ to get $\psi_\beta \in \mathscr{S}$ and $\lambda_\beta > 0$ such that $S_\beta = L_\beta \cap \{\psi_\beta > 0\} \neq \emptyset$ and $\tilde{\varphi}_{L_\beta} \vee \chi \vee (-\psi_\beta) \geqslant \lambda_\beta > 0$ on C. Then set $L_\alpha = L_\beta \setminus S_\beta$.

(ii) If α is a limit ordinal we write $L_\alpha = \bigcap_{\beta<\alpha} L_\beta$.

Since L is weak*-metrizable, there exists a countable ordinal such that $L_\gamma = \emptyset$. It follows that $L = \bigcup_{\beta<\gamma} \overline{S}^*_\beta$ and for each $\beta < \gamma$, $\tilde{\varphi}_{S_\beta} \vee \chi \vee (-\psi_\beta) \geqslant \lambda_\beta > 0$ on C while $\tilde{\varphi}_{S_\beta} \vee \chi \vee (-\psi_\beta) \leqslant 0$ on $\overline{S}^*_\beta$. Since $\overline{\delta(C)}^*$ is weak*-metrizable we can find for each $\beta < \gamma$, a countable family $(h_\ell)_\ell$ in $\mathscr{S}$ such that for every x, $\tilde{\varphi}_{S_\beta}(x) = \sup\{h_\ell(x) - \sup_{S_\beta} h_\ell ; h_\ell \in \mathscr{S}$ and $\|h_\ell\|_p \leqslant 1\}$. For a fixed $\beta < \gamma$ we can relabel the countable family $\{h_\ell - \sup_{S_\beta} h_\ell\}_\ell$ of functions in $\mathscr{S}$ to get $\{\varphi_k\}_k$ and also add to the finite family $(\psi'_i)_{i=1}^M$ the function ψ_β to obtain – after relabeling – the family $\{\psi_i\}_{i=1}^{M+1}$. It is now clear that

$$\overline{S}^*_\beta \subseteq \left(\bigcap_{k=0}^{\infty} \{\varphi_k \leqslant 0\}\right) \cap \left(\bigcap_{i=1}^{M+1} \{-\psi_i \leqslant 0\}\right)$$

while

$$(\sup_{k\in N} \varphi_k(x)) \vee (\sup_{1\leqslant i\leqslant M+1} - \psi_i(x)) \geqslant \lambda_\beta > 0 \text{ for each } x \text{ in } C.$$

By splitting in a similar fashion each L'_n obtained in step (3) we finally obtain claim (iii) of Lemma 5.3. □

Now, we can prove the implication 1) ⇒ 2) of Theorem 5.1. By Theorem 4.3, every closed bounded subset C verifies the *PSH-optimization principle*. Assume now X is separable and apply Lemma 5.3 to C equal the unit ball of X, to obtain functions $\{\varphi_{n,m} ; (n,m) \in \mathbf{N}^2\}$ and $\{\psi_{n,i} ; n \in \mathbf{N}, 1 \leqslant i \leqslant N_n\}$ in $PSH_p(X)$ verifying the conclusion of the lemma. Let $(x_n)_n$ be a dense sequence in Ball(X) and consider the functions $\varphi_n(x) = \|x - x_n\|^p$ which are also in $PSH_p(X)$. The double indexed family $(\ell_{n,i})$ required in Theorem 5.1 can be defined as follows: Let M_1, M_2 and M_3 be three independent copies of $\mathbf{N}$. Define the family $\{\ell_{n,i} ; n \in M, i \in \mathbf{N}\}$ where $M = M_1 \cup M_2 \cup M_3$ in the following fashion : If $n \in M_1$ let $\ell_{n,i} = \varphi_{n,i}$ for all $i \in \mathbf{N}$. If $n \in M_2$ let $\ell_{n,i} = \psi_{n,i}$ for $1 \leqslant i \leqslant N_n$ and 0 otherwise. If $n \in M_3$ we let $\ell_{n,i} = \varphi_n$ for all $i \in \mathbf{N}$.

Let $(x_k)_k$ be a sequence in C verifying a) and b) of assertion (2) in Theorem 5.1. Let x^* be a weak*-cluster point of $(\delta_{x_k})_k$ in $\overline{\delta(C)}^*$. We claim that $x^* \in \delta(C)$. Indeed if not, there exists n such that $x^* \in L_n$, hence $\varphi_{n,i}(x^*) \leqslant 0$ for all i and $-\psi_{n,i}(x^*) \leqslant 0$, for all $0 \leqslant i \leqslant N_n$. On the other hand from b) we get $\lim_k \sup_i \varphi_{n,i}(x_k) = \sup_i \varphi_{n,i}(x^*) = \gamma_1$ and since N_n is finite, a) gives

$$\lim_k \max_{0 \leqslant i \leqslant N_n} (-\psi_{n,i}(x_k)) = \max_{0 \leqslant i \leqslant N_n} (-\psi_{n,i}(x^*)) = \gamma_2 .$$

But $\gamma_1 \vee \gamma_2 \geqslant \lambda_n > 0$ which is a contradiction. Hence $x^* \in C$. Finally, since $\varphi_n(x_k) \to \varphi_n(x^*)$ for each n, we have $\lim_k \|x_k - x^*\| = 0$ as in Lemma 3.1.

Proof of 2) $\Rightarrow$ 1). – In view of Proposition 3.4 we need to show that uniformly bounded X-valued analytic martingales converge almost surely. Our proof will actually cover the case of all *PSH*-martingales.

Let $(F_k)_k$ be a *PSH*-martingale with values in the unit ball of X. Apply assertion (2) of Theorem 5.1 to obtain an appropriate sequence $\{\ell_{n,i};(n,i) \in \mathbf{N}^2\}$ in $PSH_p(X)$ so that $\|\ell_{n,i}\|_p \leqslant 1$ for all (n,i). We shall need the following result ([N], Lemma V.2.9).

LEMMA 5.4. – *Consider a countable family* I *of real valued submartingales* $\{(X^i_k)_k ; i \in I\}$ *such that* $\sup_k E \sup_i (X^i_k)^+ < +\infty$. *Then for each* $i \in I$, $(X^i_k)_k$ *converges a.s. to a random variable* X^i_∞. *Moreover,* $(\sup_i X^i_k)_k$ *converges to* $\sup_i X^i_\infty$ a.s.

To finish the proof of assertion (1) of Theorem 5.1, it is now enough to apply Lemma 5.4 for each n to the family of submartingales $\{(X^i_k)_k = (\ell_{n,i}(F_k))_k ; i \in \mathbf{N}\}$. The conclusions of the Lemma correspond to conditions a), b) of assertion (2) of the theorem. This implies that $(F_k)_k$ converges a.s. in X. □

Remark 5.5. – By comparing to section 3, the conclusion of Lemma 5.3 looks like a certain PSH_δ-representation of C in some compactification P^*. However the injection of X into Y^* is not holomorphic and the existence of components of the form $\bigcap_{i=0}^{N} \{-\psi_i \leqslant 0\}$ where $\psi_i \in PSH_p(X)$, destroys the symmetry in the representation of C. This is not surprising since $PSH_p(X)$ is a cone and not a vector space. However, one can remedy the situation by introducing the following concepts :

Say that a map $S: X \to Y$ between X and a Banach space Y is *pluri-quasiharmonic* if for every $y^* \in Y^*$, $y^* \circ S$ is the difference of two functions in $\widetilde{PSH}_{uc}(X)$. Say that such a map is a *symmetrized* $\widetilde{PSH}_\delta$-*injection* if $\overline{S(B_X)} \backslash S(B_X) = \bigcup_n F_n$ where each F_n is a closed set which is strictly separated from $S(B_X)$ by a function h_n verifying that $h_n \circ S$ is the difference of two functions in $\widetilde{PSH}_{uc}(B_X)$. In other words, $\inf_{S(B_X)} h_n > \sup_{L_n} h_n$ for each n.

Consider now any dense range operator $T: \ell_2 \to P$ such that $T(\ell_2^+) \subset \mathcal{S}$ where P and $\mathcal{S}$ are the space and cone considered in Lemma 5.3. Define $S = T^* \circ \delta$ where δ is the evaluation map. Note that for any positive linear functional x^* in ℓ_2, we have that $x^* \circ S \in PSH_p(X)$ so that $S: X \to \ell_2$ is a pluri-quasiharmonic map. On the other hand the functions $h_n = (\sup_m \varphi_{n,m}) \vee (\sup_{0 \leqslant i \leqslant n} -\psi_{n,i})$ appearing in Lemma 5.3 can easily be replaced by functions in $\widetilde{PSH}_p - \widetilde{PSH}_p$, which means that S is a symmetrized $\widetilde{PSH}_\delta$-injection. This combined with an adaptation of the proof of 2) $\Rightarrow$ 1) via Lemma 5.4, gives the following :

Proposition 5.6. — *Let X be a separable quasi-Banach space. The following properties are equivalent :*

1) *Every function in $H^\infty(\Delta, X)$ has radial limits a.s.*

2) *There exists a pluri-quasiharmonic symmetrized $\widetilde{PSH}_\delta$-injection from X into ℓ_2.*

Remark 5.7. — A typical example of the above is when X is an A-convex quasi-Banach lattice not containing c_0. Kalton [K1] had shown that there exists p so that the p-convexification of X is a Banach lattice Y not containing c_0. By representing Y as a function space between L^∞ and L^1 [LT], one can easily deduce that there exists a linear semi-embedding S_1 from X into L^p. Since for any Borel set $A \subset [0,1]$, the function $f \to \int_A |f|^p$ is plurisubharmonic on L^p, we obtain that the map

$S_2 : L^p \to L^1$ defined by $S_2(f) = |f|^p$ is a pluri-quasiharmonic semi-embedding. The same is true for the map $T = S_2 S_1 : X \to L^1$. Proposition 5.6 gives then another proof of the fact established in [K1], that functions in $H^\infty(\Delta, X)$ have radial limits a.s. □

In the remainder of this section, we shall discuss the problem of integral representation in terms of Jensen measures. The methods will consist of appropriate modifications and refinements of those developed by Edgar [E3], [4] in the non-compact but convex setting. So the proofs will be sketchy when the adaptations are immediate. Other integral representations – in terms of « analytic measures » – are also carried out by B. Khaoulani in [KH].

In the sequel C will be a separable closed bounded and J-convex subset of an $(A-p)$ convex quasi-Banach space X. The set $\mathscr{P}(C)$ consisting of tight Borel probability measures on C will be identified with a closed bounded convex subset of the space $LIP_p(C)^*$. It is also known that if $(\mu_\alpha)_\alpha$ is a net in $\mathscr{P}(C)$ and $\mu \in \mathscr{P}(C)$, then $\mu_\alpha \to \mu$ in the norm of $LIP_p(C)^*$ if and only if $\langle \mu_\alpha, f \rangle \to \langle \mu, f \rangle$ for all continuous and bounded functions f on C. For the proofs see Dudley [Dud]. Denote now by $J(C)$ the subset of $\mathscr{P}(C)$ consisting of the Jensen measures. It is easy to see that $J(C)$ is norm closed in $\mathscr{P}(C)$. A *Jensen-dilation* will be any Borel map T: $C \to J(C)$ such that any x in C is the J-barycenter of Tx. On $J(C)$ we define the order $\mu \prec \nu$ if $\langle \mu, f \rangle \leqslant \langle \nu, f \rangle$ for every f in $PSH_p^1(C)$.

The following proposition is well known in the case where the order is defined by the cone of convex continuous functions. A proof of that case – inspired by V. Strassen – and used in [E4] can be easily adapted to show the following.

PROPOSITION 5.8. – *Assume μ and ν in $J(C)$. The following are equivalent :*

1) $\mu \prec \nu$.

2) *There exists a Jensen dilation $T : C \to J(C)$ such that* $\langle \nu, f \rangle = \int \langle T(x), f \rangle \, d\mu(x)$ *for all continuous bounded function f on C.*

3) *There exist a probability space $(\Omega, \mathscr{F}_2, P)$, a σ-algebra $\mathscr{F}_1 \subset \mathscr{F}_2$ and two C-valued random variables (F_1, F_2) adapted to $(\mathscr{F}_1, \mathscr{F}_2)$ such that μ (resp. ν) is the distribution of F_1 (resp. F_2) and $\{F_1, F_2\}$ is a PSH-martingale.*

Sketch of proof. — 1) $\Rightarrow$ 2) Let $C_b(C)$ be the space of all continuous bounded functions on C and define for each $f \in C_b(C)$ the function $\hat{f}(x) = \inf\{\varphi(x); -\varphi \in PSH_p(X) \text{ and } \varphi \geqslant f\}$.

On the vector space S of all simple Borel functions $\theta : C \to C_b(C)$, define the sublinear functional $p : S \to \mathbf{R}$ by $p(\theta) = \int \theta(x)\hat{\ }(x)\, d\mu(x)$. Let S_0 be the subspace of S generated by the functions of the form $\chi_C \otimes f$ defined by

$$(\chi_B \otimes f)(x) = \begin{cases} f & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \notin B. \end{cases}$$

Define on S_0 the linear functional $\ell(\chi_c \otimes f) = \langle \nu, f \rangle$.

The order $\mu \prec \nu$ implies that $\ell \leqslant p$ on S_0. Let $\tilde{\ell}$ be any Hahn-Banach extension of ℓ to the whole space S. One can then check that the vector measure $m : \Sigma \to C_b(C)^*$ defined on the Borel σ-field Σ of C by $\langle m(A), h \rangle = \tilde{\ell}(\chi_A \otimes h)$ for all $A \in \Sigma$ and $h \in C_b(C)$, has average range in $\mathscr{P}(C)$: that is $\dfrac{m(A)}{\mu(A)} \in \mathscr{P}(C)$ for all $A \in \Sigma$. Since the latter has the R.N.P [E4], m has a density $T : C \to \mathscr{P}(C)$. It is easy to check that T is valued in $J(C)$ and that ν is a dilation of μ by T.

2) $\Rightarrow$ 3) Suppose $\nu = T(\mu)$. Let $\Omega = C \times C$, $\mathscr{F}_2 = \Sigma \times \Sigma$ and $\mathscr{F}_1 = \{C, \emptyset\} \times \Sigma$. Define $F_1, F_2 : \Omega \to C$ by $F_1(x,y) = x$, $F_2(x,y) = y$. Define P on $\mathscr{F}_2$ by $P(D) = \int T(x)(D_x)\, d\mu(x)$ where $D_x = \{y \in C; (x,y) \in D)\}$. The reader can easily check that $(F_1, \mathscr{F}_1)$, $(F_2, \mathscr{F}_2)$ verify the claim in 3).

The implication 3) $\Rightarrow$ 1) is immediate.

Now we can show the following :

THEOREM 5.9. — *Let X be a quasi-Banach space such that every function in $H^\infty(\Delta, X)$ has radial limits a.s. Assume C is a separable closed bounded J-convex subset of X then :*

a) *Any sequence $(\mu_n)_n$ in $\mathscr{P}(C)$ such that $\mu_1 \prec \mu_2 \prec \cdots \prec \mu_n \prec \cdots$ is convergent to μ_∞ in $\mathscr{P}(C)$ and $\mu_n \prec \mu_\infty$ for each $n \in \mathbf{N}$.*

b) *The set Jbr(C) of all Jensen boundary points is co-analytic and non-empty.*

c) *Any point in C is the barycenter of a Jensen Radon probability measure supported on $\mathcal{J}\mathrm{br}(C)$.*

Sketch of Proof. — *a*) Follows immediately from Proposition 5.8 and the convergence of *PSH*-martingale shown in Theorem 5.1.

b) That $\mathcal{J}\mathrm{br}(C) \neq \emptyset$ follows from Theorem 4.3. For the rest, note that the barycentric map r is continuous from $J(C) \to C$ and that $C \backslash \mathcal{J}\mathrm{br}(C)$ is the image by r of the set $J(C) \backslash \{\delta_x \,;\, x \in C\}$ which is clearly open. Hence $\mathcal{J}\mathrm{br}(C)$ is co-analytic in C.

c) First notice that since $\mathcal{P}(C)$ is a complete metric space, assertion a) implies that for any $\mu \in J(C)$, the family $A_\mu = \{\nu \in J(C) \,;\, \mu \prec \nu\}$ is a Zorn family. It follows that for any $x \in C$, there exists a maximal element $\bar{\nu}$ in A_{δ_x}. Since C is separable, the Von Neumann selection theorem [E3] gives a universally measurable function $S : C \to J(C)$ such that $r(S(x)) = x$ for all x while $S(x) = \delta_x$ if and only $x \in \mathcal{J}\mathrm{br}(C)$. We can assume that S is Borel measurable — modulo redefining $S(x) = \delta_x$ on a $\bar{\nu}$-nul set — and hence that S is a Jensen dilation. Since $\bar{\nu}$ is maximal we have that $\bar{\nu} = S\bar{\nu}$ which means that $\bar{\nu}(\mathcal{J}\mathrm{br}(C)) = 1$.

6. Appendix : Embedding Hardy martingales into analytic functions.

We shall now give a general result about « embedding » *analytic martingales* (and more generally *Hardy martingales*) into *analytic functions*. This procedure makes the connection between the two concepts more transparent and allows direct proofs for some related results already established by Edgar [E2] and Kalton [K1]. (See also Lemma 2.1 and Proposition 3.4.)

Let X be a quasi-Banach space. Following Garling [Garl] we shall call *Hardy martingale* any X-valued *PSH*-martingale $(M_n)_n$ on $\Omega = \mathbf{T}^N$ such that each martingale difference $d_n = M_n - M_{n-1}$ is a function on $\mathbf{T}^n$ that is analytic in the last variable. Note that the martingale $(M_n)_n$ is here adapted to the σ-fields $(\Sigma_n)_n$ where for each n, Σ_n is generated by the first n coordinates. It is clear that if X is a Banach space the above definition coincides with the one of Garling : that is $(M_n)_n$ is a martingale verifying $E[d_n e^{ik\theta_n} | \Sigma_{n-1}] = 0$ for all $k = 0, 1, 2, \ldots$. It is also clear that analytic martingales are a special kind of Hardy martingales.

THEOREM 6.1. — *Let X be an A-convex quasi-Banach space and let $(M_n)_n$ be an X-valued Hardy martingale with corresponding martingale difference $(d_n)_n$ such that d_n is a continuous function on $\mathbf{T}^n$. Then, for any sequence of positive reals $(\varepsilon_n)_n$, there exists a surjective continuous map $p : \mathbf{T} \to \mathbf{T}^N$, a sequence of positive reals $(r_n)_n$ strictly increasing to 1, and an X-valued analytic function F on Δ such that :*

a) *The image of Lebesgue measure on $\mathbf{T}$ by p is the product Lebesgue measure on $\mathbf{T}^N$.*

b) *For all $n \in \mathbf{N}$ and all $\theta \in \mathbf{T}$ we have :*

$$\|F(r_{n+1}e^{i\theta}) - F(r_n e^{i\theta}) - d_{n+1}(p(e^{i\theta}))\| < \varepsilon_n .$$

c) *If we denote by τ_n (resp. τ) the first time complex Brownian motion W_t starting at 0 hits the circle of radius r_n (resp. of radius 1) and by $q(\omega)$ the function $p(W_{\tau(\omega)}(\omega))$ then :*

$$(E\|F(W_{\tau_n}) - F(W_{\tau_{n-1}}) - d_n(q)\|^2)^{1/2} < \varepsilon_n .$$

Proof. — For simplicity we shall assume that X is a Banach space. For every positive integer n, we denote by j_n the map from $[0, 1]^n$ onto $\mathbf{T}^n$ defined by :

$$j_n(s_1, , \dots , s_n) = (e^{2i\pi s_1}, , \dots , e^{2i\pi s_n}) .$$

For $n \leqslant m$, we let $P_{m,n}$ be the natural projection from $\mathbf{T}^m$ onto $\mathbf{T}^n$ or the natural projection from $[0,1]^m$ onto $[0,1]^n$. P_n will be the projection from the infinite product $\mathbf{T}^N$ or $[0,1]^N$ onto $\mathbf{T}^n$ or $[0,1]^n$. The Lebesgue probability measure on $\mathbf{T}$ is noted λ and the measure on $\mathbf{T}^N$ given by the infinite product of copies of λ is denoted λ_∞. The metric on $\mathbf{T}^n$ or $[0,1]^n$ will be the metric given by the supremum norm. We shall need the following terminology :

Say that $\pi = ((I_\alpha, C_\alpha, \varphi_\alpha)_{\alpha \in A}), p)$ is an *n-dimensional P-family* provided :

– A is a finite set with cardinality 2^k for some integer k ;

– $(I_\alpha)_{\alpha \in A}$ is a family of closed sub-intervals of $[0, 1]$ which is a permutation of the family $\{[(j-1)2^{-k}, j2^{-k}] ; j = 1, , \dots , 2^k\}$;

– each C_α is a closed convex subset of $[0, 1]^n$ with Lebesgue measure equal to 2^{-k}, and such that

$$[0, 1]^n = \bigcup_{\alpha \in A} C_\alpha ;$$

– p is a continuous map from $\mathbf{T}$ into $\mathbf{T}^n$;

– for every $\alpha \in A$, φ_α is an affine map from I_α into C_α so that for every $s \in I_\alpha$ we have $j_n(\varphi_\alpha(s)) = p(j_1(s))$;

We shall call $\operatorname{diam}(\pi)$ the maximum of the diameters of the sets C_α for α in A. If $\rho = ((J_\beta, D_\beta, \psi_\beta)_{\beta \in B}), q)$ is an m-dimensional P-family, we say that ρ extends π if $n \leqslant m$ and if the following conditions hold:

(+) the cardinality of B is greater than the cardinality of A;

(+ +) $J_\beta \subset I_\alpha \Rightarrow P_{m,n} D_\beta \subset C_\alpha$; if furthermore J_α and I_α have a common endpoint s then $P_{m,n}\psi_\beta(s) = \varphi_\alpha(s)$.

With the above notation, one can easily see that:

(*) If ρ extends π then $|P_{m,n}q(t) - p(t)| \leqslant 2\pi \operatorname{diam}(\pi)$ for every $t \in T$.

Suppose now (π_n) is a sequence such that for every n, π_n is an n-dimensional P-family, with an associated function p_n from $\mathbf{T}$ into $\mathbf{T}^n$, while π_{n+1} extends π_n and $\operatorname{diam}(\pi_n)$ decreases to 0; it is easy to see that there exists a unique continuous mapping p from $\mathbf{T}$ onto $\mathbf{T}^N$ such that $P_n(p(t)) = \lim_{m\to\infty} P_{m,n} p_m(t)$ for every n, and the image measure $p(\lambda)$ is equal to λ_∞. We actually want to construct a sequence $(\pi_n)_n$ as above with some additional properties. First note that since d_n is a continuous function on $\mathbf{T}^n$ we can find a positive real number α_n such that $|u_1 - u_2| < 2\pi\alpha_n \Rightarrow \|d_n(u_1) - d_n(u_2)\| < \varepsilon_n$ for all u_1, u_2 in $\mathbf{T}^n$.

We now construct inductively the sequence $(\pi_n, p_n)_n$ and an increasing sequence $(r_n)_n$ in $(0,1)$ in such a way that for each n:

(i) π_n extends π_{n-1} and $\operatorname{diam}(\pi_n) < \alpha_n$.

(ii) $\|d_n(p_n(t)) - Q_n(t)\| < \varepsilon_n$ where Q_n is an X-valued complex polynomial such that $\|Q_n(rt)\| < \varepsilon_n/2^n$ for all $t \in \mathbf{T}$ and $0 \leqslant r \leqslant r_{n-1}$.

(iii) r_n is such that $(1 - r_n) < 2^{-n}$, $\sum_{j=1}^{n} \|Q_j(r_n t) - Q_j(rt)\| < \varepsilon_{n+1}$ for all $t \in \mathbf{T}$ and $r_n \leqslant r \leqslant 1$, and $\sum_{j=1}^{n} \|Q_j\|_{\mathrm{Lip}(\Delta)}[2(1 - r_n)]^{1/2} < \varepsilon_{n+1}$.

Let us first show how the conclusion of the Theorem follows once the construction is accomplished. Consider $F(z) = \sum_{n=0}^{\infty} Q_n(z)$. Since r_n

tends to 1, (ii) implies that this series converges uniformly on compact subsets of Δ. From (i) and (*) we deduce that $|P_n(p(t))-p_n(t)| < 2\pi\alpha_n$ hence

$$\|d_n(P_n(p(t)))-d_n(p_n(t))\| < \varepsilon_n. \tag{1}$$

But (ii) gives

$$\left\|\sum_{k=n}^{\infty} Q_k(r_{n-1}t)\right\| < 2\varepsilon_n \quad \text{and similarly} \quad \left\|\sum_{k=n+1}^{\infty} Q_k(r_n t)\right\| < 2\varepsilon_{n+1}. \tag{2}$$

We also get from (iii)

$$\left\|\sum_{j=1}^{n-1} (Q_j(r_n t)-Q_j(r_{n-1}t))\right\| < \varepsilon_n \quad \text{and} \quad \|Q_n(r_n t)-Q_n(t)\| < \varepsilon_n. \tag{3}$$

Adding (1), (2) and (3) we obtain

$$\|F(r_n t)-F(r_{n-1}t)-Q_n(t)\| < 6\varepsilon_n \tag{4}$$

and

$$\|F(r_n t)-F(r_{n-1}t)-d_n(P_n(p(t)))\| < 8\,\varepsilon_n. \tag{5}$$

This clearly gives statement b). For c) we only need to change the inequalities in (3); we actually have when $m \leqslant n$

$$E|W_{\tau_n}-W_{\tau_m}|^2 = r_n^2 - r_m^2 < 2(1-r_m)$$

and in the same way

$$E|W_{\tau}-W_{\tau_m}|^2 = 1 - r_m^2 < 2(1-r_m).$$

Hence

$$\begin{aligned}\left(E\left\|\sum_{j=1}^{n-1} (Q_j(W_{\tau_n})-Q_j(W_{\tau_{n-1}}))\right\|^2\right)^{1/2} &\leqslant \sum_{j=1}^{n-1} (E\|Q_j(W_{\tau_n})-Q_j(W_{\tau_{n-1}})\|^2)^{1/2}\\ &\leqslant \sum_{j=1}^{n-1} \|Q_j\|_{\mathrm{Lip}(\Delta)}(E|W_{\tau_n}-W_{\tau_{n-1}}|^2)^{1/2}\\ &\leqslant \sum_{j=1}^{n} \|Q_j\|_{\mathrm{Lip}(\Delta)}[2(1-r_{n-1})]^{1/2} < \varepsilon_n.\end{aligned}$$

We also have

$$(E\|Q_n(W_{\tau_n})-Q_n(W_\tau)\|^2)^{1/2} \leqslant \|Q_n\|_{\mathrm{Lip}(\Delta)}(1-r_n)^{1/2} < \varepsilon_n .$$

It follows as before that

$$(E\|F(W_{\tau_n})-F(W_{\tau_{n-1}})-d_n(p(W_\tau))\|^2)^{1/2} \leqslant 8\varepsilon_n .$$

Now that we have proved how all conclusions follow from the properties (i)-(iii) stated above, it remains to show how to pass form $n-1$ to n in the construction. To do that, suppose that (i)-(iii) are satisfied for $n-1$. It follows from the hypothesis that one can find an integer L_n and continuous functions $\varphi_\ell : \mathbf{T}^{n-1} \to X$ for $\ell = 1, \ldots, L_n$ such that

$$\|d_n(u,t) - \sum_{\ell=1}^{L_n} \varphi_\ell(u)t^\ell\| < \varepsilon_n/2 \text{ for every } u \in \mathbf{T}^{n-1} \text{ and } t \in \mathbf{T}.$$

Let M be an integer and $(g_\ell)_\ell$ be functions from $\mathbf{C} \to X$ such that for every $\ell = 1, \ldots, L_n$ the function $z \to z^M g_\ell(z)$ is a polynomial and :

$$\forall t \in \mathbf{T}, \quad \|g_\ell(t) - \varphi_\ell(p_{n-1}(t))\| < \varepsilon_n/2L_n . \tag{6}$$

Consider now $Q_n(z) = \sum_{\ell=1}^{L_n} g_\ell(z)z^{\ell N}$ where N is an integer of the form $N = 2^h$ for some integer h, $N > M$ large enough so that $|Q_n(rt)| < \varepsilon_n/2^n$ for all $t \in \mathbf{T}$ and $0 \leqslant r \leqslant r_{n-1}$. We can clearly find r_n close enough to 1 so that (iii) is satisfied. Next, we will use the following claim whose proof is left to the interested reader :

SUBLEMMA. — *Let C be a closed bounded convex subset of $\mathbf{R}^n (n > 1)$ with a non-empty interior and consider x, y in C. For every $\varepsilon > 0$, there exists an integer k, a family $\{C_i ; i=1, \ldots, 2^k\}$ of closed convex subsets of C and a continuous path $q : [0,1] \to C$ such that :*

— $C = \bigcup_{i=1}^{2^k} C_i$.

— *The C_i's have pairwise disjoint interiors, the same Lebesgue measure while their diameter is less than ε.*

— $q(0) = x$, $q(1) = y$, $q^{-1}(C_i) = [(i-1)2^{-k}, i2^{-k}]$ *and q is affine on each* $[(i-1)2^{-k}, i2^{-k}]$ *for all* $i = 1, \ldots, 2^k$.

Using this sublemma we can find an $(n-1)$-dimensional P-family

$\sigma_{n-1} = (I_\alpha, D_\alpha, \psi_\alpha)_{\alpha \in B_{n-1}}$ which extends π_{n-1} and such that $\operatorname{diam}(\sigma_{n-1}) < \alpha_n$. We can assume that the cardinality of B_{n-1} is larger than the previously defined $N = 2^h$, and actually we can change N and assume that N is the cardinality of B_{n-1}. Define now the n-dimensional P-family π_n in the following way :

Let k be an integer such that $2^{-k} < \alpha_n$, and consider $E_{n-1} = \{1, \ldots, 2^k\}$ and $A_n = B_{n-1} \times E_{n-1}$;

For every $\alpha \in B_{n-1}$, let u_α be the origin of the interval I_α; for $\beta \in E_{n-1}$ let $J_\beta = [(\beta-1)2^{-k}, \beta 2^{-k}]$ and set $I_{\alpha,\beta} = u_\alpha + 2^{-h} J_\beta$ and $C_{\alpha,\beta} = D_\alpha \times J_\beta$.

It is clear that $\operatorname{diam}(\pi_n) < \alpha_n$. Define now the functions $\varphi_{\alpha,\beta}$ as: $\varphi_{\alpha,\beta}(s) = (\varphi_\alpha(s), 2^h(s-u_\alpha))$ for each s in I_α. It is clear that for every $t \in T$ we have $p_n(t) = (p_{n-1}(t), t^{2^h})$, therefore $\left\| d_n(p_n(t)) - \sum_{\ell=1}^{L_n} \varphi_\ell(p_{n-1}(t)) t^{\ell 2^h} \right\| < \varepsilon_n/2$ and it follows from (6) that $\|d_n(p_n(t)) - Q_n(t)\| < \varepsilon_n$. This finishes the inductive step and the proof of the Theorem.

BIBLIOGRAPHY

[A] A. B. Alexandrov, Essays on non locally convex Hardy classes, Lecture notes in Mathematics, Vol. 864, pp. 1-89, Springer-Verlag, Berlin, New York, 1981.

[Bo] J. Bourgain, La propriété de Radon-Nikodym, Publications Mathématiques de l'Université Pierre et Marie Curie, N° 36 (1979).

[Bu] S. Bu, Quelques remarques sur la propriété de Radon-Nikodym analytique, C.R. Acad. Sci. Paris, 306 (1988), 757-760.

[BS] S. Bu, W. Schachermayer, Approximation of Jensen Measures by Image Measures under Holomorphic Functions and Applications, (1989) (to appear).

[BD] A. V. Bukhvalov and A. A. Danilevich, Boundary properties of analytic and harmonic functions with values in Banach spaces, Math Zametki, 31 (1982), 203-214; English translation, Math. Notes, 31 (1982), 104-110.

[C] S. B. Chae, Holomorphy and Calculus in normed spaces, Pure and applied Mathematics, Marcel Dekker Inc., New York and Basel, 1985.

[DGT] W. J. Davis, D. J. H. Garling, N. Tomczak-Jaegermann, The complex convexity of complex quasi-normed linear spaces, J. Funct. Anal., 55 (1984), 110-150.

[DU] J. DIESTEL, J. J. UHL, Vector measures, Math. Surveys, A.M.S. 15 (1977).

[D] P. M. DOWLING, Representable operators and the analytic Radon-Nikodym property in Banach spaces, Proc. Royal. Irish. Acad., 85A (1985), 143-150.

[Dud] R. M. DUDLEY, Convergence of Baire measure, Studia Math., 27 (1966), 251-268.

[Du] R. DURETT, Brownian motion and Martingales in Analysis, Wadsworth, 1984.

[E1] G. A. EDGAR, Complex martingale convergence, Springer-Verlag, Lecture Notes, 1116 (1985), 38-59.

[E2] G. A. EDGAR, Analytic martingale convergence, J. Funct. Analysis, 69, N° 2 (1986), 268-280.

[E3] G. A. EDGAR, Extremal Integral Representations, J. Funct. Analysis, Vol. 23, N° 2 (1976), 145-161.

[E4] G. A. EDGAR, On the Radon-Nikodym Property and Martingale Convergence, Proceedings of Conf. on Vector Space Measures and Applications, Dublin (1977).

[Et] D. O. ETTER, Vector-valued Analytic functions, T.A.M.S., 119 (1965), 352-366.

[Gam] T. W. GAMELIN, Uniform algebras and Jensen measures, Cambridge University Press, Lecture notes series (32) (1978).

[Gar] J. B. GARNETT, Bounded Analytic Functions, Pure and Applied Math., Academic Press, 1981.

[Garl] D. J. H. GARLING, On martingales with values in a complex Banach space, Proc. Cambridge. Phil. Soc. (1988) (to appear).

[GLM] N. GHOUSSOUB, J. LINDENSTRAUSS, B. MAUREY, Analytic Martingales and Plurisubharmonic Barriers in complex Banach spaces, Contemporary Math. vol. 85, 111-130 (1989).

[GM1] N. GHOUSSOUB, B. MAUREY, G_δ-embeddings in Hilbert space, J. Funct. Analysis, 61, N° 1 (1985), 72-97.

[GM2] N. GHOUSSOUB, B. MAUREY, H_δ-embeddings in Hilbert space and Optimization on G_δ-sets, Memoirs of A.M.S., 62, N° 349 (1986).

[GMS] N. GHOUSSOUB, B. MAUREY, W. SCHACHERMAYER, Pluriharmonically Dentable complex Banach spaces... Journal für die reine und angewandte Mathematik, Band 402, 39 (1989), 76-127.

[GR] N. GHOUSSOUB, H. P. ROSENTHAL, Martingales, G_δ-embeddings and quotients of L^1, Math. Annalen, 264 (1983), 321-332.

[HP] U. HAAGERUP, G. PISIER, Factorization of Analytic Functions with Values in Non-Commutative L_1-spaces and Applications, (1988) (to appear).

[JH] R. C. JAMES, A. HO, The asymptotic norming and Radon-Nikodym properties for Banach spaces, Arkiv fur Matematik, 19 (1981), 53-70.

[K1] N. J. KALTON, Plurisubharmonic functions on quasi-Banach spaces, Studia Mathematica, TLXXXIV (1986), 297-324.

[K2] N. J. KALTON, Differentiability properties of vector-valued functions, Lecture Notes in Math., Springer-Verlag, 1221 (1985), 141-181.

[K3] N. J. KALTON, Analytic functions in non-locally convex spaces and applications, Studia Mathematica, TLXXXIII (1986), 275-303.

[K4] N. J. KALTON, Compact Convex Sets and Complex Convexity, Israel J. Math., (1988) (to appear).

[Kh] B. KHAOULANI, Un Théorème de représentation intégrale (1989) (In preparation).

[Ko] P. KOOSIS, Lectures on H^p-spaces. LMS Lecture notes, Cambridge University Press, Cambridge, 1980.

[KPR] N. J. KALTON, N. T. PECK, J. W. ROBERTS, An F-space Sampler, London Math. Society Lecture Notes 89, Cambridge University Press, 1985.

[LS] J. LINDENSTRAUSS, C. STEGALL, Examples of separable spaces which do not contain ℓ^1 and whose duals are non-separable, Studia Math., 54 (1975), 81-105.

[LT] J. LINDENSTRAUSS, L. TZAFRIRI, Classical Banach Spaces II : Function Spaces, Erg. Math. Grenzgeb. 97, Berlin-Heidelberg-New York, Springer, 1979.

[N] J. NEVEU, Discrete parameter martingales, North Holland, 1975.

[SSW] W. SCHACHERMAYER, A. SERSOURI, E. WERNER, Israel J. Math., vol. 65 (1989), 225-257.

[St] C. STEGALL, Optimization of functions on certain subsets of Banach spaces, Math. Annalen., 236 (1978), 171-176.

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