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APPROXIMATION OF C∞-FUNCTIONS WITHOUT CHANGING THEIR ZERO-SET

by F. BROGLIA and A. TOGNOLI [(1)]

0. Introduction.

Let M be a compact real algebraic manifold (resp. analytic manifold) and let $\phi: M \to \mathbb{R}$ be a C^∞ function such that $Y = \phi^{-1}(0)$ is an algebraic (resp. analytic) subvariety of M.

In this paper we shall study the following problem :

Problem. — When is it possible to approximate ϕ by rational regular or polynomial (resp. analytic) functions $f: M \to \mathbb{R}$ such that $f^{-1}(0) = \phi^{-1}(0)$?

So we ask for a sharp version of the classical Weierstrass approximation theorem and of its relative versions ([T2], [T3]).

The following example proves that, in general, the answer is negative.

So we shall search for the necessary hypothesis in order to have the required approximation.

Example 1. — Let us consider the C^∞ function :

$$\phi(x,y) = x^2y^2\left(x + y - \sqrt{x^2+y^2}\right) \cdot \exp\left(-\frac{1}{x^2+y^2}\right).$$

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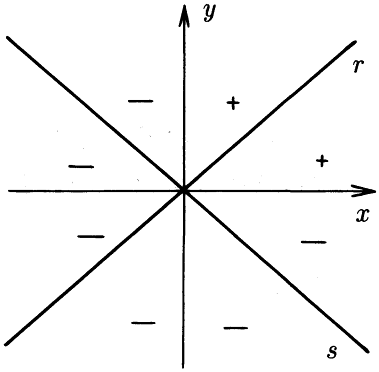
We have (see the figure below) :



$\phi^{-1}(0) = \{x \cdot y = 0\}$
$\phi(x,y) > 0$ if $x > 0$ and $y > 0$
$\phi(x,y) < 0$ if $x < 0$ or $y < 0$

Figure 1

ϕ can not be approximated, near the origin, by analytic functions f such that $f^{-1}(0) = \{x \cdot y\} = 0$; indeed, if q is the degree of the first non zero coefficient of the Taylor series of f at $(0,0)$, we deduce, (see fig.1), taking $f_{|r}$ that q is even and taking $f_{|s}$ that q is odd, since f and ϕ have the same sign.

Remark. — By the same argument we see also that any C^∞ function that approximates ϕ in the above sense must be flat at the origin.

Neverthless adding some hypothesis we find several solutions to the above problem. We state here some results.

THEOREM I. — *Let M be a compact affine real algebraic manifold or $\mathbb{R}^n$ and $\phi \in C^\infty(M)$ be such that $Y = \phi^{-1}(0)$ is an algebraic subset of M. If* codim$Y = 1$ *suppose $[Y] = 0 \in H_{m-1}(M, \mathbb{Z}_2)$. Then ϕ can be approximated in $C^0_W(M)$ by polynomial functions $f : M \to \mathbb{R}$ such that $f^{-1}(0) = \phi^{-1}(0)$ if and only if there exists a polynomial function $g : M \to \mathbb{R}$ such that $g^{-1}(0) = \phi^{-1}(0)$ and $g(x)\phi(x) \geq 0$ $\forall x \in M$.*

Moreover, if Y is "almost regular" and of pure codimension 1 then the approximation is in $C^s_W(M)$ $\forall s < \infty$.

At the end of the paper we study the "minimal" locus F of flatness of a C^∞-function that does not have a good approximation (as in example 1); we prove in the analytic case that F is often not empty.

For the statement of the result see §4.

From theorem 4.2 we obtain :

COROLLARY 4.5. — *Let M be a real analytic manifold and $\phi : M \to \mathbb{R}$ a smooth map such that $\phi^{-1}(0) = Y$ is a coherent analytic set. If Y has a finite number of irreducible components of codimension one arc-analytic connected and ϕ is nowhere flat in Y then ϕ can be approximated in $C^{\infty}_S(M)$ by analytic functions f such that $f^{-1}(0) = Y$.*

If Y is not coherent but admits global equations in M then the approximation is in $C^0_S(M)$.

The algebraic case will be considered before the analytic one and the case $M = \mathbb{R}^n$ before the general one.

We are indebted with P. Milman for his suggestions (see §4).

1. Preliminaries.

Let U be an open set of $\mathbb{R}^n$, $C^h_W(U)$, $h \in \mathbb{N}$ (resp. $h = \infty$), denotes the ring of the real functions having continuous derivatives up to order h (resp of class C^{∞}) endowed with the weak topology, i.e. the compact-open topology and $C^h_S(U)$ is the same ring endowed with the strong or Whitney topology (see [Hirs]). We denote similarly by $C^{\omega}_W(U)$ and $C^{\omega}_S(U)$ the ring of real analytic functions with the two topologies.

By a *real algebraic variety* we intend a set

$$V = \{x \in \mathbb{R}^n \mid P_1(x) = \ldots = P_q(x) = 0;\ P_i \in \mathbb{R}[X_1, \ldots, X_n]\}.$$

V is called *regular* at the point x^0 if the P_i can be chosen in such a way that $\operatorname{rank}\left(\frac{\partial P_i}{\partial x_j}\right)_{x^0} = n - \dim V$.

An *analytic set* of $\mathbb{R}^n$ is a closed set $X \subset \mathbb{R}^n$ such that there exist $f_1, \ldots, f_q \in C^{\omega}(\mathbb{R}^n)$ with

$$X = \{x \in \mathbb{R}^n \mid f_1(x) = \ldots f_q(x) = 0\}.$$

Remark that, in the literature, an analytic set in $\mathbb{R}^n$ is a closed subset of $\mathbb{R}^n$ that, locally, has analytic equations. So in our definition only the analytic sets that have global equations are considered.

An analytic set, in our sense, is not necessarily coherent, that is the subsheaf $\mathcal{I}_X$, of germs of analytic functions vanishing on X, is not $\mathcal{O}_{\mathbb{R}^n}$-coherent, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of germs of analytic functions on $\mathbb{R}^n$. But it is well known that X is the support of a coherent sheaf (the subsheaf $\hat{\mathcal{I}}_X$ of $\mathcal{O}_{\mathbf{R}^n}$ generated by $f_1, \ldots, f_q$ is coherent and $X =$ support $\mathcal{O}_{\mathbf{R}^n}/\hat{\mathcal{I}}_X$).

Let M be a regular algebraic variety : an algebraic subvariety Y of M is called *almost regular* at the point y if the ideal $\mathcal{I}_{Y,y}$ (of germs of analytic functions vanishing on Y_y) is generated by $I(Y)$, i.e. by polynomial functions vanishing on Y.

We remember that "almost regular" implies that Y is coherent as analytic space [T5].

An analytic function $f \in C^\omega(U)$ is called a *Nash function* if the graph $\Gamma_f \subset U \times \mathbf{R}$ is semialgebraic. The ring of Nash functions shall be denoted by $C^N(U)$.

A *Nash subset* of $\mathbf{R}^n$ is a closed set $X \subset \mathbf{R}^n$ such that there exist $f_1, \ldots, f_q \in C^N(\mathbf{R}^n)$ with :

$$X = \{x \in \mathbf{R}^n \mid f_1(x) = \ldots f_q(x) = 0\}.$$

Let X be a Nash set : the sheaf $\mathcal{N}_X$ (of germs of Nash functions on X) is coherent at a point x as $\mathcal{N}_X$-module if and only if the sheaf $\mathcal{O}_X$ (of germs of analytic functions) is coherent as $\mathcal{O}_X$-module ([LT]).

In §4 we shall use the following result :

THEOREM 1.1. — *Let $(Y, \mathcal{O}_Y)$ be a paracompact connected real analytic space such that* $\dim Y = n$ *and* $N = \sup_{y \in Y} \dim \mathcal{T}_y < +\infty$ *($\mathcal{T}_y$ is the Zariski tangent space at y) then the set of proper embeddings $j : Y \to \mathbf{R}^{n+N}$ is dense in $C^\infty_S(Y, \mathbf{R}^{n+N})$.*

Proof. — See [ABrT]. □

2. Admissible signatures and types changing points.

2A. *The algebraic case.*

First we consider the case $M = \mathbf{R}^n$. Let $Y \subset \mathbf{R}^n$ be a real algebraic set of codimension one.

DEFINITION 2.1. — *A signature on $\mathbf{R}^n - Y$ is a continuous map $\sigma : \mathbf{R}^n - Y \to \mathbb{Z}_2$ which associates a sign to each connected component of $\mathbf{R}^n - Y$*

Remark. — Let $\sigma : \mathbb{R}^n - Y \to \mathbb{Z}_2$ be a signature. Then there exists a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi^{-1}(0) = Y$ and σ is induced by the sign of ϕ.

Proof. — Build up a C^∞ function ψ such that $\psi^{-1}(0) = Y$ (see [BrL]); $\exp(-1/\psi^2)$ has the same property, it is non negative and it is flat at any point of Y. So $\phi = \sigma \cdot \exp(-1/\psi^2)$ is the required function. □

Let $\sigma : \mathbb{R}^n - Y \to \mathbb{Z}_2$ be a signature.

DEFINITION 2.2. — *A signature σ is called admissible :*

– *if Y is irreductible, when it is induced by one of the following polynomials : p, $-p$, p^2, $-p^2$, where p is a generator of the ideal $I(Y)$ of polynomials vanishing on Y.*

– *if $Y = \bigcup_1^k Y_i$, is the decomposition into irreducible components, when $\sigma = \Pi\sigma_i$ where σ_i is an admissible signature on $\mathbb{R}^n - Y_i, i = 1, \ldots, k$.*

DEFINITION 2.3.

– *A regular point $P \in Y$ of maximal dimension is called a change point with respect to σ (we say also that σ changes sign at P) it for any neighborhood U of P there exist $P_1, P_2 \in U - Y$ such that $\sigma(P_1) \neq \sigma(P_2)$; otherwise it is called a not change point.*

– *An irreducible component Y_j of Y is called a change component with respect to a signature σ if any regular point $P \in Y_j$ such that $\dim_P Y_j = n - 1$ is a change point.*

– *An irreducible component Y_j of Y is called a type changing component (with respect to a signature σ) if both changing and not changing points belong to Y_j.*

– *A type changing component Y_j changes type at $P \in Y_j$ if in any neighborhood U of P there are both change points and not change points of Y_j.*

– *A point $P \in Y$ is type changing if some component of Y changes type at P.*

Examples :

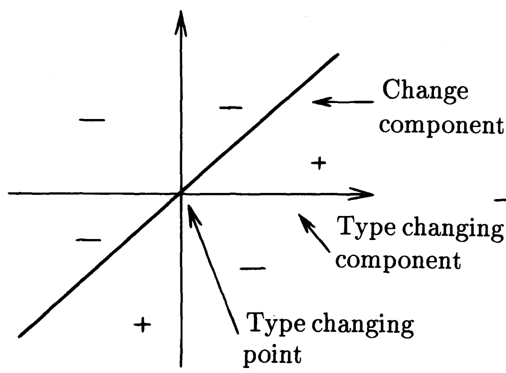
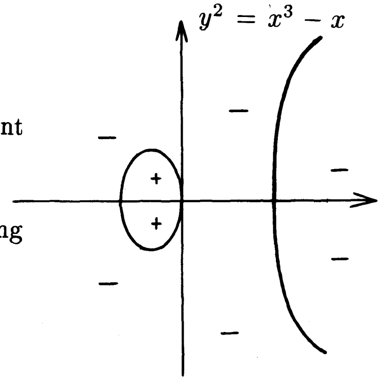


Figure 2



Type changing component without type changing point

Figure 3

– Let Y_i be an irreducible component of Y.

Define :

$$A_i = \{\text{change points in } Y_i\}$$
$$B_i = \{\text{not change points in } Y_i\}.$$

Remarks. — 1 – Both A_i and B_i are semialgebraic sets.

Indeed if $\{P_h\}$ and $\{M_k\}$ are the families of connected components respectively where the sign of σ is $+$ or $-$ then one has that P_h and M_k are semialgebraic, hence $A_i = \operatorname{int}\left(\cup(\overline{P}_h \cap \overline{M}_k)\right) \cap Y_i$ and $B_i = Y_i - \overline{A}_i$ are semialgebraic.

2 – The set of type changing points in Y_i is precisely $\overline{A}_i \cap \overline{B}_i$ and hence it is a semialgebraic set.

3 – Both A_i and B_i are open sets of Y_i; so the set of type changing points in Y_i it is not empty if Y_i is connected and $A_i \neq \emptyset$, $B_i \neq \emptyset$.

PROPOSITION 2.4. — *A signature σ is admissible if and only if it is induced by a polynomial.*

Proof. — The "only if" part is trivial.

Let $Y_1, \ldots, Y_k$ be the irreducible components of Y, and let q be a polynomial such that $q^{-1}(0) = Y \cdot q \in I(Y_i)$ for each i and so q is multiple of the generator p_i of $I(Y_i)$; let k_i be the highest integer such that $p_i^{k_i}$ divides q. Then $q = q' \cdot \prod_{i=1}^{k} p_i^{k_i}$ (since the p_i are coprime), with q' of constant sign, and so the signature of q is admissible, being induced by the product of the p_i with odd exponent. □

PROPOSITION 2.5. — *A signature σ on $\mathbb{R}^n - Y$ is admissible if and only if no irreducible component Y_j of Y is type changing with respect to σ.*

Proof. — If σ is admissible there exist generators $p_1, \ldots, p_r$ of $I(Y_1), \ldots, I(Y_r)$ such that σ is induced by $p_1^{a_1} \ldots p_r^{a_r}$, $a_i = 1$ or 2, $i = 1, \ldots, r$. If Y_j contains a change point, then necessarily $a_j = 1$ and so any point P, such that $\dim_P Y_j = n - 1$, is a change point.

Now suppose that no Y_j is type changing. Then it is either a change component or not.

Define :

$$\alpha_i = \begin{cases} 1 & \text{if } Y_i \text{ is a change component with resp. to } \sigma \\ 2 & \text{otherwise} \end{cases}$$

then σ is induced by $\pm \prod_{i=1}^{r} p_i^{\alpha_i}$, where p_i is a generator of $I(Y_i)$. □

If M is not $\mathbb{R}^n$, the definition of admissible signatures is a little more complicated, because we need global regular equations for the irreducible components of Y and they do not exist in general. Nevertheless the ring $\mathcal{O}(M)$ (of global regular rational functions on M) is factorial, and so if $\operatorname{codim} Y = 1$ and if $[Y] \in H_{n-1}(M, \mathbb{Z}_2)$ vanishes, then the ideal defining Y is principal, i.e. Y admits a global equation (see [BocCC-R]).

So we can divide Y into "minimal" homologically trivial subsets $\hat{Y}_i$, each one being union of several irreducible components.

This is done as follows :

Let M be a regular compact algebraic variety, m be the dimension of M and Y be a closed algebraic subvariety of codimension one such that $[Y] = 0$. Let $Y = \bigcup_{i=1}^{q} Y_i$ be the decomposition of Y into irreducible components.

DEFINITION 2.6. — *A decomposition* $Y = \bigcup_{j=1}^{r} \hat{Y}_j$ *shall be called a homological decomposition if :*

a) *any* $\hat{Y}_j$ *is not empty and any irreducible component* Y_i *is contained in just one* $\hat{Y}_j$ *;*

b) $[\hat{Y}_j] = 0 \in H_{m-1}(M, \mathbb{Z}_2)$ *and the* $\hat{Y}_j$ *are minimal under this condition.*

The $\hat{Y}_j$ *are called the elements of the homological decomposition.*

For any element $\hat{Y}_j$ there exists an equation ϕ_j of $\hat{Y}_j$ in M.

So the open set $M - \hat{Y}_j$ is the union of two (non necessarily connected) open sets U_j', U_j'', namely :

$$U_j' = \{x \in M \mid \phi_j(x) > 0\}$$
$$U_j'' = \{x \in M \mid \phi_j(x) < 0\}.$$

For any j we can now associate a sign σ_j to any component U_j' and U_j'' (in fact this can be done in four different ways : $(+,+)\,(+,-)\,(-,+)\,(-,-)$).

DEFINITION 2.7 (see def. 2.2). — *Let* $\sigma : M - Y \to \mathbb{Z}_2$ *be a signature. We say that* σ *is admissible if there exist an homological decomposition* $Y = \bigcup_{j=1}^{r} \hat{Y}_j$ *of* Y *and a family of associated signatures* σ_j *such that* $\sigma = \prod_{j=1}^{r} \sigma_j$.

If M is a regular real algebraic variety such that $H_{m-1}(M, \mathbb{Z}_2) = 0$, $m = \dim M$ then for any codimension one algebraic subvariety Y of M we have $[Y] = 0 \in H_{m-1}(M, \mathbb{Z}_2)$, and the unique homological decomposition of Y coincides with the decomposition into irreducible components.

PROPOSITION 2.8. — *Let* M, Y *be as before, then the admissible signatures with respect to a homological decomposition of* Y *are exactly the signatures of the polynomial functions* $p : M \to \mathbb{R}$ *such that* $p^{-1}(0) = Y$.

Proof. — If σ is admissible with respect to a homological decomposition $Y = \bigcup_{j=1}^{r} \hat{Y}_j$ and ϕ_j is a polynomial equation of $\hat{Y}_j$, then there exist admissible signatures σ_j on $M - \hat{Y}_j$ induced by $\psi_j = \pm\phi_j$ or $\psi_j = \pm\phi_j^2$. Then $\psi = \prod_j \psi_j$ is a polynomial function with signature σ.

If $p: M \to \mathbf{R}$ is a polynomial function such that $Y = p^{-1}(0)$ we have to find a homological decomposition of Y such that the signature induced by p is admissible with respect to it. This can be done in the following steps. Let $Y = \cup Y_i$ be the decomposition into irreducible components.

1) Suppose the unique homological decomposition of Y is Y itself. Then if p changes sign it must change sign along any Y_i, since the set of change points is a homologically trivial subset of Y. So, in any case, the signature induced by p is admissible.

2) If q is a polynomial equation for Y and $p = q^d$ with odd d, then p changes sign along any Y_i and so its signature is admissible with respect to any homological decomposition of Y.

3) Now let us prove that the signature induced by p is admissible by induction on the number s of elements in a homological decomposition of Y.

If $s = 1$, this is step 1.

If $s > 1$, let q be a polynomial equation for Y and h be the bigger integer such that p can be divided by q^h. Let $p_1 = p/q^h$ and $Y^1 = \{p_1 = 0\}$.

It is easy to verify that $[Y^1] = 0$ and $Y^1 \subset Y$, so we find a proper subset of Y being homological trivial. Repeat this argument until one reduces to a $Y^k = p_k^{-1}(0)$ such that $[Y^k] = 0$ but no proper subset, union of irreducible components of Y^k, is homologically trivial. So by step 1 proposition holds for p_k, hence for p_{k-1} and so on. □

Remark. — Let ϕ be an element of C^{∞} such that $\phi^{-1}(0) = Y$ and suppose that the signature induced by ϕ on $M - Y$ is not admissible. Then there is an element $\hat{Y}_j$ of a homological decomposition of Y which is type changing with respect to σ. Moreover, if $\hat{Y}_j = \bigcup_{i=1}^{s} Y_{i,j}$ is the decomposition of $\hat{Y}_j$ into irreducible components, there is at least one $Y_{i,j}$ such that $Y_{i,j}$ is type changing. Indeed the set of change points is a homologically trivial cycle in $H_{n-1}(M)$, since it is the boundary of the set $\{\phi \geq 0\}$. So it cannot be a union of irreducible components of $\hat{Y}_j$, since it is not empty nor all of $\hat{Y}_j$ and $\hat{Y}_j$ is minimal.

So again, as in proposition 2.5 :

PROPOSITION 2.9. — *If the signature σ on $M - Y$ is not admissible, then there exists at least one irreducible component of Y which changes type with repect to σ.*

2B. *The analytic case.*

If $Y \subset \mathbb{R}^n$ is a codimension one real analytic space, we can define again a signature on $\mathbb{R}^n - Y$, but to define admissible signature we need global generators for the sheaf $\mathcal{I}_Y$ (of germs of analytic functions vanishing on Y).

The following proposition shows that this is true when $\mathcal{I}_Y$ is coherent.

PROPOSITION 2.10. — *If Y has pure codimension 1 in $\mathbb{R}^n$ and $\mathcal{I}_Y$ is coherent then $\mathcal{I}_Y$ is generated by a global analytic function f.*

Proof. — This fact is locally true since for any $x \in \mathbb{R}^n$ $\mathcal{O}_{\mathbb{R}^n,x}$ is a factorial ring (see [N1]). So we obtain a covering $\mathcal{A} = \{U_i\}$ of $\mathbb{R}^n$ and for each i a generator f_i of $\mathcal{I}_{Y|U_i}$. Because of this property $g_{ij} = f_i/f_j$ is analytic and non vanishing on $U_i \cap U_j$, so $\{g_{ij}\}$ is the cocycle of an analytic line-bundle on $\mathbb{R}^n$ (associated to the divisor Y).

The isomorphism of $H^1(\mathbb{R}^n, \mathcal{O}^*)$ and $H^1(\mathbb{R}^n, \mathbb{Z}_2)$ induced by the exact sequence $0 \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ implies the analytical triviality of this line-bundle; so we can resolve the cocycle, i.e. to find $\lambda_i : U_i \to \mathbb{R}^*$, analytic, such that $f_i\lambda_i = f_j\lambda_j$ on $U_i \cap U_j$. The global function f defined by $f_{|U_i} = f_i\lambda_i$ is a generator of $\mathcal{I}_Y$. □

Remark. — If $\mathcal{I}_Y$ is not coherent, but there exists a coherent ideal sheaf $\mathcal{I}$ such that $Y = \mathrm{Supp}\mathcal{O}/\mathcal{I}$, then the same argument shows that there is a neigborhood U of $Y_{\max} = \{y \in Y \mid \dim_y Y = n - 1\}$ such that $Y \cap U$ is defined by one equation generating $\mathcal{I}_{Y_{\max}}$.

So, if Y is coherent (or the support of a coherent sheaf), we can define admissible signatures on $\mathbb{R}^n - Y$ as in def.2.2 and all definitions and remarks given for the algebraic case remain true with some obvious changes.

Now let M be a real analytic manifold and $Y \subset M$ be a coherent analytic subset with codimension Y equal to one. If $[Y] = 0$ and Y has only a finite number of irreducible components we can construct homological decompositions for Y as in the algebraic case. In fact it is enough to use the analogue of proposition 12.4.6 of [BocCC-R] in the analytic case. This follows from the fact that a topologically trivial analytic vector bundle is also analytically trivial (see [T1]).

Remarks. — 1) If σ is not admissible, the set of type changing points of Y is not empty, since an irreducible component Y_i of a real analytic set is connected.

2) If M, Y are algebraic then, in general the homological algebraic decomposition does not coincide with the analytic one.

3) Suppose to have local signature σ_i on $U_i - Y$, where $\{U_i\}$ is a locally finite open covering of M. Then the obstruction to glue together the σ_i in a signature σ on $M - Y$ is the same as to obtain a global equation for Y, i.e. an element in $H^1(M, \mathbb{Z}_2)$.

3. The main theorems.

3A. *The algebraic case.*

In this section we prove theorem I of introduction.

Let M be a compact regular real affine variety or $M = \mathbb{R}^n$. Let $Y \subset M$ be an algebraic subset; suppose $[Y] = 0 \in H_{n-1}(M, \mathbb{Z}_2)$.

Let us consider the following property :

(*) If $\phi \in C^\infty(M)$ vanishes on Y, then one can write

$$\phi = \sum \alpha_i p_i$$

where the p_i's are elements of $I(Y)$ and $\alpha_i \in C^\infty(M)$.

Property (*) is true if Y is "almost regular". In fact this condition implies that Y is coherent as analytic space and (*) is a consequence of Malgrange's theorem [M] as applied in [T4].

Let us suppose (*) is fullfilled. We have two possibilities :

a) *ϕ has constant sign* : we can suppose $\phi \geq 0$.

Consider the function $\sqrt{\phi} \in C^0(M)$. It is possible to approximate it by functions $\psi \in C^\infty(M)$ vanishing on Y (see for instance [Hirs]).

Apply (*) to ψ and write $\psi = \sum \alpha_i p_i$ where p_i are generators of $I(Y)$.

Approximate α_i by a polynomial q_i; then $q = \left(\sum q_i p_i\right)^2$ is a polynomial such that $q^{-1}(0) \supset Y$, $q \geq 0$ and q approximates ϕ. So $q + \eta \sum p_i^2$ is the required approximation for a small positive constant η.

If $\operatorname{codim} Y = 1$ this approximation can be taken in $C^s_W(M)$; in fact, always by (*), $\phi = \phi' p_Y$, where p_Y is a generator of $I(Y)$; but p_Y change sign in every point of maximal dimension while ϕ has constant sign, so we have $\phi' = \psi p_Y$ with $\operatorname{sign}\psi = \operatorname{sign}\phi$.

We can approximate the smooth function $\sqrt{\psi + \delta}$, with small positive δ, by a polynomial function p, $p(x) > 0$, in the C^s-topology. The function $p^2 \cdot p_Y^2$ gives the desired approximation of ϕ.

b) *ϕ changes sign* : this implies $\operatorname{codim} Y = 1$.

As we have seen, in this case ϕ has not a good approximation in general. But we added the hypothesis that "the signature of ϕ is admissible" (which was automatically true in case a).

By proposition 2.9 no irreducible component of Y is type changing with respect to the signature σ. Let $Y_1, \ldots, Y_n$ be the irreducible components of Y, (of course if $M \neq \mathbb{R}^n$, Y_i are the elements of a homological decomposition of Y) and $Y_1, \ldots, Y_k$ be the change components; if p_i is the generator of $I(Y_i)$, $i = 1, \ldots, n$, then the function $\psi = \phi \cdot p_1 \cdot \ldots \cdot p_k$ has constant sign. So we can apply the previous result and approximate ψ in $C^s_W(M)$ by a polynomial q such that $q^{-1}(0) = Y$: so q is divisible by $p_1, \ldots, p_k$ and $q/p_1 \ldots p_k$ is a good approximation of ϕ.

If (*) is not true we need the following lemmata :

LEMMA 3.1. — *Let $F \subset \mathbb{R}^n$ be a closed set and $\phi \in C^0(\mathbb{R}^n)$ be such that $\phi^{-1}(0) = F$. Then for any continuous function $\varepsilon : \mathbb{R}^n \to (0, +\infty)$, there exists $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi^{-1}(0) = F$ and $|\psi(x) - \phi(x)| < \varepsilon(x)$ for each $x \in \mathbb{R}^n$.*

Proof. — First remark that if F_0 and F_1 are disjoint closed sets in $\mathbb{R}^n$ then one can find $v \in C^\infty(\mathbb{R}^n)$ such that

1. $0 \leq v(x) \leq 1$

2. $v^{-1}(0) = F_0$, $v^{-1}(1) = F_1$

3. v is a flat on $F_0 \cup F_1$.

In fact if $\psi_i \in C^\infty(\mathbb{R}^n)$ are such that $\psi_i^{-1}(0) = F_i$, $i = 0,1$, then the function v is giyen by

$$v(x) = \frac{\exp(1/\psi_1^2(x))}{\exp(-1/\psi_1^2(x)) + \exp(-1/\psi_2^2(x))}.$$

Now consider $| \phi(x) | + \varepsilon(x)/4$. Since $C^\infty(\mathbb{R}^n)$ is dense into $C^0(\mathbb{R}^n)$ there exists $\psi' \in C^\infty(\mathbb{R}^n)$ such that

$$| \psi'(x) - | \phi(x) | - \varepsilon(x)/4 | < \varepsilon(x)/8.$$

In particular $\psi'(x) > 0$ for each $x \in \mathbb{R}^n$.

For any continuous function $\delta : \mathbb{R}^n \to (0, +\infty)$ denote

$$A_\delta = \Big\{ x \in \mathbb{R}^n \mid | \phi(x) < \delta(x) \Big\}.$$

Clearly A_δ is an open neighborhood of $F = \phi^{-1}(0)$.

Now choose $\delta(x) = \varepsilon(x)/8$ and consider the disjoint closed sets :

$$F_0 = F, \ F_1 = \mathbb{R}^n - A_{\varepsilon(x)/8}.$$

Let v be the function constructed at the beginning of the proof relatively to F_0 and F_1 and consider the function

$$\psi(x) = \psi'(x) \cdot v(x) \cdot \operatorname{sign} \phi(x).$$

It is easy to verify that $\psi \in C^\infty$ $(\mathbb{R}^n)$, since v is flat on $F_0 \cup F_1$. By construction one has

$$| \phi(x) - \psi(x) | < \varepsilon(x) \qquad \text{for each } x \in \mathbb{R}^n.$$

So lemma is proved. □

LEMMA 3.2. — *Let X be a metric space, and U_1, U_2 be two open sets with common boundary H such that $X = U_1 \cup H \cup U_2$. Let Y be a closed set with $H \subset Y$. If $\phi : Y \to \mathbb{R}$ is a continuous function such that $\phi^{-1}(0) = H$, $\phi_{|U_1 \cap Y} > 0$, $\phi_{|U_2 \cap Y} < 0$, then there exists a continuous extension $\tilde{\phi} : X \to \mathbb{R}$ such that $\tilde{\phi}^{-1}(0) = H = \phi^{-1}(0)$.*

Proof. — We shall construct two continuous functions : $\phi_i : \overline{U}_i \to \mathbb{R}$ such that : $\phi_1(x) \geq 0$, $\phi_1(x) = 0$ if and only if $x \in H$; $\phi_2(x) \leq 0$, $\phi_2(x) = 0$ if and only if $x \in H$. Clearly :

$$\tilde{\phi}(x) = \begin{cases} \phi_1(x) \text{ if } x \in \overline{U}_1 \\ \phi_2(x) \text{ if } x \in \overline{U}_2 \end{cases}$$

satisfies our conditions.

By Tietze's theorem one can extend $\phi_{|\overline{U}_i \cap Y}$ to continuous functions $\phi_i' : \overline{U}_i \to \mathbb{R}$. Then define

$$\begin{aligned} \phi_1(x) &= \mid \phi_1'(x) \mid + d(x, Y) \\ \phi_2(x) &= - \mid \phi_2'(x) \mid - d(x, Y). \end{aligned}$$

The lemma is proved. □

LEMMA 3.3. — *Let $Y \subset \mathbb{R}^n$ be a real algebraic subvariety or an analytic subspace with global equations.*

Then there exist an embedding $j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ and a linear subvariety H of $\mathbb{R}^{n+k}$ such that $j(Y) = j(\mathbb{R}^n) \cap H$.

Proof. — Let $g_1 = 0, \ldots, g_k = 0$ be global equations for Y $(g_1, \ldots, g_k$ are polynomials if Y is algebraic, otherwise they are analytic functions).

Consider the map $j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ defined by

$$j(x_1, \ldots, x_n) = (x_1, \ldots, x_n, g_1(x), \ldots, g_k(x)).$$

Then $j(\mathbb{R}^n)$ is a real algebraic (analytic) manifold which is isomorphic to $\mathbb{R}^n$ and

$$j(Y) = j(\mathbb{R}^n) \cap \left\{ x_{n+1} = \ldots = x_{n+k} = 0 \right\}. \qquad \square$$

Now we come back to the proof of theorem I.

Consider first the case $\operatorname{codim} Y = 1$.

Fix a compact set $K \subset M$ and an $\varepsilon > 0$.

By hypothesis the signature induced by ϕ is the signature induced by a polynomial p such that $p^{-1}(0) = Y$.

It is known (see [T4]) that there exists a polynomial function $\hat{p} : \mathbb{R}^n \to \mathbb{R}$ that extends p.

Consider the embedding $j : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $j(x) = (x, \hat{p}(x))$.

Clearly $j(\mathbb{R}^n) = V$ is algebraically isomorphic to $\mathbb{R}^n$ and $j(Y) = j(M) \cap H$ where $H = \left\{ x_{n+1} = 0 \right\}$. Let us extend the map

$\phi \circ j^{-1} : j(\mathbb{R}^n) \to \mathbb{R}$ to a continuous map $\psi : j(\mathbb{R}^n) \cup H \to \mathbb{R}$ such that $\psi_{|H} = 0$. ψ can be extended to a continuous map $\tilde{\psi} : \mathbb{R}^{n+1} \to \mathbb{R}$ vanishing only on H : in fact if ϕ has not constant sign this can be done by lemma 3.2 with $Y = j(\mathbb{R}^n) \cup H$, $U_1 = \left\{x_{n+1} > 0\right\}$, $U_2 = \left\{x_{n+1} < 0\right\}$; if ϕ has constant sign, first extend ϕ to a $\phi' : \mathbb{R}^{n+1} \to \mathbb{R}$ by Tietze's theorem, then consider :

$$\tilde{\psi}(x) = \operatorname{sign}\phi \cdot (| \psi'(x) | + d(x, j(\mathbb{R}^n) \cup H)).$$

Now by lemma 3.1 one can find a function $\phi' \in C^\infty(\mathbb{R}^{n+1})$ such that $\phi'^{-1} = H$ and $| \phi'(x) - \tilde{\psi}(x) | < \varepsilon/4$ for each $x \in \mathbb{R}^{n+1}$. This is true in particular for $x \in j(\mathbb{R}^n)$ and $\tilde{\psi}(x) = \phi \circ j^{-1}(x)$.

Remark that the signature of ϕ' on $\mathbb{R}^{n+1} - H$ is admissible since the sign of ϕ' is constant on the two half spaces. So by the previous case there is a polynomial q' such that $q'^{-1}(0) = H$ and $\| q' - \phi' \|^0_{K'} < \varepsilon/4$, where K' is a compact set in $\mathbb{R}^{n+1}$ such that $K' \cap j(\mathbb{R}^n) = j(K)$.

Finally consider the polynomial $q = q' \circ j$ which is the required approximation.

Now suppose $\operatorname{codim} Y > 1$ and $Y = \{p_1 = 0, \ldots, p_k = 0\}$.

Consider the extensions $\hat{p}_i$ of p_i to $\mathbb{R}^n$ and the embedding of lemma 3.3 $j : \mathbb{R}^n \to \mathbb{R}^{n+k}$.

Extend as before $\phi \circ j^{-1}$ to a continuous map $\tilde{\psi} : \mathbb{R}^{n+k} \to \mathbb{R}$ vanishing only on H and find $\phi' \in C^\infty(\mathbb{R}^{n+k})$ such that $\phi'^{-1}(0) = H$ and $| \phi'(x) - \tilde{\psi}(x) | \varepsilon/4$ for each $x \in \mathbb{R}^{n+k}$. Then apply again the previous case to find a polynomial q' such that $q'^{-1}(0) = H$ and $\| q' - \phi' \|^0_{K'} < \varepsilon/4$, where K' is a compact set in $\mathbb{R}^{n+k}$ such that $K' \cap j(\mathbb{R}^n) = j(K)$.

As before the polynomial $q = q' \circ j$ is the required one.

Theorem I is completely proved. □

3B. *The analytic case.*

The analytic case is very similar to the algebraic one : we remark that the approximation is now in the strong topology.

The statement is the following :

THEOREM II. — *Let M be a real analytic manifold and $Y \subset M$ be an analytic subset such that :*

1) Y has global equations in M

2) Y has only a finite number of irreducible components

3) if $\dim Y = n - 1$ then $[Y] = 0 \in H_{m-1}(M, \mathbb{Z}_2)$.

Let $\phi \in C^\infty(M)$ be such that $Y = \phi^{-1}(0)$. Then ϕ can be approximated in $C^0_S(M)$ by analytic functions f with $f^{-1}(0) = Y$ if and only if there exists an analytic function $g: M \to \mathbb{R}$ vanishing only on Y, and such that $g(x)\phi(x) \geq 0$ for any x in M.

Moreover, if $\operatorname{codim} Y = 1$ and Y is coherent then the approximation is in $C^s_S(M)$ $\forall s \leq \infty$.

In the proof we shall again consider first the case in which Y is coherent, in order to have condition (*).

If Y is not coherent Malgrange's theorem does not hold, i.e. the ideal of germs of smooth functions vanishing on Y is no more generated by the algebraic or analytic ones, as the following example shows.

Unlike Malgrange's one (see [M]), in our example the generator p of $\mathcal{I}_Y$ and the function ϕ have the same vanishing order at any point of Y. Neverless ϕ is not a multiple of p.

Example 3.4. — Let $Y \subset \mathbb{R}^n$ be the set :

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 - zy^2 = 0\}.$$

Consider a function $\alpha: \mathbb{R} \to \mathbb{R}$ with the following properties :

1. $\alpha \in C^\infty(\mathbb{R})$ and is non decreasing

2. $\alpha(t) = \begin{cases} -2 \text{ if } t \leq -2 \\ t \text{ if } t \geq -1 \end{cases}$

Define $\phi(x, y, z) = x^2 - \alpha(z) \cdot y^2$.

Clearly $\phi^{-1}(0) = Y$, but ϕ is not a multiple of $f(x, y, z) = x^2 - zy^2$ since ϕ/f does not extend to

$$\begin{cases} x = y = 0 \\ z < -2. \end{cases}$$

Remark that ϕ and f induce the same signature on $\mathbb{R}^3 - Y$. □

Proof of theorem II. — If Y is coherent the analogous of (*) in the analytic case in true, namely :

(**) If $\phi \in C^\infty(M)$ vanishes on Y, then one can write

$$\phi = \sum \alpha_i f_i$$

where $\{f_i\}$ is a system of global analytic generators for $\mathcal{I}_Y$ and $\alpha_i \in C^\infty(M)$.

The proof is exactly the same as in the algebraic case, using the fact that, when $\operatorname{codim} Y = 1$, the ideal sheaves $\mathcal{I}_{Y_i}$ have global generators f_i, changing sign at the regular points of Y_i and approximating the coefficients α_i of (**) by analytic functions q_i in $C^s_S(M)$.

If (**) is not true, we can apply the same argument as in the algebraic case, i.e. consider the embedding $(x, \hat{g}(x)) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ where $\hat{g}$ is an extension of the analytic function g which induces the signature of ϕ. The case $\operatorname{codim} Y > 1$ is exactly the same. □.

By similar arguments we obtain also the following theorem, which was known only for coherent spaces.

THEOREM 3.5. — *Let $(Y, \mathcal{O}_Y)$ be a paracompact connected real analytic space such that $\sup_{y \in Y} \dim \mathcal{I}_y < +\infty$, where $\mathcal{I}_y$ is the Zariski tangent space at y. Let $\phi : Y \to \mathbb{R}$ be a continuous function. Then ϕ can be approximated in $C^0_S(Y)$ by an analytic function g.*

Proof. — Y verifies the hypothesis of the embedding theorem 1.1, so we can suppose $Y \subset \mathbb{R}^n$. Then ϕ has a continuous extension $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$. Now apply the classical Whitney approximation theorem (see[N2]). □

3C. — *The Nash case.*

Also in this case the starting point is the analogous of property (*).

Let M be a compact affine C^ω-Nash manifold (or $\mathbb{R}^n$) and $Y \subset M$ a Nash subset.

(***) Let ϕ be a smooth map on M, vanishing on Y; then one has :

$$\phi = \sum \alpha_i f_i$$

where $\{f_i\}$ is a system of global generators for $\mathcal{I}^N_Y$ and $\alpha_i \in C^\infty(M)$.

If Y is coherent as analytic space , then it is also Nash-coherent (see [LT]).

By a result of [BeT] one has

$$\mathcal{I}_{Y,y} = \mathcal{I}^N_{Y,y} \cdot \mathcal{A}_y$$

where $\mathcal{A}_y$ denotes the algebra of germs of analytic functions at y.

This result enables as to prove as before that (***) is true when Y is coherent and there exist $f_1, \dots, f_q \in C^N(M)$ such that $f_{i|Y} = 0$ and the f_i generate the stalk $\mathcal{I}^N_{Y,y}$ for any $y \in Y$ (i.e. theorem A is true).

As in the algebraic case, one can distinguish if (***) is true or not and repeat the arguments of **3A** and **3B** to obtain the following :

THEOREM III. — *Let M be a compact affine C^ω-Nash manifold or $M = \mathbb{R}^n$. Let $Y \subset M$ be a (compact) Nash subset defined by*

$$f_1 = 0, \dots, f_k = 0$$

where $f_i : M \to \mathbb{R}$ are C^ω-Nash functions. Let $\phi : M \to \mathbb{R}$ be a smooth map such that $Y = \phi^{-1}(0)$.

Then ϕ can be approximated in $C^0_W(M)$by a Nash function f with $f^{-1}(0) = Y$ if and only if there exists a Nash function $g : M \to \mathbb{R}$ vanishing only on Y and such that $g(x)\phi(x) \geq 0$ for any x in M.

If codim $Y = 1$ *and theorem A is true for $\mathcal{I}^N_Y$ then the approximation is in $C^s(M)$ $\forall s < \infty$.*

4. Non admissible signatures and flatness.

In this paragraph we come back to non admissible signatures. Our aim is to study the behaviour of a ϕ whose signature is not admissible along a type changing component.

Let M be $\mathbb{R}^n$, or a non singular compact affine real algebraic variety, or a real analytic manifold. Let $Y \subset M$ be an algebraic or an analytic subset of codimension 1. Assume the following :

1) $[Y] = 0$ in $H_{n-1}(M, \mathbb{Z}_2)$.

2) If Y is analytic, then Y has only a finite number of irreducible components, each one admitting a global equation.

Let $\phi \in C^\infty(M)$ be such that $\phi^{-1}(0) = Y$ and let σ be the signature induced by ϕ on $M - Y$.

We begin with the following remark :

Remark. — Suppose Y be almost regular (or coherent) and σ to be non admissible. Let Y_i be a type changing component with respect to σ, and f_i a generator of $\mathcal{I}_{Y_i}$. Then one can write :

$$\phi = \phi_1 f_i^k$$

where $\phi_{1|Y_i}$ is not identically zero, and $\phi_1^{-1}(0) \cap Y_i$ is a not empty semialgebraic (or semianalytic) open set of Y_i, unless ϕ is flat at any point of Y_i.

Proof. — $\phi_{|Y_i} = 0$ and (*) or (**) is true, so ϕ can be divided by a power of f_i i.e. $\phi = \phi_1 f_i^k$.

If ϕ is not flat, one can choose a maximal finite k and so ϕ_1 is not identically zero on Y_i. If $A_i \subset Y_i$ is the set of changing points of Y_i, we know that both A_i and $B_i = Y_i - A_i$ are non empty semialgebraic (or semianalytic) subsets of Y_i, because Y_i is type changing. Since ϕ_1 must vanish on A if k is even and on B_i if k is odd, the remark is proved. □

DEFINITION 4.1. — *An irreducible algebraic variety or an analytic space Y is called arc-analytic connected if for any two points P and Q there exists an analytic arc in Y joining P and Q.*

A connected irreducible component does not need to be arc analytic connected : cf example 1.2.3. in [BiM].

The notion above and the argument proving theorem below are suggested to us by P. Milman and are presented here with his permission.

THEOREM 4.2. — *Let M, Y, ϕ, σ, be as before and suppose σ to be non admissible. If the type changing components of Y are not normal crossing, suppose moreover that at least one is arc-analytic connected. Then ϕ is flat at some type changing point of Y.*

Remark. — Unlike the analytic case, in the algebraic case the set of type changing points of Y with respect to a non admissible signature σ may be empty. Take for instance a disconnected irreducible real algebraic hypersurface Y. Take analytic equations $f_1, \dots, f_k$ for the connected components. Then $f_1^2 \cdot f_2 \cdot \dots \cdot f_k$ is an analytic function vanishing only at Y, with non admissible algebraic signature.

Proof. — Let P be a type changing point. Assume first that there are only two components Y_1, Y_2 of Y passing through P, smooth at P and normal crossing at P. Take the tangent spaces $T(Y_1)$ and $T(Y_2)$ at P we have the same situation as in the exemple 1, so one can find two non empty disjoint open sets A and B in the linear space of lines through P such that $\phi_{|\ell}$ changes sign at P if $\ell \in A$ and does not if $\ell \in B$. This is enough to prove that ϕ is flat at P by the same argument as in the example 1.

If this is not true, one can reduce to this case by a suitable suite of (global) blowing-up.

Let Y_i be a type changing component verifying the hypothesis. Let $\gamma \subset Y_i$ be an analytic arc connecting a point $a \in A_i$ and a point $b \in B_i$; then γ contains at least one type changing point P and one can suppose just one. Let U be a neighborhood of P. One can find a smooth algebraic (or analytic) subspace Z of Y, $P \in Z$, an algebraic (or analytic) manifold M and a map $\pi : M \to U$ such that :

1) $\pi : M \to U$ is surjective.

2) $\pi_{|M-\pi^{-1}(Z)} : M - \pi^{-1}(Z) \to U - Z$ is an isomorphism.

3) if $E = \pi^{-1}(Z)$ and E_j are the strict transforms of irreducible components of Y at P, then E_j and E are normal crossing (see [H]).

So $\psi = \pi \circ \phi$ is a C^∞-function. Lift γ to an analytic arc γ' in E_i; then γ' contains exactly one type changing point Q, with respect to ψ, and moreover $Q \in E$. By previous remark we have that ψ is flat at Q. At this point we can conclude by applying the following lemma :

LEMMA 4.3. — Let U, U' be open sets of $\mathbb{R}^n$ and $\pi : U \to U'$ be a smooth map such that $\pi(U)$ contains a non empty open set Ω. Then $d\pi_1 \wedge \ldots \wedge d\pi_n$ is not identically zero on U.

Proof. — π has rank n in a dense set of $\pi^{-1}(\Omega)$. □

LEMMA 4.4. — *Let* $f^* : \mathbb{R}[[u_1, \ldots, u_n]] \to \mathbb{R}[[x_1, \ldots, x_n]]$ *be the homomorphism defined by*

$$x_i = f_i(u_1, \ldots, u_n)$$

with $f_i(u_1, \ldots, u_n) \in \mathbb{R}[[u_1, \ldots, u_n]]$ *and* $f_i(0) = 0$. *Then if* f^* *is not injective* $df_1 \wedge \ldots \wedge df_n \equiv 0$.

Proof. — Suppose $\operatorname{Ker} f^* \neq \{0\}$ and choose $F \in \operatorname{Ker} f^*$ such that $F \neq 0$ and F is of minimal order. Then

$$F(f_1, \ldots, f_n) = 0.$$

Remark that not all the derivatives $\frac{\partial F}{\partial x_i}$ can be zero because if so F is constant and hence $F = 0$.

Since the derivatives have order less than F, by differentiating one finds a linear relation among $df_1, \ldots, df_n$, namely

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}_{|x_i = f_i(u)} df_i = 0$$

where not all the coefficients are equal to zero.

This is enough to conclude that the vectors df_i are lineary dependent on the quotient field of $\mathbb{R}[[u_1, \ldots, u_n]]$ and so $df_1 \wedge \ldots \wedge df_n \equiv 0$. □

In theorem 4.2 the homomorphism is induced by π. Since the image of π is an isomorphism outside $\pi^{-1}(Z)$ $d\pi_1 \wedge \ldots \wedge d\pi_n$ is not identically zero and so $\pi^* : \mathbb{R}[[u_1, \ldots, u_n]] \to \mathbb{R}[[x_1, \ldots, x_n]]$ is injective.

From theorem 4.2 and theorem II we obtain :

COROLLARY 4.5. — *Let M be a real analytic manifold and $\phi : M \to \mathbb{R}$ a smooth map such that $\phi^{-1}(0) = Y$ is a coherent analytic set. If Y has a finite number of irreducible components, say Y_i, of codimension one arc-analytic connected and ϕ is nowhere flat in Y then ϕ can be approximated in $C^\infty_S(M)$ by analytic functions f such that $f^{-1}(0) = Y$.*

If Y is not coherent but admits global equations in M then the approximation is in $C^0_S(M)$.

Remark 4.6. — Let Y be, as before, an analytic set of $\mathbb{R}^n$. For $n = 2$ or for general n when the irreducible components are normal crossing the proof of 4.2 shows that *ϕ must be flat at any type changing point* of Y. Indeed, in the case of $\mathbb{R}^2$, it is easy to see that there is an analytic arc joining changing and non changing points through any type changing point of Y.

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