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FAMILIES OF JACOBIAN MANIFOLDS AND CHARACTERISTIC CLASSES OF SURFACE BUNDLES I

by Shigeyuki MORITA

0. Introduction.

Recently E. Miller [Mi] and the author [Mo2] have defined certain cohomology classes of the mapping class groups of orientable surfaces (henceforth we call them characteristic classes of surface bundles), and proved that they are highly non-trivial. These cohomology classes are the topological version of the corresponding canonical classes in the Chow ring of the moduli space of compact Riemann surfaces of Mumford [Mu]. One of the main purposes of the present paper is to show that there is a new relation among the characteristic classes which becomes rather strong when they are combined with the previously known ones (see §3). This relation comes from an idea similar to that of various vanishing theorems in the theory of characteristic classes of foliations and will be derived along the following line of arguments.

Recall first that to any compact Riemann surface M, there is associated the Jacobian variety J(M) and if we fix a base point of M, then there is defined a holomorphic map j : M $\to$ J(M) called the Jacobi mapping (see [G][Mu2] for example).

Now let $\pi : E \to X$ be an *oriented surface bundle*, namely an oriented differentiable fibre bundle with fibre Σ_g, which is a closed oriented surface of genus g. Then as is shown in [Mo2] we can always assume that

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π is a differentiable family of compact Riemann surfaces over X so that we can write $E = \bigcup_{p \in X} M_p$, where M_p is a Riemann surface. The union $J = \bigcup_{p \in X} J(M_p)$ admits a canonical structure of a smooth manifold. If there is given a cross section $s : X \to E$, then we can define a fibre preserving map $j : E \to J$ such that the restriction of j to M_p is the Jacobi mapping of it relative to the base point $s(p)$. Topologically the canonical projection $\pi : J \to X$ admits the structure of a *flat* fibre bundle whose fibre is a $2g$ – dimensional torus T^{2g} with a prescribed symplectic form ω_0 and the structure group is the group of all linear transformation of T^{2g} which preserves the form ω_0. In another word there is defined a transversely symplectic foliation F of codimension $2g$ on J which is transverse to the fibres. Therefore there is a closed 2–form ω on J which restricts to the symplectic form on each fibre (see §1 for details). It turns out that twice of the de Rham cohomology class $[\omega] \in H^2(J; \mathbb{R})$ lifts to a canonical integral class $\Omega \in H^2(J; \mathbb{Z})$. Our main work is then to identify the cohomology class $J^*(\Omega) \in H^2(E; \mathbb{Z})$ in terms of our characteristic classes (Theorem 1.3 and Theorem 2.1). Now the relation mentioned above comes from the obvious fact that ω^{g+1} vanishes identically.

As a key step of the proof of our main result, we determine the first (co)homology groups of the mapping class groups with coefficients in the homology of the surface together with an explicit construction of their generators (see§§4-6) and we think that it has its own meaning. In fact it enables us to prove a topological version of Earle's embedding theorem [E]([Mo6]), to construct canonical group cocycles for the characteristic classes of surface bundles and to interpret the Casson invariant for homology 3–spheres as the secondary invariant associated with the first characteristic class (see [Mo7]).

The author would like to express his hearty thanks to T. Yoshida who has informed him the existence of Johnson's paper [J]. One of the motivations for the present work was to understand it from our point of view.

1. Statement of the main theorem.

A *surface bundle* is a differentiable fibre bundle $\pi : E \to X$ whose fibre is a closed orientable surface of genus g wich we denote by Σ_g. We also call such a bundle a Σ_g – bundle. Given a surface bundle $\pi : E \to X$, let

ξ be the tangent bundle along the fibres of π, namely it is the sub-bundle of the tangent bundle of E consisting of vectors which are tangent to the fibres. Assume that π is orientable and there is given an orientation on ξ. Then we have the Euler class

$$e = e(\xi) \in H^2(E; \mathbb{Z}).$$

Now let M_g be the mapping class group of Σ_g, namely it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. If we fix a base point $p_0 \in X$ and an identification $\pi^{-1}(p_0) = \Sigma_g$, then we can define a homomorphism

$$h: \ \pi_1(X, p_0) \longrightarrow M_g$$

called the holonomy homomorphism which indicates how the bundle is twisted along various closed curves. If $g \geq 2$, then the above homomorphism completely determine the isomorphism class of the surface bundle $\pi : E \to X$ (see [Mo2]). Now choose a sympletic basis $x_1, \ldots, x_g;\ y_1, \ldots, y_g$ of $H_1(\Sigma_g; \mathbb{Z})$ so that

$$x_i \cdot x_j = y_i \cdot y_j = 0$$

and

$$x_i \cdot y_j = \delta_{ij}$$

where $x_i \cdot y_j$ denotes the intersection number of the homology class x_i with y_j. With respect to the above basis, the group of all automorphisms of $H_1(\Sigma_g; \mathbb{Z})$ which preserve the intersection pairing is expressed as the Siegel modular group $\mathrm{Sp}(2g; \mathbb{Z})$ which consists of all $2g \times 2g$ matrices A with integral entries satisfying the condition

$${}^tAJA = J$$

where $J = \begin{pmatrix} O & E \\ -E & O \end{pmatrix}$. Now the action of M_g on $H_1(\Sigma_g; \mathbb{Z})$ clearly preserves the intersection pairing so that we have a representation

$$\rho: \ M_g \longrightarrow \mathrm{Sp}(2g; \mathbb{Z}).$$

For each point $p \in X$, we write E_p for the fibre $\pi^{-1}(p)$ over p. Let $J_p = H_1(E_p; \mathbb{R})/H_1(E_p; \mathbb{Z})$ be the "Jacobian manifold" of E_p. The union $J = \bigcup_{p \in X} J_p$ is naturally a smooth manifold and the canonical projection $\pi : J \longrightarrow X$ admits the structure of a fibre bundle whose fibre is $H_1(\Sigma_g; \mathbb{R})/H_1(\Sigma_g; \mathbb{Z})$ which is identified with the $2g$-dimensional torus T^{2g} via the symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$ and the structure group is $\mathrm{Sp}(2g; \mathbb{Z})$.

Let $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_g \wedge dy_g$ be a symplectic form of T^{2g}. Then the natural linear action of $\mathrm{Sp}(2g;\mathbb{Z})$ on T^{2g} preserves the form ω_0. Now the fibre bundle $\pi : J \to X$ is *flat* in the sense that the transition functions are locally constant. Hence there is a closed 2-form ω on J whose restriction to the typical fibre is ω_0. In fact the bundle $\pi : J \to X$ is nothing but the flat T^{2g}-bundle defined by the homomorphism $\rho h : \pi_1(X) \to \mathrm{Sp}(2g;\mathbb{Z})$ so that we can write

$$J = \tilde{X} \underset{\pi_1(X)}{\times} T^{2g}$$

where $\tilde{X}$ is the universal covering space of X and the action of $\pi_1(X)$ on $\tilde{X}$ is given by the deck transformations while that on T^{2g} is defined by the homomorphism ρh. Since the action of $\pi_1(X)$ on T^{2g} preserves the symplectic form ω_0, the form $q^*(\omega_0)$ on $\tilde{X} \times T^{2g}$ projects to the desired form ω on J, where $q : \tilde{X} \times T^{2g} \to T^{2g}$ is the projection. We have the de Rham cohomology class $[\omega] \in H^2(J;\mathbb{R})$. It turns out that $2[\omega]$ lifts to a canonical integral cohomology class $\Omega \in H^2(J;\mathbb{Z})$. Here we only define Ω and postpone the proof of the fact that $\Omega = 2[\omega]$ in $H^2(J;\mathbb{R})$ until §7.

First we define a nilpotent Lie group G_g as follows. As a set we put $G_g = \mathbb{R} \times H_1(\Sigma_g;\mathbb{R})$. The group law is defined by

$$(s,x)(t,y) = (s+t+x\cdot y, x+y)$$

where $s, t \in \mathbb{R}$ and $x, y \in H_1(\Sigma_g;\mathbb{R})$ so that we have a central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow G_g \longrightarrow H_1(\Sigma_g;\mathbb{R}) \longrightarrow 1.$$

G_g contains a lattice $\Gamma_g = \{(n,x) \in G_g;\ n \in \mathbb{Z},\ x \in H_1(\Sigma_g;\mathbb{Z})\}$ and we have also a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_g \longrightarrow H_1(\Sigma_g;\mathbb{Z}) \longrightarrow 1.$$

We write N_g for the associated nilmanifold G_g/Γ_g. Then we have an S^1-bundle

$$S^1 \longrightarrow N_g \longrightarrow T^{2g} = H_1(\Sigma_g;\mathbb{R})/H_1(\Sigma_g;\mathbb{Z}).$$

LEMMA 1.1. — *The linear action of* $\mathrm{Sp}(2g;\mathbb{Z})$ *on* T^{2g} *lifts to an action of it on* N_g.

Proof. — It is easy to see that the rule $A(s,x) = (s, Ax)$ where $A \in \mathrm{Sp}(2g;\mathbb{Z})$ and $(s,x) \in G_g$ defines an action of $\mathrm{Sp}(2g;\mathbb{Z})$ on G_g as automorphisms. Clearly the lattice Γ_g is invariant under this action. Hence we have the desired action of $\mathrm{Sp}(2g;\mathbb{Z})$ on N_g.

Let $N_g \to \overline{J} \to X$ be the flat N_g-bundle over X defined by the homomorphism $\rho h : \pi_1(X) \to \mathrm{Sp}(2g;\mathbb{Z})$ and the action of $\mathrm{Sp}(2g;\mathbb{Z})$ on N_g just defined above. The natural projection $\overline{J} \to J$ has the structure of an oriented S^1-bundle and we define $\Omega \in H^2(J;\mathbb{Z})$ to be the Euler class of it.

PROPOSITION 1.2. — (i) *Let $i : T^{2g} \to J$ be the inclusion of the typical fibre. Then we have $i^*(\Omega) = 2[\omega_0] \in H^2(T^{2g};\mathbb{Z})$.*

(ii) *Let $s_0 : X \to J$ be the "zero section", namely $s_0(p) = 0 \in H_1(E_p;\mathbb{R})/H_1(E_p;\mathbb{Z})$ for all $p \in X$. Then $s_0^*(\omega) \equiv 0$ and $s_0^*(\Omega) = 0$.*

Proof. — (i) It suffices to show that the Euler class of the S^1-bundle $S^1 \to N_g \to T^{2g}$ is $2[\omega_0] \in H^2(T^{2g};\mathbb{Z})$. But in terms of the group cohomology, the Euler class is represented by the 2-cocycle of $\pi_1(T^{2g}) = H_1(\Sigma_g;\mathbb{Z})$ defined by

$$(x,y) \longrightarrow x \cdot y$$

for x, $y \in H_1(\Sigma_g;\mathbb{Z})$. On the other hand the sympletic form ω_0 is defined as $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_g \wedge dy_g$ where $x_1,\ldots,x_g;\ y_1,\ldots,y_g$ is a symplectic basis of $H_1(\Sigma_g;\mathbb{Z})$ and serves simultaneously as a global coordinate for the torus $T^{2g} = H_1(\Sigma_g;\mathbb{R})/H_1(\Sigma_g;\mathbb{Z})$. If we compare the above two facts, we can conclude that the Euler class is a multiple of the de Rham cohomology class $[\omega_0] \in H^2(T^{2g};\mathbb{R})$. Now if we evaluate the above cocycle on the cycle $(x_1,y_1) - (y_1,x_1)$ we obtain 2, while the value of ω_0 on it is 1, whence (i).

(ii) Clearly $s_0^*(\omega)$ vanishes identically. For the second assertion, it is easy to construct a cross section of the S^1-bundle $S^1 \to \overline{J} \to J$ on the subset $s_0(X) \subset J$ by using the fact that the action of $\mathrm{Sp}(2g;\mathbb{Z})$ on N_g preserves the origin of N_g. As we have mentioned before, we shall later prove that $\Omega = 2[\omega]$ in $H^2(J;\mathbb{R})$ at the universal space level (see Proposition 7.4).

Next assume that our surface bundle $\pi : E \to X$ admits a cross section $s : X \to E$. Choose a Riemannian fibre metric on ξ so that each fibre E_p inherits a Riemannian metric. We can define a fibre preserving map $j : E \to J$ as follows. For each point z of E_p choose a curve ℓ from the base point $s(p)$ to z. Identify $H^1(E_p;\mathbb{R})$ with the space of harmonic 1-forms on E_p. For each harmonic 1-form $\theta \in H^1(E_p;\mathbb{R})$ consider the integral $\int_\ell \theta$. It is well defined modulo the periods of θ and we can define a map $j_p = j\,|_{E_p}\colon\ E_p \to J_p$ by

$$j_p(z) = \{\theta \to \int_\ell \theta;\ \theta \in H^1(E_p;\mathbb{R})\} \in \mathrm{Hom}(H^1(E_p;\mathbb{R}),\mathbb{R})/\{\text{periods}\}$$
$$\cong H_1(E_p;\mathbb{R})/H_1(E_p;\mathbb{Z}) = J_p.$$

We can always assume that the Riemannian metric on each fibre E_p is a hyperbolic one for $g \geq 2$ or a flat one for $g = 1$ (see [Mo2]) so that each E_p is a Riemann surface. In such a situation the mapping $j : E \to J$ is nothing but the one given in the Introduction. We shall call the map j also the Jacobi mapping.

Now $s(X)$ is a submanifold of E of codimension two and by assumption that our surface bundle is oriented, the normal bundle of $s(X)$ in E, which is just $\xi\ |_{s(X)}$, is also oriented. Hence we have the corresponding cohomology class $\nu \in H^2(E;\mathbb{Z})$. Now we can state our main theorem.

THEOREM 1.3. — *Let $\pi : E \to X$ be an oriented surface bundle with a cross section $s : X \to E$ and let $j : E \to J$ be the Jacobi mapping. Then we have the equality*

$$j^*(\Omega) = 2\nu - e - \pi^* s^*(e)$$

in $H^2(E;\mathbb{Z})$.

Remark 1.4. — The above theorem holds for all genus g. We shall prove it for $g \geq 2$ in §7. The proof for the remaining cases $g = 0$ and 1 is given in [Mo4].

2. Reformulation of the main theorem in terms of the cohomology of groups.

Assume that $g \geq 2$. Then as we shall see soon below, the classifying spaces of the bundles we concern turn out to be all Eilenberg-MacLane spaces. In this section we reformulate Theorem 1.3 in terms of cohomology of the fundamental groups of these spaces.

First we recall a few facts from [Mo2]. Let $\mathrm{Diff}_+\Sigma_g$ be the group of all orientation preserving diffeomorphisms of Σ_g equipped with the C^∞ topology. Let

$$\Sigma_g \longrightarrow \mathrm{EDiff}_+\Sigma_g \longrightarrow \mathrm{BDiff}_+\Sigma_g$$

be the universal oriented Σ_g-bundle over the classifying space $\mathrm{BDiff}_+\Sigma_g$. Rigorously speaking the definition of this Σ_g-bundle as well as the following arguments in this section should be done in the framework of semi-simplicial objects and maps between them. However to avoid unnecessary intricate expressions we work in the category of usual spaces and maps. Then the above three spaces are all Eilenberg-MacLane spaces and the associated short exact sequence of the fundamental groups is

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow M_{g,*} \longrightarrow M_g \longrightarrow 1$$

where $M_g = \pi_0(\mathrm{Diff}_+\Sigma_g)$ is the mapping class group already introduced in §1 and $M_{g,*} = \pi_0\left(\mathrm{Diff}_+(\Sigma_g,^*)\right)$, where $\mathrm{Diff}_+(\Sigma_g,^*)$ is the topological group of all orientation and base point preserving diffeomorphisms of Σ_g. Now define a space $\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g$ by the following pull back diagram

$$\begin{tikzcd}
\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \arrow[r, "\overline{\pi}"] \arrow[d, "\pi"'] & \mathrm{EDiff}_+\Sigma_g \arrow[d, "\pi"] \\
\mathrm{EDiff}_+\Sigma_g \arrow[r, "\pi"'] & \mathrm{BDiff}_+\Sigma_g.
\end{tikzcd}$$

Namely $\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g = \{(z,z') \in \mathrm{EDiff}_+\Sigma_g \times \mathrm{EDiff}_+\Sigma_g;\ \pi(z) = \pi(z')\}$ and $\pi(z,z') = z$, $\overline{\pi}(z,z') = z'$. Consider the resultant Σ_g-bundle

$$\Sigma_g \longrightarrow \overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \xrightarrow{\ \pi\ } \mathrm{EDiff}_+\Sigma_g.$$

It has a cross section $s:\ \mathrm{EDiff}_+\Sigma_g \longrightarrow \overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g$ defined by $s(z) = (z,z)$. In fact this Σ_g-bundle is the classifying bundle for such bundles. More precisely let $\pi:\ E \to X$ be a surface bundle with a cross section $s:\ X \to E$ and let

$$\begin{tikzcd}
E \arrow[r, "\overline{b}"] \arrow[d] & \mathrm{EDiff}_+\Sigma_g \arrow[d] \\
X \arrow[r, "b"'] & \mathrm{BDiff}_+\Sigma_g
\end{tikzcd}$$

be the classifying bundle map. Then we can define a bundle map

$$\begin{array}{ccc} E & \xrightarrow{\bar{b}'} & \bar{E}\mathrm{Diff}_+\Sigma_g \\ \downarrow & & \downarrow \\ X & \xrightarrow[b']{} & E\mathrm{Diff}_+\Sigma_g \end{array}$$

by $b' = \bar{b}s$ and $\bar{b}'(z) = (\bar{b}s\pi(z),\ \bar{b}(z))$ $(z \in E)$. This bundle map preserves the cross sections because $\bar{b}'s = sb'$. Hence the given surface bundle $\pi:\ E \to X$ with cross section s is isomorphic to the pull back of the universal bundle $\bar{E}\mathrm{Diff}_+\Sigma_g \to E\mathrm{Diff}_+\Sigma_g$ by the map $b':\ X \to E\mathrm{Diff}_+\Sigma_g$. As is easily shown the space $\bar{E}\mathrm{Diff}_+\Sigma_g$ is also an Eilenberg-MacLane space whose fundamental group $\overline{M}_{g,*}$ is given by the following pull back diagram

$$\begin{array}{ccc} \overline{M}_{g,*} & \xrightarrow{\bar{\pi}} & M_{g,*} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_{g,*} & \xrightarrow[\pi]{} & M_g. \end{array}$$

Namely $\overline{M}_{g,*} = \{(\phi,\psi) \in M_{g,*} \times M_{g,*};\ \pi(\phi) = \pi(\psi)\}$. Let $\pi_1(\Sigma_g) \rtimes M_{g,*}$ be the semi-direct product defined by the natural action of $M_{g,*}$ on $\pi_1(\Sigma_g)$ so that as a set it is equal to $\pi_1(\Sigma_g) \times M_{g,*}$ and the group law is given by

$$(\alpha,\phi)(\beta,\psi) = (\alpha\phi_*(\beta), \phi\psi)$$

for $\alpha,\beta \in \pi_1(\Sigma_g)$, $\phi,\psi \in M_{g,*}$. It is easy to see that the correspondence

$$\overline{M}_{g,*} \ni (\phi,\psi) \longmapsto (\psi\phi^{-1},\phi) \in \pi_1(\Sigma_g) \rtimes M_{g,*}$$

is an isomorphism. Henceforth we identify $\overline{M}_{g,*}$ with $\pi_1(\Sigma_g) \rtimes M_{g,*}$ by the above isomorphism. In short then the split extension

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \overline{M}_{g,*} \underset{s}{\overset{\pi}{\rightleftarrows}} M_{g,*} \longrightarrow 1$$

serves as the universal model for surface bundles with cross sections, where the splitting homomorphism $s: M_{g,*} \to \overline{M}_{g,*}$ is given by $s(\phi) = (1, \phi)$. Now let

$$T^{2g} \longrightarrow \mathrm{ESp}(2g; \mathbb{Z}) \longrightarrow K(\mathrm{Sp}(2g; \mathbb{Z}), 1)$$

be the universal *flat* T^{2g}-bundle with structure group $\mathrm{Sp}(2g; \mathbb{Z})$. This should be considered as the universal model for the family of Jacobian manifolds which we associated to each surface bundle in §1. More precisely if $\pi: E \to X$ is a surface bundle, then the associated bundle $\pi: J \to X$ is the pull back of the above universal flat T^{2g}-bundle by the map $X \to K(\mathrm{Sp}(2g; \mathbb{Z}), 1)$ which is defined by the homomorphism $\rho h: \pi_1(X) \to \mathrm{Sp}(2g; \mathbb{Z})$. If we apply this fact to the universal Σ_g-bundle $\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \to \mathrm{EDiff}_+\Sigma_g$ with cross section, we find that there is a fibre preserving map

$$\begin{tikzcd}
\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \arrow[r] \arrow[d] & \mathrm{ESp}(2g; \mathbb{Z}) \arrow[d] \\
\mathrm{EDiff}_+\Sigma_g \arrow[r] & K(\mathrm{Sp}(2g; \mathbb{Z}), 1)
\end{tikzcd}$$

such that for any surface bundle $\pi: E \to X$ with cross section $s: X \to E$, we have the following commutative diagram

$$\begin{tikzcd}
& J \arrow[rr] \arrow[dd] & & \mathrm{ESp}(2g; Z) \arrow[dd] \\
E \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{EDiff}_+\Sigma_g \arrow[ur] \arrow[dd] & \\
& X \arrow[rr] & & \mathrm{K}(\mathrm{Sp}(2g; Z), 1) \\
X \arrow[ur, equal] \arrow[rr] & & \overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \arrow[ur] &
\end{tikzcd}$$

By the obvious naturality of the cohomology classes in Theorem 1.3. under bundle maps, it is enough to prove our main theorem at the universal space level. Now $\mathrm{ESp}(2g;\mathbb{Z})$ is also an Eilenberg-MacLane space whose fundamental group which we denote $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$ is the semi-direct product $\overline{\mathrm{Sp}}(2g;\mathbb{Z}) = H_1(\Sigma_g;\mathbb{Z}) \rtimes \mathrm{Sp}(2g;\mathbb{Z})$ where $\mathrm{Sp}(2g;\mathbb{Z})$ acts naturally on $H_1(\Sigma_g;\mathbb{Z})$ via the symplectic basis $x_1,\dots,x_g;\ y_1,\dots,y_g$ chosen in §1. The group law on $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$ is given by $(x,A)(y,B) = (x+Ay, AB)$ $\Big(x,y \in H_1(\Sigma_g;\mathbb{Z}),\ A,B \in \mathrm{Sp}(2g;\mathbb{Z}\Big)$. Now the following commutative diagram of split extensions should be considered as the universal model for the Jacobi mapping $j:\ E \to J$ defined in §1.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r] \arrow[d] & \overline{M}_{g,*} \arrow[r, shift left, "\pi"] \arrow[d, "\overline{\rho}"'] & M_{g,*} \arrow[r] \arrow[l, shift left, "s"] \arrow[d, "\rho"'] & 1 \\
1 \arrow[r] & H_1(\Sigma_g;\mathbb{Z}) \arrow[r] & \overline{\mathrm{Sp}}(2g;\mathbb{Z}) \arrow[r, shift left, "\pi"] & \mathrm{Sp}(2g;\mathbb{Z}) \arrow[r] \arrow[l, shift left, "s"] & 1
\end{tikzcd}$$

where the homomorphism $\overline{\rho}:\ \overline{M}_{g,*} \to \overline{\mathrm{Sp}}(2g;\mathbb{Z})$ is defined by $\overline{\rho}(\gamma,\phi) = \Big([\gamma], \rho\pi(\phi)\Big)$, here $\gamma \in \pi_1(\Sigma_g)$, $\phi \in M_{g,*}$ and $[\gamma] \in H_1(\Sigma_g;\mathbb{Z})$ denotes the homology class of γ.

Now define a 2-cochain of the group $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$ by the formula

$$\Big((x,A),(y,B)\Big) \longmapsto x \cdot Ay.$$

It is easy to check that this is actually a cocycle and we denote $\Omega \in H^2(\overline{\mathrm{Sp}}(2g;\mathbb{Z});\mathbb{Z})$ for the corresponding cohomology class.

Observe that $s^*(\Omega) = 0$ in $H^2\Big(\mathrm{Sp}(2g;\mathbb{Z});\mathbb{Z}\Big)$, where $s:\ \mathrm{Sp}(2g;\mathbb{Z}) \to \overline{\mathrm{Sp}}(2g;\mathbb{Z})$ is the splitting. It is almost clear that this definition coincides with the previously defined one (§1). Next the Euler class of the universal Σ_g-bundle with cross section $\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g \to \mathrm{EDiff}_+\Sigma_g$ is clearly equal to $\overline{\pi}^*(e) \in H^2(\overline{M}_{g,*};\mathbb{Z}) = H^2(\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g;\mathbb{Z})$. Since $\overline{\pi}s\pi = \pi:\ \overline{M}_{g,*} \to M_{g,*}$ (see the pull back diagram defining $\overline{M}_{g,*}$), we have $\pi^*s^*\Big(\overline{\pi}^*(e)\Big) = \pi^*(e)$. Finally since $s(\mathrm{EDiff}_+\Sigma_g)$ is a "submanifold" of $\overline{\mathrm{E}}\mathrm{Diff}_+\Sigma_g$ of codimension two, we have the corresponding cohomology class $\nu \in H^2(\overline{M}_{g,*};\mathbb{Z})$. With these in mind the statement of Theorem 1.3 now takes the following form at the universal space level.

THEOREM 2.1. — *Let $\overline{\rho}: \overline{M}_{g,*} \to \overline{\mathrm{Sp}}(2g;\mathbb{Z})$ be the homomorphism defined above. Then we have the equality.*

$$\overline{\rho}^*(\Omega) = 2\nu - \pi^*(e) - \overline{\pi}^*(e)$$

in $H^2(\overline{M}_{g,*};\mathbb{Z})$.

Remark 2.2. — It would be interesting if one can define a natural cocycle of the group $\overline{M}_{g,*}$ which represents the cohomology class $\nu \in H^2(\overline{M}_{g,*};\mathbb{Z})$.

3. New relations among characteristic classes of surface bundles.

First we review the definition of our characteristic classes of surface bundles very briefly (see [Mo2] for details). As is explained in [Mo2] and also in §2, the short exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow M_{g,*} \xrightarrow{\ \pi\ } M_g \longrightarrow 1$$

serves as the universal Σ_g-bundle and we have the universal Euler class $e \in H^2(M_{g,*};\mathbb{Z})$. We define $e_i \in H^{2i}(M_g;\mathbb{Z})$ to be $\pi_*(e^{i+1})$ where $\pi_*: H^*(M_{g,*};\mathbb{Z}) \to H^{*-2}(M_g;\mathbb{Z})$ is the Gysin homomorphism. We write $\pi^*(e_i)$ simply by e_i. These cohomology classes define homomorphisms

$$\mathbb{Q}[e_1, e_2, \ldots] \longrightarrow H^*(M_g;\mathbb{Q})$$
$$\mathbb{Q}[e, e_1, e_2, \ldots] \longrightarrow H^*(M_{g,*};\mathbb{Q})$$

which are by no means injective and it is an important problem to determine their kernels = relations among the characteristic classes.

Now our task is to derive new relation from Theorem 2.1. For that we need

PROPOSITION 3.1. — *$\Omega^{g+1} = 0$ in $H^{2(g+1)}(\overline{\mathrm{Sp}}(2g;\mathbb{Z});\mathbb{Q})$. In fact the order of Ω^{g+1} in $H^{2(g+1)}(\overline{\mathrm{Sp}}(2g;\mathbb{Z});\mathbb{Z})$ divides $\dfrac{(2g+2)!}{2^{g+1}(g+1)!}$.*

Proof. — Over the reals this is almost clear because in the notations of §1, Ω is represented by twice of the closed 2-form ω on J (see also Proposition 7.4) and obviously ω^{g+1} vanishes identically. To obtain informations on the order of Ω^{g+1}, we use the technique of [Mo3], §6. Let us write V for the vector space $H_1(\Sigma_g;\mathbb{R})$ and let ω_0 be the symplectic form on V defined by the intersection pairing on V (see §1). The group $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$ acts on V by affine transformations : $(x,A)v = Av + x (x \in H_1(\Sigma_g;\mathbb{Z})$, $A \in \mathrm{Sp}(2g;\mathbb{Z})$ and $v \in V$. Clearly it preserves the form ω_0. Now let $K = \{K_p\}$ be the standard semi-simplicial complex of $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$. Namely $K_p = \{((x_1,A_1),\ldots,(x_p,A_p))\,;\ (x_i,A_i) \in \overline{\mathrm{Sp}}(2g;\mathbb{Z})\}$. The face and degeneracy operators are the usual ones. The fat realization $\| K \|$ of K is a $K(\overline{\mathrm{Sp}}(2g;\mathbb{Z}),1)$. Now let $A^*(K,\mathbb{Q})$ be the *rational* de Rham complex of K in the sense of Sullivan [S]. Namely a q-form on K is a compatible family of q-forms on all simplices of K such that on each simplex it is a sum of polynomials of affine coordinates times various constant forms. The integration $I : A^*(K;\mathbb{Q}) \to C^*(\overline{\mathrm{Sp}}(2g;\mathbb{Z});\mathbb{Q})$ induces an isomorphism on cohomology. Now following Dupont [D] we define a closed 2-form $\eta = \{\eta_\sigma;\ \sigma \in K_2\} \in A^2(K;\mathbb{Q})$ as follows. For each 2-simplex $\sigma = ((x,A),(y,B)) \in K_2$, which is geometrically expressed as a copy Δ^2_σ of the standard 2-simplex $\Delta^2 = (0,1,2)$, we define an affine map $k_\sigma : \Delta^2 \to V$ by $k_\sigma(0) = 0$ (the origin of V), $k_\sigma(1) = (x,A)0 = x$ and $k_\sigma(2) = (x,A)(y,B)0 = Ay + x$. We set

$$\eta_\sigma = k_\sigma^*(2\omega_0).$$

Next let $\tau = ((x,A),(y,B),(z,C)) \in K_3$ be a 3-simplex of K. We define an affine map $k_\tau : \Delta^3_\tau \to V$ by $k_\tau(0) = 0$, $k_\tau(1) = (x,A)0 = x$, $k_\tau(2) = (x,A)(y,B)0 = Ay + x$ and $k_\tau(3) = (x,A)(y,B)(z,C)0 = ABz + Ay + x$. We set

$$\eta_\tau = k_\tau^*(2\omega_0).$$

Since the action of $\overline{\mathrm{Sp}}(2g;\mathbb{Z})$ on V preserves the form ω_0, the restriction of η_τ to any face σ of τ equals η_σ. Clearly we can apply the above procedure on any q-simplex $\in K_q$ for any q and we have the desired 2-form η. From the form of ω_0, it is clear that η is a rational form. Now for each natural number n, we apply the integral operator I to η^n and obtain a $2n$-cocycle $I(\eta^n) \in Z^{2n}(\overline{\mathrm{Sp}}(2g;\mathbb{Z});\mathbb{Q})$.

Next let L be the cell complex defined by the triangulation of $\| K \| \times \| K \|$ given in [Mo3], §6 so that each simplex of L is one of the simplices of the standard triangulation of the product $\Delta^p_\sigma \times \Delta^q_\tau$ for some $\sigma, \tau \in K$. For two natural numbers m and n, we define a closed rational $2(m+n)$-form $\eta^m \times \eta^n$ on L by setting

$$(\eta^m \times \eta^n)_\lambda = p_1^*(\eta^m) \wedge p_2^*(\eta^n)$$

where $p_i (i = 1, 2)$ is the projection to the i-th factor of $\Delta^p_\sigma \times \Delta^q_\tau$. We consider the corresponding cocycle $I(\eta^m \times \eta^n) \in Z^{2(m+n)}(L; \mathbb{Q})$.

SUBLEMMA. — (i) $I(\eta) = \Omega$.

(ii) $\dfrac{(2n)!}{2^n n!} I(\eta^n)$ and $\dfrac{(2m+2n)!}{2^{m+n}(m+n)!} I(\eta^m \times \eta^n)$ are integral cocycles for all m, n.

(iii) $I(\eta^{g+1}) = 0$.

Proof. — The first two assertions follow from an elementary argument concerning the integrals of constant forms on affine spaces V and $V \times V$. The point for (ii) is the fact that $(2n)!$ times the volume of any $2n$-simplex in $\mathbb{R}^{2n}$ such that the vertices are all contained in the lattice $\mathbb{Z}^{2g}$ is an integer. (iii) follows from the obvious fact $\eta^{g+1} \equiv 0$.

Now as in [Mo3], §6 let $F_* : C_*(L) \to C_*(K) \otimes C_*(K)$ be the Alexander-Whitney map and let $G_* : C_*(K) \otimes C_*(K) \to C_*(L)$ be the Eilenberg-MacLane map. Choose a chain homotopy $H_* : C_*(L) \to C_{*+1}(L)$ so that

$$G_q F_q - id = \partial H_q + H_{q-1}\partial.$$

Define a cochain $c_{m,n} \in C^{2(m+n)-1}(K; \mathbb{Q})$ by

$$c_{m,n}(\sigma) = I(\eta^m \times \eta^n)\Big(H_{2(m+n)-1} d_*(\sigma)\Big) \quad \Big(\sigma \in K_{2(m+n)-1}\Big)$$

where $d_* : C_*(K) \to C_*(L)$ is the diagonal map. Then by exactly the same argument as that of [Mo3], §6 we can conclude

$$I(\eta^m) \cup I(\eta^n) - I(\eta^{m+n}) = \sigma c_{m,n}$$

where the cup product is the usual one. Observe that $\dfrac{(2m+2n)!}{2^{m+n}(m+n)!} c_{m,n}$ is an integral cochain. Now starting from the equation $\Omega = I(\eta)$, an inductive argument using the Sublemma shows

$$\Omega^{g+1} = \sigma\Big(c_{g,1} + c_{g-1,1} \cup \Omega + \ldots + c_{1,1} \cup \Omega^{g-1}\Big).$$

Since $\dfrac{(2g+2)!}{2^{g+1}(g+1)!}$ times the cochain in the parenthesis above is integral, we finish the proof.

If we combine Proposition 3.1 with Theorem 2.1, we obtain

COROLLARY 3.2. — $(2\nu - \pi^*(e) - \overline{\pi}^*(e))^{g+1} = 0$ in $H^{2(g+1)}(\overline{M}_{g,*};\mathbb{Q})$.

PROPOSITION 3.3. — (i) $\nu^2 = \nu\pi^*(e) = \nu\overline{\pi}^*(e)$ in $H^4(\overline{M}_{g,*};\mathbb{Z})$.

(ii) $\pi_*\overline{\pi}^*\left(e^{i+1}\right) = e_i$, where $\pi_* : H^{2(i+1)}(\overline{M}_{g,*};\mathbb{Z}) \to H^{2i}(M_{g,*};\mathbb{Z})$ is the Gysin map.

(iii) $\pi_*(\nu) = 1$.

Proof. — (i) follows the Thom isomorphism theorem applied to the image of a cross section of a surface bundle which is a submanifold of the total space of codimension two. (ii) and (iii) are clear.

Now if we apply the Gysin homomorphism $\pi_* : H^*(\overline{M}_{g,*};\mathbb{Q}) \to H^{*-2}(M_{g;*};\mathbb{Q})$ to the following equation using Proposition 3.3

$$(2\nu - \pi^*(e) - \overline{\pi}^*(e))^{g+1}\overline{\pi}^*(e^h) = 0 \quad (h \geq 0)$$

we obtain

$$\begin{aligned} &2^{g+1}e^{g+h} - \Big\{e^{g+1}e_{h-1} + \binom{g+1}{1} e^g e_h + \ldots + \\ \text{(R.1)} \qquad &\binom{g+1}{g} e e_{h+g-1} + e_{h+g}\Big\} = 0 \text{ in } H^{2(g+h)}(M_{g,*};\mathbb{Q}) \ (h \geq 0). \end{aligned}$$

Here we understand $e_{-1} = 0$ and $e_0 = 2-2g$. Similarly if we apply the Gysin homomorphism $\pi_* : H^*(M_{g,*};\mathbb{Q}) \to H^{*-2}(M_g;\mathbb{Q})$ to the equation (R.1) $\times\, e^k$, we have

$$\begin{aligned} &2^{g+1}e_{g+h+k-1} - \Big\{e_{g+k}e_{h-1} + \binom{g+1}{1} e_{g+k-1}e_h + \ldots + \\ &\binom{g+1}{g} e_k e_{h+g-1} + e_{k-1}e_{h+g}\Big\} = 0 \text{ in } H^{2(g+h+k-1)}(M_g;\mathbb{Q}) \\ \text{(R.2)} \qquad & \text{for all } h,\ k \geq 0. \end{aligned}$$

Remark 3.4. — The above relations (R.1) and (R.2) can be strengthened if we use the estimate of the order of Ω^{g+1} given in Proposition 3.3.

The above relations, when combined with the previously known ones, are very strong. For example if $g = 2$, then only the class e is non-trivial in $\tilde{H}^*(M_{2,*}; \mathbb{Q})$ and also if $g = 3$, then only the class e_1 survives in $\tilde{H}^*(M_3; \mathbb{Q})$. Thus in these cases there are no other relations.

4. Crossed homomorphisms and $H^1(M_g; H^1(\Sigma_g))$.

In the following three sections we compute the first (co)homology groups of the mapping class groups with coefficients in the (co)homology of the surface Σ_g. To avoid complicated notations, henceforth we simply write $H_1(\Sigma_g)$ for the *integral* homology of Σ_g and similarly for the cohomology. Let A be an abelian group. M_g acts on $H^1(\Sigma_g; A)$ from the left by the rule $\phi u = (\phi^{-1}) * (u)$ $(\phi \in M_g,\ u \in H^1(\Sigma_g; A))$. If we identify $H^1(\Sigma_g; A)$ with $\mathrm{Hom}(H_1(\Sigma_g), A)$, then the rule becomes $\phi u(x) = u(\phi_*^{-1}(x))$ $(x \in H_1(\Sigma_g))$. Now let $Z^1(M_g; H^1(\Sigma_g; A))$ be the set of all crossed homomorphisms $d:\ M_g \to H^1(\Sigma_g; A)$. Namely

$$Z^1(M_g; H^1(\Sigma_g; A)) = \{d:\ M_g \to H^1(\Sigma_g; A);\ d(\phi\psi) = d\phi + \phi d\psi,\ \phi, \psi \in M_g\}.$$

Let $\delta:\ H^1(\Sigma_g; A) \to Z^1(M_g; H^1(\Sigma_g; A))$ be the homomorphism defined by

$$\delta u(\phi) = \phi u - u.$$

Then as is well known we have

$$H^1(M_g; H^1(\Sigma_g; A)) = Z^1(M_g; H^1(\Sigma_g; A))/\mathrm{Im}\delta$$

(cf. K.S. Brown [Br]). Now for each crossed homomorphism $d:\ M_g \to H^1(\Sigma_g; A)$, consider the associated map

$$f_d:\ M_g \times H_1(\Sigma_g) \longrightarrow A$$

defined by $f_d(\phi, x) = d(\phi^{-1})(x)$, $\phi \in M_g$, $x \in H_1(\Sigma_g)$. It is easy to check that $f = f_d$ satisfies the following two conditions :

(i) $f(\phi, x + y) = f(\phi, x) + f(\phi, y)$.

(ii) $f(\phi\psi, x) = f(\phi, \psi_*(x)) + f(\psi, x)$ for all $\phi, \psi \in M_g$ and $x,\ y \in H_1(\Sigma_g)$. Moreover if we denote $F(M_g \times H_1(\Sigma_g), A)$ for the set of all maps $f:\ M_g \times H_1(\Sigma_g) \to A$ satisfying the above two conditions, then it is easy to see that the correspondence $d \to f_d$ defines a bijection $Z^1(M_g; H^1(\Sigma_g; A)) \cong F(M_g \times H_1(\Sigma_g), A)$. Under this bijection the coboundary δu corresponds to the map $\delta u : M_g \times H_1(\Sigma_g) \to A$ given by $\delta u(\phi, x) = u(\phi_*(x) - x)$ (we use the same letter). Henceforth we identify the above two sets and call elements of $F(M_g \times H_1(\Sigma_g), A)$ also crossed homomorphism.

Proposition 4.1. — *$H^1(M_g; H^1(\Sigma_g)) = 0$ for all $g \geq 1$.*

Before proving the above Proposition, we recall a few well-known results on the mapping class group M_g. First Lickorish [L] proved that M_g is generated by the Dehn twists along the $3g - 1$ simple closed curves $\ell_1, \ldots, \ell_g, m_1, \ldots, m_g, n_1, \ldots, n_{g-1}$ on Σ_g illustrated in Figure 1. We write λ_i, μ_i, ν_i for the isotopy classes of the *right handed* Dehn twists along ℓ_i, m_i and n_i respectively. Let θ be the isotopy class of the homeomorphism of Σ_g which moves each handle to the next one, namely it is represented by rotation by $2\pi/g$ of Σ_g to the direction indicated in Figure 1. Then clearly we have

$$\text{(R.I)} \qquad \begin{aligned} \lambda_{i+1} &= \theta\lambda_i\theta^{-1} \\ \mu_{i+1} &= \theta\mu_i\theta^{-1} \\ \nu_{i+1} &= \theta\nu_i\theta^{-1} \end{aligned}$$

(all subscripts are modulo g) so that M_g is generated by four elements λ_1, μ_1, ν_1 and θ ([L]). Next we recall several relations among the above generators from Birman and Hilden [BH].

(R.II) Let s_1 and s_2 be two mutually disjoint simple closed curves on Σ_g. Then the corresponding Dehn twists mutually commute. For example λ_1 and μ_i commute if $i \neq 1$.

(R.III) Let s_1 and s_2 be two simple closed curves on Σ_g and assume that s_1 intersects s_2 transversely in one point. Then the isotopy classes of the corresponding Dehn twists τ_1 and τ_2 satisfy $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. For example we have $\lambda_1\mu_1\lambda_1 = \mu_1\lambda_1\mu_1$.

(R.IV) Let $\tau = \mu_1\lambda_1\nu_1\lambda_2\nu_2 \ldots \lambda_{g-1}\nu_{g-1}\lambda_g\mu_g\mu_g\lambda_g\nu_{g-1}\lambda_{g-1} \ldots \nu_2\lambda_2\nu_1\lambda_1\mu_1$. Then τ commutes with μ_1.

There are still other relations than the above, but we need only (R.I) – (R.IV) for the proof of Proposition 4.1.

Next we compute the actions of λ_i, μ_i and ν_i on the homology $H_1(\Sigma_g)$. We write x_i and y_i for the homology classes of the simple closed curves ℓ_i and m_i with the orientations given in Figure 1 respectively. The result is given in Table 2. The blanks in it mean that the corresponding homology class is fixed by the corresponding homeomorphism.

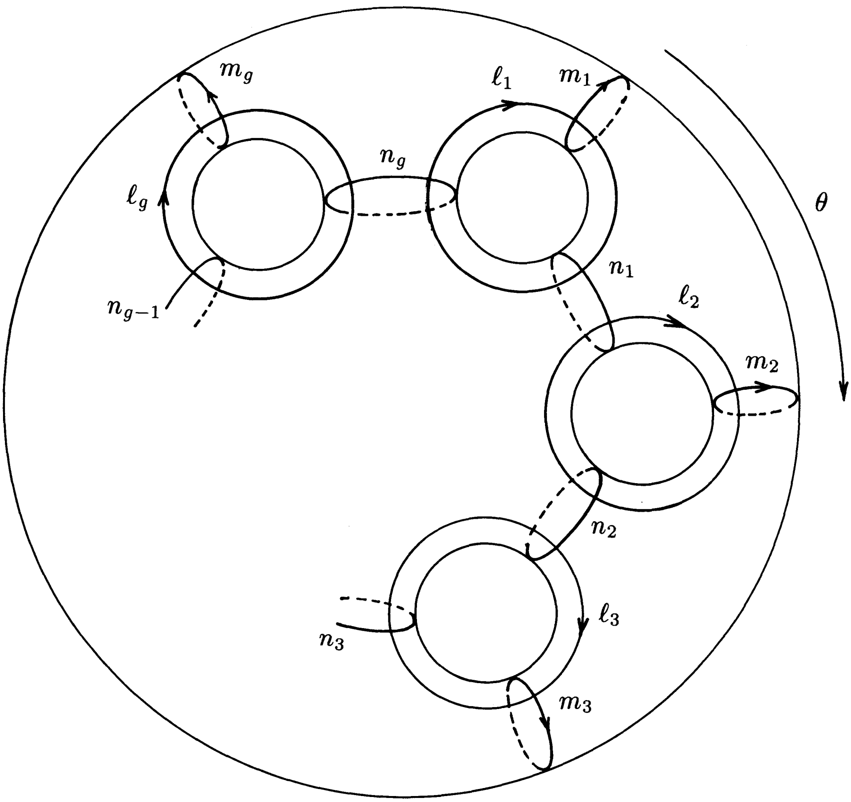

Figure 1.

Proof of Proposition 4.1. — Let

$$f:\ M_g \times H_1(\Sigma_g) \to \mathbb{Z}$$

be a crossed homomorphism. Our task is to show that f is a coboundary, namely to seek for a cohomology class $u \in H^1(\Sigma_g)$ such that $f(\phi, x) = u(\phi_*(x) - x)$ for all $\phi \in M_g$ and $x \in H_1(\Sigma_g)$. If such a class exists, then we should have

$$u(x_i) = f(\lambda_i, y_i)$$
$$u(y_i) = -f(\mu_i, x_i)$$

	x_1	x_2	$\cdots$	x_{g-1}	x_g	y_1	y_2	$\cdots$	y_{g-1}	y_g
λ_1						$x_1 + y_1$				
λ_2							$x_2 + y_2$			
$\vdots$								$\cdots$		
λ_g										$x_g + y_g$
μ_1	$x_1 - y_1$									
μ_2		$x_2 - y_2$								
$\vdots$			$\cdots$							
μ_g					$x_g - y_g$					
ν_1	$x_1 - y_1 + y_2$	$x_2 + y_1 - y_2$								
ν_2		$x_2 - y_2 + y_3$	$x_3 + y_2 - y_3$							
$\vdots$			$\cdots$	$\cdots$						
ν_{g-1}				$x_{g-1} - y_{g-1} + y_g$	$x_g + y_{g-1} - y_g$					

Table 2. The action of λ_i, μ_i, ν_i on $H_1(\Sigma_g)$

for all $i = 1, \dots, g$ (see Table 2). We define u by the above equations and prove that $f(\phi, x) = u(\phi_*(x) - x)$ for all ϕ and x. Since f is a crossed homomorphism it is enough to show this for the generators λ_i, μ_i and ν_i.

(I) $f(\lambda_i, x_j) = 0$, $f(\lambda_i, y_j) = 0$ for all $j \neq i$. To prove this we use the relation (R.II); $\lambda_i\lambda_j = \lambda_j\lambda_i$. We have $f(\lambda_i\lambda_j, x) = f(\lambda_j\lambda_i, x)$ for all $x \in H_1(\Sigma_g)$. Hence

$$f(\lambda_i, \lambda_j(x)) + f(\lambda_j, x) = f(\lambda_j, \lambda_i(x) + f(\lambda_i, x).$$

If we substitute y_j for x, we obtain $f(\lambda_i, x_j) = 0$. Next we use the relation (R.II); $\lambda_i\mu_j = \mu_j\lambda_i$ $(i \neq j)$. If we substitute x_j for x in the equation

$$f(\lambda_i, \mu_j(x)) + f(\mu_j, x) = f(\mu_j, \lambda_i(x)) + f(\lambda_i, x),$$

we obtain $f(\lambda_i, y_j) = 0$.

(II) $f(\mu_i, x_j) = 0$, $f(\mu_i, y_j) = 0$ for all $j \neq i$. This follows from the same proof as that of (I).

(III) $f(\lambda_i, x_i) = 0$, $f(\mu_i, y_i) = 0$ for all $i = 1, \dots, g$. To prove this first we use the relation (R.II); $\mu_i\nu_i = \nu_i\mu_i$. We have

$$f(\mu_i, \nu_i(x)) + f(\nu_i, x) = f(\nu_i, \mu_i(x)) + f(\mu_i, x).$$

If we substitute x_{i+1} for x, we obtain $f(\mu_i, y_i - y_{i+1}) = 0$. Hence we have $f(\mu_i, y_i) = 0$ by (II). Next the relation (R.III); $\lambda_i\mu_i\lambda_i = \mu_i\lambda_i\mu_i$ yields

$$\begin{aligned} &f(\lambda_i, \mu_i\lambda_i(x)) + f(\mu_i, \lambda_i(x)) + f(\lambda_i, x) \\ &\qquad = f(\mu_i, \lambda_i\mu_i(x)) + f(\lambda_i, \mu_i(x)) + f(\mu_i, x). \end{aligned}$$

Substituting y_i for x, we have $f(\lambda_i, x_i) = f(\mu_i, y_i)$. Hence $f(\lambda_i, x_i) = 0$ by the above.

(IV) $f(\nu_i, x_j) = 0$ for $j \neq i,\ i+1$ and $f(\nu_i, y_j) = 0$ for all j. The relation (R.II); $\nu_i\lambda_j = \lambda_j\nu_i$ $(j \neq i,\ i+1)$ yields

$$f(\nu_i, \lambda_j(x)) + f(\lambda_j, x) = f(\lambda_j, \nu_i(x)) + f(\nu_i, x).$$

If we substitute y_j for x, we obtain $f(\nu_i, x_j) = 0$. Next the relation (R.II); $\nu_i\mu_j = \mu_j\nu_i$ yields

$$f(\nu_i, \mu_j(x)) + f(\mu_j, x) = f(\mu_j, \nu_i(x)) + f(\nu_i, x).$$

If we substitute x_j for x, then we obtain

$$f(\nu_i, y_j) = f(\mu_j, x_j) - f(\mu_j, \nu_i(x_j))$$

which is zero by (II) and (III) because $x_j - \nu_i(x_j)$ is a linear combination of y_k's.

(V) $f(\nu_i, x_i + x_{i+1}) = 0$. The relation (R.III); $\lambda_i\nu_i\lambda_i = \nu_i\lambda_i\nu_i$ yields

$$f(\lambda_i, \nu_i\lambda_i(x)) + f(\nu_i, \lambda_i(x)) + f(\lambda_i, x) \\ = f(\nu_i, \lambda_i\nu_i(x)) + f(\lambda_i, \nu_i(x)) + f(\nu_i, x).$$

If we substitute x_{i+1} for x, we obtain

$$f(\lambda_i, x_{i+1}) = f(\nu_i, x_i + x_{i+1} + y_i - y_{i+1}).$$

Hence $f(\nu_i, x_i + x_{i+1}) = 0$ by (I) and (IV).

(VI) If we put $c_i = f(\nu_i, x_i) + u(y_i) - u(y_{i+1}) = f(\nu_i, x_i) - f(\mu_i, x_i) + f(\mu_{i+1}, x_{i+1})$, then we have $c_1 = \ldots = c_{g-1} = c_g$. To prove this we use the relation (R.I); $\mu_{i+1} = \theta\mu_i\theta^{-1}$ and $\nu_{i+1} = \theta\nu_i\nu^{-1}$. We have

$$\begin{aligned} f(\nu_{i+1}, x_{i+1}) &= f(\theta\nu_i\theta^{-1}, x_{i+1}) \\ &= f(\theta, \nu_i\theta^{-1}(x_{i+1})) + f(\nu_i, \theta^{-1}(x_{i+1})) + f(\theta^{-1}, x_{i+1}) \\ &= f(\theta, x_i - y_i + y_{i+1}) + f(\nu_i, x_i) + f(\theta^{-1}, x_{i+1}). \end{aligned}$$

Similarly we have

$$\begin{aligned} f(\mu_{i+1}, x_{i+1}) &= f(\theta, x_i - y_i) + f(\mu_i, x_i) + f(\theta^{-1}, x_{i+1}) \\ f(\mu_{i+2}, x_{i+2}) &= f(\theta, x_{i+1} - y_{i+1}) + f(\mu_{i+1}, x_{i+1}) + f(\theta^{-1}, x_{i+2}). \end{aligned}$$

Hence we conclude

$$\begin{aligned} c_{i+1} - c_i &= f(\theta, x_i - y_i + y_{i+1}) + f(\theta^{-1}, x_{i+1}) - f(\theta, x_i - y_i) \\ &\qquad - f(\theta^{-1}, x_{i+1}) + f(\theta, x_{i+1} - y_{i+1}) + f(\theta^{-1}, x_{i+2}) \\ &= f(\theta, x_{i+1}) + f(\theta^{-1}, x_{i+2}) \\ &= f(\theta\theta^{-1}, x_{i+2}) \\ &= 0. \end{aligned}$$

We write c for c_i.

(VII) $c = 0$. To prove this we use the relation (R.IV); $\tau\mu_1 = \mu_1\tau$, where $\tau = \mu_1\lambda_1\nu_1 \ldots \lambda_{g-1}\nu_{g-1}\lambda_g\mu_g\mu_g\lambda_g\nu_{g-1}\lambda_{g-1} \ldots \nu_1\lambda_1\mu_1$. We have $f(\tau, \mu_1(x)) + f(\mu_1, x) = f(\mu_1, \tau(x)) + f(\tau, x)$. τ acts on $H_1(\Sigma_g)$ by multiplication by -1 (see [BH]) so that if we put $x = x_1$, then we have

$$2f(\mu_1, x_1) = f(\tau, y_1).$$

Now we compute $f(\tau, y_1)$. The action of τ on y_1 is given by

$$y_1 \xrightarrow{\mu_1} y_1 \xrightarrow{\lambda_1} x_1 + y_1 \xrightarrow{\nu_1} x_1 + y_2 \xrightarrow{\lambda_2} x_1 + x_2 + y_2 \xrightarrow{\nu_2} x_1 + x_2 + y_3 \rightarrow \ldots \rightarrow x_1 + \ldots + x_{g-2} + y_g \xrightarrow{\lambda_{g-1}} x_1 + \ldots + x_{g-1} + y_{g-1} \xrightarrow{\nu_{g-1}} x_1 + \ldots + x_{g-1} + y_g \xrightarrow{\lambda_g} x_1 + \ldots + x_g + y_g \xrightarrow{\mu_g} x_1 + \ldots + x_g \xrightarrow{\mu_g} x_1 + \ldots + x_g - y_g \xrightarrow{\lambda_g} x_1 + \ldots + x_{g-1} - y_g \xrightarrow{\nu_{g-1}} x_1 + \ldots + x_{g-1} - y_{g-1} \xrightarrow{\lambda_{g-1}} x_1 + \ldots + x_{g-2} - y_{g-1} \rightarrow \ldots \rightarrow x_1 - y_2 \xrightarrow{\nu_1} x_1 - y_1 \xrightarrow{\lambda_1} - y_1 \xrightarrow{\mu_1} - y_1.$$

From this we have

$$\begin{aligned} f(\tau, y_1) =& f(\mu_1, -y_1) + f(\lambda_1, x_1 - y_1) + f(\nu_1, x_1 - y_2) + \ldots + \\ & f(\lambda_{g-1}, x_1 + \ldots + x_{g-1} - y_{g-1}) + f(\nu_{g-1}, x_1 + \ldots + x_{g-1} - y_g) \\ & + f(\lambda_g, x_1 + \ldots + x_g - y_g) + f(\mu_g, x_1 + \ldots + x_g) \\ & + f(\mu_g, x_1 + \ldots + x_g + y_g) + f(\lambda_g, x_1 + \ldots + x_{g-1} + y_g) \\ & + f(\nu_{g-1}, x_1 + \ldots + x_{g-1} + y_{g-1}) \\ & + f(\lambda_{g-1}, x_1 + \ldots + x_{g-2} + y_{g-1}) \\ & + \ldots + f(\nu_1, x_1 + y_1) + f(\lambda_1, y_1) + f(\mu_1, y_1) \\ =& 2\{f(\nu_1, x_1) + \ldots + f(\nu_{g-1}, x_{g-1}) + f(\mu_g, x_g)\}. \end{aligned}$$

Here we have used (I) – (V). Hence we have

$$2\{f(\nu_1, x_1) + \ldots + f(\nu_{g-1}, x_{g-1})\} = 2\{f(\mu_1, x_1) - f(\mu_g, x_g)\}.$$

On the other hand (VI) implies

$$f(\nu_1, x_1) + \ldots + f(\nu_{g-1}, x_{g-1}) = (g-1)c + f(\mu_1, x_1) - f(\mu_g, x_g).$$

Therefore we conclude

$$(2g-2)c = 0.$$

(We write as above instead of just $c = 0$ for later use). Now clearly (I) – (VII) imply that the crossed homomorphism f coincides with the coboundary δu. This completes the proof of Proposition 4.1.

5. $H^1(M_g; H^1(\Sigma_g; \mathbb{Z}/n))$.

In this section and the next, we will explicity determine the cohomology group $H^1(M_{g,*}; H^1(\Sigma_g; \mathbb{Z}))$. Most of the results of §§5,6 are not strictly needed for the proof of the main theorem which will be given in §7. In fact a glance at the Hochschild-Serre exact sequence in the beginning of the proof of Proposition 6.4 together with Proposition 4.1 will be sufficient for it. We include these results here because we think that they have their own meaning. As was mentioned in the introduction, they will play an important role in our subsequent papers [Mo6], [Mo7].

In the following three sections (§§5-7) we assume that $g \geq 2$.

Proposition 5.1. — *Let* $f : M_g \times H_1(\Sigma_g) \to \mathbb{Z}/n$ *be a crossed homomorphism and put* $c = f(\nu_1, x_1) - f(\mu_1, x_1) + f(\mu_2, x_2) \in \mathbb{Z}/n$.*Then this number* c *depends only on the cohomology class* $[f] \in H^1(M_g; H^1(\Sigma_g; \mathbb{Z}/n))$, $(2g-2)c = 0$ *and the correspondence* $[f] \mapsto c$ gives an isomorphism

$$H^1(M_g; H^1(\Sigma_g; \mathbb{Z}/n)) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2g-2, \mathbb{Z}/n) \qquad (g \geq 2).$$

First we observe that the arguments of (I) – (VII) in the proof of Proposition 4.1 can be equally applied to the crossed homomorphism $f : M_g \times H_1(\Sigma_g) \to \mathbb{Z}/n$ and we can conclude that there is a cohomology class $u \in H^1(\Sigma_g; \mathbb{Z}/n)$ such that the value of f on the generators λ_i, μ_i and ν_i differ from those of the coboundary δu only at (ν_i, x_i)'s and we have

$$f(\nu_i, x_i) - f(\mu_i, x_i) + f(\mu_{i+1}, x_{i+1}) = c$$

for all $i = 1, \ldots, g-1$. Obviously the cohomology class u is uniquely determined by the above properties so that the number c depends only on the cohomology class [f]. Moreover we know that $(2g-2)c = 0$. Therefore to prove Proposition 5.1, we have only to show the existence of a crossed homomorphism

$$f : M_g \times H_1(\Sigma_g) \longrightarrow \mathbb{Z}/(2g-2)$$

such that the above constant c is coprime to $(2g-2)$. Such a crossed homomorphism will be construted in the proof of Proposition 6.4.

Almost the same proof as those of Proposition 4.1 and Proposition 5.1 yields.

PROPOSITION 5.2. — *We have an isomorphism*

$$H^1(\mathrm{Sp}(2g;\mathbb{Z}); H^1(\Sigma_g;\mathbb{Z}/n)) \cong \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/n)$$

for all $g \geq 1$ and $n \geq 0$.

We have only to replace the relation (R.IV) of M_g (see §4) by (1.9) of Birman [Bi] for the group $\mathrm{Sp}(2g;\mathbb{Z})$. For $g \geq 2$, the generator of $H^1(\mathrm{Sp}(2g;\mathbb{Z}); H^1(\Sigma_g;\mathbb{Z}/2)) \cong \mathbb{Z}/2$ can be represented by the crossed homomorphism $f: \mathrm{Sp}(2g;\mathbb{Z}) \times H_1(\Sigma_g) \to \mathbb{Z}/2$ defined as

$$f(A,x) = \sum_{j=1}^{2g} \Big(\sum_{i=1}^{g} a_{ij} a_{i+g\,j} \Big) s_j \qquad \mod 2$$

for $A = (a_{ij}) \in \mathrm{Sp}(2g;\mathbb{Z})$ and $x = s_1 x_1 + \ldots + s_g x_g + s_{g+1} y_1 + \ldots + s_{2g} y_g \in H_1(\Sigma_g)$.

Remark 5.3. — It is easy to see directly that the cohomology group in Proposition 5.2 is annhilated by 2 since $-1 \in \mathrm{Sp}(2g;\mathbb{Z})$ lies in the center and acts by -1 on the coefficients. Actually this is all which is needed in the proof of Proposition 7.4. (This remark is due to the referee to whom the author would like to express his hearty thanks.)

COROLLARY 5.4. — (to proposition 5.1)

$$H_1(M_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2) \qquad (g \geq 2).$$

Proof. — This follows from Proposition 5.1 and the short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}(H_0(M_g; H_1(\Sigma_g)),\mathbb{Z}/n) \to H^1(M_g; H^1(\Sigma_g);\mathbb{Z}/n)) \\ \to \mathrm{Hom}(H_1(M_g; H_1(\Sigma_g)),\mathbb{Z}/n) \to 0 \end{aligned}$$

because we clearly have $H_0(M_g; H_1(\Sigma_g)) = 0$.

Remark 5.5. — Harer [H] has computed the second homology of the mapping class groups. However unfortunately Lemma 1.2 of [H] is false and a minor change on his results is necessary.

Namely we have

$$H_2(M_g;\mathbb{Z}) \cong \mathbb{Z} \qquad \text{for all } g \geq 5.$$

This can be shown by modifying Harer's proof using the results of his paper. However there is also a simple argument to show the above by using only the results of [H].

Now we prepare a few facts which will be used in the next section. Let Σ_g^0 be the compact surface obtained from Σ_g by subtracting the interior of an embedded disc D^2 and choose a base point $b_0 \in \Sigma_g^0$. Let $\alpha_1,\dots,\alpha_g,\beta_1\dots,\beta_g$ be the free generators of $\pi_1(\Sigma_g^0,b_0)$ as shown in Figure 3 (compare with Figure 1). We put $\zeta = [\alpha_1,\beta_1]\dots[\alpha_g,\beta_g]$ where $[\alpha_i,\beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$. It is represented by the simple closed curve ζ parallel to the boundary (see Figure 3). Now let $\overline{\alpha}_i$ and $\overline{\beta}_i$ be the elements of $\pi_1(\Sigma_g;b_0)$ represented by the same curve as α_i and β_i respectively. Then as is well known $\pi_1(\Sigma_g,b_0)$ is generated by them with a single defining relation

$$[\overline{\alpha}_1,\overline{\beta}_1]\dots[\overline{\alpha}_g\overline{\beta}_g] = 1.$$

Now let $M_{g,1}$ be the mapping class group of Σ_g^0. Namely $M_{g,1} = \pi_0(\mathrm{Diff}(\Sigma_g^0,\partial\Sigma_g^0))$, where $\mathrm{Diff}(\Sigma_g^0,\partial\Sigma_g^0)$ is the group of all diffeomorphisms of Σ_g^0 which restrict to the identity on the boundary. For a simple closed curve ℓ on Σ_g^0, we write $\tau_\ell \in M_{g,1}$ for the *right handed* Dehn twist along ℓ. The kernel of the natural homomorphism $\pi' : M_{g,1} \to M_{g,*}$ is isomorphic to $\mathbb{Z}$ whose generator is τ_ζ. Obviouly τ_ζ commutes with any element of $M_{g,1}$ and the central extension

$$0 \to \mathbb{Z} \to M_{g,1} \xrightarrow{\pi'} M_{g,*} \to 1$$

corresponds to the cohomology class $e \in H^2(M_{g,*};\mathbb{Z})$. We have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(T_1\Sigma_g) \arrow[r] \arrow[d, "\pi''"] & M_{g,1} \arrow[r] \arrow[d, "\pi'"] & M_g \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r, "\iota"] & M_{g,*} \arrow[r, "\pi"] & M_g \arrow[r] & 1
\end{tikzcd}$$

where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g (see [Mo2]). The injective homomorphism $\iota : \pi_1(\Sigma_g) \to M_{g,*}$ is given explicity as follows. For each simple closed curve γ representing an element of $\pi_1(\Sigma_g)$ (which we also

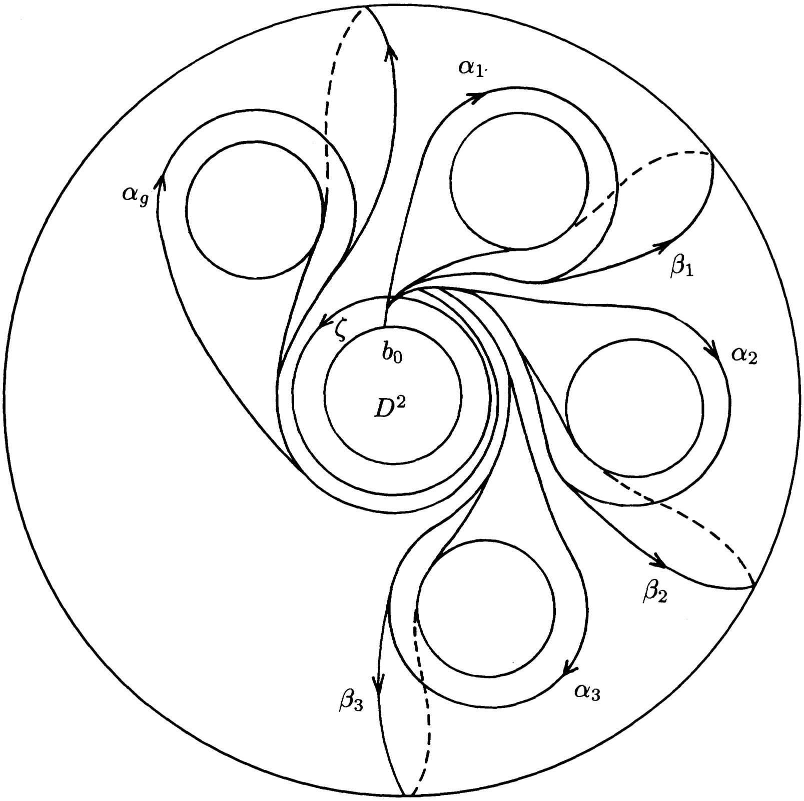

Figure 3

denote by γ), $\iota(\gamma)$ is the mapping class in $M_{g,*}$ of the homeomorphism of Σ_g which takes the base point b_0 round along the curve γ^{-1} until its original place (here γ^{-1} appears because we use the usual convention of multiplications of the groups $\pi_1(\Sigma_g)$ and $M_{g,*}$, namely the former is given by the product of paths while the latter is given by the composition of mappings). More precisely for each closed curve α_i (or β_i), choose two simple closed curves $a_i(+)$ and $a_i(-)$ $\Big(b_i(+)$ and $b_i(-)\Big)$ in Σ_g^0 as illustrated in Figure 4. We define

$$\tilde{\iota}(\overline{\alpha}_i) = \tau_{a_i(+)}\tau^{-1}_{a_i(-)}$$
$$\tilde{\iota}(\overline{\beta}_i) = \tau_{b_i(+)}\tau^{-1}_{b_i(-)}.$$

Then we have $\iota(\overline{\alpha}_i) = \pi'\tilde{\iota}(\overline{\alpha}_i)$ and $\iota(\overline{\beta}_i) = \pi'\tilde{\iota}(\overline{\beta}_i)$.

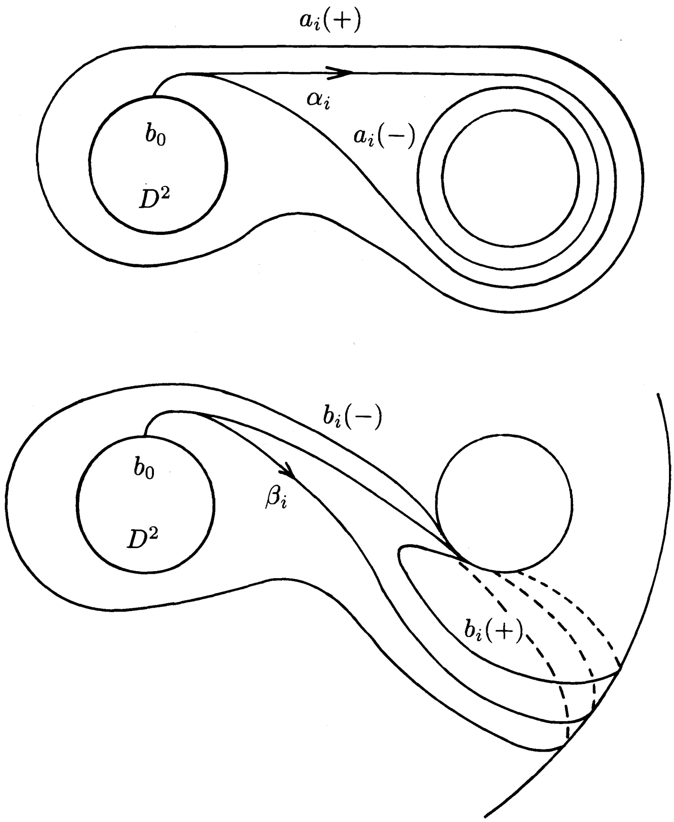

Figure 4

Now we consider μ_1, μ_2 and ν_1 as elements of $M_{g,1}$. It is easy to compute the action of these elements on $\pi_1(\Sigma_g^0)$. For example we have

$$\begin{aligned}\mu_1(\alpha_1) &= \alpha_1\beta_1^{-1}\\ \mu_2(\alpha_2) &= \alpha_2\beta_2^{-1}\\ \nu_1(\alpha_1) &= \alpha_1\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}.\end{aligned}$$

Also it is easy to determine the action of $\tilde{\iota}(\overline{\alpha}_i)$ and $\tilde{\iota}(\overline{\beta}_i)$ on $\pi_1(\Sigma_g^0)$. For example we have

$$\tilde{\iota}(\overline{\alpha}_1)(\beta_1) = \zeta^{-1}\alpha_1\beta_1\alpha_1^{-1}.$$

6. $H^1(M_{g,*}; H^1(\Sigma_g))$.

In this section we compute $H^1(M_{g,*}; H^1(\Sigma_g))$. For that we first consider $H^1(M_{g,1}; H^1(\Sigma_g))$. Recall from §5 that $\pi_1(\Sigma_g^0)$ is a free group on the generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ (see Figure 3). Now let F_2 be the free group generated by two elements α and β. Any element $\gamma \in F_2$ can be uniquely expressed as

$$\gamma = \alpha^{\varepsilon_1}\beta^{\delta_1} \dots \alpha^{\varepsilon_n}\beta^{\delta_n}$$

where ε_i and δ_i are 0, -1 or 1. Define an integer $d(\gamma)$ by

$$d(\gamma) = \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{n} \delta_j - \sum_{i=1}^{n} \delta_i \sum_{j=i+1}^{n} \varepsilon_j.$$

Now for each $i = 1, \dots, g$ let $p_i : \pi_1(\Sigma_g^0) \to F_2$ be the homomorphism defined by $p_i(\alpha_i) = \alpha$, $p_i(\beta_i) = \beta$ and p_i (other generators) $= 1$. For each element $\gamma \in \pi_1(\Sigma_g^0)$ we set $d_i(\gamma) = d(p_i(\gamma))$ and finally we define

$$d(\gamma) = \sum_{i=1}^{g} d_i(\gamma).$$

LEMMA 6.1. — *For any element γ, $\gamma' \in \pi_1(\Sigma_g^0)$, we have*

$$d(\gamma\gamma') = d(\gamma) + d(\gamma') + [\gamma] \cdot [\gamma']$$

where $[\gamma]$ denotes the homology class of γ in $H_1(\Sigma_g^0) \cong H_1(\Sigma_g)$ and $[\gamma] \cdot [\gamma']$ denotes the intersection number of $[\gamma]$ and $[\gamma']$.

Proof. — This follows directly from the definition.

LEMMA 6.2. — *For each element $\phi \in M_{g,1}$, the map $d_\phi : \pi_1(\Sigma_g^0) \to \mathbb{Z}$ defined by $d_\phi(\gamma) = d(\phi(\gamma)) - d(\gamma)$ is a homomorphism.*

Proof. — By Lemma 6.1, we have

$$\begin{aligned} d_\phi(\gamma\gamma') &= d(\phi(\gamma\gamma')) - d(\gamma\gamma') \\ &= d(\phi(\gamma)) + d(\phi(\gamma')) + [\phi(\gamma)] \cdot [\phi(\gamma')] \\ &\qquad - d(\gamma) - d(\gamma') - [\gamma] \cdot [\gamma'] \\ &= d_\phi(\gamma) + d_\phi(\gamma') \end{aligned}$$

because the action of $M_{g,1}$ on $H_1(\Sigma_g^0)$ preserves the intersection number.

In view of Lemma 6.2, we can define a map $f:\ \mathcal{M}_{g,1} \times H_1(\Sigma_g) \to \mathbb{Z}$ by

$$f(\phi, x) = d_\phi(\gamma)$$

where $\phi \in \mathcal{M}_{g,1}$, $x \in H_1(\Sigma_g)$ and $\gamma \in \pi_1(\Sigma_g^0)$ is an element such that $[\gamma] = x$.

LEMMA 6.3. — *The map* $f:\ \mathcal{M}_{g,1} \times H_1(\Sigma_g) \to \mathbb{Z}$ *defined above is a crossed homomorphism.*

Proof. — It remains to prove that $f(\phi\psi, x) = f(\phi, \psi(x)) + (\psi, x)$ for all ϕ, $\psi \in \mathcal{M}_{g,1}$ and $x \in H_1(\Sigma_g)$. But this follow easily from the definition.

Now consider the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g,1} \xrightarrow{\pi'} \mathcal{M}_{g,*} \longrightarrow 1.$$

The first three terms of the associated Gysin sequence of the cohomology with coefficients in $H^1(\Sigma_g)$ are

$$\begin{aligned} 0 \to H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g)) &\to H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g)) \\ &\to H^0(\mathcal{M}_{g,*}; H^1(\Sigma_g)) = 0. \end{aligned}$$

Hence we have an isomorphism

$$H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g)) \cong H^1(\mathcal{M}_{g,1}; H^1(\Sigma_g)).$$

In fact it is easy to see that the above crossed homomorphism f of $\mathcal{M}_{g,1}$ factors through a crossed homorphism $f:\ \mathcal{M}_{g,*} \times H_1(\Sigma_g) \to \mathbb{Z}$ (we use the same letter).

PROPOSITION 6.4. — $H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g))$ is an infinitive cyclic group generated by the cohomology class of the above crossed homomorphism f.

Proof. — Consider the exact sequence

$$1 \to \pi_1(\Sigma_g) \to \mathcal{M}_{g,*} \to \mathcal{M}_g \to 1.$$

The associated Hochschild-Serre exact sequence (see [HS]) is

$$\begin{aligned} 0 \to H^1(\mathcal{M}_g; H^1(\Sigma_g)) &\to H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g)) \\ &\to H^1\Big(\pi_1(\Sigma_g); H^1(\Sigma_g)\Big)^{\mathcal{M}_g} \to \cdots. \end{aligned}$$

We know that the first term vanishes (Proposition 4.1) and it is easy to see that the third term is isomorphic to $\mathbb{Z}$ whose generator corresponds to the intersection pairing $H_1(\Sigma_g) \times H_1(\Sigma_g) \to \mathbb{Z}$ which is invariant under the action of $\mathcal{M}_g$. To determine the image of the cohomology class $[f]$ in $H^1\Big(\pi_1(\Sigma_g); H^1(\Sigma_g)\Big)^{\mathcal{M}_g}$, we choose $\overline{\alpha}_1 \in \pi_1(\Sigma_g)$ and $\beta_1 \in \pi_1(\Sigma_g^0)$ (see §5). Then we have

$$
\begin{aligned}
f(\overline{\alpha}_1, \beta_1) &= d\Big(\tilde{\iota}(\overline{\alpha}_1)(\beta_1)\Big) - d(\beta_1) \\
&= d\Big(\zeta^{-1}\alpha_1\beta_1\alpha_1^{-1}\Big) - d(\beta_1) \\
&= 2 - 2g
\end{aligned}
$$

because $d(\zeta) = 2g$. Hence $[f]$ goes to $(2-2g)$ times the generator of $H^1\Big(\pi_1(\Sigma_g); H^1(\Sigma_g)\Big)^{\mathcal{M}_g}$. Therefore to prove Proposition 6.4, it suffices to show that $[f]$ is not divisible in $H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g))$. This condition is equivalent to the following fact. For each divisor n of $2g-2$, let $\overline{f}_n : \mathcal{M}_{g,*} \times H_1(\Sigma_g) \to \mathbb{Z}/n$ be the mod n reduction of f. Then the cohomology class of $\overline{f}_n$ is non-trivial in $H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g; \mathbb{Z}/n))$. Now it is easy to check that $\overline{f}_n$ factors through $\mathcal{M}_g$ so that it defines a crossed homomorphism $\overline{f}_n : \mathcal{M}_g \times H_1(\Sigma_g) \to \mathbb{Z}/n$ (we use the same letter). Since the homomorphism $H^1\Big(\mathcal{M}_g; H^1(\Sigma_g; \mathbb{Z}/n)\Big) \to H^1(\mathcal{M}_{g,*}; H^1(\Sigma_g; \mathbb{Z}/n))$ is injective, we have only to prove that the cohomology class $[\overline{f}_n]$ is non-trivial in $H^1(\mathcal{M}_g; H^1(\Sigma_g; \mathbb{Z}/n))$. For that we compute the constant c of the crossed homomorphism $\overline{f}_n$ (see Proposition 5.1).

As in §5 we consider the Dehn twists μ_i and ν_i as elements of $\mathcal{M}_{g,1}$. Then we have

$$
\begin{aligned}
c &= \overline{f}_n(\nu_1, [\alpha_1]) - \overline{f}_n(\mu_1, [\overline{\alpha}_1]) + \overline{f}_n(\mu_2, [\overline{\alpha}_2]) \\
&\equiv d(\nu_1(\alpha_1)) - d(\alpha_1) - d(\mu_1(\alpha_1)) \\
&\qquad + d(\alpha_1) + d(\mu_2(\alpha_2)) - d(\alpha_2) \\
&= d(\alpha_1\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}) - d(\alpha_1\beta_1^{-1}) + (\alpha_2\beta_2^{-1}) \\
&= 1.
\end{aligned}
$$

Here we have used the actions of μ_i and ν_i on $\pi_1(\Sigma_g^0)$ given in §5. This completes the proof of Proposition 6.4. Also the existence of the crossed homomorphism $\overline{f}_{2g-2} : \mathcal{M}_g \times H_1(\Sigma_g) \to \mathbb{Z}/(2g-2)$ finishes the proof of Proposition 5.1.

7. Proof of the main theorem.

In this section we prove Theorem 2.1. Recall the split extension

$$1 \to \pi_1(\Sigma_g) \to \overline{M}_{g,*} \underset{s}{\overset{\pi}{\rightleftarrows}} M_{g,*} \to 1$$

which serves as the universal model for Σ_g-bundles with cross sections (see §2). Now let $\{E_r^{p,q},\ d_r\}$ be the Hochschild-Serre spectral sequence for the integral cohomology of the above group extension. The E_2- term is given by $E_2^{p,q} = H^p(M_{g,*}; H^q(\Sigma_g))$. Our task is to prove the equality

$$\overline{\rho}^*(\Omega) = 2\nu - \pi^*(e) - \overline{\pi}^*(e)$$

in $H^2(\overline{M}_{g,*}; \mathbb{Z})$.

LEMMA 7.1. — (i) $E_\infty^{2,0} = E_2^{2,0} = H^2(M_{g,*}; \mathbb{Z})$.

(ii) $E_\infty^{1,1} = E_2^{1,1} = H^1(M_{g,*}; H^1(\Sigma_g))$.

(iii) $E_\infty^{0,2} = E_2^{0,2} = H^0(M_{g,*}; H^2(\Sigma_g)) \cong \mathbb{Z}$.

Proof. — Since our group extension splits, the induced homomorphism $\pi^* :\ H^*(M_{g,*}; \mathbb{Z}) \to H^*(\overline{M}_{g,*}; \mathbb{Z})$ is injective. Hence the differentials $d_2 :\ E_2^{p,1} \to E_2^{p+1,0}$ are all zero. (i) and (ii) follows from this (take $p = 0$ and 1 respectively). (iii) follows from the fact that $\iota^*(\nu) = 1 \in H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$.

We have the following short exact sequence

$$0 \to E_\infty^{2,0} = H^2(M_{g,*}; \mathbb{Z}) \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} H^2(\overline{M}_{g,*}; \mathbb{Z}) \to K \to 0$$

where $K = \mathrm{Cok}\pi^*$ can be naturally identified with Ker s^*.

LEMMA 7.2. — *Both of the cohomology classes $\overline{\rho}^*(\Omega)$ and $2\nu - \pi^*(e) - \overline{\pi}^*(e)$ are contained in* Ker s^*.

Proof. — $\overline{\rho}^*(\Omega) \in$ Ker s^* because $s^*(\Omega) = 0$ in $H^2(\mathrm{Sp}(2g; \mathbb{Z}); \mathbb{Z})$ where $s :\ \mathrm{Sp}(2g; \mathbb{Z}) \to \overline{\mathrm{Sp}}(2g; \mathbb{Z}))$ is the splitting (see §2). Next $s^*(2\nu - \pi^*(e) - \overline{\pi}^*(e)) = 2(s^*(\nu) - e) = 0$ because the cohomology class ν restricted to the cross section of a surface bundle is equal to the Euler class restricted there.

Next consider the following exact sequence

$$0 \longrightarrow E_\infty^{1,1} = H^1(M_{g,*}; H^1(\Sigma_g)) \longrightarrow K \stackrel{r}{\longrightarrow} E_\infty^{0,2} = H^2(\Sigma_g) \longrightarrow 0.$$

By Lemma 7.2, we may assume that both of $\overline{\rho}^*(\Omega)$ and $2\nu-\pi^*(e)-\overline{\pi}^*(e)$ are contained in K. The homomorphism r above is induced from the inclusion of the fibre. Therefore if we identify $E_\infty^{0,2} = H^2(\Sigma_g;\mathbb{Z})$ with $\mathbb{Z}$, it is easy to see that

$$r(\overline{\rho}^*(\Omega)) = 2g,\ r(\nu) = 1,$$
$$r(\pi^*(e)) = 0 \text{ and } r(\overline{\pi}^*(e)) = 2-2g$$

so that we have $r(\overline{\rho}^*(\Omega)) = r(2\nu-\pi^*(e)-\overline{\pi}^*(e))$. We know by Proposition 6.4 that $E_\infty^{1,1} = H^1(M_{g,*};H^1(\Sigma_g))$ is an infinite cyclic group. Hence to prove Theorem 2.1, we have only to check the required equality on a *single* non-trivial example of Σ_g-bundle with cross section such that the classifying map to the universal bundle induces a non-trivial map on the $E^{1,1}$-term in the spectral sequence. For this purpose it is enough to consider the following Σ_g-bundle

$$E = \Sigma_g \times \Sigma_g \underset{s}{\overset{\pi}{\rightleftarrows}} X = \Sigma_g$$

where $\pi(p,p') = p$ and $s(p) = (p,p)$ $(p,p' \in \Sigma_g)$. We have to prove that the two cohomology classes coincide in $H^2(E;\mathbb{Z})$. This cohomology group is naturally isomorphic to $\mathrm{Hom}(H_2(E;\mathbb{Z}),\mathbb{Z})$. Hence it suffices to prove that they have the same values on any 2-cycle of E. By the theorem of Künneth $H_2(E;\mathbb{Z}) \cong H_2(\Sigma_g;\mathbb{Z})\otimes 1 \oplus H_1(\Sigma_g;\mathbb{Z})\otimes H_1(\Sigma_g;\mathbb{Z}) \oplus 1\otimes H_2(\Sigma_g;\mathbb{Z})$. We already know that the two values are the same on homology of the fibre $1\otimes H_2(\Sigma_g;\mathbb{Z})$. By virtue of the obvious symetry of our example, they are also the same on the homology of the base $H_2(\Sigma_g;\mathbb{Z})\otimes 1$. Thus it remains to prove the coincidence on the cycle $[\gamma]\times[\gamma']$ for any $\gamma,\gamma' \in \pi_1(\Sigma_g)$. Now a simple computation corresponding to the argument of §2 shows that the classifying homomorphism

$$\kappa:\ \pi_1(E) = \pi_1(\Sigma_g)\times\pi_1(\Sigma_g) \longrightarrow \overline{M}_{g,*}$$

of our example is given by $\kappa(\gamma,\gamma') = (\gamma'\gamma^{-1},\gamma)$. Hence we have $\bar{\rho}\kappa(\gamma,\gamma') = ([\gamma'\gamma^{-1}],1)$. Now in terms of the group homology, the cycle $[\gamma]\times[\gamma']$ of E is represented by the 2-cycle $((\gamma,1),(1,\gamma')) - ((1,\gamma'),(\gamma,1))$ of $\pi_1(E)$. Hence we conclude

$$\begin{aligned}(\overline{\rho}\kappa)*(\Omega)([\gamma]\times[\gamma']) &= [\gamma^{-1}]\cdot[\gamma'] - [\gamma']\cdot[\gamma^{-1}]\\ &= -2[\gamma]\cdot[\gamma'].\end{aligned}$$

On the other hand the values of $\pi^*(e)$ and $\overline{\pi}^*(e)$ on $[\gamma]\times[\gamma']$ are clearly zero and the cohomology class ν is now the Poincaré dual of the diagonal

$s(\Sigma_g) \subset E$. From this it is easy to deduce

$$(2\nu - \pi^*(e) - \overline{\pi}^*(e))([\gamma] \times [\gamma']) = -2[\gamma] \cdot [\gamma'].$$

This completes the proof of Theorem 2.1.

Remark 7.3. — The final part of the above proof essentially identifies the "$E_\infty^{1,1}$ – part" of the cohomology class $\overline{\rho}^*(\Omega)$. If we compare it with the proof of Proposition 6.4, we can conclude that the cohomology class $\overline{\rho}^*(\Omega)$ is not divisible as an integral class. In particular it is *not* divisible by 2. This fact can also be proved directly by using Proposition 5.2.

Next we prove here the following fact which was promised in §1.

PROPOSITION 7.4. — *Let* π : $\mathrm{ESp}(2g;\mathbb{Z}) \to K(\mathrm{Sp}(2g;\mathbb{Z}),1)$ *be the universal* T^{2g}*-bundle with structure group* $\mathrm{Sp}(2g;\mathbb{Z})$, ω *the associated closed 2-form on* $\mathrm{ESp}(2g,\mathbb{Z})$, *and let* $\Omega \in H^2\Big(\mathrm{ESp}(2g;\mathbb{Z});\mathbb{Z}\Big)$ *be the cohomology class defined in §1. Then we have* $2[\omega] = \Omega$ in $H^2(\mathrm{ESp}(2g;\mathbb{Z});\mathbb{R})$.

Proof— Let $\{{}'E_r^{p,q},\ d'_r\}$ be the Hochschild-Serre spectral sequence for the real cohomology of $\mathrm{ESp}(2g;\mathbb{Z})$ and let $K' = \mathrm{Ker}(s_0^* : H^2(\mathrm{ESp}(2g;\mathbb{Z}),\mathbb{R}) \to H^2(K(\mathrm{Sp}(2g;\mathbb{Z}),1);\mathbb{R})$, where $s_0 : K(\mathrm{Sp}(2g;\mathbb{Z}),1) \to \mathrm{ESp}(2g;\mathbb{Z})$ is the zero-section. In view of Proposition 1.2, (ii) both of $2[\omega]$ and Ω lie in K'. Now the same argument as in the proof of Theorem 2.1 above implies a short exact sequence

$$0 \longrightarrow {}'E_\infty^{1,1} \longrightarrow K' \longrightarrow {}'E_\infty^{0,2} \longrightarrow 0.$$

Here ${}'E_\infty^{1,1} = {}'E_2^{1,1} = H^1(\mathrm{Sp}(2g;\mathbb{Z});H^1(\Sigma_g)) = 0$ by Proposition 5.2. Hence $K' = {}'E_\infty^{0,2} = \mathrm{Im}(H^2(\mathrm{ESp}(2g;\mathbb{Z});\mathbb{R}) \to H^2(T^{2g};\mathbb{R}))$. The result then follows from Proposition 1.2, (i).

8. Concluding remark.

Remark 8.1. — As stated in the Introduction, one of the motivations for the present work was Johnson's paper [J]. It turns out that there is a close relationship between Johnson's question on the homology of the Torelli group I_g and non-trivialities of some of the characteristic classes restricted to I_g. More precisely if $i : I_g \to M_g$ denotes the inclusion, then by an obvious reason we have $i^*(e_k) = 0$ for all *odd* k. However there is no reason for the classes $i^*(e_2),\ i^*(e_4),\ldots$ to vanish and in fact if Johnson's conjecture in [J] were true, then we must have non-trivial classes $i^*(e_{2k})$ for small k. We would like to discuss these points in detail in a forthcoming paper.

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