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LIPSCHITZ PROPERTIES OF SEMI-ANALYTIC SETS

par Adam PARUSINSKI

The main purpose of this paper is to prove the existence of an L-stratification for a compact semi-analytic set. The concept of L-stratification was introduced in [6] by Mostowski, where its existence for a germ of a complex analytic set was proved. An L-stratification is a stratification satisfying very strong conditions (see Definition 1.1), much stronger than Whitney's Conditions, but it ensures the constance of the Lipschitz type of the stratified set along each stratum.

The existence of an L-stratification in the real case can be deduced from the complexe one (see [7]), but in this paper we present the proof which is independent of the complex case and does not use the « quasi-wings », a machinery introduced by Mostowski in [6].

In the first section we recall Mostowski's definition of an L-stratification and introduce an equivalent definition, more convenient for us.

In Section 2, we give a brief exposition of Mostowski's theory on regular projections and its consequences. In Section 3 we derive from this theory some interesting facts about semi-analytic sets (Proposition 3.5 and Remark 3.6). Section 4 has a preparatory character for the proof of the main result, which is showed in Section 5. In section 6, we prove Key Lemma, which plays the role of « quasi-wings » in our investigations.

The reader is expected to be familiar with some basic properties of semi-analytic sets ([4] is the best reference).

The character C will stand for various constants.

Key-words : Semi-analytic sets - Lipschitz stratification (L-stratification) - Regular projections - L-regular sets - Lipschitz vector fields.

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1. L-stratifications.

Let X be a semi-analytic subset of an open subset of $\mathbb{R}^n$ or a complex analytic subset of an open subset of $\mathbb{C}^n$. By a *stratification of* X we shall mean a family $\mathscr{S} = \{S^j\}_{j=l}^m$ of closed semi-analytic subsets of X (or complex analytic) such that

$$X = S^m \supset S^{m-1} \supset \cdots \supset S^l \neq \varnothing$$

and $\mathring{S}^j = S^j \backslash S^{j-1}$, for $j = l, l+1, \ldots, m$ (we mean $S^{l-1} = \varnothing$), is a smooth manifold of pure dimension j or empty (a complex analytic manifold of pure complex dimension j). We call the connected components of $\mathring{S}^j$ *the strata of* $\mathscr{S}$. For $q \in \mathring{S}^j$ let $P_q : \mathbb{R}^n \to T_q\mathring{S}^j$ ($P_q : \mathbb{C}^n \to T_q\mathring{S}^j$) be the orthogonal projection and $P_q^\perp = I - P_q$ be the orthogonal projection onto the normal space $T_q^\perp \mathring{S}^j$. We denote the function of distance to S^j by d_j. From now on the letter l is reserved for the smallest dimension of strata of $\mathscr{S}$.

In [6] Mostowski has introduced the notion of L-stratification. Let us present his definition in a slightly shortened but an equivalent way.

Let c be a fixed constant, $c > 1$. A chain (more exactly, a c-chain) for a point $q \in \mathring{S}^j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = l$ and a sequence of points $q_{j_s} \in \mathring{S}^{j_s}$ such that $q_{j_1} = q$ and :

j_s is the greatest integer for which

$$d_k(q) \geqslant 2c^2 d_{j_s}(q) \qquad \text{for all} \qquad k < j_s,\ k \geqslant l,$$
$$|q - q_{j_s}| \leqslant c d_{j_s}(q).$$

The existence of a chain for a given point is clear. It is easy to verify the following inequalities :

$$d_{j_{s+1}}(q) \leqslant 2^n c^{2n} d_{j_s - 1}(q), \tag{1}$$

$$|q_{j_s} - q_{j_{s+1}}| \leqslant 2^{n+1} c^{2(n+1)} d_{j_s - 1}(q), \tag{2}$$

$$2 d_{j_s - 1}(q_{j_s}) \geqslant d_{j_s - 1}(q). \tag{3}$$

DEFINITION 1.1. – *We call a stratification* $\mathcal{S} = \{S^j\}_{j=l}^m$ *of* X *an L-stratification if for some constant* $C > 0$ *and every chain* $q = q_{j_1}, q_{j_2}, \ldots, q_{j_r}$ *and every* k, $1 \leqslant k \leqslant r$,

$$(4) \qquad |P^{\perp}_{q_{j_1}} P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C|q-q_{j_2}|/d_{j_k-1}(q).$$

If, further, $q' \in \mathring{S}^{j_1}$ *and* $|q-q'| \leqslant \left(\frac{1}{2c}\right) d_{j_1-1}(q)$, *then*

$$(5) \qquad |(P_{q'}-\mathrm{P}_q)\mathrm{P}_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C|q-q'|/d_{j_k-1}(q),$$

in particular,

$$(6) \qquad |P_{q'}-P_q| \leqslant C|q-q'|/d_{j_1-1}(q),$$

where $d_{l-1} \equiv 1$.

We say that a vector field v defined in a subset of X is *$\mathcal{S}$-compatible* if v is tangent to the strata of $\mathcal{S}$. The following proposition, proved first by Mostowski [6], explains the interest of L-stratifications.

PROPOSITION 1.2. – *Let* $\mathcal{S} = \{S^j\}_{j=l}^m$ *be an L-stratification of* X *and let* v *be a Lipschitz* $\mathcal{S}$*-compatible vector field on* S^j *bounded on* $S^l (l \leqslant j \leqslant m)$. *Then* v *can be extended to a Lipschitz* $\mathcal{S}$*-compatible vector field on* S^{j+1}.

Proof. – Let L denote a Lipschitz constant of v and let v be bounded on S^l by K.

Extend v to a Lipschitz vector field on $\mathbb{R}^n$ (by [1] any Lipschitz function defined in a subset of $\mathbb{R}^n$ can be extended to a Lipschitz function on $\mathbb{R}^n$ with the same Lipschitz constant). By abuse of notation we continue to write v for this extension. Define a vector field on S^{j+1} by : $w = v$ on S^j and $w(q) = P_q v(q)$ for $q \in \mathring{S}^{j+1}$. Let $q = q_{j_1}, q_{j_2}, \ldots, q_{j_r}$ be a chain for $q \in \mathring{S}^{j+1}$. Then, by (2) and (4)

$$\begin{aligned} |w(q)-w(q_{j_2})| &\leqslant |P_q(v(q)-v(q_{j_2}))| + |P^{\perp}_q v(q_{j_2})| \\ &\leqslant L|q-q_{j_2}| + |P^{\perp}_q P_{q_{j_2}} \ldots P_{q_{j_r}} v(q_{j_r})| \\ &+ \sum_{s<r} |P^{\perp}_q P_{q_{j_2}} \ldots P_{q_{j_s}}(v(q_{j_s})-v(q_{j_{s+1}}))| \leqslant C(L+K)|q-q_{j_2}|, \end{aligned}$$

for some constant C not depending on q.

For an arbitrary $q' \in S^j$ we have $|q-q'| \geqslant C|q-q_{j_2}|$ (by (1)), and consequently

$$|w(q)-w(q')| \leqslant |w(q)-w(q_{j_2})| + |w(q')-w(q_{j_2})| \leqslant C(L+K)|q-q'|$$

(recall that the character C stands for various constants).

Let $q' \in \mathring{S}^{j+1}$. If $|q-q'| \leqslant \left(\frac{1}{2c}\right) d_j(q)$, then by (2) and (5)

$$\begin{aligned}|w(q)-w(q')| &\leqslant |v(q)-v(q')| + |(P_{q'}-P_q)v(q)| \\ &\leqslant L|q-q'| + |(P_{q'}-P_q)P_{q_{j_2}} \dots P_{q_{j_r}} v(q_{j_r})| \\ &+ \sum_{s<r} |(P_{q'}-P_q)P_{q_{j_2}} \dots P_{q_{j_s}}(v(q_{j_s})-v(q_{j_{s+1}}))| \\ &\leqslant C(L+K)|q-q'|.\end{aligned}$$

If $|q-q'| \geqslant \left(\frac{1}{2c}\right) d_j(q)$, then $|q'-q_{j_2}| \leqslant C|q-q'|$ and consequently

$$|w(q')-w(q)| \leqslant |w(q)-w(q_{j_2})| + |w(q')-w(q_{j_2})| \leqslant C(L+K)|q-q'|.$$

Hence w is Lipschitz. □

Integrating Lipschitz vector fields by the same method as « controlled » vector fields in the proof of Thom's First Isotopy Lemma (see for example [5]) we obtain

COROLLARY 1.3. – *If $\mathcal{S}$ is an L-stratification of X, then for any q, q' which belong to the same stratum of $\mathcal{S}$ the germs (X,q'), (X,q) are Lipschitz homeomorphic.*

Let $\mathcal{X} = \{X_i\}$ be a family of subsets of X. We call a stratification $\mathcal{S}$ of X *compatible with* $\mathcal{X}$ if each X_i is a union of some strata of $\mathcal{S}$.

THEOREM 1.4. – *If X is a compact semi-analytic subset of $\mathbb{R}^n$ and $\mathcal{X}$ is a finite family of semi-analytic subsets of X, then there exists an L-stratification of X compatible with $\mathcal{X}$.*

Theorem 1.4 will be proved in Section 5.

Because the notion of chain is a little troublesome, we give two equivalent definitions of an L-stratification.

PROPOSITION 1.5. — *The following conditions are equivalent to the definition of an L-stratification :*

(i) *There exists* $C > 0$ *such that for every* $W \subset X$ *satisfying* $S^{j-1} \subset W \subset S^j$ *for some* j *: each Lipschitz* $\mathcal{S}$*-compatible vector field on* W *with a Lipschitz constant* L*, bounded on* $W \cap S^l$ *by* K*, can be extended to a Lipschitz* $\mathcal{S}$*-compatible vector field on* S^j *with a Lipschitz constant* $C(L+K)$.

(ii) *There exists* $C > 0$ *such that for every* j *and every* $W \subset X$ *equal to* $S^{j-1} \cup \{p\}$*, for* $p \in S^j$*, and every* $q \in \mathring{S}^j$ *: each Lipschitz* $\mathcal{S}$*-compatible vector field on* W *with a Lipschitz constant* L*, bounded on* $W \cap S^1$ *by* K*, can be extended to a Lipschitz* $\mathcal{S}$*-compatible vector field on* $W \cup \{q\}$ *with a Lipschitz constant* $C(L + K)$.

(*If* $l = 0$*, then* $K = 0$ *and the conditions* (i) *and* (ii) *are simpler.*)

Proof. — The proof of Proposition 1.2 shows that any L-stratification satisfies (i). Obviously, (i) follows (ii).

Assume that $\mathcal{S}$ satisfies (ii). We prove that $\mathcal{S}$ is an L-stratification. We give the proof only for the case $l = 0$, the case $l > 0$ is left to the reader. We show by induction on j that $\{S^p\}_{p=0}^j$ is an L-stratification of S^j. For $j = 0$ it is evident. Assume that $\{S^p\}_{p=0}^{j-1}$ is an L-stratification. Let $q \in \mathring{S}^j$ and $q = q_{j_1}, \ldots, q_{j_r}$ be a chain for q. We begin with proving (4). Take any $v \in T_{q_{j_k}} \mathring{S}^{j_k}$ such that $|v| = 1$, $1 \leqslant k \leqslant r$. The vector field w on $S^{j_k - 1} \cup \{q_{j_k}\}$, defined by: $w \equiv 0$ on $S^{j_k - 1}$ and $w(q_{j_k}) = v$, is $\mathcal{S}$-compatible and Lipschitz with constant $L = [d_{j_k - 1}(q_{j_k})]^{-1}$. By the inductive hypothesis, we extend w to a Lipschitz $\mathcal{S}$-compatible vector field on S^{j-1} with the constant CL and such that $w(q_{j_2}) = P_{q_{j_2}} \ldots P_{q_{j_k}} v$. Applying (ii) and (2) we extend this vector field on $S^{j-1} \cup \{q\}$ with the constant CL. Thus, using (3) we have

$$\begin{aligned} |P_q^{\perp} P_{q_{j_2}} \ldots P_{q_{j_k}} v| &\leqslant |P_q^{\perp}(w(q) - w(q_{j_2}))| \\ &\leqslant |q - q_{j_2}|/d_{j_k - 1}(q), \end{aligned}$$

which shows (4).

Next, we prove (6). Take any $v \in T_q\mathring{S}^j$, $|v| = 1$. The vector field w, defined by: $w \equiv 0$ on S^{j-1} and $w(q) = v$, is $\mathscr{S}$-compatible and Lipschitz with the constant $L = [d_{j-1}(q)]^{-1}$. Using (ii), we extend w on $S^{j-1} \cup \{q, q'\}$, and obtain

$$|P_{q'}^{\perp}v| = |P_{q'}^{\perp}(w(q) - w(q'))| \leqslant CL|q - q'|.$$

Since the above inequality holds for any $v \in T_q\mathring{S}^j$, $|v| = 1$,

$$|P_{q'}^{\perp}P_q| \leqslant CL\,|q - q'|.$$

Likewise,

$$|P_q^{\perp}P_{q'}| \leqslant L'|q - q'|,$$

where $L' = C[d_{j-1}(q')]^{-1}$. Note that $|P_{q'}P_q^{\perp}| = |P_q^{\perp}P_{q'}|$ ($P_{q'}$, P_q are self-adjoint). Thus

$$(7)\quad \begin{aligned}|P_q - P_{q'}| \leqslant |\mathrm{P}_{q'} - (P_{q'} + P_{q'}^{\perp})P_q| &\leqslant |P_{q'}P_q^{\perp}| + |P_{q'}^{\perp}P_q| \\ &\leqslant C(L + L').\end{aligned}$$

If $|q - q'| \leqslant \left(\frac{1}{2c}\right) d_{j-1}(q)$, then $2d_{j-1}(q') \geqslant d_{j-1}(q)$ and (6) follows from (7).

The proof of (5) is similar. First, for every $v \in T_{q_{j_k}}\mathring{S}^{j_k}$, $|v| = 1$, we construct a Lipschitz $\mathscr{S}$-compatible vector field w on S^j such that $w(q_{j_2}) = P_{q_{j_2}}, \ldots, P_{q_{j_k}}v$, $w(q) = P_q w(q_{j_2})$ and $L = C[d_{j_k-1}(q_{j_k})]^{-1}$ is a Lipschitz constant of w. Therefore, by (2) and (6),

$$\begin{aligned}|(P_{q'} - P_q)w(q_{j_2})| &\leqslant |(P_{q'} - \mathrm{P}_q)w(q)| + |(P_q - P_{q'})(w(q) - w(q_{j_2}))| \\ &\leqslant |w(q) - P_{q'}w(q') + P_{q'}(w(q') - w(q))| + CL|P_q - P_{q'}||q - q_{j_2}| \\ &\leqslant CL|q - q'|.\end{aligned}$$

This ends the proof. □

COROLLARY 1.6. – *Let $\mathscr{S}$ be an L-stratification. Then for some $C > 0$ and any $q \in \mathring{S}^j$, $q' \in \mathring{S}^k$ $(k < j)$*

$$|P_q^{\perp}P_{q'}| \leqslant C|q - q'|/d_{k-1}(q').$$

In particular, $\mathscr{S}$ satisfies the w-condition and consequently Whitney's conditions (see [9]).

Proof. — Fix $q \in \mathring{S}^j$, $q' \in \mathring{S}^k$ $(k < j)$. Using the same method as in the proof of Proposition 1.5, we find for every $v \in T_{q'}\mathring{S}^k$, $|v| = 1$, a Lipschitz $\mathcal{S}$-compatible vector field w on S^j such that $w(q') = v$ and $C[d_{k-1}(q')]^{-1}$ is a Lipschitz constant of w. Hence

$$|P_q^{\perp} P_{q'} v| = |P_q^{\perp}(w(q) - w(q'))| \leqslant C[d_{k-1}(q')]^{-1}|q - q'|.$$

This proves the corollary. □

2. Regular projections.

Let $X \subset \mathbb{C}^n$ be a germ at 0 of a hypersurface with a reduced equation $F = 0$. Fix the x_n-axis so that F does not vanishes on it. Let Ω be a neighbourhood of 0 in $\mathbb{C}^{n-1}$ such that for every $\xi \in \Omega$ the projection $\pi(\xi) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, parallel to $(\xi, 1)$, restricted to X is finite. Then, by the preparation theorem, $F(x+\lambda(\xi,1))$ is equivalent to a distinguished polynomial $W(x,\xi;\lambda)$ in λ.

Let q be a germ at 0 of a complex (real) analytic curve such that $q(0) = 0$. We say that $\pi(\xi)$ for $\xi \in \Omega$ is *ε-regular* (with respect to X) at q if there exists an integer k such that for all η, $|\eta - \xi| < \varepsilon$:

$$\begin{aligned} &\Delta_i(q(t), \eta) = 0 \qquad \text{for} \qquad i < k, \\ &\Delta_k(q(t), \eta) \neq 0 \end{aligned}$$

for $t \neq 0$, where Δ_i denotes the i'th generalized discriminant of W (see [6]).

Let $S_\varepsilon(x,\xi)$ denotes the open cone $\{x+\lambda(\eta,1); |\eta-\xi|<\varepsilon, \lambda \in \mathbb{C}^*\}$. As was proved in [6], if $\pi(\xi)$ is ε-regular at q, then for some constant C and for $t \neq 0$ sufficiently small:

(8) $S_\varepsilon(q(t),\xi) \cap X$ consists of points of the form $q(t) + \lambda_i(\eta)(\eta,1)$ $(i=1,\ldots,r)$, where λ_i are analytic for $|\eta-\xi| < \varepsilon$ and satisfy: $\lambda_i(\eta) \neq \lambda_j(\eta)$ for $i \neq j$ and all η, $|D\lambda_i| \leqslant C|\lambda_i|$.

The following proposition is an immediate consequence of Proposition 4.2 of [6] except the last statement which can be easily deduced from its proof.

PROPOSITION 2.1. — *There exist a finite subset $\{\xi_1, \ldots, \xi_N\}$ of Ω and $\varepsilon > 0$ such that for every pair p, q of germs of complex analytic curves, there is $\pi(\xi_j)$ which is ε-regular at both p and q.*

Furthermore, $\xi_1, \ldots, \xi_N$ can be taken from $\Omega \cap \mathbb{R}^{n-1}$.

Remark 2.2. — In Proposition 2.1 we can require the existence of a constant C satisfying (8) for every pair p, q of germs of real analytic curves and the associated regular projection $\pi(\xi_j)$.

Proof. — For simplicity we consider the case of single germs instead of pairs. Let $\xi_1, \ldots, \xi_N$ and ε satisfy the assertion of Proposition 2.1. Put $X^{(k)} = \{x \in \mathbb{C}^n; F \in m_x^k\}$, where m_x is the ideal of all germs at x of analytic functions vanishing at x. Then, (see [6]),

$$X^{(i)} \backslash X^{(i+1)} = \{x \in \mathbb{C}^n; \Delta_{i-1}(x,.) \not\equiv 0 \text{ and } \Delta_s(x,.) \equiv 0 \quad \text{for} \quad s < i-1\}.$$

Fix k such that $X^{(k)} \neq X^{(k+1)}$. Take an irreducible component T of $X^{(k)}$ at 0 such that $T \not\subseteq X^{(k+1)}$. For each $(x,\eta) \in T \times \Omega$ and such that $\Delta_{k-1}(x,\eta) \neq 0$, $W(x,\eta;\lambda) = 0$ has exactly $d-k$ non-zero solutions $\lambda_1, \ldots, \lambda_{d-k}$ which we consider locally as analytic functions in η. We claim that their derivates $\dfrac{\partial \lambda_i}{\partial \eta_j}$ $(i=1, \ldots, d-k, j=1, \ldots, n-1)$ are roots of a polynomial with coefficients meromorphic in a neighbourhood of $T \times \Omega$, and with denominators not vanishing identically on $T \times \Omega$. First we prove the following lemma.

LEMMA 2.3. — *Let U be an open subset of $\mathbb{C}^n$. Let $(f_1, \ldots, f_k)$ be a multivalued function on U such that each $f_1, \ldots, f_k$ satisfies*

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0 \tag{9}$$

with $a_1, \ldots, a_m$ analytic in a neighbourhood of $\bar{U}$. Then any partial derivative of $f_1, \ldots, f_k$ satisfies, on the set where it exists, an equation of the same type with coefficients meromorphic in a neighbourhood of $\bar{U}$.

Proof of the lemma. — Without loss of generality we can assume that the equation (9) is reduced. Fix a partial derivative $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z_i}$. Let $f_1, f_2, \ldots, f_m$ be all solutions of (9). Locally, outside some nowhere dense subset of U, they can be considered as analytic functions. By (9)

$$\frac{\partial f_i}{\partial z} = \frac{-\left[\dfrac{\partial a_1}{\partial z} f_i^{m-1} + \cdots + \dfrac{\partial a_m}{\partial z}\right]}{m f_i^{m-1} + \cdots + a_{m-1}}$$

and the denominator does not vanish because (9) is assumed to be reduced. So, every elementary symetric function of $\partial f_1/\partial z, \ldots, \partial f_m/\partial z$ is a symetric function in $f_1, \ldots, f_m$ with meromorphic coefficients and the lemma follows. □

In our situation, we can find a branched analytic covering $\pi : T \to U$, U an open neighbourhood of 0 in $\mathbb{C}^j$, $j = \dim T$, induced by a linear projection. Then

$$\lambda_1(\pi^{-1}(x), \eta), \ldots, \lambda_{d-k}(\pi^{-1}(x), \eta)$$

form a multivalued function on $U \times \Omega$ satisfying an equation like (9) with coefficient analytic on $U \times \Omega$. Applying to this function Lemma 2.3, we see that all $\partial\lambda_i/\partial\eta_j$ have the desired properties.

Consider the germs at 0 of sub-analytic sets

$$V_j = \{x \in T \backslash X^{(k+1)} ; \Delta_{k-1}(x,\eta) \neq 0 \text{ for } |\eta - \xi_j| \leqslant \varepsilon/2\}.$$

From the curve selection lemma (see [2] or [3]) and Proposition 2.1 we conclude that the family $\{V_j\}$ covers $T\backslash X^{(k+1)}$ near 0. Consider on $T\backslash X^{(k+1)}$ the function

$$\varphi(x) = \min \max_{|\eta - \xi_j| \leqslant \varepsilon/2} \sum_{i=1}^{d-k} |D\lambda_i(x,\eta)|^2/|\lambda_i(x,\eta)|^2,$$

where the minimum is taken over all j for which $x \in V_j$. Fix a partial derivative $\partial/\partial\eta$. Since $\partial\lambda_i/\partial\eta$, $i = 1, \ldots, d-k$, are roots of a polynomial with meromorphic coefficients, so is $g = \sum_i |(\partial\lambda_i/\partial\eta)\lambda_i^{-1}|^2$.

Hence the closure of the graph of g in $\bar{V}_j \times \{\eta : |\eta - \xi_j| \leqslant \varepsilon/2\} \times \mathbb{C}P(1)$ is sub-analytic, and so is the graph of φ in $T \times \mathbb{C}P(1)$. By Proposition 2.1, φ is bounded on any $\mathbb{R}$-analytic curve, so is bounded near 0 by the curve selection lemma ([2], [3]). This ends the proof. □

Corollary 2.4. — *Let X, Ω be as above. There exist a hypersurface of an open neighbourhood V of 0 in $\mathbb{C}^n$ representing X (call it also X), a subneighbourhood U of V, a finite set $\{\xi_1, \ldots, \xi_N\} \subset \Omega \cap \mathbb{R}^{n-1}$ and constants C, $\varepsilon > 0$ such that for every pair x, $x' \in U$, there is ξ_j such that both* $S_\varepsilon(x,\xi_j)$ and $S_\varepsilon(x',\xi_j)$ *look like in* (8).

Remark 2.5. — There exist constants ε', δ', M depending only on C, ε, n such that if X is a hypersurface in a neighbourhood of 0 in

$\mathbb{C}^n$ and (8) happens for $S_\varepsilon(x,0)$ (i.e. $\xi = 0$, then we denote $\pi(0)$ simply by π) with constant C, then:

> $S_{\varepsilon'}(x,0)$ is contained in the disjoint union of the graphs of analytic functions $\varphi_i : B(\pi(x), |\lambda_i(0)|\delta') \to \mathbb{C}$, $i = 1, \ldots, r$, where $B(y,R) = \{z \in \mathbb{C}^{n-1} : |z-y| < R\}$.
>
> Furthermore, $|D\varphi_i| \leqslant M$ for $i = 1, \ldots, r$.
>
> If X near x is the graph of an analytic function Φ, then we can also require $|D\Phi(\pi(x))| \leqslant M$.

The above remark was proved in [6] except the last statement which is obvious (X near x is outside $S_{\varepsilon'}(x,0)$).

3. Regular sets.

Our next task is to cover a compact semi-analytic set with semi-analytic sets with good metric properties. Let us start with some definitions.

DEFINITION 3.1. — *Let U be a relatively compact semi-analytic subset of $\mathbb{R}^n$ and let $\mathrm{Int}(U)$ be dense in U. We call a continuous map $F : U \to \mathbb{R}^m$ strongly semi-analytic if each coordinate function $f = F_i$ $(i=1, \ldots, m)$ satisfies an equation of the form:*

$$(10) \qquad f^d + a_1 f^{d-1} + \cdots + a_d = 0,$$

with some analytic functions $a_1, \ldots, a_d$ defined on an open neighbourhood of $\bar{U}$ and if

$$(11) \qquad |D_F(x)| \leqslant C,$$

for some constant C and any x for which DF is defined (compare with the definition of (L)-analytic surface from [4]).

Note that if f, g are strongly semi-analytic, so are $f+g$, $f-g$, fg.

DEFINITION 3.2. — *By a zero-dimensional L-regular set we mean a point. For $n > 0$, a compact n-dimensional semi-analytic subset X or $\mathbb{R}^n$ will be called to be L-regular if $\overline{\mathrm{Int}(X)} = X$ and*

$$X = \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : f(x') \leqslant x_n \leqslant g(x'),\ x' \in Y\}$$

(maybe after a linear change of coordinates in $\mathbb{R}^n$*), where* Y *is an* L*-regular subset of* $\mathbb{R}^{n-1}$ *and* f, g *are strongly semi-analytic functions on* Y, *analytic on* $\operatorname{Int}(Y)$ *and satisfying* $f < g$ *on* $\operatorname{Int}(Y)$.

For $k < n$ *we call a compact* k*-dimensional semi-analytic subset* X *of* $\mathbb{R}^n$ L*-regular if it is the graph (maybe after a linear change of coordinates) of a strongly semi-analytic map* F *defined on an* L*-regular subset of* $\mathbb{R}^k$ *and such that* F *is analytic on* $\operatorname{Int}(Y)$. *Any system of coordinates for which the above characterization occurs is called to be associated with* X.

Although an L-regular subset X of $\mathbb{R}^n$ may not be a manifold with boundary, we denote by ∂X the set of that points of X near which X is not a manifold without boundary. Particularly, if $\dim X = n$, then $\partial X = \operatorname{Fr}(X)$. If X is the graph of $F : Y \to \mathbb{R}^{n-k}$, as in Definition 3.2, then ∂X is the graph of F restricted to ∂Y. It is easy to check the following property of L-regular sets.

Remark 3.3. — Let X be an L-regular subset of $\mathbb{R}^n$. Then $X \backslash \partial X$ is homeomorphic to an open disc and for every $x, x' \in X \backslash \partial X$ there exists a smooth curve γ in $X \backslash \partial X$ joining x and x', and satisfying

$$\text{length}\,\gamma \leqslant C|x - x'|, \tag{12}$$

for some C not depending on x, x'.

Remark 3.4. — Let X be an L-regular subset of $\mathbb{R}^n$ given by the graph of $F : Y \to \mathbb{R}^{n-k}$ as in Definition 3.2. Then the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ restricted to X gives one-to-one correspondence between semi-analytic subsets of X and those of Y. Furthermore, $\pi|_X$ is a Lipschitz homeomorphism.

Proof. — The first statement follows from Łojasiewicz's version of the Seindenberg-Tarski Theorem (see [4]) and the second one is a simple consequence of (11) and (12). □

We call an L-regular set X *compatible with a semi-analytic set* Z if $X \backslash \partial X \subset Z$ or $(X \backslash \partial X) \cap Z = \varnothing$.

PROPOSITION 3.5. — *Every compact semi-analytic subset of* $\mathbb{R}^n$ *can be written as a finite union (not necessarily disjoint) of* L*-regular subsets of* $\mathbb{R}^n$ *compatible with a given semi-analytic subset* Z *of* X.

Proof. — We can assume that X is pure dimensional. Let $\dim X = k$. We show that we can cover X with k-dimensional L-regular sets. The proof is by induction on n. The case $n = 0$ is evident.

Because X is compact, it is sufficient to show that for each $x \in X$ there exist L-regular subsets of X covering a neighbourhood of x in X and compatible with Z. If $x \in \operatorname{Int}(X) \backslash \operatorname{Fr}(Z)$, it is obvious. Fix $x_0 \in \operatorname{Fr}(X) \cup \operatorname{Fr}(Z)$. Note that $\operatorname{Fr}(X) \cup \operatorname{Fr}(Z)$ is a compact semi-analytic set of dimension smaller than n. Complexify $\mathbb{R}^n$ and find a small neighbourhood U of x_0 in $\mathbb{C}^n$ and a complex hypersurface $\tilde{X}$ in U such that $Y = (\operatorname{Fr}(X) \cup \operatorname{Fr}(Z)) \cap U \subset \tilde{X}$. Without loss of generality we can assume that $\tilde{X} \cap \mathbb{R}^n$ has pure dimension $n-1$. We apply Corollary 2.4 to the germ $(\tilde{X}, x_0)$. Fix ξ one of the obtained regular projections $\xi_1, \ldots, \xi_N$. To shorten notation we assume $\pi = \pi(\xi)$ to be the standard projection $(\xi = 0)$. We can also assume that $\pi : \tilde{X} \to \pi(U) = U'$ is a branched analytic convering, so $\tilde{X}$ is defined by

$$\prod_{i=1}^{d} (x_n - f_i(x')) = 0,$$

where $x = (x', x_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $f_1, \ldots, f_d$ are analytic outside a proper analytic subset $Z' \subset U'$ (in fact, each f_i is only defined locally). Consider on $U' \backslash Z'$ the function

$$F(x') = \sum_{i=1}^{d} |Df_i(x')|^2.$$

By Lemma 2.3, $V(M) = \{x' \in (U' \cap \mathbb{R}^{n-1}) \backslash Z' : F(x') < M\}$ is open and semi-analytic for every $M \in \mathbb{R}$.

Assume that $\dim X = n$. The projection $\pi|_Y : Y \to U' \cap \mathbb{R}^{n-1}$ is a branched covering and an analytic covering outside some semi-analytic subset T of $U' \cap \mathbb{R}^{n-1}$, $\dim T < n-1$. Fix M for a moment. Apply the inductive hypothesis to $V(M)$ and its subset $W = \operatorname{Fr}(V(M)) \cup (T \cap V(M))$. Let Y_1 be one of the obtained L-regular sets. Then $\dim Y_1 = n-1$ and $W \cap \operatorname{Int}(Y_1) = \varnothing$. Put $\tilde{Y}_1 = \pi^{-1}(Y_1) \cap Y$. Over $\operatorname{Int}(Y_1)$ $\pi|_{\tilde{Y}_1} : \tilde{Y}_1 \to Y_1$ is a trivial analytic covering ($\operatorname{Int}(Y_1)$ is contractible). Thus, $\tilde{Y}_1 \cap \pi^{-1}(\operatorname{Int}(Y_1))$ is the union of the graphs of some analytic functions $g_1 < g_2 < \cdots < g_m$ from $f_1, \ldots, f_d$. By Remark 2.5, each of them satisfies (10) and (11). They are also Lipschitz by (11) and (12), so we can extend them to Lipschitz functions on Y_1. Call them also $g_1, \ldots, g_m$. It is also clear from the above construction that $\pi^{-1}(Y_1) \cap X$ is covered near x_0 with some of the sets

$$\{(x', x_n)\mathbb{R}^n : x' \in Y_1, g_i(x') \leqslant x_n \leqslant g_{i+1}(x')\},$$

and, maybe, sets of the form

$$\{(x', x_n) \in \mathbb{R}^n : x' \in Y_{1j}, x_{on} - \delta \leqslant x_n \leqslant g_1(x')\},$$
$$\{(x', x_n) \in \mathbb{R}^n : x' \in Y_{1j}, g_d(x') \leqslant x_n \leqslant x_{on} + \delta\},$$

where $x_0 = (x'_0, x_{on})$, $\delta > 0$ is sufficiently small, and $(n-1)$-dimensional L-regular sets Y_{1j} cover a neighbourhood of x'_0 in Y_1. Each of these sets is L-regular and compatible with Z. Thus, we have found the required covering of $\pi^{-1}(V(M)) \cap X$. To complete the proof for $\dim X = n$ it suffices to note that for M sufficiently large the sets $\pi^{-1}(V(M))$ constructed for all $\xi_1, \ldots, \xi_N$ cover X near x_0 (as Corollary 2.4 and Remark 2.5 state).

Assume that $\dim X = k < n - 1$. Apply the inductive hypothesis to $V(M)$ and its subset $V(M) \cap \pi(X)$, and take T one of the obtained L-regular sets. Of course $\dim T = n - 1$. Then, as we have shown above, $V = \tilde{X} \cap \mathbb{R}^n \cap \pi^{-1}(T)$ is the union of the graphs of Lipschitz semi-analytic functions $g_1 \leqslant g_2 \leqslant \cdots \leqslant g_m$. Consider the set $V \cap X \subset \pi^{-1}(\partial T)$. Apply the inductive hypothesis again to $(T \cap \pi(V), T \cap \pi(\operatorname{Fr}(Z)))$. Let Y_1 be one of the obtained L-regular sets and $\dim Y_1 = k$ (we can ignore sets of dimension smaller than k). Then Y_1 is the graph of $f: Y_2 \to \mathbb{R}^{n-k-1}$, $Y_2 \subset \mathbb{R}^k$, as in Definition 3.2 (we can assume that the standard system of coordinates is associated with Y_1). Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{k+1}$ be defined by $\pi'(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_k, x_n)$. Then the map $\pi'(\pi^{-1}(Y_1) \cap V \cap X) \to Y_2$, given by $(x_1, \ldots, x_k, x_n) \to (x_1, \ldots, x_k)$, is a branched covering, and using again the inductive hypothesis, we can assume it to be trivial over $\operatorname{Int}(Y_2)$. So, $\pi^{-1}(Y_1)$ is a finite union of the graphs of maps from Y_2 into $\mathbb{R}^{n-k}$ whose coordinates satisfy (10). To see that one of them, say F, satisfies (11) it is sufficient to consider its last coordinate F_n. We can assume that the graph of F lies on one of the graphs of g_i, say g_1. Then, $F_n(x) = g_1(x, F_{k+1}(x), \ldots, F_{n-1}(x))$. Both g_1 and $(F_{k+1}, \ldots, F_{n-1})$ are Lipschitz, so is F_n. This proves (11). Therefore, the graph of F is L-regular. Now we can repeat the argument that such sets constructed for $\xi_1, \ldots, \xi_N$ and a sufficiently large M cover X near x_0.

The similar proof works for $k = n - 1$ (the details are left to the reader). □

A slightly more detailed proof leads to the following strenghtening of Proposition 3.5.

Remark 3.6. — The assertion of Proposition 3.5 is still true if we require additionally :

a) For every $x, x' \in X$ there exist elements of the desired union Y, Y' such that $x \in Y$, $x' \in Y'$, and Y and Y' have a common associated system of coordinates, and if $k = \dim Y \geqslant \dim Y'$ and Y is the graph of $F : T \to \mathbb{R}^{n-k}$ (as in Definition 3.2) and $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the standard projection, then either $\pi(Y') = T$ or $\pi(Y') \cap (T \backslash \partial T) = \emptyset$.

b) The elements of the union are compatible with a given finite family of semi-analytic subsets of X.

4. Lifting of Lipschitz vector fields.

In proving Theorem 1.4 the main difficulty lies in showing the following fact :

FACT 4.1. — *Let X be an L-regular subset of $\mathbb{R}^{n+k}$ given by a map $F: Y \to \mathbb{R}^k$, as in Definition* 3.2. *Then there exist a stratification $\mathcal{S}$ of Y and a constant $C > 0$ such that F is analytic on every stratum of $\mathcal{S}$ and for every Lipschitz $\mathcal{S}$-compatible vector field v on Y with a Lipschitz constant L the map $A(x) = DF(x)v(x)$ is Lipschitz with a constant CL. In particular, (v, DFv) gives a Lipschitz vector field on X.*

DEFINITION 4.2. — *Given a stratification $\mathcal{S}$, we say that a stratification $\mathcal{S}'$ is compatible with $\mathcal{S}$ if each stratum of $\mathcal{S}'$ is contained in a stratum of $\mathcal{S}$ (in other words, if $\mathcal{S}'$ is compatible with $\mathcal{S}$ as a family of sets).*

Note that if $\mathcal{S}'$ is compatible with $\mathcal{S}$, then every $\mathcal{S}'$-compatible vector field is $\mathcal{S}$-compatible. For every finite family of semi-analytic sets there exists a stratification compatible with it, so each finite family of stratifications of a given set possesses a stratification compatible with each of them.

The following lemma plays the crucial role in our investigations. It states that the spaces of all symetric homogeneous polynomials is sufficiently large for carrying out some estimates concerning differentials of polynomials.

KEY LEMMA. — *For every $m \in \mathbb{N}$ there exist a constant C, a finite family $\mathcal{W}$ of real homogeneous symetric polynomials of m variables and* *a finite family* $\mathscr{V}$ *of real homogeneous polynomials of* m *variables such that :*

If $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$, $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$, *and* $DV(p)v = 0$ *if* $V(p) = 0$ *for all* $V \in \mathscr{V}$, *and*

$$|DW(p)v| \leqslant L|W(p)|, \tag{13}$$

for all $W \in \mathscr{W}$ *and some* L, *then*

$$|v_i| \leqslant CL|p_i|, \tag{14}$$

for $i = 1, \ldots, m$.

Obviously, we can require that if $V \in \mathscr{V}$, *then* $V(\sigma(p)) \in \mathscr{V}$ *for every permutation* σ.

Key Lemma will be proved in Section 6. Now, we show how it works. The Łojasiewicz Inequality ([4]) implies for each analytic function f defined in a neighbourhood of $x_0 \in \mathbb{R}^n$

$$\operatorname{dist}(x, f^{-1}(0)))|Df(x)| \leqslant C|f(x)|^\alpha,$$

for some constants C, $\alpha > 0$ in some neighbourhood of x_0. Key Lemma allows us to prove a similar result with $\alpha = 1$.

COROLLARY 4.3. — *Let* $f : U \to \mathbb{R}$ *be an analytic function satisfying* (10) *with* a_i *analytic in some neighbourhood of* $\bar{U}$ *and let* U *be open and relatively compact. Then there exist a semi-analytic subset* Y *of* $\bar{U}$ *and a constant* C *such that* dim $Y < n$ *and*

$$|Df(x)| \operatorname{dist}(x, Y) \leqslant C|f(x)|, \tag{15}$$

for every $x \in U$. *(Generally* Y *is greater than* $f^{-1}(0)$, *see for example* $f(x_1, x_2) = x_1^2 - x_2^3$).

Proof. — It is sufficient to prove (15) for x from a dense subset of U. Since the problem is local, we can work in a sufficiently small neighbourhood of some $x_0 \in \bar{U}$.

Assume that f is analytic in a neighbourhood of x_0. By the preparation theorem, we can assume that f is a distinguished polynomial

$$f(x) = x_n^d + \cdots + b_1(x')x_n^{d-1} + \cdots + b_d(x') = \prod_{i=1}^{d} (x_n - r_i(x')), \tag{16}$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$, b_i are analytic in a neighbourhood U' of $\pi(x_0)$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the standard projection. For each $x' \in U'$ and outside some proper analytic subset of U' we can treat $r_1, \ldots, r_d$ as complex-value analytic functions in some neighbourhood of x. This gives

$$|\partial f/\partial x_n| = |f(x)| \left|\sum_{i=1}^{d} (x_n - r_i(x'))^{-1}\right| \leqslant d|f(x)|(\operatorname{dist}(x', Y_n))^{-1},$$

for Y_n containing all the graphs of $\operatorname{Re} r_i$ intersected with $\bar{U}$ and $\operatorname{Fr}(U)$. Since $\operatorname{Re} r_i = \frac{1}{2}(r_i + \bar{r}_i)$, we can choose Y_n semi-analytic. In a suitable system of coordinates we obtain similar inequalities for each $\partial f/\partial x_i$ and some sets Y_i. Putting $Y = \bigcup_i Y_i$, we get (15).

Return to the general case. Let $f = f_1, f_2, \ldots, f_d$ be all solutions of (10). Let $\mathscr{W}$, $\mathscr{V}$ be as in Key Lemma with $m = d$. Each $W_i \in \mathscr{W}$ gives a real analytic function $\tilde{W}_i(x) = W_i(f_1(x), \ldots, f_d(x))$, for which there exist semi-analytic sets Y_i of dimension smaller than n and satisfying (15). Therefore, for $Y = \bigcup_i Y_i \cup Z$, where Z is the union of the zero sets of these $\tilde{V}(x) = V(f_1(x), \ldots, f_d(x))$ constructed from $V \in \mathscr{V}$, which are not identically equal to 0,

$$|\partial \tilde{W}/\partial x_j(x)| \operatorname{dist}(x, Y) \leqslant L|\tilde{W}(x)|,$$

and

$$|\partial \tilde{V}/\partial x_j(x)| \operatorname{dist}(x, Y) = 0 \quad \text{if} \quad \tilde{V}(x) = 0,$$

for all $W \in \mathscr{W}$ and $V \in \mathscr{V}$, $j = 1, \ldots, n$ and x from a dense subset of a neighbourhood of x_0. Put $p(x) = (f_1(x), \ldots, f_d(x))$,

$$v(x) = \operatorname{dist}(x, Y)((\partial f_1/\partial x_j)(x), \ldots, (\partial f_d/\partial x_j)(x)).$$

So, $|(\partial f/\partial x_j)(x)| \operatorname{dist}(x, Y) \leqslant CL|f(x)|$. This for each j gives (15). □

The following two lemmas generalize Corollary 4.3.

LEMMA 4.4. — *Let U be a relatively compact open subset of $\mathbb{R}^n$ and let $f: \bar{U} \to \mathbb{R}$ be a continuous function satisfying* (10) *with a_i analytic in a neighbourhood of $\bar{U}$. Then there exist a constant C and a stratification $\mathscr{S}$ of $\bar{U}$ such that :*

a) *f is analytic of each stratum of $\mathscr{S}$,*

b) *if v is a Lipschitz $\mathscr{S}$-compatible vector field on $\bar{U}$ with a Lipschitz constant L, then*

$$|Df(x)v(x)| \leqslant CL|f(x)|, \tag{17}$$

for every $x \in \bar{U}$.

Proof. – Induction on n. For $n = 0$ the lemma is evident. Let $n > 0$. By standard arguments, there exist a stratification satisfying a), so we concentrate on b). The idea of the proof is the same as that of the proof of Corollary 4.3.

First, assume that f is analytic in a neighbourhood of $\bar{U}$. According to Proposition 3.5, Corollary 2.4 and Remark 2.5, we can assume without loss of generality that f is a distinguished polynomial (16) and :

(i) $\bar{U} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : g_1(x') \leqslant x_n \leqslant g_2(x'), x' \in \bar{U}'\}$ is L-regular, as in Definition 3.2, $U' \subset \mathbb{R}^{n-1}$ is open,

(ii) r_i (from (16)) are well-defined continuous complex-value functions on $\bar{U}'$, and analytic on U' (the zero set of the discriminant of f does not intersect U'), and $|Dr_i| \leqslant C$ on U'.

Choose a stratification $\mathscr{S}'$ of $\bar{U}'$ satisfying the assertion of the lemma for all $\operatorname{Im} r_i$, $g_1 - \operatorname{Re} r_i$, $g_2 - \operatorname{Re} r_i$. We can do it by the inductive assumption. Let $\mathscr{S}$ be a stratification of $\bar{U}$ compatible with :

1) $\pi^{-1}(S') \cap \bar{U}$, for all $S' \in \mathscr{S}'$

2) the intersection of $\bar{U}$ and the graphs of $\operatorname{Re} r_i$

3) the graphs of g_1 and g_2.

We show that $\mathscr{S}$ satisfies the assertion of the lemma. Let v be a Lipschitz $\mathscr{S}$-compatible vector field on $\bar{U}$ with a Lipschitz constant L. We write $v(x) = (v'(x), v_n(x)) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $z = (z', z_n) \in U$ and $z' \in U'$. Consider on $\bar{U}'$ the vector field $w(y) = v'(tg_1(y) + (1-t)g_2(y))$, where t satisfies $z_n = tg_1(z') + (1-t)g_2(z')$. By (i), w is Lipschitz with a constant CL. On the other hand, 1) implies that w is $\mathscr{S}'$-compatible. Hence, for all i

$$|D \operatorname{Im} r_i(z')v'(z)| \leqslant CL|\operatorname{Im} r_i(z')|.$$

Similarly, for $i = 1, \ldots, d, j = 1, 2$

$$|D(g_j - \operatorname{Re} r_i)(z')v'(z)| \leqslant CL|g_j(z') - \operatorname{Re} r_i(z')|.$$

If $g_1(z') \leqslant \operatorname{Re} r_i(z') \leqslant g_2(z')$, then by 2)

$$\begin{aligned}|v_n(z) - D \operatorname{Re} r_i(z')v'(z)| &\leqslant |v_n(z) - v_n(z', \operatorname{Re} r_i(z'))| \\ &\qquad + |D \operatorname{Re} r_i(z')(v'(z', \operatorname{Re} r_i(z')) - v'(z))| \\ &\leqslant CL|z_n - \operatorname{Re} r_i(z')|.\end{aligned}$$

If $\operatorname{Re} r_i(z') < g_1(z')$, then

$$\begin{aligned}|v_n(z) - D \operatorname{Re} r_i(z')v'(z)| &\leqslant |v_n(z) - v_n(z', g_1(z'))| \\ &\qquad + |Dg_1(z')(v'(z', g_1(z')) - v'(z))| \\ &\qquad + |D(g_1 - \operatorname{Re} r_i)(z')v'(z)| \\ &\leqslant CL(|z_n - g_1(z')| + |g_1(z') - \operatorname{Re} r_i(z')|) \\ &\leqslant CL|z_n - \operatorname{Re} r_i(z')|.\end{aligned}$$

The same inequality holds for $\operatorname{Re} r_i(z') > g_2(z')$.

Consequently,

$$\begin{aligned}|Df(z)v(z)| &= |f(z)| \left| \sum_{i=1}^{d} \frac{v_n(z) - Dr_i(z')v'(z)}{z_n - r_i(z')} \right| \\ &\leqslant |f(z)| \left[\sum_{i=1}^{d} \left| \frac{z_n - \operatorname{Re} r_i(z')}{z_n - r_i(z')} \right| + \left| \frac{D \operatorname{Im} r_i(z')v'(z)}{z_n - r_i(z')} \right| \right] \\ &\leqslant CL|f(z)|.\end{aligned}$$

This ends the proof for f analytic.

For the general case, we use Key Lemma in the same manner as in the proof of Corollary 4.3. □

LEMMA 4.5. — *Let f be an analytic function defined in a relatively compact open subset U of $\mathbb{R}^n$ satisfying* (15) *with $a_1, \ldots, a_d$ meromorphic in a neighbourhood of $\bar{U}$. Then there exist a stratification $\mathcal{S}'$ of $\bar{U}$ and a constant C such that for each Lipschitz $\mathcal{S}$-compatible vector field v on U with a Lipschitz constant L* (17) *holds for all $x \in U$.*

Proof. — As above, it suffices to prove the lemma for $f = g/h$, where g, h are analytic in a neighbourhood of $\bar{U}$. In order to do this, we take a stratification $\mathcal{S}$ satisfying the assertion of Lemma 4.4 for h and g. Then

$$|Df(x)v(x)| \leqslant \left| \frac{(Dg(x)v(x))h(x) - (Dh(x)v(x))g(x)}{h^2(x)} \right| \leqslant CL|f(x)|,$$

and the lemma follows. □

Proof of Fact 4.1. – It suffices to prove the fact for $k = 1$. Let us denote $\operatorname{Int}(Y)$ by U. Let $\mathscr{S}$ be a stratification of Y satisfying the assertion of Lemma 4.4 for $f = F$ and Lemma 4.5 for $f = \partial F/\partial x_i$, $i = 1, \ldots, n$ (see Lemma 2.3). Let v be a Lipschitz $\mathscr{S}$-compatible vector field on Y with a constant L.

First, we prove that A is Lipschitz in U. Assume that a segment $\overline{pq}$ lies in U. Then

$$\begin{aligned}|A(p)-A(q)| &= |DF(p)v(p)-DF(q)v(q)| \\ &\leqslant |DF(p)(v(p)-v(q))| + \left|\sum_{i=1}^{n} w_i D(\partial f/\partial x_i)(q)v(q)\right| \\ &\quad + |D(\partial F/\partial w)(q)v(q)-(DF(p)-DF(q))v(q)|,\end{aligned}$$

where $w = (w_1, \ldots, w_n)$ denotes the vector $\overrightarrow{pq}$. The first two terms of the right side of the above inequality are bounded by $CL|p-q|$. To estimate the third one, we consider

$$B_i = |((\partial^2 F/\partial w\, \partial x_i)(q))v_i(q) - ((\partial F/\partial x_i)(p) - (\partial F/\partial x_i)(q))v_i(q)|.$$

By the mean value theorem $(\partial F/\partial x_i)(p) - (\partial F/\partial x_i)(q) = (\partial^2 F/\partial x_i\, \partial w)(p')$, for some $p' \in \overline{pq}$. Consequently

$$B_i = |(\partial^2 F/\partial z_i\, \partial w)(q) - (\partial^2 F/\partial z_i\, \partial w)(p')|\,|v_i(q)| \leqslant C'|p-q|^2,$$

where C' depends on the maxima on $\overline{pq}$ of $\partial^3 F/\partial z_i\, \partial^2 w$ and $v_i(q)$.

Now take x, $x' \in U$ and a smooth curve $\gamma : [0,1] \to U$ joining x and x' and such that lenght $\gamma \leqslant C|x-x'|$. If N is sufficiently large, then for $p_k = \gamma(k/N)$

$$\begin{aligned}|A(x)-A(x')| &\leqslant \sum_{k=0}^{N-1} |A(p_{k+1})-A(p_k)| \\ &\leqslant \sum_{k=0}^{N-1} (CL|p_{k+1}-p_k| + C'|p_{k+1}-p_k|^2) \\ &\leqslant CL|x-x'| + \delta.\end{aligned}$$

If N is arbitrary large, then δ is arbitrary small. So, A is Lipschitz on U.

Extend $A|_U$ to the Lipschitz function $\bar{A}$ on $\bar{U}$. We only need to show that $\bar{A} = A$. Because A is continuous on each stratum, it suffices to show this in a dense subset of each stratum. Fix j and $x_0 \in \mathring{S}^j$. By the Whitney Wings Lemma (see for instance [4]), $\mathring{S}^j$ near x_0 is contained in the closure of a semi-analytic subset Z of U such that $\dim Z = j+1$. By Proposition 3.5 we can assume that Z is L-regular, so moving x_0 a little and changing coordinates near x_0, we can write Z as the graph of a strongly semi-analytic map $g : V \to \mathbb{R}^{n-j-1}$ such that

(i) V is an open neighbourhood of 0 in

$$H = \{(y_1, \ldots, y_{j+1}) \in \mathbb{R}^{j+1} : y_{j+1} \geqslant 0\}$$

(ii) $G^{-1}(\mathring{S}^j) = V \cap \mathbb{R}^j$ and $G(0) = 0$, where $G(y) = (y, g(y))$.

Changing again x_0 and using the Puiseux Theorem (in the version from [6]), we can assume that for some natural number $r > 0$ both $g(y_1, \ldots, y_j, y_{j+1}^r)$ and $H = F \circ G(y_1, \ldots, y_j, y_{j+1}^r)$ are analytic. The vector field $w(y) = \pi(v(G(y)))$ is Lipschitz (Dg is bounded) and compatible with $V \cap \mathbb{R}^j$ (the flow generated by w preserves $V \cap \mathbb{R}^j$). Put $\tilde{v}(G(y)) = DG(y)w(y)$. The function $DF(x)\tilde{v}(x)$ is continuous on Z. In fact, $DF(x)\tilde{v}(x) = D(F \circ G)(\pi(x))w(\pi(x))$ and

$$\begin{aligned} D(F \circ G)(y)w(y) &= DH(y_1, \ldots, y_i, y_{j+1}^r)w(y) \\ &= \sum_{i=1}^{j} ((\partial H/\partial y_i)(y))w_i(y) + \frac{1}{r}\, y_{j+1}^{(1-r)/r}((\partial H/\partial y_{j+1})(y))w_{j+1}(y) \end{aligned}$$

tends to 0 if $y_{j+1} \to 0$ (w is Lipschitz, so $w_{j+1}(y) \leqslant C|y_{j+1}|$). This proves the continuity of $DF(x)\tilde{v}(x)$. Similar arguments show that $\tilde{v}(x)$ is continuous. Therefore, since DF is bounded, $DF(x)v(x) = DF(x)\tilde{v}(x) + DF(x)(v(x) - \tilde{v}(x))$ is continuous (the last term vanishes on $S^j \cap Z$). This ends the proof. □

5. Proof of Theorem 1.4.

By induction on $k = \dim X$. For $k = 0$ the theorem is obvious. It will be easier to find an L-stratification of X with $l = 0$ (see the remark after Proposition 1.5).

First, assume that X is the union of two L-regular sets X_1, X_2, given by the graphs of $F, G : Y \to \mathbb{R}^{n-k}$, as in Definition 3.2, and that $F \leqslant G$ (F can be equal to G; if $k = n$, then $F \equiv G \equiv 0$). Let $\mathcal{S}' = \{S'^j\}_{j=0}^{k-1}$ be a stratification of Y satisfying the assertion of Fact 4.1 for $F, G, F - G$, and compatible with the zero set of $F - G$. By the inductive assumption, we can assume that $\{S'^j\}_{j=0}^{k-1}$ is an

L-stratification of S'^{k-1}. Then, because every Lipschitz function in a subset of $\mathbb{R}^k$ can be extended to a Lipschitz function on $\mathbb{R}^k$ with the same Lipschitz constant ([1]), $\mathcal{S}'$ is an L-stratification of Y. Let S^j be the union of the graphs of F and G restricted to S'^j. We show that $\mathcal{S}' = \{S^j\}_{j=0}^k$ satisfies (ii) of Proposition 1.5.

Let $q \in \mathring{S}^j$, $p \in S^j$ and let v be a Lipschitz $\mathcal{S}$-compatible vector field on $W = S^{j-1} \cup \{p\}$. Put $p' = \pi(p)$, $q' = \pi(q)$, where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the standard projection.

Suppose that p and q belong to X_1 (or X_2). The distance between q and W is estimated by the distance between q' and $\pi(W)$. The vector field $x(y) = \pi v(y, F(y))$ on $\pi(W)$ is Lipschitz and $\mathcal{S}'$-compatible. Extend w to a Lipschitz $\mathcal{S}'$-compatible vector field $\tilde{w}$ on Y. By Fact 4.1, $\tilde{v}(x) = (\tilde{w}(\pi(x)), DF(\pi(x))\tilde{w}(\pi(x)))$ is Lipschitz on X_1 and $\tilde{v}(q)$ is the desired extension.

Let $p' = q'$. We can assume that $p \in X_1$, $q \in X_2$. Then the distance from q to W is estimated by $|p-q|$, and $\tilde{v}(q) = DG(p')\pi(v(p))$ is the desired extension. In fact,

$$|\tilde{v}(p) - \tilde{v}(q)| = |D(F-G)(p')\pi(v(p))| \leqslant CL|(F-G)(p')| \leqslant CL|p-q|,$$

because $\pi(v|_{X_1 \cap W})$ is Lipschitz and $\mathcal{S}'$-compatible, and can be extended on Y, so we can use Fact 4.1 again.

Let $p \in X_1$, $q \in X_2$ (or $p \in X_2$, $q \in X_1$). Put $r = (p', G(p'))$. By the above arguments, we can extend v on $W \cup \{r\}$, next on $W \cup \{r\} \cup \{q\}$, and so obtained the desired extension ($|r-q| \leqslant C|p'-q'| \leqslant C|p-q|$). Note that Lipschitz constants of the obtained extensions can be estimates by Lipschitz constants of v, so $\mathcal{S}$ is an L-stratification.

For the general case, we use Remark 3.6. Let $\mathcal{Y} = \{Y_i\}$ be the given family of L-regular sets. For each pair $Y_i, Y_j \in \mathcal{Y}$ such as in *a*) of Remark 3.6 and such that $\dim Y_i = \dim Y_j$ we fix an L-stratification $\mathcal{S}_{ij}$ of $Y_i \cup Y_j$. Let $\mathcal{S}$ be a stratification compatible with all $\mathcal{S}_{ij}$ and the family $\mathcal{Y}$. We show that $\mathcal{S}$ satisfies (ii) of Proposition 1.5 for $j = k$. Let $q \in \mathring{S}^k$, $p \in S^k$ and let v be a Lipschitz $\mathcal{S}$-compatible vector field on $W = S^{k-1} \cup \{p\}$. Let r be one of the points of W which are closest to p. Find Y, Y' for q, r as in *a*) of Remark 3.6. If $\dim Y = \dim Y' = k$, then we can find the required extension because $\mathcal{S}$ is compatible with an L-stratification of $Y' \cup Y$ (we use (i) of Proposition 1.5). If $\dim Y' < k$ ($\dim Y$ must be equal to k), then by *a*) of Remark 3.6, we can replace r by a point s of Y, such that $|p-s| \leqslant C|p-r|$, and repeat the above arguments. By b) of Remark 3.6, $\mathcal{S}$ is compatible with $\mathcal{X}$. Now we apply the inductive hypothesis to S^{k-1} and the family $\mathcal{X}' = \{S^j\}_{j=0}^{k-1}$. The obtained stratification of S^{k-1} and $\mathring{S}^k = X\backslash S^{k-1}$ form an L-stratification. In fact, it satisfies (ii) of Proposition 1.5, for $j < k$ by construction, and for $j = k$, because it is compatible with a stratification satisfying this condition. This completes the proof. □

6. Proof of Key Lemma.

Replacing v by $\frac{1}{L}v$ we may assume that $L = 1$. Let $\{\sigma_k\}_{k=0,\dots,m}$ be the family of elementary symetric polynomials

$$\sigma_k(p_1, \dots, p_m) = \sum_{i_1<\dots<i_k} p_{i_1}, \dots, p_{i_k},$$

for $k = 1, \dots, m$ and $\sigma_0 \equiv 1$. It is easy to see that

$$\begin{aligned} D\sigma_k(p)v &= \sum_{i=1}^{m} \sigma_{k-1}(p_1, \dots, \hat{p}_i, \dots, p_m)v_i \\ &= \sum_{i=1}^{m} a_{ki}(p)v_i \end{aligned}$$

and $\det(a_{ki})_{k,i=1,\dots,m} = \prod_{i<j}(p_i - p_j)$.

Defines families of polynomials $\{q_{sk}(p)\}_{k=1,\dots,m^{2s}} = \mathscr{V}(s)$ as follows: $q_{1i} = p_i$, $\{q_{sk}\} = \{q_{(s-1)i} - q_{(s-1)j}\}_{i,j=1,\dots,m^{2s-2}}$. Let $\mathscr{W}(s)$ denote the family of elementary symetric function of $\{q_{sk}(p)\}$. We shall show that $\mathscr{W} = \bigcup_{s=1}^{m} \mathscr{W}(s)$, $\mathscr{V} = \bigcup_{s=1}^{m+1} \mathscr{V}(s)$ satisfy the assertion of Key Lemma.

Suppose this is not the case. Then, by the curve selection lemma, there exist $\mathbb{R}$-analytic curves $p, v : [0, \varepsilon) \to \mathbb{C}^m$ (polynomials from $\mathscr{W}$ and $\mathscr{V}$ are homogeneous) such that

$$|DW(p(t))v(t)| \leqslant |W(p(t))|, \tag{19}$$

$$DV(p(t))v(t) \equiv 0 \quad \text{if} \quad V(p(t)) \equiv 0, \tag{20}$$

for all $W \in \mathscr{W}$, $V \in \mathscr{V}$ and

$$\lim_{t\to 0} |v_i(t)/p_i(t)| = \infty \qquad \text{for some } i. \tag{21}$$

Fix such $p(t)$, $v(t)$. Let f, g be the germ at 0 of complex-value $\mathbb{R}$-analytic functions. We write $f=O(g)$ if $\lim\limits_{t\to 0} |f(t)/g(t)| < \infty$ or $f\equiv g\equiv 0$, and $f = O(g)$ if $\lim\limits_{t\to 0} f(t)/g(t) = 0$. If $\lim\limits_{t\to 0} f(t)/g(t) = 1$ or $f \equiv g \equiv 0$ we will write $f \sim g$. Note that $\sim$ is an equivalence.

Sublemma 1. – *Let $p(t)$, $v(t)$ satisfy* (19) *and* (20) *for all $W \in \mathscr{W}(1)$, $V \in \mathscr{V}(2) \cup \mathscr{V}(1)$. If $\{p_1, \ldots, p_s\}$ is a $\sim$ class in the set of coordinate functions of $p(t)$, then*

$$(22) \qquad v_1(t) + \cdots + v_s(t) = O(p_1(t) + \cdots + p_s(t)),$$

and

$$(23) \qquad \textit{if } p_i(t) \equiv 0 \ (\textit{for some } i=1,\ldots,m), \textit{ then } v_i(t) \equiv 0.$$

Proof. – Assume, for a moment, that $p_i(t) \not\equiv p_j(t)$ for $i \neq j$. Consider the system of linear equations

$$D\sigma_k(p)v = \sum_i a_{ki}(p)v_i = \sum_i b_{ki}(p)w_i = A_k, \qquad k = 1, \ldots, m,$$

where $w_1 = v_1 + \cdots + v_s$, $w_i = v_i$ for $i > 1$. Then, $b_{k1} = a_{k1}$ and $b_{ki} = a_{ki} - a_{k1}$ for $1 < i \leqslant s$ and $b_{ki} = a_{ki}$ for $i > s$, so

$$B = \det(b_{ki}) = \det(a_{ki}) = \prod_{i<j} (p_i - p_j).$$

On the other hand,

$$A = \det(A_k; b_{ik})_{k=1,\ldots,m;\, i=2,\ldots,m} = \prod_{i<j\leqslant s} (p_i - p_j) \cdot \prod_{s<i<j} (p_i - p_j) \cdot \left(\sum_{k=1}^{m} A_k s_k(p) \right).$$

By (19) for $W \in \mathscr{W}(1)$, $|A_k| \leqslant \sigma_k(p)|$ and consequently

$$|A_k s_k(p)| \leqslant |\sigma_k(p) s_k(p)| \leqslant \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)} \left| a_\alpha \left(\prod_i p_i^{\alpha_i} \right) \right|,$$

where $\sum_i \alpha_i = s(m-s) + 1$ and $\alpha_i \leqslant s$ for $i > s$ (because p_i enters in b_{kj} or A_k with a power not larger than 1 and p_i does not enter in b_{ki}). But $p_i(t) = O(p_i(t) - p_j(t))$, for $i > s$, $j \leqslant s$, or $i \leqslant s$, $j > s$. From this, we conclude

$$A_k s_k(p) = O\left(p_1(t) \prod_{i \leqslant s < j} (p_i(t) - p_j(t))\right),$$

and (22) follows from Cramer's rule.

If $p_i(t) \equiv p_j(t)$ for some $i \neq j$, then $v_i(t) \equiv v_j(t)$ (because of (20) for $p_i - p_j$). In this case, instead of $v_1, \ldots, v_m$, we consider $v_{i_1} + \cdots + v_{i_l}$, for $i_1 < i_2 < \ldots < i_l$ such that $p_{i_1}(t) \equiv \cdots \equiv p_{i_l}(t)$ and $p_{i_1}(t) \not\equiv p_j(t)$ for $j \neq \{i_1, \ldots, i_1\}$, and the system of linear equations $D\sigma_k(p)v = A_k$, where k ranges from 1 to the number of all $\equiv$ classes in the set of coordinate functions of $p(t)$.

The rest of the proof in this case is similar to that of the special case. The detailed verification is left to the reader.

The condition (23) is an immediate consequence of (20) for $V = p_i$. $\square$

Consider a functional $l(p) = \sum_i q_{si}(p)$, where $1 \leqslant s \leqslant m$ and the sum is taken over all such i that $\{q_{si}(p)(t))\}$ is a $\sim$ class in $\{q_{sj}(p(t))\}_{j=1,\ldots,m^{2s}}$. Then, by Sublemma 1, and (19) for $W \in \mathscr{W}(s)$ and (20) for $V \in \mathscr{V}(s) \cup \mathscr{V}(s+1)$

$$l(v(t)) = Dl(p(t))v(t) = O(l(p(t))). \tag{24}$$

The same happens for $l = q_{si}$, if $q_{si}(p(t)) \equiv 0$. We call l of the above form to be admissible.

For an integer r, we denote by $K(r)$ the linear subspace of $\mathbb{C}^m$ generated by all admissible functionals l satisfying $l(p(t)) = O(t^r)$.

SUBLEMMA 2. – *If $p_i(t) = O(t^r)$ for an integer r, then $p_i \in K(r)$.*

Proof. – Suppose, by contradiction, that $p_i(t) = O(t^r)$ and $p_i \notin K(r)$. For each $s = 1, \ldots, m$ choose $q_{si(s)}$ such that $q_{si(s)}(p(t)) = O(t^r)$, $q_{si(s)} \notin K(r)$ and such that if j satisfies $q_{sj}(\mathrm{p}(t)) = O(q_{si(s)}(p(t)))$, then $q_{sj} \in K(r)$. If such $q_{si(s)}$ exists, we know that it exists at least for $s = 1$, we put $l_s(p) = \sum_j q_{sj}$, where the sum is taken over all such j that $q_{sj}(p(t)) \sim q_{si(s)}(p(t))$. We shall prove that if both l_s and l_{s+1} exist, then $l_{s-1}(p(t)) = O(l_s(p(t)))$.

In fact, if this weren't true, then $(q_{si(s)} - q_{sj}) \in K(r)$ for $q_{sj}(p(t)) \sim q_{si(s)}(p(t))$, and $l_s \in K(r)$, so $q_{si(s)} \in K(r)$, which contradicts our assumptions. The same argument shows that if l_s exists, so does l_{s+1}. Therefore $l_1, \ldots, l_m$ exist and they are lineary independent. In fact, if $\sum B_i l_i = 0$, then $\sum B_i l_i(p(t)) \equiv 0$, which implies $B_i = 0$ for all i. Thus $\{l_i\}_{i=1,\ldots,m}$ generate $K(r) = \mathbb{C}^m$, which contradicts our assumptions. This ends the proof. $\square$

We are now in a position to prove Key Lemma. Take $i \in \{1, \ldots, m\}$. Let $p_i(t) \sim At^r$, $A \neq 0$. By Sublemma 2, $p_i = \sum B_j l_j(p)$ for admissible l_j and such that $l_j(p(t)) = O(t^r)$. Therefore, by (24)

$$v_i(t)) = \sum B_j l_j(v(t)) = O(t^r) = O(p_i(t)).$$

But this contradicts (21), which ends the proof. $\square$

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