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ON DEFORMATIONS OF HOLOMORPHIC FOLIATIONS

by J. GIRBAU and M. NICOLAU

Introduction.

This paper deals with deformations of holomorphic and transversely holomorphic foliations on a compact manifold. A *transversely holomorphic foliation* on a differentiable manifold M is given by an open covering $\{U_i\}_{i\in I}$ of M, differentiable submersions $f_i : U_i \to \mathbb{C}^q$ and holomorphic isomorphisms g_{ij} from $f_j(U_i \cap U_j)$ onto $f_i(U_i \cap U_j)$ such that $f_i = g_{ij} \circ f_j$. If we assume, in addition, that the manifold M is endowed with a complex structure with respect to which the submersions f_i are holomorphic, then we obtain a *holomorphic foliation* on M.

Girbau, Haefliger and Sundararaman stated a Kuranishi theorem for transversely holomorphic foliations and for holomorphic foliations [10]. Concretely, given a transversely holomorphic foliation $\mathscr{F}$ (or a holomorphic foliation) on a compact manifold M, they proved the existence of a family of deformations of $\mathscr{F}$ (called versal family) parametrized by a germ of (non-reduced) analytic space (called Kuranishi space) from which any other family of deformations of $\mathscr{F}$ can be obtained by inverse image. Such a theorem had been obtained before in a weaker form by Duchamp and Kalka [7]. The proof given in [10] in an adaptation to the case of the foliations of a method of Douady for deformations of complex structures [4]. The framework of the present paper will be also that of Douady.

Let $\mathscr{F}$ be a holomorphic foliation on a compact complex manifold M. In particular $\mathscr{F}$ is a transversely holomorphic foliation if we forget the complex structure of M. We denote this foliation by $\mathscr{F}^{\mathrm{tr}}$. The purpose

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of this paper is to relate the deformations of $\mathscr{F}^{\mathrm{tr}}$ as transversely holomorphic foliation or (*tr*-deformations) with the deformations of $\mathscr{F}$ as holomorphic foliation (or *h*-deformations).

First of all we consider the deformations of the holomorphic foliation $\mathscr{F}$ which keep fixed its transversal type. This kind of deformations, called *f*-deformations, can be defined as follows. Given an analytic space R let Γ^f_R be the pseudo-group of local holomorphic automorphisms of $R \times \mathbb{C}^q \times \mathbb{C}^{n-q}$ (where q is the complex codimension of $\mathscr{F}$) of the form

$$f(r,z^a,z^u) = (r,f^b(z^a),f^v(r,z^a,z^u))$$

with $a,b = 1,\ldots,q$ and $u,v = 1,\ldots,n-q$. A family of *f*-deformations of $\mathscr{F}$ parametrized by R can then be defined as a Γ^f_R-structure over $R \times M$; that is an analytic structure over the topological space $X = R \times M$ given by local charts $\varphi_i : U_i \to R \times \mathbb{C}^q \times \mathbb{C}^{n-q}$ of the form $\varphi_i(r,x) = (r,\phi_i(x))$ such that $\varphi_j \cdot \varphi_i^{-1}$ belong to the pseudogroup Γ^f_R. For each $r \in R$, $M_r = M \times \{r\}$ is a complex manifold endowed with a holomorphic foliation $\mathscr{F}_r$. $\mathscr{F}_0$, where 0 is a distinguished point of R, is assumed to be isomorphic to $\mathscr{F}$. Notice that all the foliations $\mathscr{F}_r$ coincide as transversely holomorphic foliations.

Two families of *f*-deformations of $\mathscr{F}$ parametrized by R, X and X', are equivalent if there is a holomorphic isomorphism between them whose local expressions are elements of Γ^f_R. In particular X and X' are equivalent as families of *h*-deformations. One should notice, nevertheless, that equivalence as *f*-deformations is in general a relation stronger than equivalence as holomorphic deformations. The equivalence of the families of *f*-deformations X and X' implies that the transversely holomorphic foliations $\mathscr{F}^{\mathrm{tr}}_r$ and $\mathscr{F}'^{\mathrm{tr}}_r$ are in fact the same and not only isomorphic.

Families of *f*-deformations can be viewed as families of complex structures on M for which $\mathscr{F}^{\mathrm{tr}}$ becomes a holomorphic foliation. Even more, a *f*-deformation of $\mathscr{F}$ can also be defined as an unfolding of $\mathscr{F}$; i.e. a holomorphic foliation $\mathscr{F}_R$ over the complex space X of the same codimension as $\mathscr{F}$ and transverse to the projection $X \to R$ such that its restriction to $M \times \{0\}$ is isomorphic to $\mathscr{F}$. The study of the infinitesimal *f*-deformations of a holomorphic foliation has been done by Gomez-Mont in [13].

Our first result is a Kuranishi theorem for *f*-deformations (section 1). Thus, given a holomorphic foliation $\mathscr{F}$ on M, we have three Kuranishi spaces K, K^{tr} and K^f parametrizing respectively the versal families of h-deformations, tr-deformations (cf. [10]) and f-deformations of $\mathscr{F}$. Our purpose is to relate these three spaces.

Let $\Theta_{\mathscr{F}}$ be the sheaf of germs of holomorphic vector fields over M preserving $\mathscr{F}$ and denote by $\Theta^f_{\mathscr{F}}$ the subsheaf of $\Theta_{\mathscr{F}}$ consisting of those elements of $\Theta_{\mathscr{F}}$ tangent to the leaves of $\mathscr{F}$. By setting $\Theta^{\mathrm{tr}}_{\mathscr{F}} = \Theta_{\mathscr{F}}/\Theta^f_{\mathscr{F}}$ we obtain the exact sequence

$$(1) \qquad 0 \to \Theta^f_{\mathscr{F}} \xrightarrow{\alpha} \Theta_{\mathscr{F}} \xrightarrow{\beta} \Theta^{tr}_{\mathscr{F}} \to 0 .$$

The Kuranishi spaces K, K^{tr} and K^f are germs at the origin of analytic subspaces of $H^1(M,\Theta_{\mathscr{F}})$, $H^1(M,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ and $H^1(M,\Theta^f_{\mathscr{F}})$ respectively.

The versal property of K^{tr} asserts the existence of a morphism $\pi : K \to K^{\mathrm{tr}}$. This is the map which « forgets the complex structure » of the manifold M. The fibre $\pi^{-1}(0)$ of this map parametrizes the holomorphic deformations of $\mathscr{F}$ which are trivial from the transversely holomorphic point of view. Since equivalence as f-deformations is a relation stronger than equivalence as holomorphic deformations the spaces K^f and $\pi^{-1}(0)$ do not coincide in general. We prove in section 2 that K^f is isomorphic to $\pi^{-1}(0) \times \Sigma$ where Σ is the germ at the origin of a linear subspace of $H^0(M,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ supplementary to the image of the morphism $\beta_0 : H^0(M,\Theta_{\mathscr{F}}) \to H^0(M,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ induced by (1).

In section 3 we consider the problem of deciding, for a given foliation $\mathscr{F}$, if any transversely holomorphic deformation of $\mathscr{F}$ comes from a holomorphic deformation; in other words, if the map $\pi : K \to K^{\mathrm{tr}}$ is an epimorphism. Concerning this question we find two different conditions: (i) the existence of a holomorphic foliation $\mathscr{F}^{\pitchfork}$ transversal and complementary to $\mathscr{F}$, and (ii) the vanishing of the cohomology group $H^2(M,\Theta^f_{\mathscr{F}})$, which independently ensure the existence of a section of π. If this two conditions are both fulfilled then $K \cong K^f \times K^{\mathrm{tr}}$ (this is a particular case of theorem 3.2).

In section 4 we compute some examples. We consider, for instance, the case in which $\mathscr{F}$ is transverse to the fibres of a holomorphic compact bundle $p : M \to B$. Under certain cohomological assumptions the space K^f coincides with the versal space K_B of deformations of the complex manifold B and $K \cong K_B \times K^{\mathrm{tr}}$. This improves theorem 2.5 in [10]. Among the examples of section 4 we wish to remark here the foliation on the product $T^q \times \mathbb{C}P^1$ (where T^q is a complex torus of dimension q) whose leaves are the fibres of the projection $T^q \times \mathbb{C}P^1 \to T^q$. The space K^f is a certain analytic subspace of $\mathbb{C}^{3q}$. One might believe that the f-deformations have to deform the complex structure of the leaves because (by definition) they can not deform the transversal structure. This is not the case in this example since the leaves are isomorphic to $\mathbb{C}P^1$ which is rigid.

Throughout this paper we will deal with (non-reduced) analytic spaces R and with their germs $(R,0)$ at a distinguished point $0 \in R$. When there is no danger of confusion we will use the same symbol R to denote the analytic space or its germa at 0. The subscripts (or superscripts) a, b, c... will run from 1 to q and the subscripts (or superscripts) u, v, w,... will run from 1 to $n-q$, where q is the complex codimension of $\mathscr{F}$ and n is the complex dimension of the manifold M. We will use the Einstein summation convention.

1. Deformation of complex structures preserving a transversely holomorphic foliation.

Let M be a smooth (of class C^∞) manifold of dimension m. We recall that a transversely holomorphic foliation $\mathscr{F}$ on M of (complex) codimension q is given by an open cover $\{U_i\}_{i\in I}$ of M, a collection of differentiable submersions $f_i : U_i \to \mathbb{C}^q$ and holomorphic isomorphisms g_{ij} of $f_j(U_i \cap U_j)$ on $f_i(U_i \cap U_j)$ such that $f_i = q_{ij} \circ f_j$. If the submersions f_i are holomorphic with respect to a given complex structure on M then $\mathscr{F}$ is a holomophic foliation.

Denote by Γ^{tr} the pseudogroup of local differentiable automorphisms $f(z^a, x^u) = (f^a, f^u)$ of $\mathbb{C}^q \times \mathbb{R}^{m-2q}$ fulfilling

$$\frac{\partial f^a}{\partial \bar{z}^b} = \frac{\partial f^a}{\partial x^u} = 0 \quad (a,b=1,\ldots,\ldots,q\,;u=1,\ldots,m-2q).$$

Then a transversely holomorphic foliation on M is nothing but a structure of Γ^{tr}-manifold. Similarly, if $m = 2n$, a holomorphic foliation on M is given by a Γ-structure, where Γ is the pseudogroup of local holomorphic automorphisms $f(z^a, z^u) = (f^a, f^u)$ of $\mathbb{C}^q \times \mathbb{C}^{n-q}$ fulfilling

$$\frac{\partial f^a}{\partial z^u} = 0 \quad (a=1,\ldots,q\,;u=1,\ldots,n-q).$$

In the following we investigate holomorphic foliations on M that are close to a fixed holomorphic foliation $\mathscr{F}$ and that coincide with $\mathscr{F}$ as transversely holomorphic foliations.

1.1. Let $\mathscr{F}$ be a transversely holomorphic foliation of codimension q on M. Let L be the tangent bundle of $\mathscr{F}$ and $N = TM/L$ the normal bundle. The submersions f_i defining $\mathscr{F}$ induce an almost complex structure on N and thus a splitting of the complexified normal bundle in the usual way ${}^cN = N^{1,0} \oplus N^{0,1}$. The kernel F of the canonical projection ${}^cTM \to N^{1,0}$ is an involutive complex sub-bundle of ${}^cTM$ verifying $L = TM \cap F$ and ${}^cTM = F + \bar{F}$, where $\bar{F}$ denotes the complex conjugate of F. We will use the following version of the complex Frobenius theorem due to Nirenberg (cf. [21]).

THEOREM (Nirenberg). — *Let F be a complex sub-bundle of ${}^cTM$ of complex codimension q with ${}^cTM = F + \bar{F}$. Assume that F is involutive, then F is obtained from a uniquely determined transversely holomorphic foliation on M of codimension q as in the above discussion.*

Suppose now that $\mathscr{F}$ is holomorphic with respect to a given complex structure on M. If ${}^cTM = T^{1,0} \oplus T^{0,1}$ is the splitting of the complexified tangent bundle associated to this complex structure, then $T^{0,1}$ is a sub-bundle of F. From the above theorem one gets

PROPOSITION. — *There is a one-to-one correspondence between the set of holomorphic foliations on M of codimension q and the set of pairs $(T^{0,1}, F)$ of complex sub-bundles of ${}^cTM$ of codimensions n and q respectively (where $2n$ is the real dimension of M) and verifying*

(i) ${}^cTM = F + \bar{F} = T^{1,0} \oplus T^{0,1}$ (where $T^{1,0} = \overline{T^{0,1}}$)

(ii) F and $T^{0,1}$ are involutive

(iii) $T^{0,1} \subset F$.

Proof. — The bundle F defines a transversely holomorphic foliation $\mathscr{F}$ of codimension q and $T^{0,1}$ defines a complex structure ϕ on M (a transversely holomorphic foliation of codimension n is nothing but a complex structure on M). Now the inclusion $T^{0,1} \subset F$ implies that the submersions defining $\mathscr{F}$ are holomorphic with respect to ϕ. □.

From now on $\mathscr{F}$ will be a fixed codimension-q holomorphic foliation on a compact manifold M of complex dimension n and $(T^{0,1}, F)$ the pair of sub-bundles of ${}^cTM$ defining $\mathscr{F}$. The holomorphic foliations $\mathscr{F}_1$

which coincide with $\mathscr{F}$ as transversely holomorphic foliations are represented by pairs of bundles of the form $(T_1^{0,1}, F)$. We denote by $\mathrm{Fol}_f(\mathscr{F})$ the set of holomorphic foliations on M given by pairs of bundles of the form $(T_1^{0,1}, F)$ and which are close to $\mathscr{F}$ in the sense that $T_1^{0,1} \cap T^{1,0} = 0$.

Let $A^{0,k}(T)$ be the vector space of differential forms on M of type $(0,k)$ with values in $T^{1,0}$. If ${}^cTM = \overline{T_1^{0,1}} \oplus T_1^{0,1}$ is a splitting with $T_1^{0,1} \cap T^{1,0} = 0$, there is a unique element $\omega \in A^{0,1}(T)$ such that $T_1^{0,1}$ is the graph of $-\omega$; i.e., $T_1^{0,1} = (\mathrm{id}-\omega)(T^{0,1})$. We put $T_\omega^{0,1} = T_1^{0,1}$. It is well known that $T_\omega^{0,1}$ is involutive if and only if ω verifies the integrability equation

1.1.1 $$\bar{\partial}\omega - \frac{1}{2}[\omega,\omega] = 0.$$

An easy computation shows that $T_\omega^{0,1}$ is contained in F if and only if the vector 1-form ω belongs to the subspace $A^{0,1}(L)$ of $A^{0,1}(T)$ of those forms taking values only in $L^{1,0} = {}^cL \cap T^{1,0}$. Thus there is a natural injective map

1.1.2 $$a : \mathrm{Fol}_f(\mathscr{F}) \to A^{0,1}(L)$$

with image contained in the set of elements of $A^{0,1}(L)$ verifying equation 1.1.1.

Remark. — Let $\mathscr{F}'$ be an element of $\mathrm{Fol}_f(\mathscr{F})$ defined by a pair $(T_\omega^{0,1}, F)$. The bundle $T_\omega^{0,1}$ induces an almost complex structure on L defined by the splitting.

1.1.3 $${}^cL = \overline{L_\omega^{0,1}} \oplus L_\omega^{0,1}$$

where $L_\omega^{0,1} = T_\omega^{0,1} \cap {}^cL$. If the bundle morphism $\omega : T^{0,1} \to L^{1,0}$ vanishes on $L^{0,1}$ then 1.1.3 coincides with the splitting ${}^cL = \overline{L^{0,1}} \oplus L^{0,1}$ induced by $T^{0,1}$ and so any leaf in $\mathscr{F}'$ carries the same complex structure as the corresponding leaf in $\mathscr{F}$ (recall that $\mathscr{F}$ and $\mathscr{F}'$ coincide as transversely holomorphic foliations). However, the complex structures on M defined by $T^{0,1}$ and $T_\omega^{0,1}$ are different if $\omega \neq 0$. An example where this phenomenon occurs is exhibited in 4.3.

1.2. *Families of deformations.* Throughout this paper we will deal with non reduced analytic spaces. For a summary of the basic notions concerning analytic spaces we refer to [19].

Given an analytic space R we will denote by Γ_R^{tr} the pseudogroup of local differentiable automorphisms of $\mathbb{R} \times \mathbb{C}^q \times \mathbb{C}^{n-q}$ of the form $f(r,z^a,z^u) = (r,f^a,f^u)$ where f^a and f^u depend differentiably on the variables r, z^a, z^u, holomorphically on r, and such that

$$\frac{\partial f^a}{\partial \bar{z}^b} = \frac{\partial f^a}{\partial z^u} = \frac{\partial f^a}{\partial \bar{z}^u} = 0 \quad (a,b=1,\ldots,q\,;u=1,\ldots,n-q).$$

We denote by Γ_R the sub-pseudogroup of Γ_R^{tr} consisting of those elements of Γ_R^{tr} which depend holomorphically on r, z^a, z^u. Let Γ_R^f be the sub-pseudogroup of Γ_R consisting of those elements of Γ_R such that the f^a do not depend on r.

Let R be an analytic space with a distinguished point 0. A family of deformations of $\mathscr{F}$ as a transversely holomorphic foliation (briefly, a family of *tr*-deformations) parametrized by R is a topological space X with a proper projection $p: X \to R$ and a Γ_R^{tr}-structure on X such that, in the local coordinates (r,z^a,z^u) of this structure, the projection p is the canonical projection $(r,z^a,z^u) \to r$. This Γ_R^{tr}-structure induces on each fibre $M_r = p^{-1}(r)$ a Γ^{tr}-structure, that is, a transversely holomorphic foliation $\mathscr{F}_r$. We suppose that there is given a Γ^{tr}-isomorphism $i: (M,\mathscr{F}) \to (M_0,\mathscr{F}_0)$. It can be shown that all the fibres M_r in a neighbourhood of 0 are diffeomorphic.

In the same way, but using Γ_R instead of Γ_R^{tr}, we define families of holomorphic deformations (briefly, families of *h*-deformations) of $\mathscr{F}$ parametrized by R. These notions of families of *tr*-deformations and *h*-deformations are well known. We also define the notion of a family of deformations of the complex structure of M that preserve $\mathscr{F}$ as transversely holomorphic foliation (briefly, family of *f*-deformations) as a family of *h*-deformations, $p: X \to R$, in which the Γ_R-structure of X is induced by a Γ_R^f-structure.

Γ_R^{tr}-morphisms (resp. Γ_R-morphisms, Γ_R^f-morphisms) of families of *tr*-deformations (resp. *h*-deformations, *f*-deformations) are required to be compatible with the given isomorphisms $i: (M,\mathscr{F}) \to (M_0,\mathscr{F}_0)$.

We are only interested in the behaviour of families near the fibre M_0, that is, in « germs of families ». Thus, a family $p: X \to R$ will be identified with its restriction to any neighbourhood of 0 and it will be considered as parametrized by the germ $(R,0)$ of R at 0. For the sake of brevity we will denote the family $p: X \to R$ simply by X/R or X, the projection map p, the distinguished point $0 \in R$ and the isomorphism i being understood.

Notice that any family of f-deformations X/R is Γ_R^{tr}-trivial. Conversely, a family X/R of h-deformations which is Γ_R^{tr}-equivalent to the trivial family is, in fact, a family of f-deformations. This follows from the following result.

PROPOSITION. — *Let X/R and Y/R be families of tr-deformations and h-deformations respectively and let $h: X \to Y$ be a Γ_R^{tr}-isomorphism. There is a Γ_R-structure X^h on X such that*

(i) *the Γ_R^{tr}-structure* of *X is induced by X^h*

(ii) *$h: X^h \to Y$ is a Γ_R-isomorphism.*

Proof. — Let $\{(U_i, r, z_i^a, z_i^u)\}$ and $\{(V_i, r, w_i^a, w_i^u)\}$ be atlases defining, respectively, the Γ_R^{tr}-structure of X and the Γ_R-structure of Y. Assume $h(U_i) \subset V_i$. The atlas $\{(U_i, r, z_i^a, \zeta_i^u)\}$, where $\zeta_i^u = w_i^u \circ h$, defines a Γ_R-structure on X fulfilling the required conditions. □

Let Λ_R stand for any of the pseudogroups Γ_R^{tr}, Γ_R or Γ_R^f. Let X/R be a family of Λ_R-deformations of $\mathscr{F}$ and $\varphi : (R',0) \to (R,0)$ a morphism of germs of analytic spaces. The fibre product $X_\varphi = X \times_R R'$ inherits, in a natural way, a structure of a family of $\Lambda_{R'}$-deformations of $\mathscr{F}$. A family X/R of Λ_R-deformations is called versal if, for any family X'/R' of $\Lambda_{R'}$-deformations there is a morphism of germs of analytic spaces $\varphi : (R',0) \to (R,0)$ such that: (i) X'/R' and X_φ/R' are $\Lambda_{R'}$-isomorphic, and (ii) the tangent map $d_0\varphi$ of φ at 0 is unique. If the morphism φ, and not only its linear part, is unique, then X/R is called universal. If a versal family X/R exists it is unique up to isomorphisms (cf. [24], th. 1.7); then R is called the versal space.

1.3. *The Kodaira-Spencer map for families of f-deformations.* Let $\Theta_{\mathscr{F}}$ be the sheaf of germs of holomorphic vector fields on M which preserve the holomorphic foliation $\mathscr{F}$. In a local chart (z^a, z^u) of M, flat with respect to $\mathscr{F}$ (i.e. $\mathscr{F}$ is defined locally by the submersion $(z^a, z^u) \to (z^a)$), the sheaf $\Theta_{\mathscr{F}}$ can be described as the sheaf of germs of vector fields of the form

$$\xi = \xi^a(z^b)\frac{\partial}{\partial z^a} + \xi^u(z^b, z^v)\frac{\partial}{\partial z^u}$$

where ξ^a and ξ^u are holomorphic, ξ^a depends only on the coordinates z^b. Denote by $\Theta_{\mathscr{F}}^f$ the subsheaf of $\Theta_{\mathscr{F}}$ consisting of those elements of $\Theta_{\mathscr{F}}$ which are tangent to the leaves of $\mathscr{F}$ and let $\Theta_{\mathscr{F}}^{\mathrm{tr}}$ be the quotient sheaf

$\Theta_{\mathscr{F}}/\Theta^f_{\mathscr{F}}$. We obtain the exact sequence

1.3.1 $$0 \to \Theta^f_{\mathscr{F}} \to \Theta_{\mathscr{F}} \to \Theta^{\mathrm{tr}}_{\mathscr{F}} \to 0$$

which induces the long exact cohomology sequence

1.3.2 $$\ldots H^k(M,\Theta^f_{\mathscr{F}}) \xrightarrow{\alpha_k} H^k(M,\Theta_{\mathscr{F}}) \xrightarrow{\beta_k} H^k(M,\Theta^{\mathrm{tr}}_{\mathscr{F}}) \to H^{k+1}(M,\Theta^f_{\mathscr{F}}) \ldots$$

Notice that $\Theta^f_{\mathscr{F}}$ is a coherent analytic sheaf on the complex manifold M and thus $H^k(M,\Theta^f_{\mathscr{F}})$ is finite dimensional (recall that M is assumed to be compact). The Lie bracket of vector fields gives a bilinear map $[\,,]: \Theta^f_{\mathscr{F}} \times \Theta^f_{\mathscr{F}} \to \Theta^f_{\mathscr{F}}$ inducing a structure of graded Lie algebra on the cohomology groups $H^*(M,\Theta^f_{\mathscr{F}})$.

Let X/R be a family of f-deformations of $\mathscr{F}$. In a similar manner to that of tr-deformations or h-deformations (cf. : [10] or [17]), the family X/R induces a linear map

$$\rho_f \colon T_0R \to H^1(M,\Theta^f_{\mathscr{F}}),$$

which depends only on the Γ^f_R-isomorphism class of X/R, defined in the following way. Let $\{(U_i, r, \psi^a_i, \psi^u_i)\}$ be an atlas defining the Γ^f_R-structure of X. The coordinate changes will be of the form

1.3.3 $$\begin{cases} \psi^a_j = \phi^a_{ji}(\psi^b_i) \\ \psi^u_j = \phi^u_{ji}(r, \psi^b_i, \psi^u_i), \end{cases}$$

(notice that ϕ^a_{ji} does not depend on r). Given $\dfrac{\partial}{\partial r} \in T_0R$, $\rho_f\left(\dfrac{\partial}{\partial r}\right)$ is defined as the cohomology class of the cocycle associating to $M_0 \cap U_i \cap U_j$ the section of $\Theta^f_{\mathscr{F}}|(M_0 \cap U_i \cap U_j)$:

$$\theta_{ji} = \sum_{u=1}^{n-q} \left.\frac{\partial \phi^u_{ji}}{\partial r}\right|_{r=0} \left(\frac{\partial}{\partial \psi^u_j}\right).$$

This map ρ_f is called the Kodaira-Spencer map associated to the family of f-deformations X/R.

The chain rule shows that the Kodaira-Spencer maps ρ_f and ρ'_f associated to the families of f-deformations X/R and X_φ/R', where X_φ is the family induced by a morphism $\varphi : (R',0) \to (R,0)$, are related by the expression $\rho'_f = \rho_f \circ d_0\varphi$.

Assume that $\mathscr{F}$ admits a versal family X/R for f-deformations. A well known argument (cf. [24]) shows that the Kodaira-Spencer map ρ_f associated to this family is an isomorphism from T_0R onto $H^1(M,\Theta^f_{\mathscr{F}})$.

Remark. — The Kodaira-Spencer maps $\rho_h : T_0R \to H^1(M,\Theta_{\mathscr{F}})$, $\rho_{\mathrm{tr}} : T_0R \to H^1(M,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ associated to families X/R, Y/R of h-deformations and tr-deformations respectively can be defined in a similar way (cf. : [17], [10]). If ρ_f, ρ_h denote the Kodaira-Spencer maps associated to a family of f-deformations X/R when regarded as a family of f-deformations and h-deformations respectively, then one can check from the definitions that $\rho_h = \alpha_1 \circ \rho_f$ where $\alpha_1 : H^1(M,\Theta^f_{\mathscr{F}}) \to H^1(M,\Theta_{\mathscr{F}})$ is the map in 1.3.2.

1.4. *Existence of the versal space of families of f-deformations.* The existence of versal spaces (often called Kuranishi spaces) corresponding to the *tr*-deformations and h-deformations of the foliation $\mathscr{F}$ on the compact manifold M has been shown by Girbau, Haefliger and Sundararaman in [10]. In this paragraph we state the corresponding result for families of f-deformations. The proof presented here is an adaptation of the proof of Kuranishi's theorem given by Douady in [4].

THEOREM. — *Let $\mathscr{F}$ be a holomorphic foliation on a compact manifold M and let $\Theta^f_{\mathscr{F}}$ be the sheaf of germs of holomorphic vector fields on M tangent to $\mathscr{F}$.*

There is a germ of analytic space $(K^f,0)$ parametrizing a family Z^f/K^f of f-deformations of $\mathscr{F}$ which is versal with respect to f-deformations. More precisely, there is an open neighbourhood V of 0 in $H^1(M,\Theta^f_{\mathscr{F}})$ and an analytic map $\zeta_f : V \to H^2(M,\Theta^f_{\mathscr{F}})$ such that $(K^f,0)$ is isomorphic to the germ at 0 of $\zeta_f^{-1}(0)$. The jet of order two of ζ at 0 is the quadratic map $v \to [v, v]$.

If $H^0(M,\Theta^f_{\mathscr{F}}) = 0$ then $(K^f,0)$ is universal.

COROLLARY. — *If $H^2(M,\Theta^f_{\mathscr{F}}) = 0$, then the versal space of f-deformations of $\mathscr{F}$ is smooth. Indeed it is isomorphic to an open neighbourhood of 0 in $H^1(M,\Theta^f_{\mathscr{F}})$.*

For the proof of the theorem we will need the following lemma. Let X/R be a family of f-deformations and denote by $L(X)$ the disjoint union of the holomorphic vector bundles $L_r^{1,0}$ tangent to the foliations $\mathscr{F}_r$. $L(X)$ is endowed, in a natural way, with a structure of a holomorphic vector bundle over the complex space X.

LEMMA. — *Let γ_r denote the zero section of $L_r^{1,0}$. By restricting R to a small neighbourhood of 0 one can find a neighbourhood U of $\{\gamma_r(x); r \in R, x \in M\}$ in $L(X)$ and a smooth map $g : U \to M$ such that*

(i) $g(\gamma_r(x)) = x$.

(ii) *Let $g_{r,x}$ denote the restriction of g to $(L_r^{1,0})_x \cap U$. The differential of $g_{r,x}$ at $\gamma_r(x)$ is the identity of $(L_r^{1,0})_x$.*

(iii) *For any $r \in R$, $x \in M$, $g_{r,x}$ is a holomorphic isomorphism of $(L_r^{1,0})_x \cap U$ onto its image which is contained in the leaf of $\mathscr{F}_r$ through x.*

Proof. — Let $\mathscr{B}$ be the sheaf of germs at $\gamma_0(M)$ of local differentiable mappings of $L(X)$ into M verifying the conditions of the Lemma. Using local coordinates this sheaf is shown to be locally soft (« mou ») and hence also globally soft. Now the result follows from th. 3.3.1, p. 150, in [12]. □

Sketch of proof of the theorem. — Let $A^{0,k}(L)$ denote the space of differential forms on M of type $(0,k)$ with values in $L^{1,0}$. Let us consider the elliptic complex

$$A^{0,0}(L) \xrightarrow{\bar\partial} A^{0,1}(L) \xrightarrow{\bar\partial} A^{0,2}(L) \to \dots . \tag{1.4.1}$$

Choose a real analytic Hermitian metric on M and denote by ϑ the adjoint of $\bar\partial$ with respect to this metric. The Laplacian $\Delta_f = \bar\partial\vartheta + \vartheta\bar\partial$ is a real analytic elliptic operator. Set

$$N_f = \left\{\omega \in A^{0,1}(L);\ \vartheta\omega = 0 \text{ and } \vartheta\left(\bar\partial\omega - \frac{1}{2}[\omega,\omega]\right) = 0\right\}.$$

The space N_f is a finite dimensional submanifold whose tangent space at 0 is the space $H_f^1 \cong H^1(M, \Theta_{\mathscr{F}}^f)$ of harmonic elements of $A^{0,1}(L)$. It is easily checked that N_f can also be described as

$$N_f = \left\{\omega \in A^{0,1}(L)\,;\ \omega = H_f\omega + \frac{1}{2}\vartheta G_f[\omega,\omega]\right\}, \tag{1.4.2}$$

or

$$N_f = \left\{\omega \in A^{0,1}(L);\ \vartheta\left(\bar\partial\omega - \frac{1}{2}[\omega,\omega]\right) + \bar\partial\vartheta\omega = 0\right\},$$

where H_f denotes the orthogonal projection onto the space of harmonic elements and G_f is the Green operator. It follows from the last description of N_f that its elements are real analytic because they are solutions of an elliptic equation with real analytic coefficients.

The Kuranishi space $(K^f,0)$ will be the germ at 0 of the analytic subspace of N_f defined by

$$K^f = \left\{ \omega \in A^{0,1}(L);\ \vartheta\omega = 0 \text{ and } \bar{\partial}\omega - \frac{1}{2}[\omega,\omega] = 0 \right\}$$

or equivalently (cf. [18], p. 83),

1.4.3 $$K^f = \{\omega \in N_f;\ H_f[\omega,\omega] = 0\}.$$

Any element ω of K^f close to zero defines a real analytic subbundle $T_\omega^{0,1} = (\text{id-}\omega)T^{0,1}$ of ${}^cTM$ contained in F. It follows from the classical Frobenius integrability theorem in the real analytic case with complex parameters ω that $T_\omega^{0,1}$ defines a family Z^f/K^f of complex structures on M. Because of the inclusion $T_\omega^{0,1} \subset F$ this is in fact a family of f-deformations of $\mathscr{F}$.

Let $g : U \to M$, with U an open neighbourhood of the zero section of $L(Z^f)$, be the exponential map constructed in the above lemma. Let ${}^r\mathscr{D}^f$ be the family over K^f of diffeomorphisms h_ω of M of the form $x \to g(\xi(x))$, where ξ is a section of class C^r of $L_\omega^{1,0}$ contained in U. ${}^r\mathscr{D}^f$ is a Banach $\mathbb{C}$-analytic space « lisse » over K^f (cf. [4]) and there is a section $\sigma : K^f \to {}^{r+1}\mathscr{D}^f$ induced by the (differentiable) identification $K^f \times M = Z^f$.

Let ${}^rA^{0,k}(L)$ be the Banach space of differentiable forms on M of class C^r of type $(0,k)$ with values in $L^{1,0}$. Let $\theta : {}^rA^{0,1}(L) \to {}^{r-1}A^{0,2}(L)$ be the analytic map $\omega \to \bar{\partial}\omega - \frac{1}{2}[\omega,\omega]$ and denote by J the analytic subspace $\theta^{-1}(0)$ of ${}_rA_{0,1}(L)$.

For any diffeomorphism $h_\omega \in {}^{r+1}\mathscr{D}^f$, $h_\omega^*(\mathscr{F}_\omega)$ is a holomorphic foliation which, since condition (iii) in the above lemma is fulfilled by g, coincides with $\mathscr{F}_\omega$ as a transversely holomorphic foliation. So it is an element of $\text{Fol}_f(\mathscr{F})$ (cf. 1.1). We define a map

$$\mathscr{A} : {}^{r+1}\mathscr{D}^f \to J$$

by $\mathscr{A}(h_\omega) = a(h_\omega^*(\mathscr{F}_\omega))$ (cf. 1.1.2). $\mathscr{A}$ is a holomorphic map. A computation similar to that in [20], pp. 170-171 shows that the tangent map of $\mathscr{A}_0 : {}^{r+1}\mathscr{D}^f_0 \to J$ (the restriction of $\mathscr{A}$ to the fibre ${}^{r+1}\mathscr{D}^f_0$ of ${}^{r+1}\mathscr{D}^f$ over $0 \in K^f$) at the point $\sigma(0)$ coincides with the morphism $\bar{\partial} : {}^{r+1}A^{0,0}(L) \to {}^{r}A^{0,1}(L)$. Now we can adapt here the rest of the proof of Kuranishi's theorem given by Douady in [4].

2. A Description of the space K^f.

Let K and K^{tr} be the versal spaces corresponding to h-deformations and tr-deformations respectively of the holomorphic foliation $\mathscr{F}$ on the compact manifold M. Denote by Z/K and $Z^{\text{tr}}/K^{\text{tr}}$ the corresponding versal families. The fact that Z/K is in particular, a family of tr-deformations and the versality of K^{tr} give us a morphism

$$\pi : K \to K^{\text{tr}}$$

such that Z/K and Z^{tr}_{π}/K are Γ^{tr}_K-isomorphic. Recall that the tangent spaces to K and K^{tr} are $H^1(M,\Theta_{\mathscr{F}})$ and $H^1(M,\Theta^{\text{tr}}_{\mathscr{F}})$ respectively. Although the morphism π is not uniquely determined, its tangent map $d_0\pi$ is unique. It coincides with the morphism β_1:

$$H^1(M,\Theta_{\mathscr{F}}) \to H^1(M,\Theta^{\text{tr}}_{\mathscr{F}})$$

in 1.3.2.

We assume that π has been fixed once and for all and we denote by K^0 the fibre $\pi^{-1}(0)$ of π over 0. It follows from Prop. 1.2 that the restriction Z^0/K^0 of the versal family Z/K to K^0 is a family of f-deformations. In this section we establish the relation between K^f and K^0 by showing (th. 2.3) that K^f decomposes as a product of K^0 by a smooth factor Σ; i.e., $K^f \cong K^0 \times \Sigma$.

2.1. Let R be an analytic space with structural sheaf $\mathcal{O}_R$ and let 0 be a distinguished point of R. The correspondence that assigns the analytic $\mathbb{C}$-algebra $\mathcal{O}_{R,0}$ to the germ $(R,0)$ and to any morphism of germs $\varphi : (R,0) \to (R',0)$ the corresponding morphism of analytic $\mathbb{C}$-algebras $\varphi^* : \mathcal{O}_{R',0} \to \mathcal{O}_{R,0}$ is an antiequivalence between the category of germs of analytic spaces and the category of analytic $\mathbb{C}$-algebras.

We recall also that, for any germ $(R,0)$ there is an embedding of germs $i : R \to \mathbb{C}^k$ (here, $\mathbb{C}^k$ stands for the germ of $\mathbb{C}^k$ at the origin) with $k = \dim_{\mathbb{C}} T_0R$. Such a morphism is called a minimal embedding. In this situation we will identify R with its image in $\mathbb{C}^k$.

In the following proposition we obtain a sufficient condition for a morphism of germs of analytic spaces to be an isomorphism.

PROPOSITION. — *Let $\phi : S \times T \to R$ and $\varphi : R \to T$ be morphisms of analytic spaces such that.*

(i) *the tangent map $d_0\phi$ is an isomorphism,*

(ii) *the diagram*

$$\begin{tikzcd} S \times T \arrow[rr, "\phi"] \arrow[dr, "p_2"'] & & R \arrow[dl, "\varphi"] \\ & T & \end{tikzcd}$$

is commutative,

(iii) *ϕ maps $S \equiv S \times \{0\}$ isomorphically onto $\varphi^{-1}(0)$.*

Then ϕ is an isomorphism.

Note. — If S is smooth then condition (iii) is implied by (i) and (ii) (cf. [24], prop. 1.5).

Proof. — By choosing minimal embeddings $S \subset \mathbb{C}^k$, $T \subset \mathbb{C}^\ell$ and by identifying $S \times T$ with its image in R through the map ϕ, we can also assume that there is a minimal embedding $R \subset \mathbb{C}^k \times \mathbb{C}^\ell$ such that

(*a*) $S \times T$ is an analytic subspace of R

(*b*) The map $\varphi : R \to T$ is the restriction to R of the projection $\mathbb{C}^k \times \mathbb{C}^\ell \to \mathbb{C}^\ell$

(*c*) $S = R \cap (\mathbb{C}^k \times \{0\})$.

Let $x = (x_i)$, $y = (y_j)$ denote the linear coordinates of $\mathbb{C}^k$ and $\mathbb{C}^\ell$ and set $\mathcal{O}_{S,0} = \mathbb{C}\{x\}/I_S$, $\mathcal{O}_{T,0} = \mathbb{C}\{y\}/I_T$ and $\mathcal{O}_{R,0} = \mathbb{C}\{x,y\}/I_R$, with $I_S = (f_\alpha(x))$, $I_T = (g_\mu(y))$ and $I_R = (F_\varepsilon(x,y))$.

It follows from (*b*) that the ideal of $\mathbb{C}\{x,y\}$ generated by $(g_\mu(y))$ is contained in I_R. So I_R can be written in the form $I_R = (F_\varepsilon(x,y) = g_\mu(y), h_\delta(x,y))$ with $\delta = 1, \dots, p$. The inclusion $S \times T \subset R$ is equivalent to $I_R \subset (g_\mu(y), f_\alpha(x))$ and thus we can write

$$h_\delta(x,y) = a_\delta^\alpha(x,y)f_\alpha(x) + b_\delta^\mu(x,y)g_\mu(y).$$

By a suitable choice of the functions h_δ we can in fact assume that

2.1.1 $$h_\delta(x,y) = a_\delta^\alpha(x,y)f_\alpha(x).$$

Now condition (c) can be written

$$I_S = (f_\alpha) = (h_\delta(x,0)) = (a_\delta^\alpha(x,0)\cdot f_\alpha(x)).$$

One sees from 2.1.1 that there is an inclusion of ideals of $\mathbb{C}\{x,y\}$:

$$(f_\alpha(x)) \subset (h_\delta(x,y)) + (y_\beta)\cdot(f_\alpha(x)).$$

The Nakayama's lemma implies now $(f_\alpha(x)) \subset (h_\delta(x,y))$. Hence we get $I_R = (f_\alpha, g_\mu)$ and so $R = S \times T$. □

2.2. *The morphism* $\chi : K^f \to K^0$. Let $(R,0)$ be the germ of an analytic space R, $\mathcal{O}_{R,0}$ the associated analytic $\mathbb{C}$-algebra and m_R the maximal ideal of $\mathcal{O}_{R,0}$. The k^{th} infinitesimal neighbourhood of 0 in R is the analytic subspace $R^{(k)}$ of R defined by the analytic algebra $\mathcal{O}_{R,0}/m_R^{k+1}$. If X/R is a family of deformations, its restriction to the subspace $R^{(k)}$ will be denoted by $X^{(k)}/R^{(k)}$.

Following Wavrik (cf. [23]) we describe a basis of the finite dimensional $\mathbb{C}$-algebra $\mathcal{O}_{R,0}/m_R^{k+1}$ in the following way. Fix a minimal embedding of $(R,0)$ in $(\mathbb{C}^m,0)$ and let $r = (r_1, \dots, r_m)$ be the linear coordinates of $\mathbb{C}^m$ (or their restriction to R). The ideal m_R^ℓ is generated by the r^J, where J denotes a multiindex with $|J| = \ell$. Choose, for each ℓ, a set $\mathcal{J}_\ell$ of multiindices with $|J| = \ell$ such that the classes of r^J $(J \in \mathcal{J}_\ell)$ provide a basis of $m_R^\ell/m_R^{\ell+1}$. Then $\{r^J; J \in \mathcal{J}_\ell, \ell = 0, \dots, k\}$ is a basis of $\mathcal{O}_{R,0}/m_R^{k+1}$.

This basis of $\mathcal{O}_{R,0}/m_R^{k+1}$ can be used to represent sections of sheaves over $X^{(k)}$ as a generalization of Taylor series expansion. For instance, if V is an open set in M, then every holomorphic function f on $V \times R^{(k)}$ may be uniquely written in the form $f = \sum_{\ell=0}^{k} \sum_{J \in \mathcal{J}_\ell} f_J r^J$, where f_J are holomorphic functions on V. This «Taylor polynomial» representation reduces many proofs to simple calculations.

Proposition. — *Let* $\varphi : R \to K^{\text{tr}}$ *be a morphism of the germ* R *into the versal family* K^{tr}. If Z_φ^{tr}/R *is* Γ_R^{tr}*-isomorphic to the trivial family then* φ *is the zero map.*

Note. — The corresponding statements for families of h-deformations or f-deformations are also true.

Proof. — Let Y/R be the trivial family and denote by $G^{(k)}$ and $G_f^{(k)}$ the sheaves of germs of local $\Gamma_{R^{(k)}}$-automorphisms and

$\Gamma^f_{R^{(k)}}$-automorphisms respectively of $Y^{(k)}/R^{(k)}$. $G_f^{(k)}$ is a sheaf of normal subgroups of $G^{(k)}$. Set $G_{\text{tr}}^{(k)} = G^{(k)}/G_f^{(k)}$. By virtue of th. 2.2 in [23] it suffices to prove that any element of $H^0(M, G_{\text{tr}}^{(k-1)})$ can be extended to an element of $H^0(M, G_{\text{tr}}^{(k)})$.

Let $\{(U_i, z_i^a, z_i^u)\}$ be an atlas of M flat with respect to the holomorphic foliation $\mathscr{F}$. The coordinate changes F_{ji} which define the trivial family Y/R can be written in the form $F_{ji}(r, z_i^b, z_i^v) = (r, z_j^a, z_j^u)$ with $z_j^a = g_{ji}^a(z_i^b)$ and $z_j^u = g_{ji}^u(z_i^b, z_i^v)$. Let $\mathscr{V}_{\text{tr}}^{(k)}$ denote the sheaf over M of germs of vertical, transversely holomorphic vector fields on $Y^{(k)}/R^{(k)}$ vanishing on $Y^{(0)} = M$. The sections of $\mathscr{V}_{\text{tr}}^{(k)}$ over an open set $V \subset M$ are described by morphisms $\xi_i : (V \cap U_i) \times R^{(k)} \to \mathbb{C}^q$ such that its restriction to $(V \cap U_i) \times R^{(0)} \cong V \cap U_i$ vanishes and verifying

$$\xi_j^a \circ F_{ji} = \left(\frac{\partial g_{ji}^a}{\partial z_i^b}\right) \xi_i^b \quad (a, b = 1, \ldots, q).$$

The sheafs $G_{\text{tr}}^{(k)}$ and $\mathscr{V}_{\text{tr}}^{(k)}$ are isomorphic (cf. [23], th. 3.2). In particular $H^0(M, G_{\text{tr}}^{(k)}) = H(M, \mathscr{V}_{\text{tr}}^{(k)})$.

Now, by means of the « Taylor polynomial » representation one easily checks that any global section of $\mathscr{V}_{\text{tr}}^{(k-1)}$ can be extended to a global section of $\mathscr{V}_{\text{tr}}^{(k)}$. □

The versal family of f-deformations Z^f/K^f (cf. 1.4) is in particular a family of h-deformations. Thus the versality of K induces a morphism of germs $\chi : K^f \to K$ such that Z^f/K^f and Z_χ/K^f are Γ_{K^f}-isomorphic. The above proposition asserts that $\pi \circ \chi : K^f \to K^{\text{tr}}$ is the zero map. Hence, the universal property of the fibre $K^0 = \pi^{-1}(0)$ implies that the morphism χ factors through the subspace K^0 giving the following commutative diagram:

$$\begin{tikzcd}
 & K^0 \arrow[d, hook] \\
K^f \arrow[ur, "\chi"] \arrow[r, "\chi"] \arrow[dr, "0"'] & K \arrow[d, "\pi"] \\
 & K^{\text{tr}}
\end{tikzcd}$$

2.3. *Existence of an isomorphism* $K^f \cong K^0 \times \Sigma$. Let us choose a splitting of the exact sequence of bundles

$$0 \to F \to {}^cTM \to N^{1,0} \to 0$$

(see 1.1 for the definition of F and $N^{1,0}$). This gives a direct sum decomposition $T^{1,0} = L^{1,0} \oplus N^{1,0}$. Let $\{(U_i, z_i^a, z_i^u)\}$ be an atlas of M, flat with respect to $\mathscr{F}$. In a coordinate neighbourhood U_i, the vector bundle $N^{1,0}$ will be generated by vector fields

$$Z_{i,a} = \frac{\partial}{\partial z_i^a} + t_{i,a}^u \frac{\partial}{\partial z_i^u} \quad (a=1,\ldots,q,\ u=1,\ldots,n-q)$$

where $t_{i,a}^u$ are differentiable functions on U_i such that, on $U_i \cap U_j$

$$t_{j,a}^u = \frac{\partial z_i^v}{\partial z_j^a} \cdot \frac{\partial z_j^u}{\partial z_i^v} + t_{i,b}^v \frac{\partial z_i^b}{\partial z_j^a} \cdot \frac{\partial z_j^u}{\partial t_i^v}.$$

One easily checks that, also on $U_i \cap U_j$,

2.3.1 $$Z_{j,a} = \frac{\partial z_i^b}{\partial z_j^a} \cdot Z_{i,b}.$$

An element $\xi \in H^0(M, \Theta_{\mathscr{F}}^{\mathrm{tr}})$ will be locally written in the form

2.3.2 $$\xi = \xi_i^a \left[\frac{\partial}{\partial z_i^a}\right],$$

where $\xi_i^a = \xi_i^a(z_i^b)$ are holomorphic functions depending only on the coordinates z_i^b and verifying $\xi_j^a = \dfrac{\partial z_j^a}{\partial z_i^b} \xi_i^b$.

It follows from 2.3.1 that the expression

2.3.3 $$\xi' = \xi_i^a \cdot Z_{i,a} = \xi_i^a \frac{\partial}{\partial z_i^a} + \xi_i^a t_{i,a}^u \frac{\partial}{\partial z_i^u}$$

defines a global differentiable vector field on M which preserves $\mathscr{F}$ as a transversely holomorphic foliation.

Let $\delta : H^0(M, \Theta_{\mathscr{F}}^{\mathrm{tr}}) \to H^1(M, \Theta_{\mathscr{F}}^{f})$ be the connecting homomorphism associated to the exact sequence 1.3.1. If $\xi \in H^0(M, \Theta_{\mathscr{F}}^{\mathrm{tr}})$ is given locally by 2.3.2, then $\delta(\xi)$ is the cohomology class of the cocycle $\{\varepsilon_{ji}\}$ defined by

2.3.4 $$\varepsilon_{ji} = \xi_i^b \frac{\partial z_j^u}{\partial z_i^b} \frac{\partial}{\partial z_j^u}.$$

THEOREM. — *Let Σ be the germ at zero of a linear subspace* $\tilde{\Sigma}$ of $H^0(M,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ *complementary to* $\operatorname{Ker}\delta$. *There is an isomorphism of germs of analytic spaces $\kappa : K^0 \times \Sigma \to K^f$ making commutative the following diagram :*

$$\begin{array}{ccc} K^0 \times \Sigma & \xrightarrow{\kappa} & K^f \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle \chi} & \\ & K^0 & \end{array} \tag{2.3.5}$$

In particular, if α_1: $H^1(M,\Theta^f_{\mathscr{F}}) \to H^1(M,\Theta_{\mathscr{F}})$ is injective, then $K^f \cong K^0$.

Proof. — Let us fix a basis $\eta_1, \cdots, \eta_m$ of $\tilde{\Sigma}$ and identify $\mathbb{C}^m \cong \tilde{\Sigma}$ through the isomorphism $s = (s_1, \cdots, s_m) \to \sum_{\mu=1}^{m} s_\mu \eta_\mu$. Let η'_μ denote the vector field on M obtained from η_μ as in 2.3.3 by means of a fixed splitting $T^{1,0} = L^{1,0} \oplus N^{1,0}$. For any $s \in \mathbb{C}^m$, $\zeta^s = \sum_\mu (s_\mu . \eta'_\mu + \bar{s}_\mu . \bar{\eta}'_\mu)$ is a real vector field whose exponential, $h^s = \exp \zeta^s$, is a diffeomorphism preserving $\mathscr{F}$ as a transversely holomorphic foliation.

The family Z^0/K^0 is trivial as family of *tr*-deformations. By means of a $\Gamma^{\mathrm{tr}}_{K^0}$-isomorphism $\varphi : Z^0 \to M \times K^0$ we can extend h^s to a $\Gamma^{\mathrm{tr}}_{K^0}$-isomorphism $\tilde{h}^s$ of Z^0; that is, $\tilde{h}^s$ is the automorphism of the family of *tr*-deformations Z^0/K^0 making commutative the diagram

$$\begin{array}{ccc} M \times K^0 & \xrightarrow{h^s \times \mathrm{id}} & M \times K^0 \\ \varphi \downarrow & & \downarrow \varphi \\ Z^0 & \xrightarrow{\tilde{h}^s} & Z^0 \end{array}$$

The family of $\Gamma^{\mathrm{tr}}_{K^0}$-isomorphism $\{\tilde{h}^s\}_{s \in \mathbb{C}^m}$ induces now a $\Gamma^{\mathrm{tr}}_{K^0 \times \Sigma}$-isomorphism, $h : Z^0_{p_1} \to Z^0_{p_1}$, defined by the condition that h coincides with $\tilde{h}^s$ on $K^0 \times \{s\}$. Notice that h is the identity over $K^0 \times \{0\}$.

Let $\{(\tilde{U}_i, r, \psi^a_i, \psi^u_i)$ be an atlas defining the $\Gamma^f_{K^0}$-structure of Z^0 with coordinate changes of the form 1.3.3 and assume $\tilde{U}_i \cap M_0 = U_i$ and $\psi^a_i | U_i = z^a_i, \psi^u_i | U_i = z^u_i$. With respect to these coordinates the isomorphism h will be written in the form

$$(r, s, \psi^a_i, \psi^u_i) \;\to\; (r, s, h^a_i(s, \psi^b_i), h^u_i(r, s, \psi^b_i, \psi^v_i))$$

(where h_i^u are only differentiable functions) and the $\Gamma^f_{K^0\times\Sigma}$-structure on $Z^0_{p_1}$, that we denote $(Z^0_{p_1})^h$, constructed in Prop. 1.2, is given by the local charts $(r,s,\zeta_i^a=\psi_i^a, \zeta_i^u=h_i^u(r,s,\psi_i^b,\psi_i^v))$.

Let us compute the Kodaira-Spencer map ρ_f associated to the family of f-deformations $(Z^0_{p_1})^h/K^0\times\Sigma$. The restriction of this family to $K^0\times\{0\}$ coincides with Z^0/K^0. It follows from this and Remark 1.3 that $\rho_f|T_0(K^0\times\{0\})$ is an isomorphism from $T_0K^0\cong \operatorname{Im}\alpha_1$ onto a subspace of $H^1(M,\Theta^f_{\mathscr{F}})$ complementary to $\operatorname{Ker}\alpha_1$.

Now let $\dfrac{\partial}{\partial s_\mu}$ be an element of $T_0(\{0\}\times\Sigma)$. Its image by ρ_f is the class in $H^1(M,\Theta^f_{\mathscr{F}})$ of the cocycle $\{\theta_{ji}\}$ given by

$$\theta_{ji}=\dot\zeta_j^u\frac{\partial}{\partial\zeta_j^u}-\dot\zeta_i^v\frac{\partial}{\partial\zeta_i^v}=\left(\dot\zeta_j^u-\dot\zeta_i^\alpha\frac{\partial\zeta_j^u}{\partial\zeta_i^\alpha}\right)\frac{\partial}{\partial\zeta_j^u}$$

where $\alpha\in\{a,u\}$ and $\dot\zeta_i^\alpha=\dfrac{\partial}{\partial s_\mu}\zeta_i^\alpha(0,s,z)|_{s=0}$ (cf. [20], pp. 148-149). In our case, $\dot\zeta_i^a=0$, $\dot\zeta_i^u=\eta'^u_{\mu i}=\eta^a_{\mu i}t^u_{i,a}$ and $\dfrac{\partial\zeta_j^u}{\partial\zeta_i^v}=\dfrac{\partial\psi_j^u}{\partial\psi_i^v}h^0=\dfrac{\partial z_j^u}{\partial z_i^v}\cdot$ Thus

$$\begin{aligned}\theta_{ji}&=\left(\eta^a_{\mu j}t^u_{j,a}-\eta^b_{\mu i}\cdot t^v_{i,b}\cdot\frac{\partial z_j^u}{\partial z_i^v}\right)\frac{\partial}{\partial z_j^u}\\&=\left[\eta^c_{\mu i}\cdot\frac{\partial z_j^a}{\partial z_i^c}\cdot\left(\frac{\partial z_i^v}{\partial z_j^a}\frac{\partial z_j^u}{\partial z_i^v}+t^v_{ib}\frac{\partial z_i^b}{\partial z_j^a}\frac{\partial z_j^u}{\partial z_i^v}\right)-\eta^b_{\mu i}t^v_{i,b}\frac{\partial z_j^u}{\partial z_i^v}\right]\frac{\partial}{\partial z_j^u}\\&=-\eta^b_{\mu i}\frac{\partial z_j^u}{\partial z_i^b}\frac{\partial}{\partial z_j^u}.\end{aligned}$$

Hence, $\rho_f\left(\dfrac{\partial}{\partial s_\mu}\right)=-\delta(\eta_\mu)$ (cf. 2.3.4) and we conclude that ρ_f is an isomorphism. Thus the morphism $\kappa':K^0\times\Sigma\to K^f$ given by the versal property of K^f is an immersion (its tangent map is an isomorphism).

Let $\iota: K^0\to K^0\times\Sigma$ be the canonical inclusion and set $\psi=\chi\circ\kappa'\circ\iota$. Since the families Z^0/K^0 and Z^0_ψ/K^0 are $\Gamma^h_{K^0}$-isomorphic one has $d_0\psi=\mathrm{id}$ and thus ψ is an automorphism of K^0 (cf. [24], prop. 1.4). Clearly $\kappa=\kappa'\circ(\psi\times\mathrm{id}_\Sigma)$ is an immersion which makes commutative the diagram 2.3.5. Now prop. 2.1 implies that κ is in fact an isomorphism. □

Example. — We construct here a holomorphic foliation $\mathscr{F}$ on a compact manifold W for which the space Σ in the above theorem does not reduce to a point.

Denote $\tilde{U} = \mathbb{C} \times (\mathbb{C}^2 - \{0\})$ and let $\alpha \in \mathbb{C}$, with $0 < |\alpha| < 1$, be fixed. Let $h : \tilde{U} \to \tilde{U}$ be the holomorphic isomorphism defined by

$$h(u, z_1, z_2) = (u, \alpha z_1 + u z_2, \alpha z_2)$$

and set $U = \tilde{U}/G$ where $G = \{h^n | n \in \mathbb{Z}\}$. We get a fibre space $\pi_{\mathcal{U}} : U \to \mathbb{C}$ whose fibres $H_u (u \in \mathbb{C})$ are Hopf surfaces. Notice that $H_u \cong H_{u'}$ if u, $u' \in \mathbb{C}^*$ and $H_0 \not\cong H_u$ if $u \neq 0$ (cf. [25]). Let U' be another copy of U and glue them together in the following way: the map

$$\begin{aligned} \mathbb{C}^* \times (\mathbb{C}^2 - \{0\}) &\to \mathbb{C}^* \times (\mathbb{C}^2 - \{0\}) \\ (u, z_1, z_2) &\to (1/u, a z_1 + b z_2, a u^2 z_2) \end{aligned}$$

where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, commutes with the action of G and thus induces a holomorphic isomorphism $\pi_U^{-1}(\mathbb{C}^*) \to \pi_{U'}^{-1}(\mathbb{C}^*)$ commuting with the projections π_U and $\pi_{U'}$. By means of this identification we obtain a compact manifold W endowed with a canonical projection $p : W \to \mathbb{C}P^1$.

Let $\mathcal{F}$ denote the holomorphic foliation on W whose leaves are the fibres of p. Using the description of the holomorphic vector fields on Hopf surfaces given in [25] one easily checks that $H^0(W, \Theta^f_{\mathcal{F}}) \cong \mathbb{C}^4$, $H^0(W, \Theta_{\mathcal{F}}) \cong \mathbb{C}^4$ and $H^0(W, \Theta^{tr}_{\mathcal{F}}) \cong \mathbb{C}^3$. Hence α_0 is an isomorphism, β_0 is the nul map and Σ is the germ at zero of $H^0(W, \Theta^{tr}_{\mathcal{F}}) \cong \mathbb{C}^3$.

3. Sufficient conditions for the versal map $\pi : K \to K^{tr}$ to be an epimorphism.

The versal map $\pi : K \to K^{tr}$ is not an epimorphism in general (cf. [15]). In this section we investigate the image of π and give sufficient conditions for it to be the whole space K^{tr}. We also show that, under suitable cohomological assumptions, the versal space K decomposes as the product of K^{tr} by the fibre $K^0 = \pi^{-1}(0)$; that is, $K \cong K^0 \times K^{tr}$.

3.1. *Extensions of families of deformations.* Let R be a germ of an analytic space and denote by Y/R the trivial family of h-deformations of $\mathcal{F}$ parametrized by R. As in the proof of Prop. 2.2, let $G^{(k)}$ (resp. $G^{(k)}_{tr}$) denote the sheaf of germs of local $\Gamma_{R^{(k)}}$-automorphisms (resp. $\Gamma^f_{R^{(k)}}$-automorphisms) of $Y^{(k)}/R^{(k)}$ and set $G^{(k)}_{tr} = G^{(k)}/G^{(k)}_f$. (Notice that the sheaf $G^{(k)}_{tr}$ is also a quotient of the sheaf of germs of local $\Gamma^{tr}_{R^{(k)}}$-automorphisms of $Y^{(k)}/R^{(k)}$.) So we have an exact sequence of sheaves of (non Abelian) groups over M

3.1.1 $$1 \to G_f^{(k)} \xrightarrow{\sigma} G^{(k)} \xrightarrow{\tau} G_{\mathrm{tr}}^{(k)} \to 1 .$$

The morphism $P: G^{(k)} \to G^{(k-1)}$ which maps any element of $G^{(k)}$ to its restriction to $Y^{(k-1)}/R^{(k-1)}$ is clearly an epimorphism. If we denote by $G^{[k]}$ the kernel of P we obtain the exact sequence

3.1.2 $$1 \to G^{[k]} \xrightarrow{N} G^{(k)} \xrightarrow{P} G^{(k-1)} \to 1 .$$

In a similar way we denote by $G_f^{[k]}$ and $G_{\mathrm{tr}}^{[k]}$ the respective kernels of the restriction epimorphisms $P: G_f^{(k)} \to G_f^{(k-1)}$ and $P: G_{\mathrm{tr}}^{(k)} \to G_{\mathrm{tr}}^{(k-1)}$.

Let m_R be the maximal ideal of $\mathcal{O}_{R,0}$ and set $m_R^{[k]} = m_R^k/m_R^{k+1}$. By means of the « Taylor polynomial » representation one can show

PROPOSITION. – *$G^{[k]}$, $G_f^{[k]}$ and $G_{\mathrm{tr}}^{[k]}$ are sheaves of Abelian groups contained in the center of $G^{(k)}$.* $G_f^{(k)}$ and $G_{\mathrm{tr}}^{(k)}$ *respectively. Furthermore, there are natural isomorphisms*

3.1.3 $$\begin{aligned} G^{[k]} &\cong \Theta_{\mathscr{F}} \otimes m_R^{[k]} \\ G_f^{[k]} &\cong \Theta_{\mathscr{F}}^f \otimes m_R^{[k]} \\ G_{\mathrm{tr}}^{[k]} &\cong \Theta_{\mathscr{F}}^{\mathrm{tr}} \otimes m_R^{[k]} . \end{aligned}$$

The morphisms of sheaves σ and τ in 3.1.1 commute with the restriction morphism P. So they induce morphisms $\sigma : G_f^{[k]} \to G^{[k]}$ and $\tau : G^{[k]} \to G_{\mathrm{tr}}^{[k]}$ which coincide, through the identifications 3.1.3, with $\alpha \otimes 1 :\ \Theta_{\mathscr{F}}^f \otimes m_R^{[k]} \to \Theta_{\mathscr{F}} \otimes m_R^{[k]}$ and $\beta \otimes 1 :\ \Theta_{\mathscr{F}} \otimes m_R^{[k]} \to \Theta_{\mathscr{F}}^{\mathrm{tr}} \otimes m_R^{[k]}$. (See 1.3.1 for the definition of α and β.)

From 3.1.1, 3.1.2 and the exact sequences analogous to 3.1.2 which correspond to $G_f^{(k)}$ and $G_{\mathrm{tr}}^{(k)}$ one gets the following commutative and exact diagram of cohomology sets (cf. [9])

3.1.4 $$\begin{array}{ccccccc} 0\to H^1(M,\Theta_{\mathscr{F}}^f)\otimes m_R^{[k]} & \xrightarrow{N} & H^1(M,G_f^{(k)}) & \xrightarrow{P} & H^1(M,G_f^{(k-1)}) & \xrightarrow{Q} & H^2(M,\Theta_{\mathscr{F}}^f)\otimes m_R^{[k]} \\ \alpha_1\otimes 1 \downarrow & & \sigma\downarrow & & \sigma\downarrow & & \alpha_2\otimes 1\downarrow \\ 0\to H^1(M,\Theta_{\mathscr{F}})\otimes m_R^{[k]} & \xrightarrow{N} & H^1(M,G^{(k)}) & \xrightarrow{P} & H^1(M,G^{(k-1)}) & \xrightarrow{Q} & H^2(M,\Theta_{\mathscr{F}})\otimes m_R^{[k]} \\ \beta_1\otimes 1 \downarrow & & \tau\downarrow & & \tau\downarrow & & \beta_2\otimes 1\downarrow \\ 0\to H^1(M,\Theta_{\mathscr{F}}^{\mathrm{tr}})\otimes m_R^{[k]} & \xrightarrow{N} & H^1(M,G_{\mathrm{tr}}^{(k)}) & \xrightarrow{P} & H^1(M,G_{\mathrm{tr}}^{(k-1)}) & \xrightarrow{Q} & H^2(M,\Theta_{\mathscr{F}}^{\mathrm{tr}})\otimes m_R^{[k]} \end{array}$$

where Q is the connecting map. The injectivity of N can be shown by the same argument as in the proof of Prop. 2.2.

The set of isomorphism classes of families of h-deformations (resp. f-deformations, tr-deformations) parametrized by $R^{(k)}$ can be identified, in a natural way, with the cohomology set $H^1(\mathrm{M},G^{(k)})$ (resp. $H^1(M,G_f^{(k)})$, $H^1(M,G_{\mathrm{tr}}^{(k)})$); the class of the trivial family corresponding to the distinguished point in the cohomology set. If a family of deformations $X^{(k-1)}/R^{(k-1)}$ is represented by a cohomology class γ, then $Q(\gamma)$ is the obstruction of $X^{(k-1)}/R^{(k-1)}$ to be extended to a family over $R^{(k)}$.

Notice that if a family $X^{(k-1)}/R^{(k-1)}$ of h-deformations can be extended as a family of tr-deformations, then their obstruction to be extended as a family of h-deformations lies in $H^2(M,\Theta_{\mathscr{F}}^f) \otimes m_R^{[k]}$.

3.2. *A Theorem of decomposition.* In this paragraph we proof that, if $H^2(M,\Theta_{\mathscr{F}}^f) = 0$ then K decomposes as the product $K^0 \times K^{\mathrm{tr}}$. This implies when $\alpha_1 : H^1(M,\Theta_{\mathscr{F}}^f) \to H^1(M,\Theta_{\mathscr{F}})$ is injective that $K \cong K^f \times K^{\mathrm{tr}}$.

PROPOSITION. — *Assume that the versal space K is smooth (for instance if $H^2(M,\Theta_{\mathscr{F}})=0$) and that β_1 is surjective, then K^{tr} and K^0 are smooth (and so is K^f) and $K \cong K^0 \times K^{\mathrm{tr}}$.*

Proof. — It follows from prop. 1.5 in [24] and the implicit function theorem. □

THEOREM. — *Assume that $H^2(M,\Theta_{\mathscr{F}}^f) = 0$. Then $K^0 = \pi^{-1}(0)$ is smooth (and so is K^f) and there is an isomorphism $\phi : K^0 \times K^{\mathrm{tr}} \to K$ making commutative the following diagram*

3.2.1
$$\begin{tikzcd} K^0 \times K^{\mathrm{tr}} \arrow[rr] \arrow[dr, "p_2"'] & & K \arrow[dl, "\pi"] \\ & K^{\mathrm{tr}} & \end{tikzcd}$$

Proof. — Let S denote the germ at 0 of the vector space $\ker d_0\pi = \ker \beta_1 \subset H^1(M,\Theta_{\mathscr{F}})$. We want to construct, for any $k \in \mathbb{N}$, a morphism of germs $\phi^{(k)} : (S \times K^{\mathrm{tr}})^{(k)} \to K^{(k)}$ making commutative the diagram

3.2.2
$$\begin{tikzcd} (S \times K^{\mathrm{tr}})^{(k)} \arrow[rr, "\phi^{(k)}"] \arrow[dr, "p_2^{(k)}"'] & & K^{(k)} \arrow[dl, "\pi^{(k)}"] \\ & K^{\mathrm{tr}(k)} & \end{tikzcd}$$

and in such a way that $\phi^{(1)}$ is an isomorphism and $\phi^{(k)}$ is an extension of $\phi^{(k-1)}$ for any k. For $k = 1$, we simply define

$$\phi^{(1)}: (S \times K^{\mathrm{tr}})^{(1)} = \ker \beta_1 \times \operatorname{Im} \beta_1 \to H^1(M, \Theta_{\mathscr{F}}) = K^{(1)}$$

to be the product $i \times \varepsilon$ where $i : \ker \beta_1 \to H^1(M, \Theta_{\mathscr{F}})$ is the canonical inclusion and $\varepsilon : \operatorname{Im} \beta_1 \to H^1(M, \Theta_{\mathscr{F}})$ is a section of β_1.

Assume that we have defined $\phi^{(1)}, \cdots, \phi^{(k-1)}$ and let us construct $\phi^{(k)}$. We will use the sheaves defined in paragraph 3.1 for $R = S \times K^{\mathrm{tr}}$.

Let $Z^{\mathrm{tr}}/K^{\mathrm{tr}}$ be the versal family and $Z^{\mathrm{tr}}_{p_2}/S \times K^{\mathrm{tr}}$ the family induced by the projection $p_2 : S \times K^{\mathrm{tr}} \to K^{\mathrm{tr}}$. For any $m \in \mathbb{N}$, the restriction of this last family to $(S \times K^{\mathrm{tr}})^{(m)}$ determines a cohomology class $\zeta^{(m)} \in H^1(M, G^{(m)}_{\mathrm{tr}})$.

Let $\xi^{(k-1)} \in H^1(M, G^{(k-1)}_{\mathrm{tr}})$ be the cohomology class associated to the family of h-deformations over $(S \times K^{\mathrm{tr}})^{(k-1)}$ obtained from Z/K by pull-back of the morphism $\phi^{(k-1)}$. The commutativity of 3.2.2 for $k - 1$ implies $\tau(\xi^{(k-1)}) = \zeta^{(k-1)}$.

Since $\zeta^{(k-1)}$ is not obstructed, that is $Q(\zeta^{(k-1)}) = 0$, it follows from the commutativity of 3.1.4 and the injectivity of $\beta_2 \otimes 1$ that $Q(\xi^{(k-1)}) = 0$. Thus there is an element $\xi^{(k)} \in H^1(M, G^{(k)})$ extending $\xi^{(k-1)}$, which determines a family of h-deformations $X^{(k)}/(S \times K^{\mathrm{tr}})^{(k)}$. Let $\tilde{\phi}^{(k)}: (S \times K^{\mathrm{tr}})^{(k)} \to K^{(k)}$ be the versal map associated to that family. Since $X^{(k)}/(S \times K^{\mathrm{tr}})^{(k)}$ is an extension of $Z_{\phi^{(k-1)}}/(S \times K^{\mathrm{tr}})^{(k-1)}$ we can assume that $\tilde{\phi}^{(k)}$ is an extension of $\phi^{(k-1)}$ (cf. [6], prop. 1, VIII, 3).

The morphism $p_1^{(k)} \times (\pi^{(k)} \circ \tilde{\phi}^{(k)}) : (S \times K^{\mathrm{tr}})^{(k)} \to S^{(k)} \times K^{\mathrm{tr}(k)}$ maps $(S \times K^{\mathrm{tr}})^{(k)}$ isomorphically onto itself. This automorphism of $(S \times K^{\mathrm{tr}})^{(k)}$, that we denote by μ, is the identity over $(S \times K^{\mathrm{tr}})^{(k-1)}$. We define $\phi^{(k)} = \tilde{\phi}^{(k)} \circ \mu^{-1}$. This morphism extends $\phi^{(k-1)}$ and makes commutative the diagram 3.2.2.

The morphisms of local $\mathbb{C}$-algebras, $(\phi^{(k)})^* : \mathcal{O}_{K,0}/m_K^{k+1} \to \mathcal{O}_{S \times T,0}/m^{k+1}_{S \times K^{\mathrm{tr}}}$, induce a morphism between the completions of the local algebras $\mathcal{O}_{K,0} \mathcal{O}_{S \times K^{\mathrm{tr}},0}$

$$\hat{\phi} = \varprojlim (\phi^{(k)})^* : \hat{\mathcal{O}}_{K,0} \to \hat{\mathcal{O}}_{S \times K^{\mathrm{tr}},0}$$

such that the diagram

$$\begin{tikzcd} \hat{\mathcal{O}}_{K,0} \arrow[rr, "\hat{\phi}"] \arrow[dr, "\hat{\pi}^*"'] & & \hat{\mathcal{O}}_{S \times K^{\mathrm{tr}},0} \arrow[dl, "\hat{p}_2^*"] \\ & \hat{\mathcal{O}}_{K^{\mathrm{tr}},0} & \end{tikzcd}$$

is commutative.

A theorem of M. Artin [1] shows that, in this situation, we can find a morphism of germs of analytic spaces $\phi : S \times K^{\mathrm{tr}} \to K$ which is an immersion and such that the composition $\pi \circ \phi$ coincides with the projection $p_2 : S \times K^{\mathrm{tr}} \to K^{\mathrm{tr}}$. In particular ϕ maps $S \cong S \times \{0\}$ into K^0 and, since S is smooth, this is in fact an isomorphism of S onto K^0. Now the statement follows from proposition 2.1. □

Remarks. — If $\Omega^j(L^{1,0})$ denotes the sheaf of germs of $L^{1,0}$-valued holomorphic j-forms (see 1.1 for the definition of $L^{1,0}$), then $\Theta^f_{\mathscr{F}} = \Omega^0(L^{1,0})$. When M is Kähler and $L^{1,0}$ is k-negative one has $H^i(M,\Omega^j(L^{1,0})) = 0$ for $i + j < q - k + 1$ (q=codimension of $\mathscr{F}$). (See [22], def. (3.1) and th. (6.32).)

3.3. *Deformations of $\mathscr{F}$ when it admits a transversal holomorphic foliation $\mathscr{F}^{\pitchfork}$.* In this paragraph we study the deformations of $\mathscr{F}$ in the special case in which there is another holomorphic foliation $\mathscr{F}^{\pitchfork}$ on M, transversal and complementary to $\mathscr{F}$.

With this assumption the exact sequence of sheaves 1.3.1 splits giving rise to the exact sequence

$$0 \to H^1(M,\Theta^f_{\mathscr{F}}) \to H^1(M,\Theta_{\mathscr{F}}) \to H^1(M,\Theta^{\mathrm{tr}}_{\mathscr{F}}) \to 0.$$

In particular K^f coincides with the fibre K^0 of π over zero and it follows from prop. 3.2 that, if K is smooth, then K^f and K^{tr} are smooth and $K \cong K^f \times K^{\mathrm{tr}}$.

PROPOSITION 1. — *Assume that there is a holomorphic foliation $\mathscr{F}^{\pitchfork}$ which is transversal and complementary to $\mathscr{F}$. Then there is a morphism $\lambda : K^{\mathrm{tr}} \to K$ which is a section of π. In particular π is an epimorphism.*

Proof. — Let $\{(U_i, z^a_i, z^u_i)\}$ be an atlas of M such that the foliations $\mathscr{F}$ and $\mathscr{F}^{\pitchfork}$ are locally given by the holomorphic submersions $(z^a_i, z^u_i) \to (z^a_i)$ and $(z^a_i, z^u_i) \to (z^u_i)$ respectively. Assume that the $\Gamma^{\mathrm{tr}}_{K^{\mathrm{tr}}}$-structure of the versal family Z^{tr} is given by local chart $(\tilde{U}_i, r, \psi^a_i, \psi^u_i)$ with $U_i = \tilde{U}_i \cap M_0$ and such that $\psi^a_i | U_i = z^a_i$ and $\psi^u_i | U_i = z^u_i$. Then the family of charts $\{(\tilde{U}_i, r, \zeta^a_i = \psi^a_i, \zeta^u_i = z^u_i)\}$ defines a $\Gamma_{K^{\mathrm{tr}}}$-structure $(Z^{\mathrm{tr}})^{\perp}$ on Z^{tr} which underlies its $\Gamma^{\mathrm{tr}}_{K^{\mathrm{tr}}}$-structure. In this way we get, from the versality of K, a morphism $\tilde{\lambda} : K^{\mathrm{tr}} \to K$ such that $Z_{\tilde{\lambda}}/K^{\mathrm{tr}}$ and $(Z^{\mathrm{tr}})^{\perp}/K^{\mathrm{tr}}$ are isomorphic as families of h-deformations.

The families $Z^{\mathrm{tr}}_{\pi\circ\tilde\lambda}/K^{\mathrm{tr}}$ and $Z^{\mathrm{tr}}/K^{\mathrm{tr}}$ are $\Gamma^{\mathrm{tr}}_{K^{\mathrm{tr}}}$-equivalent and the versality of K^{tr} implies that $d_0(\pi\circ\tilde\lambda) = \mathrm{Id}$. It follows from this that $\pi\circ\tilde\lambda$ is an automorphism of K^{tr} (cf. [24], prop. 1.4). Now, $\lambda = \tilde\lambda\circ(\pi\circ\tilde\lambda)^{-1}$ is a section of π. $\square$

As a corollary of th. 3.2 we obtain

PROPOSITION 2. — *If there exists a holomorphic foliation $\mathscr{F}^{\pitchfork}$ transversal and complementary to $\mathscr{F}$ and $H^2(M,\Theta^f_{\mathscr{F}}) = 0$ then K^f is smooth and $K \cong K^f \times K^{\mathrm{tr}}$.*

We end this paragraph by pointing out an alternate way of computing the versal space K^f. When there is a transversal holomorphic foliation $\mathscr{F}^{\pitchfork}$ this new approach is sometimes useful to describe the relations among the spaces K^f, K and K^{tr}.

Let $\hat\Theta_{\mathscr{F}}$ be the sheaf of germs of vector fields ξ on M whose expression in a flat local chart is

$$\xi = \xi^a\frac{\partial}{\partial z^a} + \xi^u\frac{\partial}{\partial z^u} + \xi^{\bar a}\frac{\partial}{\partial z^{\bar a}} + \xi^{\bar u}\frac{\partial}{\partial z^{\bar u}}$$

with $\xi^a = \xi^a(z^b)$ and $\xi^u = \xi^u(z^b,z^v)$ holomorphic functions. The sheaf $\Theta_{\mathscr{F}}$ is a quotient of $\hat\Theta_{\mathscr{F}}$ by a fine subsheaf, so their cohomologies coincide except in degree 0. We denote by

$$0 \to \hat\Theta_{\mathscr{F}} \to \phi^0 \xrightarrow{D} \phi^1 \xrightarrow{D} \phi^2 \to \dots$$

the resolution of the sheaf $\hat\Theta_{\mathscr{F}}$ used by Kodaira and Spencer in [17] to study the holomorphic deformations of $\mathscr{F}$. It can be described at follows.

Let $A^*(M)$ denote the algebra of $\mathbb{C}$-valued differential forms on M and let I_F (resp. I_T) be the ideal of $A^*(M)$ formed by those forms whose restrictions to F (resp. $T^{0,1}$) vanish. Let $\mathscr{C}^k$ be the space of complex derivations δ of $A^*(M)$ of degree k such that $\delta(I_F) \subset I_F$ and $\delta(I_T) \subset I_T$. The derivations $\delta \in \mathscr{C}^k$ are given, locally, by couples (φ,ζ) of vector valued differential forms of degrees k and $k+1$ and where φ is globally defined. The sheaf ϕ^k is the sheaf of germs of elements of $\mathscr{C}^k$ and D is the differential operator defined by $D(\delta) = [d,\delta]$ where d is the exterior derivative and $[\,,]$ is the bracket of derivations.

Let us consider also the resolution of $\Theta^{\mathrm{tr}}_{\mathcal{F}}$ used by Duchamp and Kalka in [7] to study the *tr*-deformations of $\mathcal{F}$.

$$0 \to \Theta^{\mathrm{tr}}_{\mathcal{F}} \to \Omega^0 \xrightarrow{d_F} \Omega^1 \xrightarrow{d_F} \Omega^2 \to \dots .$$

The sheaf Ω^k is the sheaf of germs of elements of $A^k_F(N)$, where $A^k_F(N)$ denotes the space of sections of the vector bundle $\Lambda^k(F^*) \otimes N^{1,0}$.

Let $p: \mathcal{C}^k \to A^k_F(N)$ be the projection defined as follows. Given $\delta \in \mathcal{C}^k$ represented locally by couples (φ,ζ), then $p(\delta)$ is the element of $A^k_F(N)$ defined by

$$p(\delta)(\xi_1, \dots, \xi_k) = p_N(\varphi(\xi_1, \dots, \xi_k))$$

where $\xi_1, \dots, \xi_k$ are sections of F and $p_N: {}^cTM \to N^{1,0}$ is the canonical projection. These maps induce a projection $p: \phi^* \to \Omega^*$ of resolutions whose kernel ϕ^*_f can also be used to construct the Kuranishi space for the *f*-deformations of $\mathcal{F}$ (cf. [11] for details). Indeed, the correspondence which maps $\omega \in A^{0,k}(L)$ to the derivation δ_ω defined locally by the couples $(\omega, d\omega - \bar{\partial}\omega)$ is an injective homomorphism of $A^{0,k}(L)$ into the space of sections $\mathcal{C}^k_f$ of the sheaf ϕ^k_f. This correspondence induces isomorphisms at the cohomology level (except for degree zero).

If there exists a holomorphic foliation $\mathcal{F}^{\pitchfork}$ transversal and complementary to $\mathcal{F}$ it can be used to obtain a decomposition ${}^cTM = F \oplus N^{1,0}$ and the elements of $A^*_F(N)$ can be viewed as vector valued differential forms on M. The correspondence

$$s: A^k_F(N) \to \mathcal{C}^k$$

defined by $s(\psi) = \delta_\psi$, where δ_ψ is the derivation locally defined by the couples $(\psi, d\psi - d_F\psi)$, is a section of p. Thus we get the direct sum decomposition of complexes $\mathcal{C}^* = \mathcal{C}^*_f \oplus s(A^*_F(N))$ (cf. [11]).

If one uses a Hermitian metric on M with respect to which $\mathcal{F}$ and $\mathcal{F}^{\pitchfork}$ are orthogonal, then the Laplacians Δ, Δ'_f and Δ'_{tr}, associated to the operator D in the complexes $\mathcal{C}^*$, $\mathcal{C}^*_f$ and $s(A^*_F(N))$ respectively, verify $\Delta = \Delta'_f + \Delta'_{\mathrm{tr}}$. In particular, we have an orthogonal decomposition of the corresponding spaces of harmonic derivations

$$H^k = H'^k_f \oplus H'^k_{\mathrm{tr}}. \qquad (3.3.1)$$

Then one can construct the versal spaces K, K^f and K^{tr} by expressions analogous to 1.4.2 and 1.4.3 (cf. [10] and [11]). These decompositions of Δ and H^k motivate the use of the complexes $\mathscr{C}^*$, $\mathscr{C}_f^*$ and $s(A_F^*(N))$ for the computation of the versal spaces K, K^f and K^{tr}, particularly when one is interested in describing the relations among these spaces (cf. example 4.4). One can easily see for instance that if there exists a holomorphic foliation $\mathscr{F}^{\pitchfork}$ transversal and complementary to $\mathscr{F}$ and the versal space K is smooth, then K^f and K^{tr} are also smooth and $K \cong K^f \times K^{\mathrm{tr}}$.

4. Some examples.

In this section we study the deformation of some specific examples of holomorphic foliations on compact manifolds.

4.1. *Flows over Hopf manifolds.* A Hopf manifold is a complex manifold W_h obtained as the quotient of $W = \mathbb{C}^n - \{0\}$ $(n \geqslant 2)$ by a polynomial automorphism h of $\mathbb{C}^n$ such that: (i) h fixes the origin, (ii) the derivative $h'(0)$ at 0 has eigenvalues $\mu = (\mu_1, \ldots, \mu_n)$ inside the unit circle (so h is a contractive map), and (iii) h is μ-resonant; i.e., h commutes with the diagonal linear map

$$d_\mu = \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix}$$

Let W_h be a Hopf manifold defined by an automorphism h of $\mathbb{C}^n$ as above. Any holomorphic vector field ξ on W_h without zeroes defines a holomorphic foliation $\mathscr{F}$ whose leaves are the orbits of ξ. In this situation one has

PROPOSITION. — *$H^1(W_h, \Theta_{\mathscr{F}}^f) = \mathbb{C}$ and $H^2(W_h, \Theta_{\mathscr{F}}^f) = 0$. In particular the versal space K^f is smooth of dimension* 1 *and $K \cong K^0 \times K^{\mathrm{tr}}$ (recall that K^0 denotes the fibre over* 0 *of the versal map $\pi : K \to K^{\mathrm{tr}}$) where K^0 is also smooth.*

Proof. — Let $\mathcal{O}$ and $\mathcal{O}_h$ denote the sheaves of germs of holomorphic functions on W and W_h respectively. Let $\mathscr{U} = \{U_i\}$ be a covering of W_h by 1-connected Stein open subsets and set $\tilde{\mathscr{U}} = \{\tilde{U}_i\}$ where $\tilde{U}_i = p^{-1}(U_i)$ and $p : W \to W_h$ is the canonical projection. $\tilde{\mathscr{U}}$ is an open covering of W by (non-connected) Stein open subsets invariant by h.

An argument of Douady [3] shows that the sequence of complexes

$$0 \to \check{C}^*(\mathscr{U},\mathscr{O}_h) \xrightarrow{p^*} \check{C}^*(\tilde{\mathscr{U}},\mathscr{O}) \xrightarrow{1-h^*} \check{C}^*(\tilde{\mathscr{U}},\mathscr{O}) \to 0$$

is exact, so it gives rise to a long exact sequence

$$\ldots \to H^k(W_h,\mathscr{O}_h) \xrightarrow{p^*} H^k(W,\mathscr{O}) \xrightarrow{1-h^*} H^k(W,\mathscr{O}) \to H^{k+1}(W_h,\mathscr{O}_h) \to \ldots$$

In a similar manner to that of Haefliger in [14] one can see that $(1-h^*): H^k(W,\mathscr{O}) \to H^k(W,\mathscr{O})$ is an isomorphism for $k \geqslant 1$ and that it has a cokernel of dimension 1 for $k = 0$. So $H^1(W_h,\mathscr{O}_h) = \mathbb{C}$ and $H^2(W_h,\mathscr{O}_h) = 0$. Now the statement follows from the observation that the sheaves $\mathscr{O}_h$ and $\Theta^f_{\mathscr{F}}$ are isomorphic and from th. 2.3 and the corollary to th. 3.2. □

Remark. — Let us consider the special case for which $n = 2$, h is the homothety $h(z_1,z_2) = (az_1,az_2)$ with $0 < |a| < 1$ and $\xi = z_1\dfrac{\partial}{\partial z_1} + z_2\dfrac{\partial}{\partial z_2}$. In this situation $\mathscr{F}$ is a fibration $W_h \to \mathbb{C}P^1$ with fibre a complex torus. By means of a representation of $H^1(W_h,\Theta_{\mathscr{F}})$ and $H^1(W_h,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ as spaces of vector 1-forms harmonic with respect to a metric on W_h one easily checks that these spaces have complex dimension 7 and 6 respectively. Furthermore, the map $H^0(W_h,\Theta_{\mathscr{F}}) \to H^0(W_h,\Theta^{\mathrm{tr}}_{\mathscr{F}})$ is an epimorphism. A computation similar to that of Duchamp and Kalka in [8] shows that the versal space K^{tr} is not smooth.

4.2. *Foliations tranverse to the fibres of a bundle.* Let $\mathscr{F}$ be a holomorphic foliation transverse to the fibres P of a holomorphic bundle $p: M \to B$, where M and B are compact connected complex manifolds. The bundle and the foliation $\mathscr{F}$ are completely determined by the global holonomy

$$H : \pi_1(B,b) \to \mathrm{Aut}_{\mathbb{C}}(P)$$

where $\mathrm{Aut}_{\mathbb{C}}(P)$ denotes the group of holomorphic automorphisms of P (cf. [10]).

It has been shown by Girbau, Haefliger and Sundaraman in [10] that the versal space K^{tr} is isomorphic to the versal space of $\pi_1(B,b_0)$-equivariant deformations of the complex structure of P (where $\pi_1(B,b_0)$ acts on P through H) as defined by Cathelineau in [2].

The homomorphism H induces an action of $\pi_1(B,b_0)$ on $H^1(P,\mathcal{O}_P)$ where $\mathcal{O}_P$ is the sheaf of germs of holomorphic functions on P. Let $\mathcal{H}^1(P,\mathcal{O}_P)$ denote the locally constant sheaf on B with fibre $H^1(P,\mathcal{O}_P)$ associated to this representation. Let Θ_B denote the sheaf of germs of holomorphic vector fields on B. In this situation we obtain the following result which improves th. 2.5 in [10].

THEOREM. – *Assume that the sheaf $\Theta_B \otimes \mathcal{H}^1(P,\mathcal{O}_P)$ has no non trivial sections over B. Then K^f is isomorphic to the versal space K_B of deformations of the complex structure of B. Furthermore, $K \cong K_B \times K^{\mathrm{tr}}$.*

Note. – The hypothesis of the theorem are fulfilled if $H^1(P,\mathcal{O}_P) = 0$ (for instance if P is a 1-connected Kähler manifold) or if H acts trivially on $H^1(P,\mathcal{O}_P)$ and $H^0(B,\Theta_B) = 0$ (for instance if B is a Riemann surface of genus >1).

Proof. – Any *tr*-deformation of $\mathcal{F}$ is given by a deformation of the holonomy H. So any *tr*-deformation gives rise to a *h*-deformation of $\mathcal{F}$ uniquely determined by the requeriment that the complex structure on B remains fixed and the projection p is still holomorphic. Thus we get a commutative diagram of germs of analytic spaces

$$\begin{tikzcd} K_B \times K^{\mathrm{tr}} \arrow[rr, "\phi"] \arrow[dr, "p_2"'] & & K \arrow[dl, "\pi"] \\ & K^{\mathrm{tr}} & \end{tikzcd}$$

From the spectral sequence of the fibration $p: M \to B$ we obtain the exact sequence

$$0 \to H^1(B,\Theta_B) \xrightarrow{\alpha} H^1(M,\Theta^f_{\mathcal{F}}) \longrightarrow H(B,\mathcal{O}_P)) \to$$
$$\longrightarrow H^2(B,\Theta_B) \xrightarrow{\beta} H^2(M,\Theta^f_{\mathcal{F}}).$$

By the hypothesis made, α is an isomorphism and β is injective. The restriction ϕ' of ϕ to $K_B \cong K_B \times \{0\}$ is an immersion (its tangent map is α) with image contained in $K^f = \pi^{-1}(0)$.

Let X/K^f be the trivial family of deformations of the complex structure of B $(X=B\times K^f)$ and let $G_B^{(k)}$ denote the sheaf of germs of holomorphic automorphisms of the family $X^{(k)}/(K^f)^{(k)}$. Let Y/K^f be the trivial family of f-deformations of $\mathcal{F}$ $(Y=M\times K^f)$ and let $G_f^{(k)}$ denote the sheaf of germs of $\Gamma^f_{K^f}$-isomorphisms of $Y^{(k)}/(K^f)^{(k)}$.

Denote by $m_f^{[k]}$ the quotient m_f^k/m_f^{k+1} where m_f is the maximal ideal of $\mathcal{O}_{K^f,0}$. Since, any holomorphic deformation of B induces a f-deformation of $\mathcal{F}$, we obtain, in a similar manner as that used in 3.1, the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & H^1(B,\Theta_B)\otimes m_f^{[k]} & \to & H^1(B,G_B^{(k)}) & \to & H^1(B,G_B^{(k-1)}) & \to & H^2(B,\Theta_B)\otimes m_f^{[k]} & \\ & \downarrow{\scriptstyle \alpha\otimes 1} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta\otimes 1} & \\ 0 \to & H^1(M,\Theta_{\mathcal{F}}^f)\otimes m_f^{[k]} & \to & H^1(M,G_f^{(k)}) & \to & H^1(M,G_f^{(k-1)}) & \to & H^2(M,\Theta_{\mathcal{F}}^f)\otimes m_f^{[k]}. & \end{array}$$

Since β is injective, we can repeat the argument used in the proof of th. 3.2 to construct a morphism $\varphi : K^f \to K_B$ which is a section of ϕ'. This implies that ϕ' is an isomorphism (cf. [24], prop. 1.4). Now the second part of the statement follows from prop. 2.1. □

4.3. Let M be the product $T^q \times \mathbb{C}P^1$, where T^q is a complex torus of dimension q, and let us consider on M the holomorphic foliation $\mathcal{F}$ whose leaves are the fibres of the projection $p_1 : T^q \times \mathbb{C}P^1 \to T^q$.

Since the leaves of $\mathcal{F}$ are simply connected, $K_{\mathcal{F}}$ is isomorphic to the versal space $K_T \cong \mathbb{C}^{q^2}$ of deformations of the complex structure T^q (cf. [10], th. 4.3). In particular, the holomorphic projection p_1 is preserved by tr-deformations of $\mathcal{F}$.

The space K^f is isomorphic to the analytic subspace of $H^1(M,\Theta_{\mathcal{F}}^f) \cong \mathbb{C}^{3q}$ defined by the $3q(q-1)$ equations

$$\begin{aligned} x_j y_k - x_k y_j &= 0 \\ x_j z_k - x_k z_j &= 0 \quad (1 \leqslant j < k < 2q) \\ y_j z_k - y_k z_j &= 0 \end{aligned}$$

where $x_i,\ y_i,\ z_i\ (i=1,\dots,q)$ are the linear coordinates of $\mathbb{C}^{3q}$ (cf. [11] for explicit computations). Notice that the f-deformations of $\mathcal{F}$ do not change the complex structure of the leaves (they are isomorphic to $\mathbb{C}P^1$ which is rigid). This is an example of the situation described in Remark 1.1.

Although the foliation $\mathcal{F}$ is not in the hypothesis of th. 3.2 or th. 4.2 one has $K \cong K^f \times K^{\mathrm{tr}}$. One can see this by means of the method indicated at the end of paragraph 3.3 (cf. [11] for details). The versal space K coincides also with the versal space K_M of deformations of the complex structure of M (cf. [16] part. II, § 16).

4.4. *Flows over a complex torus.* Let T^n be the complex torus defined by a subgroup Λ of $\mathbb{C}^n$ generated by $2n$ $\mathbb{R}$-linearly independent elements of $\mathbb{C}^n$, i.e. $T^n = \mathbb{C}^n/\Lambda$. Any vector field ξ on T^n is induced by a linear vector field $\tilde{\xi}$ on $\mathbb{C}^n$. We can take coordinates $z, w^1, \ldots, w^{n-1}$ on $\mathbb{C}^n$ which are linear functions of the canonical coordinates of $\mathbb{C}^n$, with $\tilde{\xi} = \frac{\partial}{\partial z}$ and such that $\frac{\partial}{\partial z}, \frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^{n-1}}$ are orthogonal with respect to the Euclidean metric.

Let $\mathscr{F}$ be the foliation on T^n induced by a vector field $\xi \neq 0$. The sheaf $\Theta^f_{\mathscr{F}}$ is isomorphic to the sheaf $\mathscr{O}_T$ of germs of holomorphic functions on T^n. So $\dim_{\mathbb{C}} H^1(T^n, \Theta^f_{\mathscr{F}}) = \dim_{\mathbb{C}} H^1(T^n, \mathscr{O}_T) = n$. Indeed, $\left\{ d\bar{z} \otimes \frac{\partial}{\partial z}, d\bar{w}^a \otimes \frac{\partial}{\partial z} ; a = 1, \ldots, n-1 \right\}$ is a basis of the space of harmonic vector 1-forms H^1_f defined in 1.4. One easily checks that $[\omega, \omega] = 0$ for any $\omega \in H^1_f$ and now it follows from 1.4.2 and 1.4.3 that $K^f = N_f = H^1_f \cong H^1(T^n, \Theta^f_{\mathscr{F}})$.

Let $\mathscr{O}^{\mathrm{tr}}_T$ denote the subsheaf of $\mathscr{O}_T$ formed by the germs of those functions which are constant along the leaves. The exact sequence of sheaves

$$0 \to \mathscr{O}^{\mathrm{tr}}_T \to \mathscr{O}_T \xrightarrow{L_\xi} \mathscr{O}_T \to 0,$$

where L_ξ denotes the Lie derivative with respect to ξ, induces the long exact cohomology sequence

$$\cdots \to H^k(T^n, \mathscr{O}^{\mathrm{tr}}_T) \to H^k(T^n, \mathscr{O}_T) \xrightarrow{L^\#_\xi} H^k(T^n, \mathscr{O}_T) \to H^{k+1}(T^n, \mathscr{O}^{\mathrm{tr}}_T) \to$$

By representing the elements of $H^k(T^n, \mathscr{O}_T)$ by harmonic forms one can see that $L^\#_\xi$ is the zero map. So $\dim_{\mathbb{C}} H^1(T^n, \mathscr{O}^{\mathrm{tr}}_T) = \dim_{\mathbb{C}} H^0(T^n, \mathscr{O}_T) + \dim_{\mathbb{C}} H^1(T^n, \mathscr{O}_T) = n + 1$. Since $\Theta^{\mathrm{tr}}_{\mathscr{F}}$ is isomorphic to $(\mathscr{O}^{\mathrm{tr}}_T)^{n-1}$, $\dim H^1(T^n, \Theta^{\mathrm{tr}}_{\mathscr{F}}) = n^2 - 1$. Hence

$$\dim_{\mathbb{C}} H^1(T^n, \Theta_{\mathscr{F}}) = n^2 + n - 1.$$

The space $H^1(T^n, \Theta_{\mathscr{F}})$ is isomorphic to the space of harmonic derivations $H^1 \cong H'^1_f \oplus H'^1_{\mathrm{tr}}$ (cf. 3.3.1). H'^1_f is the space of derivations $\delta_\omega = (\omega, d\omega - \bar{\partial}\omega)$, where ω belongs to H^1_f, and H'^1_{tr} is the space of derivations $\delta_\psi = (\psi, d\psi - d_F\psi)$, where ψ belongs to the subspace of $A^1_F(N)$ spanned by $dz \otimes \frac{\partial}{\partial w^a}$, $d\bar{z} \otimes \frac{\partial}{\partial w^a}$, $d\bar{w}^b \otimes \frac{\partial}{\partial w^a}$ $(a, b = 1, \ldots, n-1)$. One has $[\delta_\omega, \delta_\omega] = [\delta_\omega, \delta_\psi] = [\delta_\psi, \delta_\psi] = 0$ for any δ_ω H'^1_f and $\delta_\psi \in H'^1_{\mathrm{tr}}$. From this and from the descriptions of K and K^{tr} analogous to that of K^f given in 1.4.3 (cf. [10]) it follows that the versal spaces K and K^{tr} are smooth and that $K \cong K^f \times K^{\text{tr}}$.

We can collect these results in the following statement.

PROPOSITION. — *Let $\mathscr{F}$ be the foliation, on a complex torus T^n, whose leaves are the orbits of a non zero holomorphic vector field. Then the versal spaces K, K^f and K^{tr} are smooth of dimensions $n^2 + n - 1$, $n^2 - 1$ and n respectively and $K \cong K^f \times K^{\text{tr}}$.*

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