ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 37, nº 4 (1987), p. 191-205

<http://www.numdam.org/item?id=AIF_1987__37_4_191_0>

Ann. Inst. Fourier, Grenoble
37, 4 (1987), 191-205.

SOME REMARKS ON KOSZUL ALGEBRAS AND QUANTUM GROUPS

by Yu. I. MANIN

Introduction.

This note is devoted to algebras defined by quadratic relations. Such algebras arise in various contexts and have interesting homological properties whose investigation is based upon Koszul complexes : cf. [2], [3], [6].

My interest in them is connected with recent work of L. D. Faddeev and collaborators, V. G. Drinfeld and M. Jimbo who have introduced some remarkable Hopf algebras, non commutative and non cocommutative, or « quantum groups » : cf. [1], Drinfeld's Berkeley talk with ample bibliography. The coordinate rings of these groups are defined mainly by quadratic relations of very specific type.

The main observation of this note (Section 2, Th. 4) consists in realization, that the category of quadratic algebras is endowed with a natural « pseudo tensor product » $\bullet$ and a corresponding internal *Hom* functor. After a dualization this leads to large supply of « quantum semi-groups » of coendomorphisms of arbitrary quadratic algebras together with a supply of their comodules (Section 2, n. 9). At the same time a Koszul complex is associated with each morphism of quadratic algebras (Section 2, n. 5).

Section 1 gives a detailed description of quantum GL(2) as an automorphism « group » of a quantum plane. This description discovered by Yu. Kobozev was the starting point for this work. Section 3 gives two constructions of quantum determinants. In Section 4 we explain how to carry over the previous constructions into certain tensor categories of linear spaces (« the Yang-Baxter categories »).

Key-words : Koszul algebra - Tensor category - Quadratic algebra - Quantum groups - Yang-Baxter equation.

1. Quantum group GL(2): a motivating example.

1. *Notation.* — We fix once and for all a ground field k over which all tensor products are taken. For a linear k-space V, T(V) means its tensor algebra. For a subset $R \subset T(V)$ we denote by $T(V)/(R)$ the quotient algebra with respect to the ideal generated by R. If $V = \bigoplus_i kx_i$, we write

$$T(V)/(R) = k[x_i] \quad \text{with relations} \quad r = 0 \quad \text{for} \quad r \in R$$

etc. As in [1], it is sometimes suggestive to imagine a ring $A = T(V)/(R)$ as a coordinate ring, i.e. a ring of functions on an imaginary space of « noncommutative geometry », or « quantum space » Spec A.

2. *Two quantum planes.* — Let $q \in k$, $q \neq 0$. The quantum plane Spec $A_q(2|0)$ is defined by the ring

$$A_q(2|0) = k[x, y] \quad \text{with relation} \quad xy = q^{-1}yx. \tag{1}$$

We shall need also the quantum plane

$$\begin{aligned} &A_q(0|2) = k[\xi, \eta] \quad \text{with relations} \\ &\xi\eta = -q\eta\xi, \qquad \xi^2 = \eta^2 = 0. \end{aligned} \tag{2}$$

Since relations (1), (2) are homogeneous, both rings are graded, with generators of degree 1. The dimensions of their homogeneous components are the same ones, as for commutative polynomials of two variables in case (1) and for a grassmannian algebra of two generators in case (2).

3. *Quantum matrices.* — The coordinate ring of the manifold of quantum matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by the relations

$$\begin{aligned} &M_q(2) = k[a,b,c,d], \\ &ab = q^{-1}ba, \qquad ac = q^{-1}ca, \qquad cd = q^{-1}dc, \qquad bd = q^{-1}db, \\ &bc = cb, \qquad ad - da = (q^{-1} - q)bc. \end{aligned} \tag{3}$$

These relations first emerged in a fairly indirect way: cf. [1], formulas (16)-(19). The next Proposition shows that they can be naturally interpreted as defining the quantum space of linear endomorphisms of relations (1) and (2).

4. PROPOSITION. – *Let* (x,y) (resp. (ξ,η)) *be the generic solutions of* (1), (resp. (2)). *Let* a, b, c, d *commute with* x, y, ξ, η. *Put*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

If $q^2 \neq -1$, *the following conditions are equivalent :*

(i) (x',y') *and* (x'',y'') *verify* (1).

(ii) (x',y') *verify* (1) *and* (ξ',η') *verify* (2).

(iii) (a,b,c,d) *verify* (3).

Proof. – The relation $x'y' = q^{-1}y'x'$ means

$$(ax+by)(cx+dy) = q^{-1}(cx+dy)(ax+by).$$

Taking into account that a, b, c, d commute with x, y and comparing coefficients, we get

$$(3)' \qquad \begin{aligned} x^2 &: ac = q^{-1}ca, \\ y^2 &: bd = q^{-1}db, \\ xy &: ad - da = q^{-1}cb - qbc. \end{aligned}$$

Exchanging here b and c we get the relations, equivalent to $x''y'' = q^{-1}y''x''$:

$$(3)'' \quad ab = q^{-1}ba, \quad cd = q^{-1}dc, \quad ad - da = q^{-1}bc - qcb.$$

Comparing the last relations in $(3)'$ and $(3)''$ we obtain

$$(q+q^{-1})(bc-cb) = 0 \Rightarrow bc = cb, \quad \text{if} \quad q^2 \neq -1.$$

Hence $(3)'$ and $(3)''$ together are equivalent to (3).

Finally, a similar direct calculation shows that the relations (2) for ξ', η' are equivalent to $(3)''$.

This proposition, due to Yu. Kobozev, shows without further calculations the two main properties of relations (3) :

5. *Multiplicativity.* – Let (a, b, c, d) and (a', b', c', d') separately verify (3) and pairwise commute among themselves. Then the matrix elements of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ also verify (3).

6. *Quantum determinant.* — In the same conditions put

$$\mathrm{DET}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc = da - qcb. \tag{4}$$

Then

$$\mathrm{DET}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \mathrm{DET}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathrm{DET}_q\begin{pmatrix} a' & b' \\ c' & b' \end{pmatrix}.$$

In fact, if (3) (or even only (3)″) hold, we have in notation of Proposition 4 :

$$\xi'\eta' = \mathrm{DET}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}\xi\eta\,.$$

7. *Quantum group* $\mathrm{G'L}_q(2)$. — Its quantum coordinate ring can be obtained from that of $\mathrm{M}_q(2)$ by inverting $\mathrm{DET}_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Similarly, adding the relation $\mathrm{DET}_q = 1$, we get quantum SL(2).

8. *Language of Hopf algebras.* — Proposition 4 in more intrinsic form says that the map

$$\Delta : \mathrm{M}_q(2) \to \mathrm{M}_q(2) \otimes \mathrm{M}_q(2)\,, \qquad \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot$$

makes of $\mathrm{M}_q(2)$ a Hopf algebra (without antipode), and the maps

$$\delta : \mathrm{A}_q(2|0) \to \mathrm{M}_q(2) \otimes \mathrm{A}_q(2|0), \qquad \delta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\delta^t : \mathrm{A}_q(2|0) \to \mathrm{M}_q(2) \otimes \mathrm{A}_q(2|0), \qquad \delta^t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\delta^\pi : \mathrm{A}_q(0|2) \to \mathrm{M}_q(2) \otimes \mathrm{A}_q(0|2), \qquad \delta^\pi\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

define comodules over this Hopf algebra.

2. Quadratic algebras.

1. *Quadratic algebras.* — Quadratic algebra is an associative **Z**-graded k-algebra $\mathrm{A} = \sum_{i=0}^{\infty} \mathrm{A}_i$ with the following properties :

a) $\mathrm{A}_0 = k$, $\dim \mathrm{A}_1 < \infty$.

b) A is generated by A_1 over k, and the ideal of relations between elements of A_1 (i.e. the kernel of the homomorphism $T(A_1) \to A$) is generated by a subspace $R(A) \subset A_1 \otimes A_1$.

(Priddy [2] calls such a ring « homogeneous prekoszul algebra » and endows A_1 with an additional **Z**-grading).

Morphism of quadratic algebras $f: A \to B$ is a k-homomorphism preserving gradings. There is a bijection between such morphisms and k-linear maps $f_1 : A_1 \to B_1$ for which $(f_1 \otimes f_1)(R(A)) \subset R(B)$. We denote by QA the category of quadratic algebras.

It is often convenient to write A as

$$A \leftrightarrow \{A_1,\ R(A) \subset A_1 \otimes A_1\}.$$

For example,

$$A_q(2|0) \leftrightarrow \{kx \otimes ky,\ k(x \otimes y - q^{-1} y \otimes x)\}.$$

Algebras $A_q(0|2)$ and $M_q(2)$ are also quadratic.

2. *Operations on quadratic algebras.* — Let A, B be two quadratic algebras. Put

$$A \circ B \leftrightarrow \{A_1 \otimes B_1,\ S_{(23)}(R(A) \otimes B_1^{\otimes 2} + A_1^{\otimes 2} \otimes R(B))\} \tag{5}$$

$$A \bullet B \leftrightarrow \{A_1 \otimes B_1,\ S_{(23)}(R(A) \otimes R(B))\} \tag{6}$$

$$A^! \leftrightarrow \{A_1^*,\ R(A)^\perp\}. \tag{7}$$

In (6) we have $R(A) \otimes R(B) \subset A_1 \otimes A_1 \otimes B_1 \otimes B_1$ while $R(A \bullet B) \subset A_1 \otimes B_1 \otimes A_1 \otimes B_1$, and $S_{(23)}$ in the relevant rearrangement operator (for $\sigma \in \mathfrak{S}_p$ we write here

$$S_\sigma(a_1 \otimes \ldots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}).$$

For the similar reason we put $S_{(23)}$ in (5). In (7) V^* means a dual space of V, and the corresponding pairing $\langle v^*, v \rangle$ is also called a contraction. We identify $(V \otimes W)^*$ with $V^* \otimes W^*$, etc. By definition, $R(A)^\perp$ is then the annihilator of $R(A)$.

3. *Properties of operations.* — A usual tensor product of quadratic algebras (over k) is again a quadratic algebra :

$$A \otimes B \leftrightarrow \{A_1 \oplus B_1,\ R(A) \oplus R(B) \oplus [A_1, B_1]\},$$

where $[A_1, B_1]$ is generated by

$$a \otimes b - b \otimes a \in A_1 \otimes B_1 \oplus B_1 \otimes A_1 .$$

However, $\circ$ and $\bullet$ are in some respects more relevant in QA, e.g. the comultiplication for natural quadratic Hopf algebras is a morphism $A \to A \circ A$. (Since there is a canonical ring homomorphism $A \circ B \to A \otimes B$ doubling the degree, this defines also a comultiplication in the usual sense).

Both products are endowed with functorial associativity and commutativity constraints :

$$(A \circ B) \circ C = A \circ (B \circ C), \qquad (A \bullet B) \bullet C = A \bullet (B \bullet C),$$
$$A \circ B \xrightarrow{\sim} B \circ A, \qquad A \bullet B \xrightarrow{\sim} B \bullet A,$$

which are induced by the standard maps on generators of degree 1. Morphisms $f : A \to A'$ and $g : B \to B'$ define morphisms $f \circ g : A \circ B \to A' \circ B'$ and $f \bullet g : A \bullet B \to A' \bullet B'$, mapping $a \otimes b$ to $f(a) \otimes g(b)$ for $a \in A_1$, $b \in B_1$.

A morphism $f : A \to B$ defines a morphism $f^! : B^! \to A^!$, for which $\langle f^!(b), a \rangle = \langle f(a), b \rangle$ if $a \in A_1$, $b \in B_1^*$. This defines an equivalence of categories $! ;$ $QA \to QA^{0p}$. Besides, there are natural isomorphisms

$$(A \bullet B)^! = A^! \circ B^!, \qquad (A \circ B)^! = A^! \bullet B^! .$$

Finally, since $R(A) \otimes R(B) = R(A) \otimes Bo_1^{\otimes 2} \cap A_1^{\otimes 2} \otimes R(B)$, there is a canonical morphism $A \bullet B \to A \circ B$.

The following simple fact is our main result. It shows that $(QA, \bullet)$ is a (non additive) tensor category with internal *Hom* in the terminology of [5] (cf. also Section 4 below).

4. THEOREM. — *a) There is a functorial isomorphism*

$$\mathrm{Hom}\,(A \bullet B, C) = \mathrm{Hom}\,(A, B^! \circ C)$$

identifying a map $f : A_1 \otimes B_1 \to C_1$ *with a map* $g : A_1 \to B_1^* \otimes C_1$ *if* $\langle g(a), b \rangle = f(a \otimes b)$ *for all* $a \in A_1$, $b \in B_1$ (l.h.s. *denotes contraction with respect to* B_1).

b) Let $K = k[\varepsilon]$, $\varepsilon^2 = 0$. *Then* K *is a unit object of* $(QA, \bullet)$. *In particular, for all* A *there is a canonical isomorphism* $K \bullet A \xrightarrow{\sim} A$.

Proof. — *a*) We must check that if f, g are related as in the statement the following conditions are equivalent :

$$(f\otimes f)\,S_{(23)}(R(A)\otimes(B)) \subset R(C)),$$
$$(g\otimes g)R(A) \subset S_{(23)}(R(B)^{\perp} \otimes C_1^{\otimes 2} + B_1^{*\otimes 2} \otimes R(C).$$

But they are respectively equivalent to

$\langle R(C)^{\perp}, (f\otimes f)S_{(23)}(R(A)\otimes R(B))\rangle = 0$ (contraction w.r.t. $C_1 \otimes C_1$)
$\langle R(B)\otimes R(C)^{\perp}, (g\otimes g)R(A)\rangle = 0$ (contraction w.r.t. $B_1\otimes B_1\otimes C_1^*\otimes C_1^*$).

In their turn, each of these last orthogonality relations mean that if we start with an element of $R(A)$, apply $g \otimes g$ and then contract consecutively with arbitrary elements of $R(B)$ and $R(C)^{\perp}$, we shall get zero.

b) By definition

$$K\bullet A \leftrightarrow \{\varepsilon\otimes A_1, S_{(23)}(\varepsilon^{\otimes 2}\otimes R(A))\},$$

whence an isomorphism $a \mapsto \varepsilon \otimes a$ for $a \in A_1$. □

The rest of this Section is devoted to formal consequences of this theorem.

5. *Generalized Koszul complexes.* — Put $A = K$ in Th. 4*a*. We get

$$\mathrm{Hom}\,(B,C) = \mathrm{Hom}\,(K, B^! \circ C).$$

Denote by d_f the image of $\varepsilon \in K$ under the morphism $K^! \to B^! \circ C$ corresponding to $f: B \to C$. We have $d_f^2 = 0$. Therefore with each morphism f in QA is associated a complex

$$K^\bullet(f) = \{B^! \otimes C, \text{ right multiplication by } d_f\}.$$

For $B = C = A$, $f = \mathrm{id}$ we get one of Koszul complexes associated with a quadratic algebra A :

$$K^\bullet(A) = \{A^! \otimes A, \delta\}.$$

In order to define the second complex put $A' = \bigoplus_i (A_i^!)*$ where $A_i^!$ is the i-th component of $A^!$. This space is a right $A^!$-module. Put

$$K_\bullet(A) = A \otimes A' \subset \mathrm{Hom}_{\text{mod-}A}\,(A^! \otimes A, A)$$

and define the differential d by $(df)(a) = f(a\delta)$, $a \in A^! \otimes A$.

The following result is proved in [2] (cf. also [3]):

6. THEOREM-DEFINITION. — *A quadratic algebra* A *is called a Koszul algebra if the following equivalent properties are verified :*

a) $K_\bullet(A)$ *is acyclic.*

b) *Algebra* $\mathrm{Ext}^*_A(k,k)$ *is generated by* $\mathrm{Ext}^1_A(k,k) \simeq A^*_1$.

c) *Algebra* $\mathrm{Ext}^*_A(k,k)$ *is naturally isomorphic to* $A^!$. □

Using a sufficient condition for this property found by Priddy, the existence of a PBW-basis ([2], Th. 5.3), one can prove that coordinate rings of quantum spaces and their (co)endomorphism spaces considered below are Koszul algebras.

7. *Internal Hom.* — As in general formalism of tensor categories (see [5]) we put $\mathit{Hom}(B,C) = B^! \circ C$. In particular,

$$B^! = \mathit{Hom}(B,K^!).$$

This can be checked directly using $K^! = k[t]$, or by dualization of Th. 4*b*.

8. *Internal multiplication.* — By the general properties of Hom, the following internal product maps are defined:

$$\begin{aligned} &\mathit{Hom}(B,C) \bullet B \to C, \\ &\mathit{Hom}(B,C) \bullet \mathit{Hom}(C,D) \to \mathit{Hom}(B,D) \end{aligned} \tag{8}$$

with evident associativity properties (cf. [5], (1.6.2)).

9. *Dualization and internal comultiplication.* — We define the algebra of internal cohomomorphisms by

$$\mathit{hom}(B,C) = \mathit{Hom}(B^!,C^!)^! = B^! \bullet C.$$

Applying ! to (8), we get internal comultiplication maps

$$\begin{aligned} &\Delta_{BC} : C \to \mathit{hom}(B,C) \circ B, \\ &\Delta_{BCD} : \mathit{hom}(B,D) \to \mathit{hom}(B,C) \circ \mathit{hom}(C,D). \end{aligned} \tag{9}$$

In particular, $\mathit{hom}(A,A) = \mathit{end}(A)$ is endowed with a Hopf algebra structure

$$\Delta_A = \Delta_{AAA} : \mathit{end}(A) \to \mathit{end}(A) \circ \mathit{end}(A),$$

while algebras A, *hom* (A, B) and their homogeneous components furnish plenty natural comodules over *end* (A).

This is our generalization of the main construction of Section 1. More precisely, *end* $A_q(2|0)$ is defined by a half of relations, defining $M_q(2)$. The complementary relations stem from *end* $A_q(2|0)^!$ since $A_q(2|0)^! \simeq A_q(0|2)$.

3. Determinant.

In this section we shall consider two natural constructions of determinants for quantum semigroups *end* (A). The first one is applicable to algebras A, which are « similar » to usual Grassmann algebras and imitates the conventional definition of det. The second one is universal and imitates the homological description of Berezinian in commutative superalgebra, but its properties are poorly known. (In the context of Yang-Baxter categories it was considered by Lyubashenko [4].)

1. *Quantum grassmannian algebras (q.g.a).* — We shall call a quadratic algebra A a q.g.a. of dimension n, if $\dim A_n = 1$ and $A_m = 0$ for $m > n$. From the construction of the morphism (9), $\Delta : A \to end\,(A) \circ A$ one sees that $\Delta(A_i) \subset (end\,(A))_i \otimes A_i$ for any i. In particular, we can define an element $DET_A \in (\mathrm{end}\,(A))_n$ for a q.g.a. of dimension n by the formula $\Delta(a) = DET_A \otimes a$, where $a \in A_n$ is a generator. It enjoys the comultiplicativity property $\Delta\,(DET_A) = DET_A \otimes DET_A$.

2. *Example.* — Let $A = k[X_1, \ldots, X_n]$ with relations $X_i^2 = 0$, $X_iX_j = -qX_jX_i$ for $i < j$. One easily sees that A is an n-dimensional q.g.a. The determinant DET_A lies in the ring *end* (A) generated by Y_i^j corresponding to $X_i \otimes X^j$, where $\{X^j\}$ is the dual basis to $\{X_i\}$. The formula for calculating DET_A is

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{n} Y_i^j X_j\right) = DET_A \cdot \prod_{j=1}^{n} X_j \tag{10}$$

where $[Y_i^j, X_k] = 0$. Therefore

$$DET_A = \sum_{s \in \mathfrak{S}_n} (-q)^{l(s)} Y_1^{s(1)} \ldots Y_1^{s(n)}$$

(cf. [4] and the formula in [1] following (19). However *end* (A) has fewer relations than (16)-(19), as our discussion in Sec. 1 shows).

3. *Example* (V. G. Drinfeld [7]). — Let k be a quotient field of a local ring O. Consider anticommutation relations

$$X^k X^l + X^l X^k = \sum_{i,j} c_{ij}^{kl} X^i X^j \tag{11}$$

where c_{ij}^{kl} are some elements of the maximal ideal in O enjoying the symmetry conditions $c_{ij}^{kl} = c_{ij}^{lk} = - c_{ji}^{kl}$. Clearly, for $c_{ij}^{kl} = 0$ we get a usual Grassmann algebra. In [7] Drinfeld calculated conditions on c_{ij}^{kl} which should be verified in order that (11) define a q.g.a. over O (and *a fortiori* over k).

Put

$$b_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \sum_j c_{i_1 j}^{j_1 j_2} c_{i_2 i_3}^{j j_3},$$

$$b = (b_{i_1 i_2 i_3}^{j_1 j_2 j_3}), \quad a = b(1 - b/3)^{-1}.$$

Then

$$\text{(11) define a q.g.a. over } O \Leftrightarrow \text{alt}\,(i)\ \text{sym}\,(j)\ a_{i_1 i_2 i_3}^{j_1 j_2 j_3} = 0.$$

The dimension of A in this case equals n and we obtain a quantum determinant in *end* (A).

In [7] it is shown also that q.g.a. (11) are Koszul.

4. *Question.* — Let A be a q.g.a. Can one make of *end* $(A)/(\mathrm{DET}_A - 1)$ a Hopf algebra with antipode, i.e. a quantum group SL(A)?

One can similarly define a DET with respect to any component A_n if $\dim A_n = 1$.

5. *Homological determinant.* — Let A be a quadratic algebra. Consider $A^! \otimes A$ as a quadratic algebra, put $E(A) = end\,(A^! \otimes A)$ and construct the structure comultiplication

$$\Delta : A^! \otimes A \to E(A) \circ (A^! \otimes A) \to E(A) \otimes A^! \otimes A.$$

Put $\Delta(\delta) = \delta'$, where $\delta \in A_1^* \otimes A_1$ corresponds to id. In $E(A) \otimes A^! \otimes A$ there is also an element $1 \otimes \delta$ with square zero. Although δ' and $1 \otimes \delta$ do not coincide, there exists a minimal ideal $I \subset E(A)$ with the property

$$\delta' \equiv 1 \otimes \delta \mod I \otimes A^! \otimes A.$$

Put $\tilde{E}(A) = E(A)/I$ and further

$$H(A) = H^*((A^! \otimes A), \delta),$$
$$\tilde{E}(A) \otimes H(A) = H^*(\tilde{E}(A) \otimes A^! \otimes A, 1 \otimes \delta).$$

Finally, define the cohomological determinant as a map

$$\mathrm{DETH}_A = H^*(\Delta \bmod I) : H(A) \to \tilde{E}(A) \otimes H(A).$$

If $\dim H(A) = 1$, we can consider this as an element

$$\mathrm{DETH}_A \in \tilde{E}(A).$$

Therefore a natural problem arises : to study the class of quadratic algebras A for which the Koszul complex $K^\bullet(A)$ has one-dimensional cohomology (the same in Yang-Baxter categories : cf. below).

4. Tensor categories and quadratic algebras.

1. *Tensor categories.* – Recall that a tensor category [5] is a couple $(C, \otimes)$ consisting of a category C, a functor $\otimes : C \times C \to C$, $(X, Y) \mapsto X \otimes Y$ and two additional functor isomorphisms

$$\psi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

(associativity constraint) and

$$\varphi_{X,Y} : X \otimes Y \to Y \otimes X, \qquad \varphi_{X,Y}\varphi_{Y,X} = \mathrm{id}$$

(commutativity constraint). Several axioms are imposed on this data, the most important for us being a compatibility diagram. A unit object of a tensor category consists of an object $U \in C$ and an isomorphism u : $U \to U \otimes U$ such that the functor $X \to U \otimes X$ is an equivalence. Internal *Hom* (X, Y) represents the functor $T \mapsto \mathrm{Hom}\,(T \otimes Y, Y)$. For further details see [5].

2. *YB-categories.* – A tensor category $\mathcal{S} = (C, \otimes)$ is called a YB-category (or Yang-Baxter category, or vectorsymmetry, cf. Lyubashenko [4]) if the following conditions are fulfilled :

a) C is a subcategory of finite-dimensional vector spaces over a field k.

b) $\otimes$ in C coincides with tensor product over k, and $\psi_{X,Y,Z}(x \otimes (y \otimes \mathscr{Z})) = (x \otimes y) \otimes z$ is the usual associativity constraint.

c) $(k, 1 \mapsto 1 \otimes 1)$ is a unit object, for all $X,Y \in C$ there exists *Hom* (X,Y) and the structure diagram *Hom* $(X,Y) \otimes X \to Y$ is isomorphic to the standard diagram $Y \otimes X^* \otimes X \to Y$.

It follows that essentially new data distinguishing the YB-category $(C, \otimes)$ from C consists of a family of isomorphisms $S_{(1,2)}$: $X \otimes Y \to Y \otimes X$ (former $\varphi_{X,Y}$) verifying the equations

$$S_{(1,2)}S_{(1,2)} = \mathrm{id},$$
$$S_{(1,2)(3)}S_{(1)(2,3)}S_{(1,2)(3)} = S_{(1)(2,3)}S_{(1,2)(3)}S_{(1)(2,3)}.$$

Physicists call the last equation the Yang-Baxter, or triangle, relation. In the context of tensor categories it is essentially the compatibility diagram for two constraints.

Each object X of a YB-category $\mathscr{S}$ over a field of characteristics zero defines a quadratic algebra « $\mathscr{S}$-symmetric algebra of X ».

3. Proposition. — *a*) *For each n one can define a representation* $\rho_n^{\mathscr{S}}: \mathfrak{S}_n \to GL(X^{\otimes n})$ *by the following prescription : to calculate* $\rho_n(s)$, *decompose s into a product of transpositions and construct the product of the corresponding commutativity operators* S.

b) *Put*

$$A \leftrightarrow \{X, \mathrm{Im}(1 - S_{(1,2)}) \subset X \otimes X\}.$$

Then the natural map $T(X) \to A$ *induces isomorphisms* $(X^{\otimes n})^{\mathfrak{S}_n} \xrightarrow{\sim} A_n$. □

These symmetric algebras in YB-categories are the closest analogs of polynomial rings in noncommutative algebra. Their internal coendomorphism algebras should be considered as (coordinate rings of) quantum matrices.

4. *Quadratic $\mathscr{S}$-algebras.* — There is however a more interesting way to connect the constructions of Sec. 2 with YB-categories. One can first change definitions and then the resultats in order to generalize, say, th. 2.4 to arbitrary YB-category. It was in this way that the main notions of superalgebra were formed.

Fix $\mathscr{S}$ and call a quadratic $\mathscr{S}$-algebra such a quadratic algebra A that $A_1 \in \mathscr{S}$ and $R(A) \subset A_1 \otimes A_1$ is a subobject in $\mathscr{S}$. By definition, an $\mathscr{S}$-morphism of quadratic $\mathscr{S}$-algebras should be induced by a morphism $A_1 \to B_1$ in $\mathscr{S}$.

We now define $\mathcal{S}$-operations on the quadratic $\mathcal{S}$-algebras $A \underset{\mathcal{S}}{\bigcirc} B$, $A \underset{\mathcal{S}}{\bullet} B$ by the same formulas (5), (6) where however the operator $\overset{\mathcal{S}}{S}_{(23)}$ should now be understood as the commutativity constraint in $\mathcal{S}$. The definition of $A^!$ changes in an indirect way via a new identification of $(V \otimes W)^*$ with $V^* \otimes W^*$. All the main statements and constructions of Sec. 2 remain valid in the new context.

In particular, we get the Koszul $\mathcal{S}$-complexes which are also complexes in the usual sense. However, the new Hopf-$\mathcal{S}$-algebras will not in general be usual Hopf algebras since $\underset{\mathcal{S}}{\bigcirc}$ and $\underset{\mathcal{S}}{\otimes}$ in general will not coincide with $\bigcirc$ and $\otimes$ (the multiplication rule in the product algebra changes in a nontrivial way due to the S-operator).

5. *Example.* – Let $V = \bigoplus_{i=1}^{n} kX_i$ and A is generated by X_i subject to the relations $X_iX_j = a_{ij}X_jX_i$ for all i, j where $a_{ij} \in k^*$. It will be an $\mathcal{S}_3$-symmetric algebra in the relevant YB-category $\mathcal{S}_3$ generated by V with the symmetry operator $X_i \otimes X_j \mapsto a_{ij}X_j \otimes X_i$, if $a_{ij}a_{ji} = 1$ for $i < j$ and $a_{ii} = \pm 1$. But its Hopf algebra of internal coendomorphisms $end_{\mathcal{S}}A = A^! \underset{\mathcal{S}}{\bullet} A$ can be naturally considered in at least three different YB-categories $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$.

Version 1. $\mathcal{S}_1$ = the category of finite dimensional vector spaces with the standard symmetry.

$$(12) \quad A \leftrightarrow \{V; r_{ij} = X_i \otimes X_j - a_{ij}X_j \otimes X_i, i < j; \quad r_{ii} = X_i \otimes X_i \text{ for } a_{ii} = -1\}$$

$$(13) \quad A^! \leftrightarrow \{V = \otimes kX^i; r^{kl} = X^k \otimes X^l + a_{kl}^{-1}X^l \otimes X^k; \quad r^{kk} = X^k \otimes X^k \text{ for } a_{kk} = 1\}.$$

Denote by Y_i^k the image of $X^k \otimes X_i$ in $A^! \bullet A$. By definition, we get the following relations for Y_i^k:

$S_{(23)}(r^{kk} \otimes r_{ii})$: $(Y_i^k)^2 = 0$ for $a_{ii} = -1$, $a_{kk} = 1$.

$S_{(23)}(r^{kl} \otimes r_{ii})$: $Y_i^kY_i^l + a_{kl}^{-1}Y_i^lY_i^k = 0$ for $k < l$, $a_{ii} = -1$.

$S_{(23)}(r^{kk} \otimes r_{ij})$: $Y_i^kY_j^k - a_{ij}Y_j^kY_i^k = 0$ for $i < j$, $a_{kk} = 1$.

$S_{(23)}(r^{kl} \otimes r_{ij})$: $Y_i^kY_j^l + a_{kl}^{-1}Y_i^lY_j^k - a_{ij}Y_j^kY_i^l - a_{ij}a_{kl}^{-1}Y_j^lY_i^k = 0$

for $i < j$, $k < l$.

Version 2. $\mathcal{S}_2$ = category of Z_2-graded vector spaces, morphisms = linear maps conserving grading, $s_{(12)}(v \otimes w) = (-1)^{\tilde{v}\tilde{w}} w \otimes v$, where $v \in V_{\tilde{v}}$, $w \in W_{\tilde{w}}$.

In the previous notation put $\tilde{X}_i = 0$ for $a_{ii} = 1$, 1 for $a_{ii} = -1$. As earlier, put $\langle X^k, X_i \rangle = \delta_i^k$, $\tilde{X}_k = \tilde{X}^k$, $\tilde{Y}_i^k = \tilde{X}_i + \tilde{X}_k$. For simplicity we shall write $\tilde{i} = \tilde{X}_i$, $\widetilde{ik} = \tilde{X}_i \tilde{X}_k$. Then $A^! \underset{\mathcal{S}_2}{\bullet} A$ is defined by the relations

$$S_{(23)}(r^{kk} \otimes r_{ii}) \colon \ (Y_i^k)^2 = 0 \ \text{for} \ \tilde{X}_i = 1, \ \tilde{X}_k = 0.$$

$$S_{(23)}(r^{kl} \otimes r_{ii}) \colon \ (-1)^{\tilde{l}} Y_i^k Y_i^l + (-1)^{\tilde{k}} a_{kl}^{-1} Y_i^l Y_i^k = 0 \ \text{for} \ k < l, \quad \tilde{X}_i = 1.$$

$$S_{(23)}(r^{kk} \otimes r_{ij}) \colon \ Y_i^k Y_j^k - a_{ij} Y_{jl}^k Y_i^k = 0 \ \text{for} \ i < j, \ \tilde{X}_k = 0.$$

$$S_{(23)}(r^{kl} \otimes r_{ij}) \colon \ (-1)^{\widetilde{li}} Y_i^k Y_j^l + (-1)^{\widetilde{ki}} a_{kl}^{-1} Y_i^l Y_j^k - (-1)^{\widetilde{lj}} a_{ij} Y_j^k Y_i^l$$
$$- (-1)^{\widetilde{kj}} a_{ij} a_{kl}^{-1} Y_j^l Y_i^k = 0$$

for $i < j$, $k < l$.

As in Sec. 1, we can add to these relations those ones which correspond to the transposed matrix Y_i^k (in version 1) or to the supertransposed one (in version 2): $(Y^{st})_i^k = (-1)^{\widetilde{ik} + \tilde{i}} Y_k^i$. In this way we shall get various versions of the quantum (super) matrix semigroup.

We leave to the reader the last version. We recall only that first one should calculate in $\mathcal{S}_3$ a commutation rule $X^j \otimes X_i \mapsto A_{ij} X_i \otimes X^j$, whose coefficients are defined by the condition that the contraction $V^* \otimes V \to k$ be a morphism in $\mathcal{S}_3$.

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