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COMPLEX-SYMMETRIC SPACES

by Ralf LEHMANN

0. Introduction.

Let M be a complete complex Hermitian manifold. Then M is called a Hermitian symmetric space, if and only if for every $x \in M$ there exists a holomorphic isometry of order two which has x as an isolated fixed point.

This condition immediately implies that the group of holomorphic isometries acts transitively on M and that the Hermitian metric is Kählerian.

Moreover, each compact Hermitian symmetric space can be written as a product of a compact torus and a homogeneous projective rational manifold which is Hermitian symmetric. These manifolds are classified by using the classification of semisimple Lie algebras (see e.g. [Hel] and [Ca]).

Generalizations of symmetric spaces to the infinite-dimensional case have turned out to be very interesting (see e.g. [K]).

We are concerned here with manifolds where the isometry condition is dropped. These were first considered by Borel ([Bo]).

Let X be a reduced compact complex space and G a complex subgroup of the complex group of all holomorphic automorphisms. The pair (X,G) is called *complex-symmetric with respect to* G or *Borel-symmetric*, if and only if for every $x \in X$ there exists $s_x \in G$ of order two such that x is an isolated fixed point of s_x. We call s_x a *holomorphic symmetry* at x.

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Our purpose here is to describe the classification of complex-symmetric spaces with an algebraic action of a reductive group.

If (X,G) is complex-symmetric, the group G^0 has an open orbit and the number of orbits is finite. Thus (X,G) is almost G^0-homogeneous ([Bo]). If we assume in addition that G^0 is acting transitively, then X is Hermitian symmetric ([Bo]).

Of course each compact Hermitian symmetric space is complex-symmetric. Non-homogeneous examples can be produced by blowing up appropriate points of Hermitian symmetric spaces.

For example, $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in the four fixed points of the product $(\mathbb{C}^*)^2$-action is a non-homogeneous complex-symmetric manifold.

Minimal examples can be obtained as follows (§ 1):

Let $\mathscr{P}$ be a homogeneous $\mathbb{C}^*$-principal bundle over a compact torus and X the natural compactification of $\mathscr{P}$ as a $\mathbb{P}_1$-bundle. Then X is complex-symmetric, if and only if $\mathscr{P} \otimes \mathscr{P}$ is holomorphically trivial (Lemma 1.2).

On the other hand, the compactification as a $\mathbb{P}_1$-bundle of a homogeneous $\mathbb{C}$-principal bundle over a compact torus is always complex-symmetric (Lemma 1.3).

Note that in these cases the Albanese fibration is not trivial.

Since almost homogeneous manifolds of dimension two are well-known we easily obtain a classification of two-dimensional complex-symmetric manifolds (Proposition 1.7):

A minimal two-dimensional complex-symmetric manifold is either a Hermitian symmetric space, a Hirzebruch surface, or one of the $\mathbb{P}_1$-bundles over a torus introduced above. A Hopf surface is never complex-symmetric.

In general, the compactification of a positive line bundle over a Hermitian symmetric space is complex-symmetric.

The examples stated above indicate that in the Kählerian case it is reasonable to start the examination of complex-symmetric spaces by looking at the fiber of the Albanese map, i.e. by looking at algebraic manifolds with an algebraic group action.

In order to state our results, it is necessary to recall the notion of a torus embedding.

A compact torus embedding X is a compactification of an algebraic torus $T \cong (\mathbb{C}^*)^k$ such that the action of T on itself by translations can be extended to an action of T on X. A compact torus embedding is called complex-symmetric, if X is complex-symmetric with respect to a Lie group G with $G^0 = T$.

Now we are able to state our main results (Theorem 2.16, Theorem 2.22 and Theorem 3.13):

THEOREM. – (1) *Let X be a normal compact algebraic space which is complex-symmetric with respect to a Lie group G. Assume that G^0 is reductive and that the action of G is algebraic. Then X is the direct product of a normal complex-symmetric torus embedding and a Hermitian symmetric space which is homogeneous via a semisimple group.*

(2) *Let X be a smooth complex-symmetric torus embedding. Then X can be written as a direct product of the following examples:*

$Y_1 := \mathbb{P}_1$;
$Y_2 := \mathbb{P}_1 \times \mathbb{P}_1$ *blown up in two points;*
$Y_3 := \mathbb{P}_1 \times \mathbb{P}_1$ *blown up in four points;*
$Y_4 := Y_3$ *blown up in four suitable points.* □

Since Hermitian symmetric spaces are completely classified, this Theorem yields a complete classification in the case of reductive G^0 and smooth X.

The first to analyse this situation was Ahiezer ([A]). He assumed that X is complex-symmetric with respect to G with G^0 semisimple. It is shown that there exists a decomposition $G^0 = S_1 \cdot \ldots \cdot S_n$ such that the S_i-orbits in X are either Stein or compact Hermitian symmetric spaces.

Using the classification of two-orbit-varieties he showed that for $n = 1$ and X not G^0-homogeneous the only possibility is the compact quadric Q_m on which $SO(m,\mathbb{C})$ acts with two orbits ([A]). But Q_m is homogeneous via $SO((m+1),\mathbb{C})$ and Hermitian symmetric.

We also first consider the case where X is complex-symmetric with respect to G with G^0 semisimple. The classification is carried out inductively. Using geometric methods we show that one only has to consider the case where the open G^0-orbit is Stein. It follows that the closure of each G^0-orbit is again complex-symmetric and that the non-compact S_i-orbits are affine quadrics. We then show that we have a naturally defined biholomorphic mapping from X to a product of compact quadrics.

Next we prove that a normal space which is complex-symmetric with respect to a reductive group is a product of a Hermitian symmetric space and a normal complex-symmetric torus embedding.

We show that the projection of X onto the set of fixed points of a Borel subgroup $B < S$ realizes X as a trivial bundle where the fiber is a Hermitian symmetric space and the base is a normal complex-symmetric torus embedding. Our proof here follows an idea of Domingo Luna whom we want to thank very much for his support. Using his methods our original proof, which used elementary methods but was quite long, could be shortened a lot.

The classification of complex-symmetric torus embeddings which is carried out in § 3 makes use of the fundamental fact that a torus embedding which is a normal variety can be described by systems of convex rational cones satisfying certain conditions (« fans »).

A necessary and sufficient condition for a non-singular fan Σ to define a smooth complex-symmetric torus embedding can be stated as follows :

For each cone $\sigma \in \Sigma$ there exists a fan automorphism φ_σ of order two such that

$$\varphi_\sigma | \sigma = \mathrm{Id},$$

and the induced isomorphism on the fan defined by the torus embedding $\overline{\text{orb}(\sigma)}$ is $-\mathrm{Id}$.

This condition immediately implies a classification in dimension two.

For the general classification we show that the system of vectors defining the fan of a complex-symmetric torus embedding is the root system of a Coxeter group, i.e. a finite group generated by reflections. The corresponding variety is the product of lower-dimensional varieties, if and only if the Coxeter group is reducible. Proceeding by induction we make use of the explicit description of the root systems of Coxeter groups and determine the possibilities for the fans defining the T-stable divisors. We then show that an irreducible Coxeter group cannot define a complex-symmetric torus embedding except in dimension two.

Finally we show that in the singular case new examples arise.

This paper is a presentation of the Main Results of my Ph. D. Thesis ([L 1]). I want to thank Professor Klaus-Werner Wiegmann for his constant support of my work and Professor Domingo Luna for many improvements of the proofs in § 2.

More especially, I want to thank Professor Alan T. Huckleberry for suggesting the problem and for many useful and interesting discussions. His strong support and encouragement helped me very much to finish this project successfully.

1. Examples of complex-symmetric spaces.

Of course each Hermitian symmetric space is complex-symmetric. Thus tori, projective spaces and Grassmannians are complex-symmetric.

Before giving some non-homogeneous examples we state the following basic Lemma which is implicitly contained in [Bo] (see [L1] for a detailed proof).

LEMMA 1.1. — *Assume that (X,G) is complex-symmetric and that the open G^0-orbit Ω is of the form G^0/Γ (Γ discrete). Then Ω is an abelian Lie group and the symmetries for Ω are of the form*

$$g \mapsto g_0 g^{-1} \, (g, g_0 \in \Omega). \qquad \square$$

Let X be an almost homogeneous $\mathbb{P}_1$-bundle over a compact torus $T := \mathbb{C}^n/\Gamma$ which is not holomorphically trivial. Then $G^0 := \mathrm{Aut}_{\mathcal{O}}(X)^0$ is abelian ([H-O], p. 93). Either the open G^0-orbit Ω is a $\mathbb{C}$-principal bundle over T and X is obtained by adding the infinity-section E, or Ω is a $\mathbb{C}^*$-principal bundle and X is obtained by adding the zero-section N and the infinity-section E. In the latter case X contains the two line bundles

$$L := N \cup \Omega \quad \text{and} \quad L^* := E \cup \Omega .$$

LEMMA 1.2. — *The compactification of a $\mathbb{C}^*$-principal bundle as a $\mathbb{P}_1$-bundle X over T is complex-symmetric, if and only if L and L^* are equivalent bundles.*

Proof. — Assume that X is complex-symmetric. The projection $\pi : X \to T$ is $\mathrm{Aut}_{\mathcal{O}}(X)$-equivariant ([H-O], p. 83). Thus the symmetries σ_x (resp. σ_y) for $x \in \Omega$ (resp. $y \in N$) induce a symmetry for $\pi(x)$ (resp.

$\pi(y))$ which is unique by Lemma 1.1. So if we assume $\pi(x) = \pi(y)$ then $\sigma_x \circ \sigma_y$ induces the identity on T and exchanges N and E. Thus L and L^* are equivalent bundles.

On the other hand, there exists a homomorphism $\alpha : \Gamma \to \mathbb{C}^*$ such that $X = (\mathbb{C}^n \times \mathbb{P}_1)/\sim$, where

$$(z,[x_0 : x_1]) \sim (z',[x'_0 : x'_1]) \Leftrightarrow \exists \gamma \in \Gamma \quad \text{such that} \quad z' = z + \gamma \quad \text{and} \quad [x'_0 : x'_1] = [x_0 : \alpha(\gamma)x_1] \text{ ([Ma]).}$$

The bundles L and L^* are equivalent, if and only if $\alpha(\gamma) = \alpha(-\gamma), \forall \gamma \in \Gamma$. Then the mappings

$$(z,[x_0 : x_1]) \mapsto (-z,[x_1 : x_0])$$
$$(\text{resp. } (z,[x_0 : x_1]) \mapsto (-z,[-x_0 : x_1])),$$

induce well-defined symmetries for points of the open orbit (resp. of N and E) in X. □

LEMMA 1.3. – *The compactification of a $\mathbb{C}$-principal bundle as a $\mathbb{P}_1$-bundle X over T is always complex-symmetric.*

Proof. – There exists a homomorphism $\rho : \Gamma \to \mathbb{C}$ such that $X = (\mathbb{C}^n \times \mathbb{P}_1)/\sim$ where

$$(z,[x_0 : x_1]) \sim (z',[x'_0 : x'_1]) \Leftrightarrow \exists \gamma \in \Gamma \quad \text{such that} \quad z' = z + \gamma \quad \text{and} \quad [x'_0 : x'_1] = [x_0 + \rho(\gamma)x_1 : x_1].$$

It is easy to see that the mapping $\tilde{\sigma} : \mathbb{C}^n \times \mathbb{P}_1 \to \mathbb{C}^n \times \mathbb{P}_1$ defined by

$$\tilde{\sigma}(z,[x_0 : x_1]) := (-z,[-x_0 : x_1])$$

induces a well-defined mapping $\sigma : X \to X$ of order two which has isolated fixed points in each G^0-orbit. □

Remark. – If the open G^0-orbit Ω is a $\mathbb{C}^*$-principal bundle, then Ω has non-constant holomorphic functions. If Ω is a $\mathbb{C}$-principal bundle, then Ω can be a Cousin group ([L1]).

Now consider a Hermitian symmetric space Q of the compact type and a non-trivial $\mathbb{C}^*$-principal bundle $\mathscr{P}$ over Q. By adding the zero-section N and the infinity-section E we obtain a $\mathbb{P}_1$-bundle X over Q which contains the line bundles $L := \mathscr{P} \cup N$ and $L^* := \mathscr{P} \cup E$. We choose L^* such that $\Gamma_{\mathscr{O}}(Q,L^*) = (0)$. Then we have

LEMMA 1.4. – *The compactification of a $\mathbb{C}^*$-principal bundle $\mathscr{P}$ over Q as a $\mathbb{P}_1$-bundle X is complex-symmetric, if and only if $\Gamma_{\mathcal{O}}(Q,L) \neq (0)$.*

Proof. – Let $\pi : X \to Q$ be the projection and $S := \mathrm{Aut}_{\mathcal{O}}(Q)$. Then $G := \mathrm{Aut}_{\mathcal{O}}(X)^0 \cong (\tilde{S}\cdot\mathbb{C}^*) \ltimes \Gamma_{\mathcal{O}}(Q,L)$, where $\tilde{S}$ is a finite cover of S stabilizing E and N, $\Gamma_{\mathcal{O}}(Q,L)$ acts on L (hence on X) by $\tilde{\sigma}(x) := x + \sigma(\pi(x))$ and the $\mathbb{C}^*$-action on X is induced by the natural $\mathbb{C}^*$-action on $\mathscr{P}$ ([H-O], p. 99).

If $\Gamma_{\mathcal{O}}(Q,L) = \Gamma_{\mathcal{O}}(Q,L^*) = (0)$, then the G-orbits on X are $\mathscr{P}$, E and N. Assume that X is complex-symmetric. The base of the normalizer-fibration of the open orbit is Q ([H-O], p. 81), and each symmetry $\tilde{\sigma}$ for $\mathscr{P}$ is a bundle automorphism by Lemma 1.1. Thus $\tilde{\sigma}(E) = N$ (again by Lemma 1.1) and $\pi(\tilde{\sigma}(x)) = \sigma(\pi(x))$ for a suitable $\sigma \in \mathrm{Aut}_{\mathcal{O}}(Q)$. Moreover, $\sigma \in S$ ([Hel], p. 305). On the other hand, σ induces a bundle automorphism stabilizing E and N. As in the proof of Lemma 1.2 we see that L and L^* are equivalent bundles. This is impossible since $H^2(Q,\mathbb{Z}) \cong H^1(Q,\mathcal{O}^*)$ has no torsion. Thus X is not complex-symmetric.

Now let $\Gamma_{\mathcal{O}}(Q,L) \neq (0)$. Then G has two orbits, and we only have to find symmetries for points in E and N. Let $q \in Q$ and σ be a symmetry for q in Q. Then $\sigma \in S$ ([Hel], p. 305), and therefore σ induces a bundle automorphism $\tilde{\sigma}$ of X. We can assume that $\tilde{\sigma}$ stabilizes E and N. Then $p_1 := N \cap \pi^{-1}(q)$ and $p_2 := E \cap \pi^{-1}(q)$ are fixed points of $\tilde{\sigma}$. By combining $\tilde{\sigma}$ with an appropriate element of the center $\mathbb{C}^*$ of G we obtain an automorphism ψ such that the differential of ψ in p_1 (resp. p_2) is $-\mathrm{Id}$. The automorphism ψ^2 induces the identity on Q and stabilizes N, E and $\pi^{-1}(q)$ pointwise. It is easy to see that $\psi^2 = \mathrm{Id}$ and that ψ is a symmetry for p_1 and p_2. □

COROLLARY 1.5. – *The Hirzebruch surfaces Σ_n $(n \geq 1)$ are complex-symmetric.* □

We now give some examples which are not complex-symmetric.

LEMMA 1.6. – *A Hopf surface X is never complex-symmetric.*

Proof. – Homogeneous Hopf surfaces cannot be complex-symmetric since a homogeneous complex-symmetric space is Hermitian symmetric ([Bo]). The non-homogeneous Hopf surfaces are G-equivariant compactifications of abelian groups G. If we consider the induced action of the universal covering $\tilde{G} \cong \mathbb{C}^2$ on the universal covering $\tilde{X} = \mathbb{C}^2\setminus\{(0,0)\}$, the open $\tilde{G}$-orbit $\tilde{\Omega}$ is either $\mathbb{C}^* \times \mathbb{C}$ or $\mathbb{C}^* \times \mathbb{C}^*$.

If we assume that X is complex-symmetric, the symmetries can be lifted to automorphisms of $\tilde{X}$. But these are induced be the mapping $g \mapsto g^{-1}$ by Lemma 1.1. Thus the lifted automorphisms don't induce automorphisms of $\tilde{X}$. Contradiction. □

We finish this paragraph by giving a complete classification of minimal complex-symmetric surfaces. This is an immediate consequence of a Theorem due to Potters ([P]) and of the preceding Lemmas.

PROPOSITION 1.7. — *Let X be a minimal complex-symmetric surface. Then X is one of the following*

(1) *a product of two compact homogeneous Riemann surfaces ;*

(2) *a two-dimensional complex torus ;*

(3) *the projective plane $\mathbb{P}_2$;*

(4) *the Hirzebruch surface Σ_n, $n \geqslant 2$;*

(5) *an almost-homogeneous $\mathbb{P}_1$-bundle over a complex torus T, where the open orbit $\mathscr{P}$ is a $\mathbb{C}$-principal bundle over T ;*

(6) *an almost-homogeneous $\mathbb{P}_1$-bundle over T, where $\mathscr{P}$ is a $\mathbb{C}^*$-principal bundle over T with $\mathscr{P}^2$ trivial.* □

2. Classification of complex-symmetric varieties with G^0 reductive.

In this paragraph we consider the case where (X,G) is a complex-symmetric space and G^0 is *reductive*. We always assume that X is normal and algebraic and that the action of G on X is algebraic.

Let $G^0 = S \cdot T$ (finite intersection) where S is semisimple and T is an algebraic torus and let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$ be the corresponding decomposition of the Lie algebra of G^0.

Now let $x \in X$ and $s_x \in G$ be a symmetry for x. Then S and T are normalized by s_x, and s_x induces a symmetry for the S-orbit $S(x)$. At first we consider the structure of such an S-orbit.

There exists a maximal compact subgroup of $S \cup s_x(S)$ containing the symmetry s_x. The identity-component of this group is denoted by K, its Lie algebra by $\mathfrak{k}$. It follows that K is maximal compact in S ([Mo]), hence semisimple.

Since s_x defines an involutive automorphism σ of $\mathfrak{k}$, the K-orbit $K(x)$ carries the structure of a Riemannian symmetric space of the compact type ([Hel]).

Let $\mathfrak{k} = \tilde{\mathfrak{k}}_1 \oplus \cdots \oplus \tilde{\mathfrak{k}}_m$ be the decomposition of $\mathfrak{k}$ in simple ideals. A factor $\tilde{\mathfrak{k}}_i$ is either stabilized by σ or two isomorphic factors are exchanged by σ. So we have a decomposition

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_n$$

in minimal σ-stable ideals. Note that this decomposition depends on the S-orbit.

Denoting the restriction of σ to $\mathfrak{k}_i$ by σ_i and letting $\mathfrak{u}_i$ (resp. $\mathfrak{e}_i$) be the eigenspaces for the eigenvalues $+1$ (resp. -1) of σ_i we have

$$\mathfrak{k}_i = \mathfrak{u}_i \oplus \mathfrak{e}_i,$$

and the pair $(\mathfrak{k}_i, \sigma_i)$ is an irreducible, orthogonal symmetric Lie algebra ([Hel], p. 309).

If $\mathfrak{s}_i := \mathfrak{k}_i^{\mathbb{C}}$ and S_i is the normal subgroup of S associated to $\mathfrak{s}_i$, then the following Lemma due to Ahiezer ([A]) holds (see [L1] for a detailed proof).

LEMMA 2.1. — *Consider the S_i-orbit S_i/H_i through x, define $\mathfrak{h}_i := \mathrm{Lie}(H_i)$. Then either $\mathfrak{h}_i = \mathfrak{u}_i^{\mathbb{C}}$ is maximal reductive and the S_i-orbit is a Stein manifold, or $\mathfrak{h}_i$ is a maximal parabolic subalgebra containing $\mathfrak{u}_i^{\mathbb{C}}$ and the S_i-orbit is a Hermitian symmetric space of the compact type.* □

Remark. — By construction the symmetry s_x induces a symmetry for the S_i-orbit $S_i(x)$.

Now look at an S-orbit S/H. Then for $\mathfrak{h} := \mathrm{Lie}\,(H)$ we have the inclusion

$$\mathfrak{h} \supset \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n .$$

Indeed it can be shown ([A] ; [L1] following an idea of Ahiezer) that we have

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n .$$

Consider the decomposition of $\mathfrak{s}$ in simple ideals

$$\mathfrak{s} = \tilde{\mathfrak{s}}_1 \oplus \cdots \oplus \tilde{\mathfrak{s}}_m .$$

If the automorphism σ induced by a symmetry s_x exchanges $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$, then the fixed point set of σ is the diagonal in $\tilde{\mathfrak{s}}_i \oplus \tilde{\mathfrak{s}}_j$. Therefore $\dim_{\mathbb{C}} \widetilde{S}_i(x) = \dim_{\mathbb{C}} \tilde{\mathfrak{s}}_i$ and $\widetilde{S}_i(x)$ has maximal possible dimension. So it follows that for each element y of the open G^0-orbit Ω the symmetry s_y also exchanges $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$, and we have the following

LEMMA 2.2. — *If (X,G) is complex-symmetric and $G^0 = S \cdot T$ is reductive, there exists a decomposition $S = S_1 \cdot \ldots \cdot S_n$ of the semisimple part S such that for each S-orbit S/H in X we have*

$$S/H^0 = S_1 \cdot \ldots \cdot S_n / H_1 \cdot \ldots \cdot H_n .$$

This decomposition is minimal for symmetries of the open G^0-orbit. □

In the sequel we will always consider this decomposition.

LEMMA 2.3 ([A]). — *Consider an S-orbit S/H and let $H_i := H^0 \cap S_i$. Denote the normalizer of H_i^0 in S_i by $N_{S_i}(H_i^0)$. Then the factor groups*

$$N_{S_i}(H_i^0)/H_i^0 \qquad \text{and} \qquad N_S(H^0)/H^0$$

are finite. □

In order to analyse the structure of a variety X which is complex-symmetric with respect to a reductive group we need the following

DEFINITION (e.g. [B-L-V]; [L-V]). — *Let a reductive group G^0 act algebraically on an algebraic variety X. Then X is called spherical, if and only if a Borel subgroup (and hence every Borel subgroup) of G^0 has a dense orbit in X.*

It was proved by Vust ([V]) that the quotient of a reductive group by the set of fixed points of an involutive automorphism is spherical. Since the compactification of a spherical variety is again spherical by definition, we have

LEMMA 2.4. — a) *Let (X,G) be complex-symmetric and $G^0 = S \cdot T$ reductive. Then X is spherical.*

b) *Let $S = S' \cdot S''$ be a decomposition of S into normal factors which are stabilized by all symmetries. Then for $x \in X$ the analytic sets $\overline{S'(x)}$ (resp. $\overline{S''(x)}$) are spherical.* □

A summary of some of the basic results concerning spherical varieties can be found in [B-P]. For our arguments the following properties are important.

PROPOSITION 2.5. — *Let X be spherical, G^0 reductive and B a Borel subgroup of G^0 having a dense open orbit in X. Then each irreducible G^0-stable subvariety of X is again spherical and the groups G^0 and B have only finitely many orbits in X.*

Moreover, if X is normal, the closure of each G^0-orbit in X is again a normal variety. □

As a first consequence we note

LEMMA 2.6. — *Let $S = S' \cdot S''$ be a decomposition of S into normal factors which are stabilized by all symmetries. Then for each $x \in X$ the analytic sets $\overline{S'(x)}$ (resp. $\overline{S''(x)}$) contain a unique closed S'-orbit (resp. S''-orbit).*

Proof. — It follows from Lemma 2.4 and Proposition 2.5 that $\overline{S'(x)}$ consists of finitely many S'-orbits. By Lemma 2.1 and Lemma 2.2 each S'-orbit has one end in the sense of Freudenthal.

Now if there exists more than one closed S'-orbit in $\overline{S'(x)}$, there is a minimal S'-orbit $E \subset \overline{S'(x)}$ having more than one closed S'-orbit in its closure. This orbit has more than one end ([A]). Contradiction. □

The following Lemma on spherical varieties will be used later.

LEMMA 2.7. — *Let X be a normal complete spherical variety with open orbit G/H. If $N_G(H)/H$ is not finite, then G has more than one closed G-orbit in X.*

Proof. — Normal spherical varieties can be described using G-invariant discrete normalized valuations of the field $\mathbb{C}(G/H)$ ([L-V]); see [B-P] for a summary). The set of all such valuations is denoted by $\mathcal{V}(G/H)$ and can be considered as the set of integral points of a convex rational cone $\mathcal{C}\mathcal{V}(G/H)$ in a finite-dimensional $\mathbb{Q}$-vector space V.

Now if $N_G(H)/H$ is not finite, the cone $\mathcal{C}\mathcal{V}(G/H)$ is not strongly convex ([B-P], corollaire 5.3). But a variety with exactly one closed orbit is described by a strongly convex cone $\mathcal{C} \subset V$ satisfying certain properties ([B-P], 2.6; [L-V]). The completeness of X is equivalent to the fact that $\mathcal{C}\mathcal{V}(G/H) \subset \mathcal{C}$ ([B-P], 2.7; [L-V]). Therefore, if there is only one closed orbit, a cone which is not strongly convex is contained in a cone which is strongly convex. Contradiction.

Thus there is more than one closed G-orbit. □

The following Lemmas will be important for the classification of complex-symmetric spaces (X,G) with $G^0 = S$ semisimple.

LEMMA 2.8. — *Let (X,G) be complex-symmetric, normal and $G^0 = S$ semisimple. Let $S = S' \cdot S''$ as above and assume that the S'-orbits in the open S-orbit Ω are compact. Then X is S-equivariantly biholomorphic to $S'(x) \times Y$ $(x \in \Omega)$, where Y is a normal complex-symmetric space with respect to a group $\hat{S}$ with $\hat{S}^0 = S''$.*

Proof. — We first show that all S'-orbits in X are biholomorphic. Let $x \in \Omega$. Then $S'(x) = S'/P'$ where P' is maximal parabolic in S'. Since parabolic groups are self-normalizing we have $S'(x) \cap S''(x) = \{x\}$. Moreover, $S'(x) = K'/L'$ where K' is a maximal compact subgroup of S' ([Mon]) and L' is maximal in K' ([Hel]).

Now if $S(x)$ is not compact, then $X \backslash S(x)$ contains an S-orbit E_1 of codimension one in X, since $S(x)$ is holomorphically convex. The normality of X implies that $S(x) \cup E_1$ is smooth. Thus by the differentiable slice theorem ([Jän]) for $y \in E_1$ it follows that L' is conjugate to a subgroup of $\mathrm{Iso}_y(K')$. Since L' is maximal it is either conjugate to $\mathrm{Iso}_y(K')$ or $\mathrm{Iso}_y(K') = K'$. The latter case implies that y is a fixed point of S' which is impossible due to faithful linearization. Thus all S'-orbits in $S(x) \cup E_1$ are biholomorphic.

Now if E_1 is not the unique closed S-orbit, the analytic set $\bar{E}_1$ is again a normal variety (Proposition 2.5) containing an S-orbit E_2 of codimension one in $\bar{E}_1$. Again it follows that the S'-orbits in E_1 and E_2 are all biholomorphic. Repeating this procedure we finally arrive at the unique closed S-orbit, and it is clear that all S'-orbits in X are biholomorphic.

Moreover, we have that $K'(x) = S'(x)$ for all $x \in X$. Thus the equivalence relation on X defined by the S'-action is proper and the geometric quotient Y of X by the S'-action exists and carries the structure of a normal complex space induced by the projection $\pi : X \to Y$ ([Ho]; [Car]).

Let $P' := \mathrm{Iso}_x(S')$ and let F be the analytic set of fixed points of P'. Then $\pi|F : F \to Y$ is a bijective holomorphic map. Since Y is normal the inverse mapping $\tau : Y \to F$ is a holomorphic section and the mapping

$$\psi : Y \times S'/P' \to X$$

defined by

$$\psi(y,sP') := s(\tau(y)) \quad (s \in S')$$

is well-defined and biholomorphic.

Since the symmetries respect the decomposition $S' \cdot S''$ of S the Lemma is proved. □

Remark. – The Lemma remains valid if G^0 is reductive. The only difficulty is the fact that then there is more than one closed G^0-orbit.

By Lemma 2.8 the classification of complex-symmetric spaces (X,G) with $G^0 = S$ semisimple is reduced to the case where the open S-orbit is Stein.

LEMMA 2.9. – *Let (X,G) be complex-symmetric with $G^0 = S = S_1 \cdot \ldots \cdot S_n$ semisimple. Assume that X is normal and that the open S-orbit Ω is Stein. Let E be an S-orbit of codimension one in X. Then there exists exactly one factor S_i of S such that*

$$\overline{S_i(x)} \backslash S_i(x) \subset E, \quad \forall x \in \Omega.$$

Moreover, $\overline{S_i(x)}$ consists of two S_i-orbits and is smooth.

Proof. – Let $y \in E$ and $J = J_1 \cdot \ldots \cdot J_n := \mathrm{Iso}_y(S)^0$. At first we show that J is not reductive.

Assume that J is reductive. Since $\Omega \cup E$ is smooth by faithful linearization of the J-action near y we can find a neighbourhood $U(y)$ of y in X and a one-dimensional complex subspace $L \subset U(y)$ containing y which is stable under a maximal compact subgroup of J and with $L \cap \Omega = L \backslash \{y\}$ ([H-O], p. 12).

Since $\dim_{\mathbb{C}} J = \dim_{\mathbb{C}} (\mathrm{Iso}_x(S)) + 1$ ($x \in \Omega$ arbitrary) the action of J on L cannot be trivial, and for every $x \in L$ it follows that J contains $\mathrm{Iso}_x(S)^0$. But then J contains a parabolic factor by Lemma 2.1. Contradiction.

Next we show that exactly one factor of J is parabolic. If we assume that more than one factor is parabolic, there exists a factor S_j which has closed and non-closed orbits of the same dimension, since the S_j-orbits in Ω are of highest dimension. But each maximal compact subgroup K_j of S_j acts transitively on the compact S_j-orbits ([Mon]). Since the K_j-orbits in Ω are of highest dimension the S_j-orbits in Ω are compact. This is absurd since Ω is assumed to be Stein.

So we may assume $J = J_1 \cdot \tilde{J}$ with J_1 parabolic and $\tilde{J}$ reductive. Correspondingly $S = S_1 \cdot S'$.

Since $\tilde{J}$ is reductive the action of $\tilde{J}$ can be faithfully linearized near y. Thus there exists a one-dimensional complex subspace L near y stable under a maximal compact subgroup of $\tilde{J}$ with $\Omega \cap L = L \backslash \{y\}$. It is clear that $\tilde{J}$ acts trivially on L. By Lemma 2.3 the set of fixed points of $\tilde{J}^0$ in Ω consists of finitely many S_1-orbits. Thus $\overline{L \cap \Omega}$ is contained in an S_1-orbit $S_1(x) \subset \Omega$. Thus it follows that $y \in \overline{S_1(x)}$.

Moreover, it is clear that S_1 is the only factor that contains y in the closure of its orbits in Ω.

By construction the set $\overline{S_1(x)}$ is locally the set of fixed points of the reductive group $\tilde{J}$. Since $\Omega \cup E$ is smooth it follows that $\overline{S_1(x)}$ is smooth. Moreover, the orbit $S_1(x)$ is the only orbit with $y \in \overline{S_1(x)}$. Otherwise the set of fixed points is not locally irreducible.

Since all symmetries respect the decomposition $S = S_1 \cdot \ldots \cdot S_n$ of S the set $\overline{S_1(x)}$ is complex-symmetric. □

Remark. – The same result can be proved, if G^0 is reductive ([L1], p. 33 ff).

Corollary 2.10. – *Let (X,G) be complex-symmetric, X normal, $G^0 = S = S_1 \cdot \ldots \cdot S_n$ semisimple, and let the open S-orbit Ω be Stein. Let E be an S-orbit of codimension one. Then $\bar{E}$ is normal and complex-symmetric.*

Proof. – The normality follows from Proposition 2.5. By Lemma 2.9 there exists a factor S_1 of S such that S_1 has compact orbits in E and $S_2, \ldots, S_n$ have non-compact orbits in E. It is clear that E is the unique S-orbit of codimension one with this property. Since the factors S_i are stable under all symmetries by construction, each symmetry has to stabilize E and thus $\bar{E}$. □

Example 2.11. – Consider the compact quadric

$$Q_n := \left\{ [z_0 : z_1 : \cdots : z_n] \in \mathbb{P}_n(\mathbb{C}) \,|\, z_0^2 = \sum_{i=1}^{n} z_i^2 \right\}.$$

This quadric is a Hermitian symmetric space for the group $SO(n+1,\mathbb{R})$, and there is an action of $SO(n,\mathbb{C})$, defined by

$$A([z_0 : z_1 : \cdots : z_n]) = A([z_0 : z']) := [z_0 : Az']$$

having two orbits. The closed orbit is a compact quadric $Q_{n-1} = Q_n \cap \{z_0 = 0\}$ and the complement is an affine quadric $Q_{(n)}$ which is known to be simply connected. It can be shown that Q_n is complex-symmetric with respect to $SO(n, \mathbb{C})$, but not homogeneous ([A]).

Moreover, the normalizer-fibration of the open orbit $Q_{(n)} \subset Q_n$ can be extended to a holomorphic mapping $\varphi : Q_n \to \mathbb{P}_{n-1}$, where $\varphi | Q_{(n)} : Q_{(n)} \to (\mathbb{P}_{n-1} \backslash Q_{n-1})$ is $2:1$ and the closed $SO(n, \mathbb{C})$-orbit in Q_n is mapped biholomorphically onto Q_{n-1}.

However, $\mathbb{P}_{n-1}$ with this orbit structure is not complex-symmetric.

The following Lemma was proved by Ahiezer for G^0 simple ([A]). By Lemma 2.9 it is not difficult to prove it for G^0 semisimple.

LEMMA 2.12. — *Let (X, G) be complex-symmetric, X normal, $G^0 = S = S_1 \cdot \ldots \cdot S_n$, and let the open S-orbit Ω be Stein. Then $\mathfrak{s}_i \cong \mathfrak{so}(m_i, \mathbb{C})$, the compact S_i-orbits are compact quadrics and the non-compact S_i-orbits are affine quadrics.*

Proof. — Let E_1 be an S-orbit of codimension one. By Lemma 2.9 there exists a factor S_1 of S such that $\overline{S_1(x)}$ is a smooth complex-symmetric two-orbit variety for $x \in \Omega$. It was proved by Ahiezer that $\mathfrak{s}_1 \cong \mathfrak{so}(m_1, \mathbb{C})$ and that $\overline{S_1(x)} \cong Q_{m_1}$ with the orbit structure introduced above.

Now $\bar{E}_1$ is normal (Proposition 2.5) and $\bar{E}_1 \cong Q_{m_1 - 1} \times Y_1$ where Y_1 is complex-symmetric and $S_2 \cdot \ldots \cdot S_n$ has a Stein open orbit in Y_1 (Lemma 2.8 and Corollary 2.10). Thus we can apply Lemma 2.9 again for Y_1. Thus there exists a factor S_2 such that $\mathfrak{s}_2 \simeq \mathfrak{so}(m_2, \mathbb{C})$, such that the S_2-orbits in E_1 are isomorphic to $Q_{(m_2)}$ and such that the compact S_2-orbits are compact quadrics $Q_{m_2 - 1}$. Note that the arguments in Lemma 2.8 imply that all compact S_i-orbits in X are biholomorphic.

It follows that the S_2-orbits in Ω are affine quadrics since the S_2-orbits in E_1 are simply connected finite covers of the S_2-orbits in Ω by the arguments in Lemma 2.9.

By repeating this procedure it follows that all S_i-orbits in Ω are affine quadrics.

By applying the same arguments for all S-orbits of codimension one it follows that all non-compact S_i-orbits are affine quadrics. □

Lemma 2.13. — *Let (X,G) be complex-symmetric, X normal, $G^0 = S$ semisimple, and let Ω be Stein. Then for each $i \in \{1,\dots,n\}$ there exists exactly one S-orbit E of codimension one such that the S_i-orbits in E_i are compact.*

Proof. — The uniqueness of E_i is clear since affine quadrics have one end. So we only have to prove the existence.

Let $E_1,\dots,E_s$ be the S-orbits of codimension one in X. Proceeding by contradiction assume that the S_i-orbits in $\Omega \cup \left(\bigcup_{j=1}^{s} E_j\right)$ are affine quadrics S_i/H_i.

For $x \in \Omega$ the isotropy subalgebra $\mathfrak{iso}_x(S_i)$ is an element of a certain Grassmannian Z. The mapping

$$\tau : \Omega \to Z, \qquad x \mapsto \mathfrak{iso}_x(S_i)$$

is S-equivariant and $\tau(\Omega)$ is isomorphic to $S_i/N_{S_i}(H_i)$. By Lemma 2.3 it follows that $\tau(\Omega)$ is Stein.

It follows from Lemma 2.9 that τ is holomorphically extendable to $\Omega \cup \left(\bigcup_{j=1}^{s} E_j\right)$ and that $\tau\left(\Omega \cup \left(\bigcup_{j=1}^{s} E_j\right)\right) = \tau(\Omega)$.

Thus there exists a non-constant holomorphic function on $X\backslash A$ where A is an analytic set of codimension two.

This is absurd since X is normal. □

The following Lemma is due to Domingo Luna (private communication). It is essential for the proofs of the Main Theorems in this paragraph and simplifies our original arguments very much.

Lemma 2.14. — *Let (X,G) be complex-symmetric, X normal and $G^0 = T\cdot S$ reductive. Let $S = S'\cdot S''$ be a decomposition of S into normal factors which are stable under all symmetries. For $B' < S'$ a Borel subgroup of S' denote the set of fixed points of B' in X by $X^{B'}$.*

Then $X^{B'}$ is connected and normal, and for each $x \in X$ the analytic set $\overline{S'(x)}$ contains a unique fixed point of B'.

Moreover, the natural G^0-equivariant mapping φ', which assigns to $x \in X$ the fixed point of B' in $\overline{S'(x)}$, is algebraic.

Proof. – Let $B' = T' \rtimes U'$ be the Levi-decomposition of B'. It is a standard fact that the fixed point set $X^{U'}$ of U' is connected.

For $x \in X^{U'}$ it follows that the isotropy subgroup $\mathrm{Iso}_x(S')$ does not contain a reductive factor. This can be seen using the Bruhat-decomposition of $\mathrm{Iso}_x(S')$ and Lemma 2.3.

Thus $x \in X^{U'}$ implies that $\mathrm{Iso}_x(S')$ is parabolic (Lemma 2.1 and Lemma 2.2). Therefore $X^{B'} = X^{U'}$ is connected. Since $\overline{S'(x)}$ contains a unique closed S'-orbit for each $x \in X$ by Lemma 2.6 and B' has exactly one fixed point in S'/P' (if P' is parabolic in S') the algebraic set $\overline{S'(x)}$ contains a unique point $\varphi'(x) \in X^{B'}$. Thus we have a well-defined mapping φ'. We will prove that φ' is algebraic.

Let $y \in X^{B'}$ and $P^+ := \mathrm{Iso}_y(S')$. Consider a Levi-decomposition $P^+ = L \rtimes R_u(P^+)$ and choose an opposite parabolic subgroup P^-. It follows that $P^+ \cap P^- = L$ and that we can assume $P^- = L \rtimes R_u(P^-)$. By [B-L-V] (Théorème 1.4) it follows that there exists an affine P^--stable neighbourhood U_y of y and an L-stable locally closed subset W_y of U_y such that

$$\begin{aligned} \tau : W_y \times R_u(P^-) &\to U_y \\ (y',u) &\mapsto u(y') \quad \text{(group action)} \end{aligned}$$

is an algebraic isomorphism. Thus we have an algebraic mapping $p := \mathrm{pr}_1 \circ \tau^{-1} : U_y \to W_y$ where pr_1 denotes the projection onto the first factor.

Since W_y is affine, normal and L-stable we can consider the categorical quotient $W_y/\!\!/L$ which is a normal affine variety. Let π be the projection $W_y \to W_y/\!\!/L$ onto the categorical quotient. Then $\pi \circ p : U_y \to W_y/\!\!/L$ is algebraic.

We claim that for $x \in U_y$ arbitrary the set $\pi \circ p(\overline{S(x)} \cap U_y)$ is a point in $W_y/\!\!/L$. It is clear that $\overline{S(x)} \cap U_y$ is connected and consists of finitely many P^--orbits since $\overline{S(x)}$ is spherical. Thus $p(\overline{S(x)} \cap U_y)$ consists of finitely many L-orbits by construction and $\overline{p(\overline{S(x)} \cap U_y)}$ has a unique closed L-orbit. So it is clear that $\pi \circ p(\overline{S(x)} \cap U_y)$ is a point.

Now choose y such that $A_y := \{x \in X | \varphi'(x) \in X^{B'} \cap U_y\}$ is dense in X. This is possible since $X^{B'}$ is compact.

Then $A_y \cap U_y$ is dense in U_y and it is enough to prove that $\varphi'|U_y$ is algebraic since for $x \in A_y$ there exists $\tilde{x} \in A_y \cap U_y$ such that $\tilde{x} = s'(x)$ $(s' \in S')$.

Let $x \in U_y$ with $\varphi'(x) \in X^{B'} \cap U_y$. By the above remarks is is clear that $\pi \circ p \circ \varphi'(x) = \pi \circ p(x)$. Moreover $\pi \circ p\,(X^{B'} \cap U_y)$ is injective. Let $y_1 \neq y_2 \in X^{B'} \cap U_y$. Then $S'(y_1) \cap S'(y_2) = \emptyset$ and for x_1 (resp. x_2) with $\varphi'(x_1) = y_1$ (resp. $\varphi'(x_2) = y_2$) the analytic sets $\overline{S'(x_1)}$ and $\overline{S'(x_2)}$ are disjoint. Thus $\pi \circ p(x_1) \neq \pi \circ p(x_2)$ which proves the injectivity.

Since $\pi \circ p(X^{B'} \cap U_y) = \pi \circ p(A_y)$ is Zariski-dense and closed in $W_y /\!\!/ L$ it follows that

$$\pi \circ p\ (X^{B'} \cap U_y): \quad X^{B'} \cap U_y \rightarrow W_y /\!\!/ L$$

is bijective. Thus $X^{B'} \cap U_y$ is normal and there exists an inverse mapping $\mu: W_y /\!\!/ L \rightarrow X^{B'} \cap U_y$.

Moreover, by construction

$$\mu \circ \pi \circ p(x) = \mu \circ \pi \circ p \circ \varphi'(x) = \varphi'(x) \quad \forall x \in U_y$$

and therefore φ'/U_y is algebraic and $A_y \cap U_y = U_y$.

By the preceding arguments, in order to prove that φ' is algebraic and that $X^{B'}$ is normal, it is enough to show that for $\tilde{y} \in X^{B'}$ arbitrary the set $A_{\tilde{y}} \cap U_{\tilde{y}}$ (constructed analogously to $A_y \cap U_y$) is Zariski-dense in $U_{\tilde{y}}$. Choose $\tilde{y} \in X^{B'}$ arbitrary and construct $U_{\tilde{y}}$ as above. Then $X^{B'} \cap U_{\tilde{y}} \cap U_y$ is Zariski-dense in $X^{B'} \cap U_{\tilde{y}}$ and $\varphi'^{-1}(X^{B'} \cap U_{\tilde{y}} \cap U_y) \cap U_{\tilde{y}}$ is Zariski-dense in U_y since $\varphi'|U_y$ is algebraic. Then $A_{\tilde{y}} \cap U_{\tilde{y}}$ is Zariski-dense in $U_{\tilde{y}}$. □

Remark 2.15. – By construction φ' is surjective and each fiber of φ' contains exactly one closed S'-orbit. Since $X^{B'}$ is normal it can be identified locally with the categorical quotient of X by the S'-action ([Kr], p. 105). Thus φ' is G^0-equivariant ([Kr], p. 139). □

Now it is possible to classify spaces which are complex-symmetric with respect to a semisimple group.

THEOREM 2.16. – *Let (X,G) be complex-symmetric, X normal, $G^0 = S = S_1 \cdot \ldots \cdot S_n$ semisimple and assume that the open S-orbit Ω is Stein. Then Ω is the product $Q_{(m_1)} \times \ldots \times Q_{(m_n)}$ of affine quadrics and X is correspondingly the product $Q_{m_1} \times \cdots \times Q_{m_n}$ of compact quadrics.*

Proof. – We proceed by induction on the number n of factors in the decomposition $S = S_1 \cdot \ldots \cdot S_n$. The case $n = 1$ is Lemma 2.12. By Lemma 2.12 it follows that we can assume

$$S = SO(m_1, \mathbb{C}) \times \cdots \times SO(m_n, \mathbb{C}).$$

Let $S = S_1 \times S'$ and let $B_1 < S_1$ (resp. $B' < S'$) be Borel subgroups of S_1 (resp. of S'). The corresponding sets of fixed points are denoted by X^{B_1} (resp. $X^{B'}$).

Now let E_i be the unique S-orbit of codimension one in X where S_i has compact orbits. By Corollary 2.10 we can apply induction on $\bar{E}_i$. Thus $\bar{E}_i$ is a product of compact quadrics, and it follows that

$$X^{B_1} \cong Q_{m_2} \times \cdots \times Q_{m_n} \quad \text{and} \quad X^{B'} \cong Q_{m_1}.$$

Next, let φ_1 (resp. φ') be the projection on X^{B_1} (resp. $X^{B'}$) introduced in Lemma 2.14.

From this Lemma it follows that we have an algebraic mapping

$$\varphi = (\varphi', \varphi_1) : X \to Q_{m_1} \times (Q_{m_2} \times \cdots \times Q_{m_n}) = : \tilde{X}$$

which is surjective and S-equivariant. Note that the S-action on $\tilde{X}$ is induced by the S_i-action on Q_{m_i} introduced in Example 2.11.

If the open S-orbit in X is denoted by $\Omega = S/H$, then the open S-orbit $\tilde{\Omega}$ in $\tilde{X}$ is of the form $\tilde{\Omega} = S/\hat{H}$ and the set $\Gamma : = \hat{H}/H$ is finite.

But $\varphi(\Omega) = \tilde{\Omega}$ is a product of affine quadrics, hence simply connected (Example 2.11). Thus $\Omega \cong \tilde{\Omega}$.

Now for each $i \in \{1, \ldots, n\}$ there exists a unique S-orbit E_i (resp. $\tilde{E}_i$) in X (resp. $\tilde{X}$) such that

$$\overline{S_i(x)} \backslash S_i(x) \subset E_i, \quad \forall x \in \Omega$$
$$(\text{resp. } \overline{S_i(x)} \backslash S_i(x) \subset \tilde{E}_i, \ \forall x \in \tilde{\Omega}).$$

This follows from Lemma 2.9 for $x \in \Omega$ (resp. from the product structure of $\tilde{X}$ for $x \in \tilde{\Omega}$).

Thus $\varphi(E_i) = \tilde{E}_i$ and it is easy to see that

$$\varphi | E_i : E_i \to \tilde{E}_i \text{ is biholomorphic.}$$

Repeating the same arguments for the lower-dimensional S-orbits we see that the restriction of φ to each S-orbit in X is bijective. Thus it follows that φ is bijective and therefore a bialgebraic mapping. So the Theorem is proved by induction. □

COROLLARY 2.17. – *Let (X,G) be complex-symmetric. Assume that X is normal and that G^0 is semisimple. Then X is homogeneous and therefore a Hermitian symmetric space of the compact type.* □

After having classified spaces which are complex-symmetric with respect to a semisimple group, we now come back to the case (X,G) complex-symmetric, $G^0 = S \cdot T$ reductive and X normal. First we recall the definition of a complex-symmetric torus embedding from the introduction.

DEFINITION. – *A compact torus embedding is an algebraic compactification X of an algebraic torus $T \simeq (\mathbb{C}^*)^k$ such that the natural action of T on T given by the group action extends to an algebraic action of T on X.*

A compact torus embedding X is called complex-symmetric, if X is complex-symmetric with respect to G and $G^0 = T$.

So a complex-symmetric torus embedding is a complex-symmetric space where the symmetries respect the decomposition into T-orbits.

We will prove that a normal space which is complex-symmetric with respect to a reductive group is biholomorphic to a product of a normal complex-symmetric torus embedding and a space which is complex-symmetric with respect to a semisimple group.

The proof follows an idea of Domingo Luna. The original proof was much more complicated and much longer than the one presented here.

Now let $B < S$ be a Borel subgroup of S and $Y := X^B$ be the set of fixed points of B in X. From Lemma 2.14 it follows that Y is normal and that we have a G^0-equivariant surjective algebraic mapping $\varphi : X \to Y$ by projecting $x \in X$ on the unique fixed point of B in $\overline{S(x)}$.

It is clear that Y is T-stable and consists of finitely many T-orbits. Moreover, a symmetry s_y for $y \in Y$ leaves Y invariant. Thus

LEMMA 2.18. – *The set $Y = X^B$ is a normal complex-symmetric torus embedding.* □

Now the set Y^T of fixed points of T in Y is finite by Proposition 2.5. Let $a \in Y^T$ be arbitrary. Then we claim

LEMMA 2.19. – *The torus T acts trivially on the set $\varphi^{-1}(a)$.*

Proof. – Since φ is G^0-equivariant the set $\varphi^{-1}(a)$ is G^0-invariant. Thus an irreducible component E of $\varphi^{-1}(a)$ consists of finitely many G^0-orbits and is normal by Proposition 2.5. Moreover, the irreducible component E has only one compact G^0-orbit since $\varphi^{-1}(a) \cap Y = \{a\}$ and each compact S-orbit has exactly one point of intersection with Y.

Now if $\varphi^{-1}(a) \not\subset X^T$, the torus T does not act trivially on the open G^0-orbit $\Omega_a = G^0/H$ of E. Therefore $N_{G^0}(H)/H$ is not finite and $\bar{\Omega}_a = E$ contains more than one compact G^0-orbit by Lemma 2.7. Contradiction. Thus $\varphi^{-1}(a) \subset X^T$. □

By a Theorem of Sumihiro ([Su], p. 8) it follows that each fixed point a of T in Y has an open T-invariant affine neighbourhood V_a in Y. There exists an algebraic one-parameter subgroup $\lambda_a : \mathbb{C}^* \to T$ with $\lambda_a(\mathbb{C}^*) = : T_a$ such that λ_a is injective and $Y^{T_a} = Y^T$ is discrete.

Moreover, by [T-E] (Theorem 1′, p. 8) it follows that λ_a can be chosen such that $V_a = \{y \in Y | \lim_{t \to 0} \lambda_a(t) \cdot y = a\}$.

Defining $U_a : = \varphi^{-1}(V_a)$ it follows from the G^0-equivariance of φ that U_a is G^0-stable and that

$$U_a = \{x \in X | \lim_{t \to 0} \lambda_a(t) \cdot x \in \varphi^{-1}(a)\}.$$

It is clear that $\varphi^{-1}(a)$ is a connected component of X^T. Since U_a is dense in X it follows from [Ko] (p. 296) that $\varphi^{-1}(a)$ is irreducible.

Now it is possible to analyse the fibers of φ in detail.

LEMMA 2.20. – *All fibers of φ are smooth, consist of finitely many S-orbits, and have the same dimension.*

Proof. – Let a be a fixed point of T. Then $Z_a : = \varphi^{-1}(a)$ is irreducible and consists of finitely many G^0-orbits. Thus Z_a is normal and complex-symmetric. Moreover, Z_a is smooth by Corollary 2.17.

Now the set of all points where the fibers are normal and reduced (resp. smooth) is Zariski-open in Y ([EGA], IV, § 12, 1.7) and T-invariant. Since T acts with finitely many orbits in Y each fiber of φ is normal (resp. smooth).

Moreover, it follows from Remark 2.15 and [Kr] (Satz 4, p. 141) that the set of fibers which consist of finitely many orbits is T-invariant and Zariski open in X. Thus all fibers of φ consist of finitely many orbits.

From [G-R] (p. 160) it follows that there exists a fixed point of T such that Z_a is maximal-dimensional among all fibers of φ. Thus the open S-orbit $\Omega_a \subset Z_a$ has the same dimension as the generic S-orbits in X. Since a generic fiber of φ contains a generic S-orbit the generic fibers have the same dimension as Z_a. Thus all fibers have the same dimension. □

It follows from a Theorem of Konarski ([Ko], Theorem 2; in the smooth case cf. Bialynicki-Birula ([BB])) that the mapping

$$\psi_a : U_a \to \varphi^{-1}(a) = Z_a \quad \text{defined by} \quad x \mapsto \lim_{t\to 0} \lambda_a(t)\cdot x$$

is algebraic.

We then have

LEMMA 2.21. – *The product mapping*

$$\varphi \times \psi_a : U_a \to V_a \times Z_a$$

is biregular.

Proof. – By construction it is clear that $\varphi \times \psi_a$ is algebraic and surjective.

So it is enough to prove that $\varphi \times \psi_a$ is injective.

Let $Z_y = \varphi^{-1}(y)$ be a fiber of $\varphi | U_a$. Then Z_y is smooth and S acts with finitely many orbits on Z_y by Lemma 2.20. If we prove that $\psi_a | Z_y : Z_y \to Z_a$ is bijective for $y \in V_a$ arbitrary, it is clear that $\varphi \times \psi_a$ is injective.

We will first show that the open S-orbit Ω_y of Z_y is mapped onto the open S-orbit $\Omega_a \subset Z_a$. It is clear that for a point x in the open G^0-orbit $\Omega \subset X$ the image $\psi_a(x)$ is contained in the open S-orbit $\Omega_a \subset Z_a$. Let $H := \mathrm{Iso}_x(S)$. Then $\overline{T(x)}$ is an irreducible component of the set of fixed points of H since $N_S(H)/H^0$ is finite by Lemma 2.3. Again by Lemma 2.3 the image $\psi_a(\overline{T(x)})$ consists of a point $z \in \Omega_a$ with $\mathrm{Iso}_z(S)^0 = H^0$. Thus $\psi_a(\overline{T(x)} \cap U_a) = \{z\}$ and it follows that for each $y \in V_a$ the open S-orbit $\Omega_y \subset Z_y$ is mapped onto the open orbit $\Omega_a \subset Z_a$.

Now we will prove that $\psi_a|Z_y: Z_y \to Z_a$ is bijective. For the open S-orbit Ω_y of Z_y we have $\psi_a(\Omega_y) = \Omega_a$ and both sets have the same dimension. Since Ω_a is simply connected by Corollary 2.17, it follows that Ω_y is mapped bijectively onto Ω_a.

Let E_a be an S-orbit of codimension one in Z_y. Then there exists an S-orbit E_y of codimension one in Z_y with $\psi_a(E_y) = E_a$. Since E_a is simply connected we have $E_y \cong E_a$. From Lemma 2.8 and Lemma 2.9 (applied to Z_y) it follows that Z_y cannot have more S-orbits of codimension one than Z_a (for the orbit structure of Z_y we don't need that Z_y is complex-symmetric. We only need the structure of the S-orbits and the normality of Z_y).

This proves that the mapping ψ_a restricted to $\Omega_y \cup \{S\text{-orbits of codimension one}\}$ is bijective onto its image.

The arguments above can be repeated for the lower-dimensional S-orbits since all S-orbits in Z_a are simply connected by Corollary 2.17, and the orbit structures of Z_a and Z_y coincide by Lemma 2.8 and Lemma 2.9.

This proves that $\psi_a: Z_y \to Z_a$ is bijective.

Hence $\varphi \times \psi_a: U_a \to V_a \times Z_a$ is bijective. □

We are now able to prove

THEOREM 2.22. – *Let (X,G) be a complex-symmetric normal variety and let $G^0 = T \cdot S$ be reductive. Then*

$$X \cong Y \times Z \quad (G\text{-}equivariantly),$$

where Y is a normal complex-symmetric torus embedding and Z is a smooth variety which is complex-symmetric with respect to a group with identity component S.

Proof. – Let $Y := X^B$ (B a Borel subgroup of S) and let $\varphi : X \to Y$ be the algebraic mapping introduced in Lemma 2.14. By Lemma 2.18 the set Y is a normal complex-symmetric torus embedding and by Lemma 2.21 there exists a finite covering $(V_a)_{a \in Y^T}$ with Zariski-open sets such that

$$\varphi^{-1}(V_a) \cong V_a \times Z$$

where Z is a smooth variety which is complex-symmetric with respect to a group with identity component S.

The Theorem is proved, if we show that this fibration is globally trivial.

Now the transition automorphisms have to respect the S-orbits of Z. By Theorem 2.16 and Corollary 2.17 such automorphisms are induced by automorphisms of the quadric Q_n that respect the orbit structure given by the $SO(n,\mathbb{C})$-action on Q_n as introduced in Example 2.11. Since $\mathrm{Aut}_{\mathcal{O}}(Q_n) = SO(n+1,\mathbb{C})$ and $N_{SO(n+1,\mathbb{C})}(SO(n,\mathbb{C}))/SO(n,\mathbb{C})$ is finite the structure group of the fibration is finite.

Since for a finite group $\mathcal{G}$ and a covering $\mathcal{U} = (U_i)_{i \in I}$ with U_{ij} connected we have $H^1(\mathcal{U},\mathcal{G}) = 0$ the fibration is trivial. $\square$

Final Remark. – By Theorem 2.16 and Theorem 2.22 the classification of normal varieties which are complex-symmetric with respect to a reductive group is reduced to the classification of normal complex-symmetric torus embeddings.

This will be done for smooth embeddings in the next paragraph.

3. Classification of smooth complex-symmetric torus embeddings.

In this paragraph all complex-symmetric torus embeddings which are smooth varieties will be classified (cf. § 2 for the definition). By Theorem 2.22 this will give us a detailed description of all smooth varieties which are complex-symmetric with respect to a reductive group.

In order to make the ideas of our proof clearer many calculations which are basic Linear Algebra are not carried out in detail. These can be found in [L1] and [L2].

Our proof makes use of the fundamental fact that a normal torus embedding (i.e. a normal variety which is a torus embedding) can be described by a system of convex rational cones satisfying certain conditions. These systems are called fans or finite rational partial polyhedral decompositions. We will briefly summarize the basic facts that will be needed (for details cf. [Dan], [T-E] or [Oda]).

Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus, M the group of characters of T and N the group of algebraic one-parameter subgroups of T. The groups N and M are free abelian groups of rank k.

DEFINITION. — (1) *A cone* $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{R}} \mathbb{R}$ *is called a convex rational polyhedral cone, if there exist primitive vectors* $v_1, \ldots, v_s \in N$ *such that*

$$\sigma = \{x \in N_{\mathbb{R}} \mid x = \sum_{i=1}^{s} \lambda_i v_i, \lambda_i \geqslant 0, i = 1, \ldots, s\} =: \langle v_1, \ldots, v_s \rangle .$$

(2) *For* $v_1, \ldots, v_s \in N$ *the cone* $\sigma = \langle v_1, \ldots, v_s \rangle$ *is called simplicial, if the vectors* $v_1, \ldots, v_s$ *are linear independent over* $\mathbb{R}$. *It is called non-singular, if the vectors are part of a* $\mathbb{Z}$*-basis of* $N_{\mathbb{R}}$ *or if* $\sigma = \langle 0 \rangle$.

(3) *A fan in* $N_{\mathbb{R}}$ *is a finite system* $\Sigma = \{\sigma\}$ *of convex rational polyhedral cones satisfying the following conditions :*

(a) *each cone does not contain any linear subspace except* $\{0\}$;

(b) *if* $\sigma \in \Sigma$ *and* τ *is a face of* σ *(we write* $\tau < \sigma$*), then* $\tau \in \Sigma$;

(c) *if* $\sigma_1, \sigma_2 \in \Sigma$, *then* $\sigma_1 \cap \sigma_2 \in \Sigma$, *and* $\sigma_1 \cap \sigma_2$ *is a face of* σ_1 *and as well of* σ_2.

The following Proposition makes use of the fact that normal torus embeddings are covered by T-stable affine open subsets ([Su], p. 8).

PROPOSITION 3.1. — (1) *There exists a bijection between*

$$\{\text{fans in } N_{\mathbb{R}}\}$$

and

$$\{\text{normal torus embeddings } X_\Sigma \text{ of } T\} \quad \text{([Oda], p. 16).}$$

(2) *There is a bijection between*

$$\{\text{cones } \sigma \in \Sigma\}$$

and

$$\{T\text{-stable open affine subsets } X_\sigma \text{ of } X_\Sigma\} \quad \text{([T-E], p. 24).}$$

(3) *A T-equivariant isomorphism* $f\colon X_\Sigma \to X_{\Sigma'}$ *between normal torus embeddings induces an isomorphism* $\varphi : N \to N$ *such that its scalar extension* $\varphi : N_{\mathbb{R}} \to N_{\mathbb{R}}$ *satisfies*

$$\varphi(\Sigma) = \Sigma' :\Leftrightarrow \varphi(\sigma) =: \sigma' \in \Sigma' \quad \forall \sigma \in \Sigma \quad \text{([Oda], p. 10).}$$

(4) *There exists a map*

$$\text{orb} : \Sigma \to \{T\text{-orbits in } X_\Sigma\}$$

such that

$$\text{orb}(\langle 0 \rangle) = T, \qquad \tau_1 < \tau_2 \Leftrightarrow \text{orb}(\tau_2) \subset \overline{\text{orb}(\tau_1)},$$
$$\dim \tau + \dim \text{orb}(\tau) = \dim T = k \quad \text{([Oda], p. 11).}$$

(5) *The variety* X_Σ *is smooth, if and only if each cone* $\sigma \in \Sigma$ *is non-singular. If* X_Σ *is smooth, then* $\overline{\mathrm{orb}(\sigma)}$ *is again smooth* ([T-E], p. 14).

(6) *The variety* X_Σ *is complete, if and only if* $\mathrm{supp}(\Sigma) := \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$ ([Dan], p. 114).

(7) *The variety* X_Σ *can be written as the direct product of lower-dimensional torus embeddings, if there exist fans* Σ' *and* Σ'' *such that*

$$\Sigma = \Sigma' \times \Sigma'' := \{\sigma' \times \sigma'' \mid \sigma' \in \Sigma', \sigma'' \in \Sigma''\} \text{ ([Oda], p. 39).} \qquad \square$$

DEFINITION. — *Let* X_Σ *be a smooth torus embedding. Then the one-dimensional cones* $\sigma = \langle v \rangle$ *are called rays. The system of all rays is called the skeleton of* Σ

$$\mathrm{Sk}(\Sigma) := \{v \in N \mid v \text{ primitive}, \langle v \rangle \in \Sigma\}.$$

The following description of the direct product will be used later.

LEMMA 3.2. — *Let* Σ *be a complete non-singular fan in* $V := N_{\mathbb{R}}$ *(i.e. all cones* $\sigma \in \Sigma$ *are non-singular). Then* Σ *is the direct product*

$$\Sigma = \Sigma' \times \Sigma''$$

of non-singular lower-dimensional fans, if and only if there exist non-trivial subspaces V', V'' *of* V *such that*

(a) $V = V' \oplus V''$

(b) $\mathrm{Sk}(\Sigma) = (\mathrm{Sk}(\Sigma) \cap V') \cup (\mathrm{Sk}(\Sigma) \cap V'')$.

Moreover, we then have

$$\mathrm{Sk}(\Sigma) \cap V' = \mathrm{Sk}(\Sigma') \quad \text{and} \quad \mathrm{Sk}(\Sigma) \cap V'' = \mathrm{Sk}(\Sigma'').$$

Proof. — If Σ is a product, then the existence of subspaces with these properties is obvious. So we only have to prove the other direction. Each cone $\sigma \in \Sigma$ can be written as

$$\sigma = \langle v_1', \ldots, v_r', v_1'', \ldots, v_s'' \rangle \quad \text{with} \quad v_i' \in V', \quad v_j'' \in V''.$$

Thus $\sigma \cap V' = \langle v_1', \ldots, v_r' \rangle \in \Sigma$ is a non-singular cone in V' since $v_1', \ldots, v_r'$ is part of a $\mathbb{Z}$-basis of V'.

Therefore $\Sigma' := V' \cap \Sigma = \{\sigma \in \Sigma \mid \sigma \subset V'\}$ and $\Sigma'' := V'' \cap \Sigma$ are well-defined complete non-singular fans, and each cone $\sigma \in \Sigma$ is contained in

$$\tilde{\Sigma} := \{\sigma' \times \sigma'' \mid \sigma' \in \Sigma', \sigma'' \in \Sigma''\}.$$

But since Σ is complete we have $\Sigma = \tilde{\Sigma}$. $\square$

Remark. – Neither the completeness nor the non-singularity of Σ can be dropped (cf. [L2], p. 14 ; Example 3.15).

Let Σ be a non-singular fan and $\sigma \in \Sigma$. Then it follows that $\overline{\mathrm{orb}(\sigma)}$ is again a smooth torus embedding (Proposition 3.1) for the torus T' which is obtained as the quotient of T by the connected ([L2], Lemma 16) ineffectivity of the T-action on $\mathrm{orb}(\sigma)$.

The set of all cones in Σ which contain σ is called the star of σ

$$\mathrm{St}(\sigma) := \{\tau \in \Sigma | \tau > \sigma\} \text{ ([Dan], p. 115).}$$

We then have

PROPOSITION 3.3 ([Dan], p. 115). – *The fan Σ of the torus embedding $\overline{\mathrm{orb}(\sigma)}$ is obtained by projecting* $\mathrm{St}(\sigma)$ *on the factor space $N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$, where $(\sigma)_{\mathbb{R}}$ is the smallest linear subspace of $N_{\mathbb{R}}$ containing σ.* □

We call a normal compact algebraic torus embedding X a complex-symmetric torus embedding, if for each $x \in X$ there exists a holomorphic automorphism s_x of order two such that

(a) x is isolated fixed point of s_x

(b) s_x is T-equivariant.

It follows from Proposition 3.1 (3) that each symmetry induces a fan automorphism. These automorphisms will be described in Theorem 3.7.

We first note

LEMMA 3.4. – *Let X be a smooth complete torus embedding. Then there always exist symmetries for the fixed points of the T-action.*

Proof. – Let $p \in X$ be a fixed point. By [Su] (p. 8) there exists a T-invariant affine neighbourhood of p. Since X is smooth this neighbourhood is isomorphic to $\mathbb{C}^k$ with the standard $(\mathbb{C}^*)^k$-action (Proposition 3.1 (5)) and p corresponds to $0 \in \mathbb{C}^k$. The automorphism $x \mapsto -x$ of $\mathbb{C}^k$ is a symmetry for 0. Since this automorphism is induced by the T-action it can be extended to all of X. □

Symmetries for the other points in X cannot be found so easily : A fan Σ is called centrally symmetric, if $\varphi = -\mathrm{Id}$ is a fan automorphism, i.e. $\langle v_1, \ldots, v_s\rangle \in \Sigma \Leftrightarrow \langle -v_1, \ldots, -v_s\rangle \in \Sigma$.

Since a complete torus embedding is a compactification of an abelian group we immediately deduce from Lemma 1.1.

Lemma 3.5. — *Let X be a complete normal torus embedding with open T-orbit $\Omega = T$. Then there exist symmetries for $x \in \Omega$, if and only if the associated fan Σ is centrally symmetric.* □

The following Lemma is proved very easily using faithful linearization and the fact that the differential of a symmetry at a fixed point x of a smooth variety is $-\mathrm{Id}$. However, the Lemma is very important for our results and it is not clear whether it is valid in the singular case.

Lemma 3.6. — *Let X be a complete smooth complex-symmetric torus embedding and E a T-orbit in X. Then $\bar{E}$ is again a complete smooth complex-symmetric torus embedding via the symmetries for the ambient space.* □

It is clear that a symmetry for $x \in X_\Sigma$ induces a fan automorphism of order two, i.e. an automorphism $\varphi : N \to N$ of order two such that its scalar extension satisfies $\varphi(\sigma) \in \Sigma \,\forall \sigma \in \Sigma$. We now establish a necessary and sufficient condition for a fan to be the fan of a complex-symmetric torus embedding.

Theorem 3.7. — *Let Σ be a non-singular complete fan in $N_{\mathbb{R}}$. Then Σ is the fan of a complex-symmetric smooth torus embedding, if and only if Σ fulfills the following condition :*

For each cone $\sigma \in \Sigma$ there exists a fan automorphism φ_σ of order two with

$$\varphi_\sigma | (\sigma)_{\mathbb{R}} = \mathrm{Id}$$

such that for the induced automorphism

$$\tilde{\varphi}_\sigma : N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \to N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$$

we have

$$\tilde{\varphi}_\sigma = -\mathrm{Id}.$$

Proof. — (1) First assume that X is complete, smooth and complex-symmetric. Let $E = \mathrm{orb}(\sigma) = \mathrm{orb}(\langle v_1, \ldots, v_s\rangle)$ be a T-orbit, $x \in E$, s_x the symmetry for x, and φ the induced fan automorphism of order two. Since X is smooth we have $\bar{E} = \overline{\mathrm{orb}(\langle v_1\rangle)} \cap \ldots \cap \overline{\mathrm{orb}(\langle v_s\rangle)}$ where $\overline{\mathrm{orb}(\langle v_i\rangle)}$ is a T-invariant divisor. Now each subspace of the tangent space $T_x(X)$ is stabilized by the differential $d_x s_x$, since $d_x s_x = -\mathrm{Id}$. Thus $\varphi(\langle v_i\rangle) = \langle v_i\rangle$ and $\varphi(\sigma) = \sigma$. Moreover, we have $\varphi(v_i) = v_i$ because the v_i are primitive elements of N. This gives $\varphi | (\sigma)_{\mathbb{R}} = \mathrm{Id}$.

The symmetry s_x induces $-\mathrm{Id}$ on the complete fan corresponding to the torus embedding $\bar{E}$ (by Lemma 3.5). This fan is obtained as the projection of $\mathrm{St}(\sigma)$ on $N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$. So for the induced automorphism $\tilde{\varphi}$ of the factor space we have $\tilde{\varphi} = -\mathrm{Id}$.

Thus the condition is necessary.

(2) Now assume that the fan Σ fulfills the condition in the Theorem. We begin by showing that for a one-dimensional cone $\sigma = \langle v \rangle$ the fan automorphism φ_σ induces a symmetry for a point in the one-codimensional orbit $E := \mathrm{orb}(\langle v \rangle)$.

We have $E \cong T' := T/v$ (v also denotes a one-parameter subgroup of T by construction). Let f_σ be an automorphism of X induced by φ_σ (this automorphism is unique up to translations). Then E is stabilized by f_σ, since $\varphi_\sigma(\sigma) = \sigma$. Since for $\tilde{\varphi}_\sigma : N_{\mathbb{R}}/(v)_{\mathbb{R}} \to N_{\mathbb{R}}/(v)_{\mathbb{R}}$ we have $\tilde{\varphi}_\sigma = -\mathrm{Id}$ it follows that $f_\sigma|E$ is of the form $t' \to t'_0 t'^{-1}$ $(t', t'_0 \in T')$. Thus f_σ has a fixed point $x \in E$ which is isolated in E. Since $\varphi_\sigma^2 = \mathrm{Id}$ it is clear that $f_0^2 \in v$. Thus the group S generated by f_σ and v is reductive and there exists a decomposition $T_x(X) = T_x(\bar{E}) \oplus W$ of $T_x(X)$ such that for each $t \in S$ the differential $d_x t$ stabilizes $T_x(\bar{E})$ and W.

Now v does not act trivially on W. Hence we can find an element $t \in v$ such that $\hat{f}_\sigma := t \circ f_\sigma$ satisfies

$$d_x \hat{f}_\sigma = -\mathrm{Id} \qquad \text{and} \qquad d_x \hat{f}_\sigma^2 = \mathrm{Id}.$$

By faithful linearization it is clear that x is an isolated fixed point of $\hat{f}_\sigma$ and that $\hat{f}_\sigma^2 = \mathrm{Id}$. Thus we have found a symmetry for x.

(3) Now we can prove the sufficiency of the condition in the Theorem by induction. Assume that for each non-singular fan Σ which fulfills the condition in the Theorem the corresponding fan automorphism induces symmetries for the T-orbits of codimension smaller than s.

Let $\sigma = \langle v_1, \ldots, v_s \rangle \in \Sigma$ and $E := \mathrm{orb}(\sigma)$ be the corresponding T-orbit of codimension s. Let $D_1 := \overline{\mathrm{orb}(\langle v_1 \rangle)}$ be a T-invariant divisor containing E. Then φ_σ induces a fan automorphism $\hat{\varphi}_\sigma$ of the fan $\Sigma_{\langle v_1 \rangle}$ belonging to D_1.

By induction $\hat{\varphi}_\sigma$ induces a symmetry in D_1 for $x \in E$. Thus there exists $x \in E \subset X$ and an automorphism f_σ of X such that $f_\sigma(x) = x$ and $d_x f_\sigma | D_1 = -\mathrm{Id}$.

Now the same arguments as in (2) show that we can find an automorphism $\hat{f}_\sigma$ which is a symmetry in X for $x \in E$. Thus the sufficiency is proved by induction. □

DEFINITION. — *A non-singular complete fan which fulfills the condition in Theorem* 3.7 *will be called a complex-symmetric fan.*

Remark. — If X_Σ is a singular but normal complete complex-symmetric torus embedding, then it is easy to see that Σ has the following property:

For each cone $\sigma \in \Sigma$ there exists a fan automorphism φ_σ of order two with $\varphi_\sigma(\sigma) = \sigma$ such that for the induced automorphism $\tilde{\varphi}_\sigma : N_{\mathbf{R}}/(\sigma)_{\mathbf{R}} \to N_{\mathbf{R}}/(\sigma)_{\mathbf{R}}$ we have

$$\tilde{\varphi}_\sigma = -\,\mathrm{Id}\,.$$

However, it is not clear that $\varphi_\sigma|(\sigma)_{\mathbf{R}} = \mathrm{Id}$.

As a first application we will classify all smooth two-dimensional complex-symmetric torus embeddings.

THEOREM 3.8. — *There are exactly four non-isomorphic smooth complete complex-symmetric torus embeddings of dimension two :*

1) $\mathbb{P}_1 \times \mathbb{P}_1$;

2) *the del Pezzo-surface, i.e.* $\mathbb{P}_1 \times \mathbb{P}_1$ *blown up in two suitable fixed points of the standard* $(\mathbb{C}^*)^2$*-action on* $\mathbb{P}_1 \times \mathbb{P}_1$;

3) $Y := \mathbb{P}_1 \times \mathbb{P}_1$ *blown up in all four fixed points of the standard* $(\mathbb{C}^*)^2$*-action on* $\mathbb{P}_1 \times \mathbb{P}_1$;

4) Y *blown up in four suitable fixed points of the* $(\mathbb{C}^*)^2$*-action on* Y.

Proof. — Let Σ be a two-dimensional complex-symmetric fan and $\mathcal{G}$ be the finite group generated by $\{\varphi_\sigma | \sigma \in \Sigma\}$. Then it is clear from Theorem 3.7 that $\mathcal{G}$ acts transitively on the set of two-dimensional cones in Σ and with at most two orbits on the set of one-dimensional cones ([L2], Lemma 23).

If $\sigma = \langle v \rangle$ is a one-dimensional cone, then σ is a face of two maximal-dimensional cones $\langle v_1, v \rangle$ and $\langle v_2, v \rangle$ and the integer λ defined by $v_1 + v_2 = (-\lambda).v$ is the self-intersection number of the divisor $\overline{\mathrm{orb}(\langle v \rangle)}$ ([Oda], Proposition 6.7). Note that all vectors are primitive and that all cones are smooth by assumption.

Now let X_Σ be a minimal two-dimensional complex-symmetric torus embedding, i.e. X_Σ cannot be blown down T-equivariantly onto a manifold. Then $X_\Sigma = \mathbb{P}_1 \times \mathbb{P}_1$ since minimal two-dimensional torus embeddings are classified (e.g. [Oda], Theorem 8.2) and the fan of $\mathbb{P}_1 \times \mathbb{P}_1$ is the only centrally symmetric one.

So we assume that X_Σ is not minimal. Let

$$\mathrm{Sk}(\Sigma) = \{v_0, \ldots, v_{k-1}, v_k = -v_0, \ldots, v_{2k-1}\}$$

(the vectors ordered counterclockwise in $\mathbb{R}^2$). We may assume that $v_0 = e_1$, $v_1 = e_2$ and that $\overline{\mathrm{orb}(\langle e_2\rangle)}$ has self-intersection number -1. The self-intersection number of $\overline{\mathrm{orb}(\langle e_1\rangle)}$ is denoted by λ. Since the first coordinate of v_{2k-1} is strictly positive and

$$v_1 + v_{2k-1} = -\lambda v_0 \Leftrightarrow e_2 + v_{2k-1} = -\lambda e_1,$$

it is clear that $\lambda < 0$.

We claim that $-3 \leqslant \lambda$.

Indeed, if $\lambda \neq -1$ all divisors with self-intersection number -1 can be blown down equivariantly one after the other since blowing down $\overline{\mathrm{orb}(\langle v_i\rangle)}$ only affects the self-intersection number of $\overline{\mathrm{orb}(\langle v_{i+1}\rangle)}$ and $\overline{\mathrm{orb}(\langle v_{i-1}\rangle)}$ ([Oda], p. 43 and Proposition 6.7). It follows from [Oda] (p. 43) that the skeleton of the fan $\tilde{\Sigma}$ obtained in this way is

$$\mathrm{Sk}(\tilde{\Sigma}) = \{v_0, v_2, \ldots, v_{2k-2}\}$$

and that all self-intersection numbers are $\lambda + 2$.

Now if $\lambda < -3$, the variety $X_{\tilde{\Sigma}}$ is minimal and all self-intersection numbers are negative. But such a variety does not exist ([Oda], Theorem 8.2).

So we only have to consider the cases $\lambda = -1$, $\lambda = -2$ and $\lambda = -3$. For $\lambda = -1$ we have

$$\mathrm{Sk}(\Sigma) = \{e_1, e_2, -e_1 + e_2, -e_1, -e_2, +e_1 - e_2\}.$$

For $\lambda = -2$ we have

$$\mathrm{Sk}(\Sigma) = \{\pm e_1, \pm e_2, \pm(2e_1 - e_2), \pm(e_1 - e_2)\}.$$

For $\lambda = -3$ we have

$$\mathrm{Sk}(\Sigma) = \{\pm e_1, \pm e_2, \pm(3e_1 - e_2), \pm(e_1 - e_2), \pm(2e_1 - e_2)\}.$$

In each case, by a direct application of Theorem 3.7, we can prove that Σ is a complex-symmetric fan ([L2], pp. 24-28). From [Oda] (p. 43) it is easy to see that the fans defined above are exactly the fans of the varieties in the Theorem. $\square$

In order to classify all smooth complex-symmetric torus embeddings we make use of the classification of Coxeter groups.

Recall that a finite subgroup $\mathcal{G}$ of the orthogonal group $O(\mathbb{R}^n)$, which is generated by reflections at linear hyperplanes, is called a *Coxeter group* ([G-B]).

For each reflection $S \in \mathcal{G}$ there are two unit vectors orthogonal to the hyperplane stabilized pointwise by S. These are called the *roots* of S, and the set of all roots is called the *root system* Δ of $\mathcal{G}$.

A *base* π of Δ is a set $\{r_1, \ldots, r_n\}$ of roots which is a basis of $\mathbb{R}^n$ such that each element of Δ can be written as a linear combination of the r_i with all coefficients either non-negative or non-positive ([G-B], p. 37). The reflections corresponding to a fixed base π are called *fundamental reflections* and are denoted by S_{r_i} $(r_i \in \pi)$. The group $\mathcal{G}$ is generated by these reflections. A Coxeter group is called *irreducible*, if its base cannot be written as union of two non-empty orthogonal subsets ([G-B], p. 56). Equivalently, an irreducible Coxeter group cannot be written as a direct product of proper subgroups which are Coxeter groups.

By using the classification of positive definite Coxeter graphs ([G-B]) it is possible to classify all irreducible Coxeter groups ([G-B], Theorem 5.12; [Cox]) and to determine their root systems explicitly up to linear equivalence. We will make use of this explicit description. But before we have to establish a link between complex-symmetric fans and Coxeter groups.

THEOREM 3.9. — *Let Σ be a complex-symmetric fan and $\mathcal{G}$ be the group generated by $\{\varphi_\sigma \circ (-\mathrm{Id}) | \sigma = \langle v \rangle;\, v \in \mathrm{Sk}(\Sigma)\}$ (φ_σ as in Theorem 3.7). Then by choosing a suitable inner product $\mathcal{G}$ is a Coxeter group with root system*

$$\Delta = \left\{ \frac{v}{\|v\|} \,\middle|\, v \in \mathrm{Sk}(\Sigma) \right\}.$$

Moreover, the group $\mathcal{G}$ is reducible, if and only if Σ is the product $\Sigma' \times \Sigma''$ of lower-dimensional fans.

Proof. – Since $-\mathrm{Id} = \varphi_{\langle 0 \rangle}$, the group $\mathscr{G}$ is a subgroup of the group generated by $\{\varphi_\sigma | \sigma \in \Sigma\}$, which is a finite group since Σ is a finite set.

Hence, by a suitable linear change of coordinates we may assume that this group is a subgroup of the orthogonal group. Note that this change of coordinates perhaps destroys the property that the vectors $v \in \mathrm{Sk}(\Sigma)$ are (primitive) elements of $N \cong \mathbb{Z}^k$.

Now for a one-dimensional cone $\sigma = \langle v \rangle$ the mapping $\varphi_\sigma \circ (-\mathrm{Id}) =: \hat{\varphi}_\sigma$ is a reflection by Theorem 3.7 with roots $\pm \dfrac{v}{\|v\|}$. Thus $\mathscr{G}$ is a Coxeter group and it is clear that $\mathscr{G}$ does not act trivially on any linear subspace of $V := N_\mathbb{R}$.

Now let $\mathscr{G}$ be reducible. Then Δ is the union of two non-empty mutually orthogonal subspaces. Thus there exists a decomposition $V = V' \oplus V''$ of V such that $\Delta = (\Delta \cap V') \cup (\Delta \cap V'')$.

Therefore

$$\mathrm{Sk}(\Sigma) = (\mathrm{Sk}(\Sigma) \cap V') \cup (\mathrm{Sk}(\Sigma) \cap V''),$$

and by Lemma 3.2 we have $\Sigma = \Sigma' \times \Sigma''$.

Now assume that Σ is reducible. By Lemma 3.2 there exist non-trivial subspaces V' and V'' such that

$$V = V' \oplus V'' \quad \text{and} \quad \mathrm{Sk}(\Sigma) = (\mathrm{Sk}(\Sigma) \cap V') \cup (\mathrm{Sk}(\Sigma) \cap V'').$$

Thus

$$\Delta = (\Delta \cap V') \cup (\Delta \cap V'') =: \Delta' \cup \Delta''.$$

If we show that Δ' is orthogonal to Δ'', it follows that $\mathscr{G}$ is reducible.

Let $\dfrac{v}{\|v\|} \in \Delta'$ and E_v be the hyperplane stabilized pointwise by $\hat{\varphi}_\sigma$. Then $\hat{\varphi}_\sigma(\Delta') = \Delta'$, $\hat{\varphi}_\sigma(\Delta'') = \Delta''$ and $\hat{\varphi}_\sigma(V'') = V''$. Since $\hat{\varphi}_\sigma$ is a reflection it follows that $V'' \subset E_v$. Therefore each element $v \in \Delta'$ is orthogonal to V'' and thus to Δ''. □

Remark. – If Σ is a complete fan defining a singular but normal complex-symmetric torus embedding, it follows from the Remark after Theorem 3.7 that the group $\mathscr{G}$ generated by

$$\{\hat{\varphi}_\sigma | \sigma = \langle v \rangle \in \Sigma\}$$

is a Coxeter group with root system

$$\Delta = \left\{ \frac{v}{\|v\|} \,\middle|\, v \in \mathrm{Sk}(\Sigma) \right\}.$$

However, it may happen that Σ is irreducible although $\mathscr{G}$ is reducible (cf. Example 3.15).

We will now explain how the classification of Coxeter groups is used to classify complex-symmetric torus embeddings. Let Σ be a complex-symmetric fan of dimension k which is not the product $\Sigma' \times \Sigma''$ of lower-dimensional fans. We associate an irreducible Coxeter group of rank k to Σ as in Theorem 3.9. If $k \geqslant 3$, then there is only a finite number of non-isomorphic Coxeter groups of rank k, and we will show that the root system of an irreducible Coxeter group of rank $k \geqslant 3$ cannot define a complex-symmetric fan.

This will be done as follows.

Let Δ be the root system of an irreducible Coxeter group of rank $k \geqslant 3$. Then Δ can be described explicitly and in each case there exists $r \in \Delta$ such that the stabilizer subgroup $\mathrm{Stab}_r(\mathscr{G})$ is a Coxeter group of smaller rank (either $k-1$ or $k-2$).

If we construct the fan $\Sigma_{\langle r \rangle}$ corresponding to $\overline{\mathrm{orb}\,(\langle r \rangle)}$ by projecting the cones of $\mathrm{St}\,(\langle r \rangle)$ orthogonally on $(r)_{\mathbb{R}}^{\perp}$, then $\mathrm{Stab}_r(\mathscr{G})$ acts on $\Sigma_{\langle r \rangle}$ as a group of orthogonal transformations. By induction we know that the fan $\Sigma_{\langle r \rangle}$ is the product of one- and two-dimensional fans. Since all fan automorphisms of Σ can be chosen to be orthogonal it follows that the irreducible factors of $\Sigma_{\langle r \rangle}$ are orthogonal to each other.

It is difficult to decide which vectors of Δ define cones in $\mathrm{St}\,(\langle r \rangle)$. Thus we consider the orthogonal projection $\tilde{\Delta}$ of all vectors of Δ on $(r)_{\mathbb{R}}^{\perp}$. We then determine which subsets of $\tilde{\Delta}$ are stabilized by the group $\mathrm{Stab}_r(\mathscr{G})$. In this way we are able to list all possibilities for $\mathrm{St}\,(\langle r \rangle)$. In some cases the set $\tilde{\Delta}$ will not contain $\mathrm{Stab}_r(\mathscr{G})$-stable subsets which define complex-symmetric fans of dimension $k-1$. In some other cases such subsets of $\tilde{\Delta}$ exist. Using the properties of complex-symmetric fans we then determine the possibilities for $\mathrm{St}\,(\langle r \rangle)$. In each case we will obtain a contradiction.

If one looks at the classification of Coxeter groups of rank $k \geqslant 3$, there are three infinite series $\mathscr{A}_k$, $\mathscr{B}_k$ and $\mathscr{D}_k$ $(k \geqslant 4)$ called classical groups and some exceptional groups which are denoted by $\mathscr{I}_3$, $\mathscr{I}_4$, $\mathscr{F}_4$, $\mathscr{E}_6$, $\mathscr{E}_7$, $\mathscr{E}_8$. Each case will be treated separately.

The following Lemmas will be useful in order to restrict the possible cases for $\Delta_{\langle r \rangle}$.

LEMMA 3.10. — *Let $\mathcal{G}$ be a Coxeter group of rank $k \geqslant 4$ with root system Δ and $r \in \Delta$ such that* $\mathrm{Stab}_r(\mathcal{G})$ *is an irreducible Coxeter group of rank $k-1$. If $\Sigma_{\langle r \rangle}$ corresponds to a product $\Delta_1 \times \cdots \times \Delta_s$ of irreducible root systems of rank smaller than three, then each Δ_i has rank one.*

Proof. — Assume that Δ_1 is two-dimensional. Let $\{v_1, v_2\}$ be a base of Δ_1. Then v_1 and v_2 are not orthogonal. Since $\mathrm{Stab}_r(\mathcal{G})$ is irreducible of rank $k-1$ and $k \geqslant 4$ there exists $\varphi \in \mathrm{Stab}_r(\mathcal{G})$ such that $\varphi(v_1) \notin \Delta_1$. Then it follows that $\varphi(v_2) \notin \Delta_1$ and that $\varphi(\Delta_1) \perp \Delta_1$, since φ is orthogonal.

Moreover, we may assume that φ is a reflection. But then $\dim(E_\varphi \cap (\Delta_1)_{\mathbb{R}}) \geqslant 1$,where E_φ is the hyperplane stabilized pointwise by φ. This contradicts $\varphi((\Delta_1)_{\mathbb{R}}) \cap (\Delta_1)_{\mathbb{R}} = \{0\}$.

Therefore all factors Δ_i have rank one. □

With the same methods it can be shown :

LEMMA 3.11. — *Let $\mathcal{G}$ be a Coxeter group of rank $k \geqslant 5$ with root system Δ and $r \in \Delta$ such that* $\mathrm{Stab}_r(\mathcal{G})$ *contains an irreducible Coxeter group of rank $k-2$. If $\Sigma_{\langle r \rangle}$ corresponds to a product $\Delta_1 \times \cdots \times \Delta_s$ of irreducible root systems of rank smaller than three, then each Δ_i has rank one.* □

Now we will prove

THEOREM 3.12. — *Let Σ be a complex-symmetric fan of dimension $k \geqslant 2$. Then Σ can be written as a product*

$$\Sigma_{i_1} \times \cdots \times \Sigma_{i_\ell} \times (\Sigma_0)^s \, (s + 2\ell = k),$$

where Σ_{i_j} is a two-dimensional complex-symmetric fan and Σ_0 is the one-dimensional complex-symmetric fan.

Proof. — We proceed by induction on k. The case $k = 2$ is Theorem 3.8. The different types of Coxeter groups will be treated separately. Let $\mathcal{G}$ be an irreducible Coxeter group, Δ its root system, π a base, $r \in \Delta$ chosen such that $\mathrm{Stab}_r(\mathcal{G})$ is a Coxeter group and $\tilde{\Delta}$ the orthogonal projection of Δ on $(r)^{\perp}_{\mathbb{R}}$.

Case I : $\mathcal{G} = \mathcal{A}_k$:

$\Delta = \{\pm(e_i - e_j) | 1 \leqslant j < i \leqslant k+1\} \subset (1, \ldots, 1)^{\perp}_{\mathbb{R}} \subset \mathbb{R}^{k+1}$;
$\pi = \{r_i := e_{i+1} - e_i | 1 \leqslant i \leqslant k\}$ ([G-B], p. 71, p. 76);
$r = e_2 - e_1$; $\mathrm{Stab}_r(\mathcal{G}) \cong \mathcal{A}_{k-2} = \langle S_{r_3}, \ldots, S_{r_k} \rangle$ ([L2], Lemma 47);
$\tilde{\Delta} = \{\pm(e_i - e_j) | 3 \leqslant j < i \leqslant k+1\}$

$$\cup \left\{ \pm \left[\frac{1}{2}(e_1 + e_2) - e_j \right] \middle| 3 \leqslant j \leqslant k+1 \right\} \text{ ([L2], Lemma 50).}$$

Case Ia : $k = 3$:

In this case we have

$$\tilde{\Delta} = \left\{ \pm(e_3 - e_4), \pm\left[\frac{1}{2}(e_1 + e_2) - e_3\right], \pm\left[\frac{1}{2}(e_1 + e_2) - e_4\right] \right\}.$$

Then the only $\mathrm{Stab}_r(\mathcal{G})$-stable subsets that define a complete fan are

$$\Delta_1 = \tilde{\Delta} \quad \text{or} \quad \Delta_2 = \left\{ \pm\left[\frac{1}{2}(e_1 + e_2) - e_j\right] \middle| j = 3,4 \right\},$$

since S_{r_3} exchanges e_3 and e_4.

But Δ_2 is not complex-symmetric since it has to be reducible by induction and the generating vectors of Δ_2 have to be orthogonal to each other. Note that we always assume that all fan automorphisms are orthogonal.

If we assume that $\mathrm{Sk}(\Sigma_{\langle r \rangle}) = \Delta_1$, then

$$\sigma_1 := \left\langle \frac{1}{2}(e_1 + e_2) - e_3, \frac{1}{2}(e_1 + e_2) - e_4 \right\rangle$$

and

$$\sigma_2 := \left\langle \frac{1}{2}(e_1 + e_2) - e_4, e_3 - e_4 \right\rangle$$

are cones in $\Sigma_{\langle r \rangle}$.

Since $\frac{1}{2}(e_1 + e_2) - e_j$ is the projection of $e_1 - e_j$ or $e_2 - e_j$ ([L2], p. 56) an easy calculation shows that the cone σ_1 can only be obtained as the projection of

$$\tau_1 := \langle e_2 - e_1, e_2 - e_3, e_2 - e_4 \rangle$$

(a cone of the form $<e_2-e_1, e_1-e_3>$ cannot exist in Σ!). Similarly, σ_2 is obtained as the projection of

$$\tau_2 := <e_2-e_1, e_2-e_4, e_3-e_4> .$$

By Theorem 3.7 and by [Oda] (p. 35) there exists an orthogonal fan automorphism φ with

$$\varphi(e_2-e_1)=e_2-e_1, \qquad \varphi(e_2-e_4)=e_2-e_4, \qquad \varphi(e_2-e_3)=e_3-e_4 .$$

Now $1 = (e_2-e_1, e_2-e_3) = (\varphi(e_2-e_1), \varphi(e_2-e_3)) = 0$.

Contradiction. So the root system of $\mathscr{A}_3$ does not define a complex-symmetric fan.

Case Ib : $k \geqslant 4$:

For $k \geqslant 5$ the fan $\Sigma_{\langle r\rangle}$ can only be the fan of $(\mathbb{P}_1)^{k-1}$ by Lemma 3.11. For $k = 4$ the fan $\Sigma_{\langle r\rangle}$ is either the fan of $(\mathbb{P}_1)^3$ or the fan of a product of $\mathbb{P}_1$ with a two-dimensional complex-symmetric torus embedding by induction.

In the last case $\tilde{\Delta}$ contains a vector v such that

$$\varphi(v) = \pm v \quad \forall \varphi \in \mathrm{Stab}_r(\mathscr{G}) .$$

But it can easily be shown that such a vector does not exist ([L2], Lemma 51).

In the other case $\tilde{\Delta}$ contains a vector v such that for all $\varphi \in \mathrm{Stab}_r(\mathscr{G})$ either $\varphi(v) = \pm v$ or $\varphi(v)$ is orthogonal to v. But such a vector does not exist, either ([L2], Lemma 52). Thus the root system of $\mathscr{A}_k$ does not define a complex-symmetric fan.

Case II : $\mathscr{G} = \mathscr{B}_k$, $k \geqslant 3$:

$\Delta = \{\pm e_i | 1 \leqslant i \leqslant k\} \cup \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant k\}$;
$\pi = \{r_1 := e_1, r_i := e_i - e_{i-1} | 2 \leqslant i \leqslant k\}$;
$r = e_k$, $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{B}_{k-1} = (S_{r_1}, \ldots, S_{r_{k-1}})$ ([L2], Prop. 45);
$\tilde{\Delta} = \{\pm e_i | 1 \leqslant i \leqslant k-1\} \cup \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant k-1\}$.

Case IIa : $k = 3$:

In this case we have $\tilde{\Delta} = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$.

Since S_{r_2} exchanges e_1 and e_2 the only $\mathrm{Stab}_r(\mathscr{G})$-stable subsets of

$\tilde{\Delta}$ defining a complex-symmetric fan are

$$\Delta_1 = \tilde{\Delta} \qquad \text{and} \qquad \Delta_2 = \{\pm e_1, \pm e_2\}.$$

If $\mathrm{Sk}(\Sigma_{\langle r\rangle})$ corresponds to Δ_1, then $\langle e_1, e_1+e_2\rangle$ and $\langle e_1+e_2, e_2\rangle$ are cones in $\Sigma_{\langle r\rangle}$. Then $\langle e_3, e_1+e_3, e_1+e_2\rangle$ and $\langle e_3, e_1+e_2, e_2+e_3\rangle$ are cones in Σ since $e_i \in \Sigma_{\langle r\rangle}$ can be obtained as the projection of e_i, $e_i - e_3$ and $e_i + e_3$ $(i=1,2)$. On the other hand, it follows that $\langle e_1, e_1+e_2\rangle \in \Sigma$, and it can be shown that $\langle e_1, e_1+e_2, e_1+e_3\rangle$ or $\langle e_1, e_1+e_2, e_2+e_3\rangle$ are cones in Σ. In the first case, by Theorem 3.7 and [Oda] (p. 35), there exists an orthogonal map φ such that

$$\varphi(\langle e_3, e_1+e_2, e_1+e_3\rangle) = \langle e_1, e_1+e_2, e_1+e_3\rangle$$

with $\varphi|\langle e_1+e_2, e_1+e_3\rangle = \mathrm{Id}$ and $\varphi(e_3) = e_1$.

But this mapping is not orthogonal.

In the second case we can construct a similar counterexample.

If $\mathrm{Sk}(\Sigma_{\langle r\rangle})$ corresponds to Δ_2, then $\langle e_1+e_3, e_2+e_3, e_3\rangle \in \Sigma$. By Theorem 3.7 there exists an orthogonal map with one eigenvalue -1 such that $\varphi|\langle e_2+e_3, e_1+e_3\rangle = \mathrm{Id}$. But then it follows that $\varphi(e_3) = \frac{2}{3}(e_1+e_2) + \frac{1}{3}e_3$. But no multiple of the last vector is an element of Δ.

So the root system of $\mathscr{B}_3$ does not define a complex-symmetric fan.

Case IIb : $k \geqslant 4$:

By Lemma 3.10 and by induction the fan $\Sigma_{\langle r\rangle}$ has to be the fan of $(\mathbb{P}_1)^{k-1}$. The only subset of $\tilde{\Delta}$ which can define a complex-symmetric fan is

$$\Delta_1 = \{\pm e_1, \pm e_2, \ldots, \pm e_{k-1}\} \text{ ([L2], Lemma 54, Lemma 55).}$$

Therefore $\langle e_1, \ldots, e_{k-1}\rangle \in \Sigma_{\langle r\rangle}$ and

$$\langle e_1+e_k, \ldots, e_{k-1}+e_k, e_k\rangle \in \Sigma \text{ ([L2], Lemma 56).}$$

Thus there exists an orthogonal fan automorphism φ with one eigenvalue -1 such that $\varphi(e_i+e_k) = e_i+e_k$, $i = 1, \ldots, k-1$. It follows that

$$\varphi(e_k) = \frac{2}{k}(e_1+\cdots+e_{k-1}) + \frac{k-2}{k}e_k \text{ ([L2], Lemma 57).}$$

Therefore φ does not stabilize Δ and it follows that the root system of $\mathscr{B}_k$ does not define a complex-symmetric fan.

Case III : $\mathscr{G} = \mathscr{D}_k$, $k \geqslant 4$:

$\Delta = \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant k\}$;

$\pi = \{r_1 := e_1 + e_2,\ r_i = e_i - e_{i-1} | 2 \leqslant i \leqslant k\}$;

$r = e_{k-1} + e_k$; $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{D}_{k-2} \times \mathscr{A}_1 = (S_{r_1}, \ldots, S_{r_{k-2}}, S_r)$

$(\mathscr{D}_3 \cong \mathscr{A}_3,\ \mathscr{D}_2 \cong \mathscr{A}_1 \times \mathscr{A}_1)$ ([L2], p. 46);

$$\tilde{\Delta} = S_1 \cup S_2 \cup S_3 = \left\{\pm \frac{1}{2}(e_k - e_{k-1}) \pm e_i | 1 \leqslant i \leqslant k-2\right\}$$
$$\cup \{\pm (e_k - e_{k-1})\} \cup \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant k-2\} \text{ ([L2], Lemma 59).}$$

Case IIIa : $k = 4$:

By induction $\Sigma_{\langle r \rangle}$ is a reducible fan. It follows that the only $\mathrm{Stab}_r(\mathscr{G})$-stable subset of $\tilde{\Delta}$ that can define a complex-symmetric fan is $\{\pm(e_1+e_2), \pm(e_1-e_2), \pm(e_3-e_4)\}$ ([L2], p. 55).

Therefore $\langle e_1+e_2, e_1-e_2, e_3-e_4\rangle \in \Sigma_{\langle r \rangle}$ can only be obtained as the orthogonal projection of

$$\tau := \langle e_1+e_2, e_1-e_2, e_3-e_4, e_3+e_4\rangle \text{ ([L2], p. 55).}$$

Now τ contains $\langle e_1+e_3\rangle$, but not as a face. This contradicts the definition of a complex-symmetric fan.

Thus the root system of $\mathscr{D}_4$ does not define a complex-symmetric fan.

Case IIIb : $k \geqslant 5$:

By Lemma 3.11 and by induction $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^{k-1}$. But there is no subset of $\tilde{\Delta}$ that defines the fan of $(\mathbb{P}_1)^{k-1}$ ([L2], Lemma 60). Note that $\Sigma_{\langle r \rangle}$ has to contain an orthogonal basis of $\mathbb{R}^{k-1}$.

Thus the root system of $\mathscr{D}_k$ does not define a complex-symmetric fan.

Case IV : $\mathscr{G} = \mathscr{I}_3$:

Let $\alpha = \cos\frac{\pi}{5}$ and $\beta = (4\alpha)^{-1}$. We have $4\alpha^2 = 2\alpha + 1$. Then

$$\Delta = \{\pm e_i | i = 1,2,3\} \cup \{\beta(\pm(2\alpha+1), \pm 1, \pm 2\alpha)\}$$
$$\cup \{\beta(\pm 2\alpha, \pm(2\alpha+1), \pm 1)\} \cup \{\beta(\pm 1, \pm 2\alpha, \pm(2\alpha+1))\};$$
$$\pi = \{r_1 := \beta(2\alpha+1, 1, -2\alpha),\ r_2 := \beta(-2\alpha-1, 1, 2\alpha),\ r_3 := \beta(2\alpha, -2\alpha-1, 1)\};$$
$$r = \beta(1, 2\alpha, 2\alpha+1);\ \mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{A}_1 \times \mathscr{A}_1 \cong (S_{r_1}, S_{r_3}).$$

The set $\{r, r_1, r_3\}$ is an orthogonal basis of $\mathbb{R}^3$ and we obtain $\tilde{\Delta}$ as orthogonal projection on $(r_1, r_3)_{\mathbb{R}}$. In *these* coordinates we have

$$\begin{aligned}\tilde{\Delta} = \{\pm(0,1), \pm(1,0)\} &\cup \{\pm(\alpha, 1/2), \pm(-\alpha, 1/2)\} \\ &\cup \{\pm(1/2, \beta), \pm(-1/2, \beta)\} \\ &\cup \{\pm(\beta, \alpha), \pm(-\beta, \alpha)\} = : S_1 \cup S_2 \cup S_3 \cup S_4 .\end{aligned}$$

The sets S_i are $\mathrm{Stab}_r(G)$-stable subsets of $\tilde{\Delta}$. Note that for $v \in S_i$, $w \in S_j$, we have $\|v\| = \|w\| \Leftrightarrow i = j$.

Now S_1 is the only possibility for a complex-symmetric fan with four vectors since such a fan has to contain an orthogonal basis of $(r_1, r_3)_{\mathbb{R}}$. Note that by assumption all induced fan automorphisms are orthogonal.

The set $\tilde{\Delta}$ does not contain a complex-symmetric fan with six vectors since all vectors of such a subset have to be of equal length. Similarly, $\tilde{\Delta}$ does not contain a complex-symmetric fan with eight vectors since there do not exist two vectors $v_1, v_2 \in \tilde{\Delta}$ with $\sphericalangle (v_1, v_2) = \frac{\pi}{4}$.

Finally, $\tilde{\Delta}$ does not contain a complex-symmetric fan with twelve vectors since in $\tilde{\Delta}$ there do not exist more than four vectors of equal length.

So we only have to analyse the case $\mathrm{Sk}(\Sigma_{\langle r \rangle}) = S_1$. But then $\sigma = \langle r_1, r_3, r \rangle \in \Sigma$. Since the generating vectors of σ are orthogonal and the group generated by all fan automorphisms acts transitively on the set of maximal-dimensional cones ([L2], Lemma 20; Theorem 3.7 above) the fan Σ has to contain the cone $\tau = \langle e_1, e_2, e_3 \rangle$ since $\pm e_i$ are the only vectors in Δ orthogonal to e_1. Now τ contains $\langle r \rangle$, but not as a face. Contradiction.

Thus the root system of $\mathscr{I}_3$ does not define a complex-symmetric fan.

Case V : $\mathscr{G} = \mathscr{I}_4$:

In this case we find $r \in \Delta$ such that $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{I}_3$ ([G-B], p. 80). Then by Lemma 3.10 and by induction the fan $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^3$. The group of all fan automorphisms of $\Sigma_{\langle r \rangle}$ has the order $2^3 \cdot 3! = 48$ ([L2], Lemma 62). Moreover, we have a faithful representation of $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{I}_3$ as a subgroup of the group of fan automorphisms of $\Sigma_{\langle r \rangle}$. This is impossible since $|\mathscr{I}_3| = 120$ ([G-B], p. 80).

Thus the root system of $\mathscr{I}_4$ does not define a complex-symmetric fan.

Case VI : $\mathscr{G} = \mathscr{F}_4$:

$$\Delta := \{\pm e_i | 1 \leqslant i \leqslant 4\} \cup \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant 4\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\};$$

$$\pi := \left\{r_1 := -\frac{1}{2}(e_1 + e_2 + e_3 + e_4), r_2 := e_1, r_3 := e_2 - e_1, r_4 := e_3 - e_2\right\};$$

$r = e_4$; $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{B}_3 = (S_{r_2}, S_{r_3}, S_{r_4})$;

$$\tilde{\Delta} = \{\pm e_i | 1 \leqslant i \leqslant 3\} \cup \{\pm e_i \pm e_j | 1 \leqslant j < i \leqslant 3\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)\right\}.$$

Since $\mathrm{Stab}_r(\mathscr{G})$ is irreducible the fan $\Sigma_{\langle r\rangle}$ is the fan of $(\mathbb{P}_1)^3$ by Lemma 3.10 and by induction. The only possibility for $\mathrm{Sk}(\Sigma_{\langle r\rangle})$ is $\mathrm{Sk}(\Sigma_{\langle r\rangle}) = \{\pm e_1, \pm e_2, \pm e_3\}$ ([L2], Lemma 64). Thus $\langle e_1, e_2, e_3\rangle \in \Sigma_{\langle r\rangle}$ and $\langle e_1 + e_4, e_2 + e_4, e_3 + e_4, e_4\rangle \in \Sigma$ ([L2], Lemma 65). By Theorem 3.7 there exists an orthogonal mapping φ of order two with double eigenvalue -1 such that

$$\varphi | \langle e_1 + e_4, e_2 + e_4\rangle = \mathrm{Id}.$$

It follows that $\varphi(e_1) = \frac{1}{3}(e_1 - 2e_2 + 2e_4)$ ([L2], Lemma 66). Therefore φ does not stabilize Δ.

Thus the root system of $\mathscr{F}_4$ does not define a complex-symmetric fan.

Case VII : $\mathscr{G} = \mathscr{E}_6$, $\mathscr{G} = \mathscr{E}_7$:

In the first case the root system Δ contains $r \in \Delta$ such that $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{A}_5$ ([G-B], p. 80). Thus $\mathrm{Stab}_r(\mathscr{G})$ is irreducible and $\Sigma_{\langle r\rangle}$ is the fan of $(\mathbb{P}_1)^5$ by Lemma 3.10 and by induction. But an easy calculation shows that there is no subset of the orthogonal projection $\tilde{\Delta}$ of Δ on $\langle r\rangle_{\mathbb{R}}^{\perp}$ defining the fan of $(\mathbb{P}_1)^5$ which is stabilized by $\mathrm{Stab}_r(\mathscr{G})$ ([L2], p. 65). Note that $\Sigma_{\langle r\rangle}$ has to contain an orthogonal basis of $\mathbb{R}^5$.

In the second case Δ contains $r \in \Delta$ such that $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{D}_6$ ([G-B], p. 80). Thus $\Sigma_{\langle r\rangle}$ is the fan of $(\mathbb{P}_1)^6$ and the same argument as in the first case shows that this is impossible. Thus the root systems of $\mathscr{E}_6$ and $\mathscr{E}_7$ do not define a complex-symmetric fan.

Case VIII : $\mathscr{G} = \mathscr{E}_8$:

In this case we find $r \in \Delta$ such that $\mathrm{Stab}_r(\mathscr{G}) \cong \mathscr{E}_7$ ([G-B], p. 80). Thus by Lemma 3.10 and by induction the fan $\Sigma_{\langle r\rangle}$ is the fan of $(\mathbb{P}_1)^7$. Now the group of all fan automorphisms of $\Sigma_{\langle r\rangle}$ has the order $2^7 \cdot 7!$.

Moreover, we have a faithful representation of $\mathrm{Stab}_r(\mathscr{G})$ as a subgroup of this group. This is impossible since $|\mathscr{E}_7| > 2^7 \cdot 7\,!$ ([G-B], p. 80).

Thus the root system of $\mathscr{E}_8$ does not define a complex-symmetric fan.

This case completes the proof of Theorem 3.12. □

Remark. – Obviously the most difficult cases are the cases $k = 3, 4$. The case $k = 3$ can be treated using triangulations of spheres that are obtained as the intersection of a sphere with a three-dimensional fan Σ ([L2], § 3.1 ; [Oda]).

From Theorem 3.6 and Theorem 3.12 we obtain

THEOREM 3.13. – *Let X be a smooth complex-symmetric torus embedding of dimension k. Then X can be written as a product*

$$X_{i_1} \times \cdots \times X_{i_\ell} \times (\mathbb{P}_1)^s \, (s+2\ell=k),$$

where X_{i_j} is a two-dimensional complex-symmetric torus embedding. □

COROLLARY 3.14. – *Each smooth complex-symmetric torus embedding is (T-equivariantly) projective algebraic and can equivariantly be blown down to $(\mathbb{P}_1)^k (k=\dim X)$.* □

We finish by giving an Example which shows the difficulties that occur if X is only assumed to be a normal complex-symmetric torus embedding.

Example 3.15. – Consider the two-dimensional complete fan Σ with

$$\mathrm{Sk}(\Sigma) = \{e_1+e_2, -e_1+e_2, -e_1-e_2, e_1-e_2\}.$$

This corresponds to a complete torus embedding X_Σ with four singular points belonging to the maximal-dimensional cones, and it is not difficult to see that X_Σ is complex-symmetric ([L2], p. 66). By the Remark following Theorem 3.9 the set $\mathrm{Sk}(\Sigma)$ also corresponds to the root system of a Coxeter group $\mathscr{G}$. In this case $\mathscr{G} = \mathscr{A}_1 \times \mathscr{A}_1$. But $\mathscr{G}$ is reducible while X_Σ cannot be written as a product of one-dimensional torus embeddings.

Thus irreducible torus embeddings do not correspond to irreducible Coxeter groups in the singular case.

Moreover, there are at least two non-isomorphic varieties which have $\mathscr{A}_1 \times \mathscr{A}_1$ as corresponding Coxeter group. In the smooth case it follows from our classification that there is at most one variety corresponding to a fixed Coxeter group.

So in the singular case a classification seems to be much more difficult. This classification will be treated in a forthcoming paper.

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