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AN F. AND M. RIESZ THEOREM FOR BOUNDED SYMMETRIC DOMAINS

by R. G. M. BRUMMELHUIS (*)

to the memory of my mother

1. Introduction.

In [11] J. H. Shapiro has given new proofs of the classical F. and M. Riesz theorem for the circle group $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and of Bochner's generalization of F. and M. Riesz to the torus $\mathbf{T} \times \mathbf{T}$. These proofs were based on a study of the duals of certain subspaces of $L^p(\mathbf{T})$, respectively $L^p(\mathbf{T} \times \mathbf{T})$ for p's between 0 and 1.

In this paper Shapiro's methods are generalized to arbitrary compact groups. As a result, we obtain in section 3 a general F. and M. Riesz theorem for compact groups whose center contains a circle group.

A typical special case of our F. and M. Riesz theorem is the unit sphere S in $\mathbf{C}^n$: $S = S_{2n-1} = U(n)/U(n-1)$, where $U(n)$ is the unitary group. For the formulation we have to recall some definitions from harmonic analysis on S, cf. [7], chapter 12. Let $H(p,q)$ be the set of restrictions to S of harmonic polynomials in z and $\bar{z}$ which are homogeneous of degree p in z and of degree q in $\bar{z}$. Let σ denote the $U(n)$-invariant measure on S with total mass 1. The spaces $H(p,q)$ span $L^2(S,\sigma)$ and are pairwise orthogonal. Let π_{pq} denote the orthogonal projection of $L^2(S,\sigma)$ onto $H(p,q)$. The map $f \to (\pi_{pq}f)(z)(z \in S)$ can be represented as the inner product in L^2 of f with an element K_z in $H(p,q)$. Hence we can define $\pi_{pq}\mu \in H(p,q)$ for any finite Borel measure μ on S. Let $\operatorname{spec}\mu = \{(p,q) \in \mathbf{N} \times \mathbf{N} : \pi_{pq}\mu \neq 0\}$.

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1.1. THEOREM. — *Let* $\Delta \subseteq \mathbf{N} \times \mathbf{N}$ *satisfy the following two conditions :*

(i) *For each* $m \in \mathbf{Z}$ *the set* $\{(p,q) \in \Delta : p-q=m\}$ *is finite.*

(ii) *The set* $\{p-q : (p,q) \in \Delta\}$ *is bounded from below (or above).*

Let μ *be a finite Borel measure on* S *such that* $\operatorname{spec} \mu \subseteq \Delta$. *Then* μ *is absolutely continuous with respect to* σ. □

Examples of sets Δ which satisfy conditions (i) and (ii) of 1.1 are the sets $\Delta_\alpha = \{(p,q) \in \mathbf{N} \times \mathbf{N} : q \leqslant \alpha p\}$ for $\alpha < 1$. The singular measures τ_m defined by

$$\int_S f \, d\tau_m = \int_{-\pi}^{\pi} f(e^{i\theta}\zeta)e^{im\theta} \, d\theta, \qquad f \in C(S),$$

($\zeta \in S$ fixed) show that condition 1.1 (ii) by itself is not sufficient. Similary, the existence of a singular pluriharmonic measure μ (that is,

$$\operatorname{spec} \mu \subseteq \mathbf{N} \times \{0\} \cup \{0\} \times \mathbf{N},$$

cf. Aleksandrov [1] or Rudin [8]) shows that some condition on the set $\{p-q : (p,q) \in \operatorname{spec} \mu\}$ is necessary. Cf. also remark 3.4 below.

Another application of our F. and M. Riesz theorem is made to the Bergman-Shilov boundary S of a bounded symmetric domain Ω : we get another proof of the known result that an H^1 function on Ω can be written as the Poisson integral of an L^1 function on S. Finally, our F. and M. Riesz theorem contains the classical results of the Riesz brothers and of Bochner as special cases.

Kanjin [5] has proved an F. and M. Riesz theorem for zonal (i.e. $U(n-1)$-invariant) measures on S: such a measure μ is absolutely continuous with respect to σ if $\operatorname{spec} \mu \subseteq \{(p,q) \in \mathbf{N} \times \mathbf{N} : \min(p,q) \leqslant N\}$ for some $N \in \mathbf{N}$. I do not know if Kanjin's result can be proved (and extended) by the methods in this paper.

As in the classical case, if μ is a measure such that spec μ satisfies 1.1 (i) and (ii) then not only is μ absolutely continuous with respect to σ but σ is absolutely continuous with respect to μ as well. This will be shown in the final section of this paper.

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2. Absolute continuity and the existence of L^p continuous linear functionals, $p < 1$.

2.1. *Notations.* – If X is a compact topological space, let C(X) denote the space of complex valued continuous functions on X, with the sup norm. M(X) denotes the dual of C(X), the space of finite Borel measures on X.

Throughout this paper, K will denote a compact group with a countable basis of neighborhoods at *e*. By dk we denote the Haar measure on K, normalized to total mass 1; $L^p(K,dk) = L^p(K)$ and $\|f\|_p (0<p<\infty)$ have their usual meaning. If $\mu \in M(K)$ we write (as usual) $\mu \ll dk$, $\mu \perp dk$ for « μ is absolutely continuous with respect to dk », respectively, « μ is singular with respect to dk ».

2.2 *Fourier transform on* K. – Let $\hat{K}$ be the unitary dual of K, i.e. $\hat{K}$ is the set of (equivalence classes of) irreducible unitary continuous representations of K. For τ in $\hat{K}$, let $H(\tau)$ denote the representation space of τ, and d_τ the complex dimension of $H(\tau)$, the degree of τ. The Fourier transform $\hat{\mu}$ of $\mu \in M(K)$ is defined as the following (operator valued) function on K:

$$\hat{\mu}(\tau) = \int_K \tau(x^{-1})\, d\mu(x).$$

Let T(K) be the space of trigonometric polynomials on K; i.e. T(K) is the set of finite linear combinations of functions $k \to (\tau(k)v,w)$ where $v,w \in H(\tau)$, $\tau \in \hat{K}$ and $(.,.)$ is the inner product of $H(\tau)$. If χ_τ denotes the character of τ then for $F \in T(K)$

$$F(k) = \sum_\tau d_\tau(\chi_\tau * F)(k) = \sum_\tau d_\tau \mathrm{Tr}\,\{\hat{F}(\tau)\tau(k)\}$$

where $*$ denotes convolution on K and Tr means trace.

For $\tau \in \hat{K}$ let $T_\tau(K)$ denote the linear span of all functions $k \to (\tau(k)v,w)$, where $v,w \in H(\tau)$. The map $f \to d_\tau\chi_\tau * f$ is the L^2-orthogonal projection of $L^2(K)$ onto $T_\tau(K)$. For $\mu \in M(K)$, the Fourier-Stieltjes series of μ is defined as the formal series

$$\sum_{\tau \in \hat{K}} d_\tau(\chi_\tau * \mu)(k) = \Sigma d_\tau\, \mathrm{Tr}\,\{\hat{\mu}(\tau)\tau(k)\}.$$

2.3. *The spectrum of a measure.* — For $\mu \in M(K)$, let spec μ be the support of $\hat{\mu}$: spec $\mu = \{\tau \in \hat{K} : \hat{\mu}(\tau) \neq 0\}$.
Clearly, spec $\mu = \{\tau \in \hat{K} : \chi_\tau * \mu \neq 0\}$.

Let X_μ be the subspace of T(K) defined by

$$X_\mu = \{F * \mu : F \in T(K)\}.$$

X_μ determines spec μ completely, since spec $\mu = \{\tau \in \hat{K} : X_\mu \cap T_\tau(K) \neq 0\}$. Conversely, spec μ does not determine X_μ in general : note that X_μ is spanned by the functions

$$k \to (\tau(k)v,w), \qquad v \in \text{Range}\, \hat{\mu}(\tau), \qquad w \in H(\tau) :$$

a short computation shows that for w_1, $w_2 \in H(\tau)$ one has

$$((\tau(.)w_1,w_2) * \mu)(k) = (\tau(k)\hat{\mu}(\tau)w_1,w_2).$$

If K is abelian, $\hat{K}$ can be identified with the character group of K and then X_μ is the linear span of spec μ.

It is expedient to use X_μ instead of spec μ when generalizing Shapiro's results to non-abelian K.

Recall that a space of functions Y on K is called invariant under left translation if $f \in Y$ implies ${}^k f \in Y$, where ${}^k f(x) := f(kx)$. Note that X_μ is invariant under left translation.

If Y is a subspace of T(K), let Y^p denote the closure of Y in $L^p(K)$.

2.4. THEOREM. — *Let μ in M(K) be singular with respect to dk. Then X_μ^p has no nonzero continuous linear functionals if $0 < p < 1$.*

Compare [11], theorem 2.1. For the proof we need some lemmas.

2.5. LEMMA. — *There exists a sequence $\{F_n\}$ of trigonometric polynomials, with $\{\|F_n\|_1\}$ bounded such that*

(i) *if $\mu \in M(K)$, $\mu \perp dk$, then $F_n * \mu \to 0$ in Haar measure as $n \to \infty$;*

(ii) *if $f \in L^1(K)$, then $F_n * f \to f$ in $L^1(K)$ as $n \to \infty$.*

Proof. — Let $\{V_n : n \in N\}$, $V_{n+1} \subseteq V_n$, be a countable basis of neighborhoods of e (the identity element) in K. Let h_n be the characteristic function of V_n, divided by the Haar measure of V_n. A rather straightforward argument shows that (i) and (ii) hold with F_n replaced by h_n, cf. [11], proof of lemma 1.1.

Since each h_n is in $L^2(K)$, there exists $F_n \in T(K)$ such that $\|F_n - h_n\|_1 < 2^{-n}$. Hence (ii) holds. Furthermore, $(F_n - h_n) * \mu \to 0$ in Haar measure as $n \to \infty$ for every $\mu \in M(K)$ since $\|(F_n - h_n) * \mu\|_1 \to 0$. Hence (i) follows. □

2.6. Lemma. — *Let $\{f_n\}$ be a sequence of functions in $L^1(K)$ which converges to 0 in Haar measure. Suppose there exists a $C > 0$ such that $\|f_n\|_1 \leqslant C$ for all n. Then $\|f_n\|_p \to 0$ as $n \to \infty$ for all $p \in (0,1)$.*

Proof. — It is enough to observe the following: if $E \subseteq K$ is Borel measurable then if $|E|$ denotes the Haar measure of E,

$$\int_E |f_n|^p \, dk = \int |f_n|^p \chi_E \, dk \leqslant \|f_n\|_1^p |E|^{1-p}$$
$$\leqslant C^p . |E|^{1-p}$$

by Hölder's inequality with exponent $1/p$. Now take for $E = E_n$ the set where $|f_n| > \varepsilon$; for large n, it will have small measure. □

Proof of 2.4. — Fix p, $p \in (0,1)$ and write X for X_μ^p. Let $\{F_n\}$ be as in lemma 2.5.

Then $f_n := F_n * \mu \to 0$ in $L^p(K)$ by 2.5 (i), 2.6 and the fact that $\|F_n * \mu\|_1 \leqslant \|F_n\|_1 \|\mu\| \leqslant C\|\mu\|$ for all n.

Suppose Φ is an L^p continuous linear functional on X. Then Φ is L^1 continuous on $L^1(K) \cap X$, since $\| \ \|_p \leqslant \| \ \|_1$. By Hahn-Banach and the fact that the dual of $L^1(K)$ is $L^\infty(K)$, there exists a φ in $L^\infty(K)$ such that

$$\Phi(f) = \int_K f(x)\varphi(x^{-1}) \, dx, \qquad f \in X \cap L^1(K).$$

By the left translation invariance of X_μ, ${}^k f_n \in X_\mu$ for all $k \in K$ and ${}^k f_n \to 0$ in $L^p(K)$ as $n \to \infty$ since dk is left invariant. Therefore

$$\Phi({}^k f_n) = (f_n * \varphi)(k) = ((F_n * \mu) * \varphi)(k) \to 0, \qquad n \to \infty.$$

Since $\mu * \varphi \in L^1(K)$, $F_n * (\mu * \varphi) \to (\mu * \varphi)$ in $L^1(K)$ (2.5(ii)). Hence $\mu * \varphi = 0$ a.e.. Since

$$\Phi(F * \mu) = ((F * \mu) * \varphi)(e), \qquad F \in T(K),$$

$\Phi = 0$ on X. □

2.7. Theorem. — *Let $0 < p < 1$ and let $Y \subseteq L^p(K)$ be a closed subspace, invariant under left translation. Suppose that $Y \cap T(K)$ has sufficiently many L^p continuous linear functionals to separate points. Let μ in $M(K)$ be such that $X_\mu \subseteq Y$. Then μ is absolutely continuous with respect to dk.*

Compare [11], corollary 5.2.

Proof. — Let $\mu = fdk + \nu$ be the Lebesgue decomposition of μ, $f \in L^1(K)$, $\nu \perp dk$.

Choose $\{F_n\}$ as in lemma 2.5. Then by 2.5 (i) and (ii) and 2.6,

$$F_n * \mu \to f \quad \text{in} \quad L^p \quad \text{as} \quad n \to \infty \quad (0<p<1).$$

This implies that $f \in Y$, since $X_\mu \subseteq Y$.

Let V be the closed subspace of $L^1(K)$ spanned by the left translates of f. Then $V \subseteq Y$ since Y is closed under left translation. Hence $F * f \in V \subseteq Y$ for all $F \in T(K)$. Also $F * \mu \in Y$ for all $F \in T(K)$. Hence $X_\nu \subseteq Y \cap T(K)$. But this implies $\nu = 0$, by theorem 2.4. □

3. A general F. and M. Riesz theorem.

3.1. Our main theorem concerns compact groups K (with a countable neighborhood basis at e) whose center $Z(K)$ contains a circle group $\mathbf{T}$. Throughout this section, let K be such a group, and fix an identification $\mathbf{T} \to Z(K)$, so that $e^{i\theta}$ denotes an element of K as well as of $\mathbf{T}$. By Schur's lemma, there exists for each $\tau \in \hat{K}$ an $n(\tau) \in \mathbf{Z}$ such that

$$\tau(e^{i\theta}) = e^{in(\tau)\theta}.\mathrm{Id}, \qquad \theta \in \mathbf{R}.$$

We can now formulate our main result.

3.2. Theorem. — *Let $\Delta \subseteq \hat{K}$ satisfy the following two conditions:*

(i) *For each $m \in \mathbf{Z}$ the set $\{\tau \in \Delta : n(\tau)=m\}$ is finite.*

(ii) *The set $\{n(\tau) : \tau \in \Delta\}$ is bounded from below.*

Let $\mu \in M(K)$ be such that spec $\mu \subseteq \Delta$. *Then μ is absolutely continuous with respect to dk.*

In condition (ii) of 3.2 « from below » may be replaced by « from above »: just replace μ by $\bar{\mu}$.

Proof. – Let Y be the linear span of the $T_\tau(K)$'s with $\tau \in \Delta$. By theorem 2.7 it is sufficient to show that for $p < 1$

(3.1*a*) $$Y^p \cap T(K) = Y,$$

(3.1*b*) Y has sufficiently many L^p continuous linear functionals to separate points.

In the proof of (3.1*a*) and (3.1*b*) we will use the following lemma :

3.3. LEMMA. – *For $m \in \mathbf{Z}$ define the projection $\Pi_m : T(K) \to T(K)$ by*

$$\begin{aligned}\Pi_m f(k) &= \int_{-\pi}^{\pi} f(e^{i\theta}k)e^{-im\theta}\, d\theta/2\pi \\ &= \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k).\end{aligned}$$

If Y *is a subspace of* T(K) *such that the set* $\{n(\tau) : \exists f \in Y : \chi_\tau * f \neq 0\}$ *is bounded from below, then* Π_m *is* L^p *continuous on* Y *for all* $p > 0$. (*The interesting case is of course* $0 < p < 1$.)

Proof. – For $k \in K$, $f \in T(K)$ define the « slice function » f_k on **T** by $f_k(e^{i\theta}) := f(e^{i\theta}k)$. Obviously

$$\begin{aligned}f_k(e^{i\theta}) &= \sum_{m \in \mathbf{Z}} \left(\sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) \right) e^{im\theta} \\ &= \sum_{m \in \mathbf{Z}} \Pi_m f(k) e^{im\theta}.\end{aligned}$$

Let $N \in \mathbf{Z}$ be such that $n(\tau) \geqslant N$ for all τ for which $d_\tau \chi_\tau * f \neq 0$ for some $f \in Y$. Suppose first that $N \geqslant 0$. Then $f_k(e^{i\theta})$ is an *analytic* trigonometric polynomial for each f in Y. Hence, by a result from one variable H^p theory due to Hardy and Littlewood (cf. [3], theorem 6.4; cf. also [2], p. 68, for a short proof) for each $p > 0$ and each $m \in \mathbf{Z}$ there exists a constant $C = C(p,m)$ such that

$$|\Pi_m f(k)|^p \leqslant C \int_{-\pi}^{\pi} |f(e^{i\theta}k)|^p \, d\theta/2\pi.$$

Integration over K yields the lemma when $N \geqslant 0$.

If $N < 0$ then for each $f \in Y$, $f_k(e^{i\theta}) = e^{iN\theta}.F(e^{i\theta})$ where F is again an analytic trigonometric polynomial on **T**. Apply the one variable result mentioned above to F and note that $|F| = |f_k|$ on **T**. □

We now return to the proof of theorem 3.2. To prove (3.1*a*) suppose that $f_n \in Y$, $f \in T(K)$ such that $f_n \to f$ in $L^p(K)$ as $n \to \infty$. By lemma 3.3 applied to $Y' = \text{span}\{Y, f\}$, $\Pi_m(f_n) \to \Pi_m(f)$ in $L^p(K)$ for all $m \in \mathbf{Z}$. Since Y is invariant under left translation, $\Pi_m(f_n)$ belongs to $Y \cap \oplus\{T_\tau(K) : n(\tau) = m\}$. The latter is a finite dimensional subspace of T(K) by condition 3.2(i). Since all vector space topologies on a finite dimensional vector space are complete, $\Pi_m(f) \in Y$ for all $m \in \mathbf{Z}$, which implies that $f \in Y$.

For (3.1*b*) it is sufficient to show that for each $\sigma \in \Delta$ and each $k \in K$ the linear functional

$$f \to d_\sigma(\chi_\sigma * f)(k) \tag{3.2}$$

is L^p continuous on Y. Take a σ in Δ. Clearly, the linear functional (3.2) is equal to the composition of the projection $\Pi_{n(\sigma)}$ with the restriction of (3.2) to $\oplus\{T_\tau(K) : \tau \in \Delta,\ n(\tau) = n(\sigma)\}$. Since this subspace is finite dimensional, the L^p continuity of (3.2) follows from the L^p continuity of $\Pi_{n(\sigma)}$. This proves the theorem. □

3.4. *Remark.* – Recall that a subset Σ of $\hat{K}$ is called a $\Lambda(1)$ set (Rudin [10]) if there exists a $p < 1$ and a constant C such that for all f in $\oplus\ \{T_\tau(K) : \tau \in \Sigma\}$,

$$\|f\|_1 \leqslant C\|f\|_p,$$

i.e. if the L^1 and L^p topologies coincide on $\text{span}\{T_\tau(K) : \tau \in \Sigma\}$.

We can replace condition (i) of 3.2 by the following weaker condition :

(i)′ For each $m \in \mathbf{Z}$ the set $\{\tau \in \Delta : n(\tau) = m\}$ is a $\Lambda(1)$ subset of $\hat{K}$.

The proof remains essentially the same : instead of the finite dimensionality of the subspaces $\oplus\{T_\tau(K) : \tau \in \Delta,\ n(\tau) = m\}$ we now use the equivalence, for some $p < 1$, of the L^1 and L^p topologies on these subspaces, and the L^1 continuity of the linear functionals (3.2) (for all $\sigma \in K$).

Similarly, we may also replace condition (ii) by

(ii)′ The set $\{n(\tau) : \tau \in \Delta\}$ is a $\Lambda(1)$ subset of $\mathbf{Z}$ (considered as the dual of $\mathbf{T}$).

In this case the analogue of lemma 3.3 becomes trivial.

Note, by the way, that for arbitrary compact K the conclusion of the F. and M. Riesz theorem holds for all $\Lambda(1)$ subsets of $\hat{K}$: this follows immediately from theorem 2.7 and the definition of a $\Lambda(1)$ set.

3.5. *Example.* – Let $K = \mathbf{T} \times \mathbf{T}$ and identify $\mathbf{T}$ with a subgroup of K via the map $e^{i\theta} \to (e^{i\theta},1)$, $\theta \in (-\pi,\pi]$. The irreducible unitary representations of K are the characters $\chi_{p,q} : (e^{i\theta},e^{i\psi}) \to e^{i(p\theta+q\psi)}$ and $n(\chi_{p,q}) = p$. In this case theorem 3.2 contains Bochner's theorem where the spectrum is required to lie in an angle to the right of opening less than π (cf. [9], theorem 8.2.5 for the precise formulation).

According to the remarks made in 3.4 it suffices to require in condition (i) that for each p the set $\{q \in \mathbf{Z} : (p,q) \in \operatorname{spec} \mu\}$ is a $\Lambda(1)$ set. We refer to the appendix of [2] for another strenghtening of Bochner's theorem which only requires that for each p these sets are either bounded from above or from below. This can easily be proved by the method of proof of theorem 3.2 if we note that $H^p(\mathbf{T}) \cap T(\mathbf{T})$ consists of analytic polynomials.

3.6. *F. and M. Riesz for homogeneous spaces.* – Let H be a closed subgroup of K. Functions and measures on K/H can be identified with functions and measures on K which are right H-invariant. If $\mu \in M(K/H)$ is a right H- invariant measure on K, then $\pi_\tau \mu := d_\tau \chi_\tau * \mu$ is again right H-invariant. Let σ be the K-invariant measure on K/H, normalized to 1. The map $\pi_\tau : f \to d_\tau \chi_\tau * f (\tau \in \hat{K})$ is an L^2 orthogonal projection of $L^2(K/H,\sigma) = L^2(\sigma)$ which is different from zero iff τ occurs in the left regular representation of K on $L^2(\sigma)$. As in the case of the unit sphere, π_τ can be represented by an integral operator with continuous kernel.

Theorem 3.2 can now be formulated for measures on K/H entirely in terms of π_τ and σ :

3.7. THEOREM. – *Let $\Delta \subseteq \hat{K}$ be such that all $\tau \in \Delta$ occur in the left regular representation of* K *on* $L^2(\sigma)$ *and suppose Δ satifies conditions* (i) *and* (ii) *of* 3.2. *Let $\mu \in M(K/H)$ be such that $\pi_\tau \mu = 0$ if $\tau \notin \Delta$. Then μ is absolutely continuous with respect to σ.*

If we take $K = U(n)$, $H = U(n-1)$ then $K/H = S$ and we get theorem 1.1 : Z(K) contains the multiplications by $e^{i\theta}$. If τ_{pq} denotes the restriction of the left regular representation of $U(n)$ on $L^2(\sigma)$ to $H(p,q)$, i.e. $\tau_{pq}(U)f(\zeta) = f(U^{-1}\zeta)$, $f \in H(p,q)$, $U \in U(n)$, $\zeta \in S$, then τ_{pq} is irreducible, $n(\tau_{pq}) = q - p$, the τ_{pq} are pairwise inequivalent and they represent all irreducible representations of $U(n)$ which occur in $L^2(\sigma)$ (cf. for example [7], chapter 12).

In case $H = \mathbf{T}$ theorem 3.7 becomes trivial: $n(\tau) = 0$ for all τ which occur in $K/\mathbf{T}$ and (i) then implies that Δ is finite.

3.8. *Application to bounded symmetric domains.* — Let $\Omega \subseteq \mathbf{C}^n$ be a bounded symmetric domain. (Cf. [6], [4] for the relevant facts.) We may assume that Ω is convex and circular (i.e. $z \in \Omega$ implies $e^{i\theta}.z \in \Omega$ for all $\theta \in \mathbf{R}$). Let K be the stabilizer of 0 in the component of the identity of the group of holomorphic automorphisms of Ω. The action of K on Ω incorporates multiplication by $e^{i\theta}$; in particular, $\mathbf{T} \subseteq Z(K)$. Let S denote the Bergman-Shilov boundary of Ω. Then K acts transitively on S and we can apply the principal theorems 3.2, 3.7 to S. As above, let σ be the normalized K-invariant measure on S.

Let $H^2(S)$ be the closure in $L^2(S,\sigma) = L^2(\sigma)$ of the holomorphic polynomials, restricted to S. Obviously $H^2(S)$ is K-invariant under the left regular representation of K on $L^2(\sigma)$. Let $\hat{K}_{Hol}$ be the set of irreducible representations of K which occur in $H^2(S)$; for a description of $\hat{K}_{Hol}$, cf. [12].

We claim that $\hat{K}_{Hol}$ satisfies conditions (i) and (ii) of 3.2. For let $H(p)$ be the space of holomorphic polynomials which are homogeneous of degree p, restricted to S. By a well known theorem of H. Cartan (cf. [7], theorem 2.1.3) K acts on Ω by complex linear transformations. Hence each $H(p)$ is K-invariant and decomposes as a finite sum of representations in $\hat{K}_{Hol}$. Obviously, $n(\tau) \leqslant 0$ if τ is in $\hat{K}_{Hol}$ and $n(\tau) = -p$ if τ occurs in $H(p)$. This proves the claim.

By theorem 3.7 a measure μ in $M(K/H)$ for which $\operatorname{spec} \mu \subseteq \hat{K}_{Hol}$ is absolutely continuous with respect to σ. By a familiar weak-$*$ compactness argument this implies the following result:

3.9. COROLLARY. — *If f is in the Hardy space* $H^1(\Omega)$ *of* Ω *then f can be written as the Poisson integral of a function in* $L^1(\sigma)$. (*Cf.* [6], [14] *for the definitions of Hardy space and Poisson kernel.*) □

The analogue of 3.9 for generalized Siegel half-planes is due to E. M. Stein [13]; 3.9 can be deduced from his result by using the generalized Cayley transform, cf. [6], p. 189. Corollary 3.9 can also be proved more directly, by using Bochner's method of slicing and the Hardy-Littlewood inequality for the radial maximal function from one dimensional H^p theory.

4. A supplement to theorem 3.2.

4.1. THEOREM. — *Let* K *be a compact Lie group whose center contains the circle group* **T** *and let* H *be a closed subgroup of* K *such that* K/H *is connected. Suppose* f *in* $L^1(K/H,\sigma)$ *is such that* spec f *satisfies conditions* (i) *and* (ii) *of theorem* 3.7. *Then either* $f = 0$ *or* $f(\xi) \neq 0$ *a.e.* $[\sigma]$.

In particular, if $\mu \in M(K/H)$ is as in theorem 3.7 and $\mu \neq 0$ then $\sigma \ll \mu$ as well as $\mu \ll \sigma$. The case $K = \mathbf{T}$, $H = \{1\}$ of this theorem is classical (cf. for example [3], theorem 2.2) and will be used in the proof of 4.1.

Proof. — Let f in $L^1(K/H,\sigma)$ satisfy the conditions of the theorem and suppose that $f = 0$ on a (Borel) set of nonzero measure. Identify f with a right H-invariant L^1 function on K, also denoted by f.

For almost all $k \in K$ the slice function $f_k(e^{i\theta}) = f(e^{i\theta}k)$ is in $L^1(\mathbf{T})$. Set $c_m(k) := \hat{f}_k(m)$, the m-th Fourier coefficient of f_k. Then $c_m \in L^1(K)$ and a calculation of $d_\tau\chi_\tau * c_m$ shows that

$$c_m(k) = \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) \tag{4.1}$$

(note that the sum is finite). The case $K = \mathbf{T}$ of 4.1 now shows that for almost all k in K

$$f_k = 0 \quad \text{or} \quad f_k(e^{i\theta}) \neq 0 \quad \text{a.e.}[d\theta]. \tag{4.2}$$

Since $f = 0$ on a set of nonzero Haar measure there exists an $F \subseteq K$ of strictly positive Haar measure such that for all k in F, $f_k = 0$ on a subset of **T** of nonzero (one-dimensional) Lebesgue measure. By (4.2) $f_k = 0$ for almost all k in F and hence, by (4.1)

$$\sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) = 0 \qquad \text{for all } m \in \mathbf{Z}, \quad \text{a.a. } k \in F.$$

Each $\tau \in \hat{K}$ is an analytic function on K and therefore $d_\tau\chi_\tau * f$ is an analytic function on the analytic manifold K/H. It is not difficult to prove that the zero set of a nonzero analytic function on a connected analytic manifold has Lebesgue measure zero. Hence

$$\sum_{n(\tau)=m} d_\tau\chi_\tau * f = 0$$

for all m and therefore $d_\tau\chi_\tau * f = 0$ for all $\tau \in \hat{K}$, i.e. $f = 0$. □

It is easy to find a counterexample to 4.1 if K/H is not connected. Take $K = \mathbf{T} \times F$ and $H = \{e\}$, with F a finite non-commutative group. Let τ be an irreducible representation of F and choose a matrix coefficient τ_{mn} of τ such that $\tau_{mn}(e) = 0$, $\tau_{mn}(x) \neq 0$ for some $x \in F$. Now take $f = (1 \otimes \tau)_{mn}$: spec f consists of one point but $f = 0$ on the component of the unit element of K.

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