Annales de l'institut Fourier

Edward Bierstone
P. D. Milman

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Annales de l'institut Fourier, tome 37, nº 2 (1987), p. 49-77
<http://www.numdam.org/item?id=AIF_1987__37_2_49_0>

Ann. Inst. Fourier, Grenoble
37, 2 (1987), 49-77.

RELATIONS AMONG ANALYTIC FUNCTIONS II

by E. BIERSTONE (1) and P. D. MILMAN (2)

Contents (*)

(*) *Chapters* 0 *et* 1 *published in the first issue of volume* **37** (1987).

(1) Research partially supported by NSERC operating grant A9070.
(2) Supported by NSERC University Research Fellowship and operating grant U0076.

Key-words: Morphism of analytic spaces - Module of formal relations - Hilbert-Samuel function - Diagram of initial exponents - Zariski semicontinuity - Division and composition of $\mathscr{C}^\infty$ functions.

CHAPTER II

DIFFERENTIABLE FUNCTIONS

10. Ideals generated by analytic functions.

We give an elementary proof of the theorem of Malgrange [27, Ch. VI]. Let N be a real analytic manifold. Put $\mathcal{O} = \mathcal{O}_N$. Let A be a $p \times q$ matrix of real analytic functions on N, and let $A\cdot : \mathcal{C}^\infty(N)^q \to \mathcal{C}^\infty(N)^p$ denote the $\mathcal{C}^\infty(N)$-homomorphism defined by multiplication by A.

THEOREM 10.1. — $A\cdot\mathcal{C}^\infty(N)^q = (A\cdot\mathcal{C}^\infty(N)^q)\hat{}$.

Remark 10.2. — Let $Z \subset Y$ be closed subanalytic subsets of N. Suppose that $f \in \mathcal{I}(N;Z)^p$ and, for all $a \in Y$, there exists $G_a \in \hat{\mathcal{O}}_a^q$ such that $\hat{f}_a = A_a \cdot G_a$. The following proof shows, moreover, that there exists $g \in \mathcal{I}(N;Z)^q$ such that $f - A\cdot g \in \mathcal{I}(N;Y)^p$ (cf. [7, Thm. 0.1.1]).

Proof of Theorem 10.1. — Let $\mathcal{A}$ denote the sheaf of submodules of $\mathcal{O}^p$ generated by the columns $\varphi^1, \dots \varphi^q$ of A. Let $\mathcal{B}$ be the subsheaf of $\mathcal{O}^q$ of (germs of) relations among the columns of A. Then $\mathcal{B}$ is coherent.

We can assume that N is an open subset of $\mathbf{R}^n$. If $a \in N$, we identify $\hat{\mathcal{O}}_a$ with $\mathbf{R}[[y]]$, $y = (y_1, \dots, y_n)$. By Lemma 7.2 and Remark 7.3, we can suppose there is a filtration of N by closed analytic subsets,

$$N = X_0 \supset X_1 \supset \cdots \supset X_{r+1} = \emptyset,$$

such that, for each $k = 0, \dots r$:

(1) $X_k - X_{k+1}$ is smooth.

(2) $\mathfrak{N}(\hat{\mathcal{A}}_a)$ and $\mathfrak{N}(\hat{\mathcal{B}}_a)$ are constant on $X_k - X_{k+1}$. We write $\mathfrak{N}_k(\mathcal{A}) = \mathfrak{N}(\hat{\mathcal{A}}_a)$ and $\mathfrak{N}_k(\mathcal{B}) = \mathfrak{N}(\hat{\mathcal{B}}_a)$, $a \in X_k - X_{k+1}$.

(3) Let (β_i, j_i), $i = 1, \dots, t$, denote the vertices of $\mathfrak{N}_k(\mathcal{A})$. Then, for each i, there exists ψ^i in the submodule of $\mathcal{O}(X_k)[[y]]^p$ generated by

(the elements induced by) the φ^j (cf. Remark 7.3), such that, for all $a \in X_k - X_{k+1}$, $\nu(\psi^i(a;\cdot)) = (\beta_i, j_i)$ and $\psi^i_a \in \mathscr{A}_a$, where $\psi^i_a(y) = \psi^i(a;y)$.

(4) There exist σ^ℓ in the submodule of $\mathcal{O}(X_k)[[y]]^q$ induced by $\mathscr{B}(N)$ such that the $\nu(\sigma^\ell(a;\cdot))$ are the vertices of $\mathfrak{N}_k(\mathscr{B})$, for all $a \in X_k - X_{k+1}$.

Fix k. Let $\{\Delta_i, \Delta\}$ denote the decomposition of $N^n \times \{1, \ldots, p\}$ determined by the vertices (β_i, j_i) of $\mathfrak{N}_k(\mathscr{A})$, as in § 6. Let $a \in X_k - X_{k+1}$. By the formal division algorithm (Theorem 6.2) and Remark 6.7, there exist unique $r^i_a \in \mathcal{O}^p_a$ and $q_{i\ell,a} \in \mathcal{O}_a$, $\ell = 1, \ldots, t$, such that $\operatorname{supp} r^i_a \subset \Delta$, $(\beta_\ell, j_\ell) + \operatorname{supp} q_{i\ell,a} \subset \Delta_\ell$, and

$$(10.3) \qquad y^{\beta_i, j_i} = \sum_{\ell=1}^{t} q_{i\ell,a}(y)\psi^\ell_a(y) + r^i_a(y).$$

Put $\theta^i_a(y) = y^{\beta_i,j_i} - r^i_a(y)$, $i = 1, \ldots, t$; then the $\theta^i_a \in \mathscr{A}_a$ (cf. Corollary 7.7). The coefficients $\theta^i_{\beta,j}(a)$ of $\theta^i_a(y) = \sum_{\beta,j} \theta^i_{\beta,j}(a) y^{\beta,j}$, as well as the coefficients of the $q_{i\ell,a}$, are analytic on $X_k - X_{k+1}$, and extend to X_k as quotients of analytic functions by products of powers of the $\psi^\ell_{\beta_\ell,j_\ell}(a)$, where $\psi^\ell_a(y) = \sum_{\beta,j} \psi^\ell_{\beta,j}(a) y^{\beta,j}$. There exist analytic functions θ^i defined in a neighborhood of $X_k - X_{k+1}$, whose power series expansions at each $a \in X_k - X_{k+1}$ are the θ^i_a (cf. Corollary 7.7(3)).

Suppose that $f \in (A \cdot \mathscr{C}^\infty(N)^q)\hat{}$ and that f is flat on X_{k+1}. It suffices to find $h \in \mathscr{I}(N; X_{k+1})^q$ such that $f - A \cdot h \in \mathscr{I}(N; X_k)^p$.

Let $a \in X_k - X_{k+1}$. Then $\hat{f}_a \in \hat{\mathscr{A}}_a$. By the formal division algorithm, there are unique $G_{i,a} \in \hat{\mathcal{O}}_a$, $i = 1, \ldots, t$, such that $(\beta_i, j_i) + \operatorname{supp} G_{i,a} \subset \Delta_i$ and

$$(10.4) \qquad \hat{f}_a = \sum_{i=1}^{t} G_{i,a}\theta^i_a.$$

Put $G_{i,a} = 0$ if $a \in X_{k+1}$.

We claim there exist $g_i \in \mathscr{I}(N; X_{k+1})$ such that $G_{i,a} = \hat{g}_{i,a}$ for all $a \in X_k$: Write $G_{i,a} = \sum_\beta G_{i,\beta}(a) y^\beta$. By the formal division algorithm and Łojasiewicz's inequality [27, IV.4.1], each $G_{i,\beta}$ is the restriction to X_k of a $\mathscr{C}^\infty$ function which is flat on X_{k+1}. Let $a \in X_k - X_{k+1}$. Since f is $\mathscr{C}^\infty$ and the θ^i are analytic, then, regarding both a and y as variables in N, we have

$$(10.5)\qquad \frac{\partial \hat{f}_a(y)}{\partial a_j} = \frac{\partial \hat{f}_a(y)}{\partial y_j}, \quad \frac{\partial \theta^i_a(y)}{\partial a_j} = \frac{\partial \theta^i_a(y)}{\partial y_j},$$

$j = 1, \dots, n$ (« Taylor expansion commutes with differentiation »). If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$, write $D_{\lambda,a} = \sum \lambda_j \partial/\partial a_j$; $D_{\lambda,a}$ is the directional derivative with respect to the a variables in the direction λ. If $D_{\lambda,a}$ is tangent to $X_k - X_{k+1}$ at a, then $D_{\lambda,a}G_{i,a}(y)$ is well-defined, and, by (10.4) and (10.5), $\sum_{i=1}^{t} (D_{\lambda,a}G_{i,a} - D_{\lambda,y}G_{i,a}) \cdot \theta^i_a = 0$. For each i, $(\beta_i, j_i) + \operatorname{supp}(D_{\lambda,a}G_{i,a} - D_{\lambda,y}G_{i,a}) \subset \Delta_i$ (where supp is with respect to y). Therefore, by the uniqueness of formal division, for each $i = 1, \dots, t$,

$$(10.6)\qquad D_{\lambda,a}G_{i,a} = D_{\lambda,y}G_{i,a}.$$

Choose local coordinates $(u,v) = (u_1, \dots, u_m, v_1, \dots, v_{n-m})$ near $a \in X_k - X_{k+1}$ such that $X_k - X_{k+1}$ is given by $v = 0$. Write $G_{i,a}$ as

$$G_{i,a}(u,v) = \sum_{\beta \in \mathbf{N}^{n-m}} \left(\sum_{\alpha \in \mathbf{N}^m} G_i^{\alpha,\beta}(a) \frac{u^\alpha}{\alpha!} \right) \cdot \frac{v^\beta}{\beta!}.$$

Then (10.6) implies that $\sum_\alpha G_i^{\alpha,\beta}(a) u^\alpha/\alpha!$ is the formal Taylor series of $G_i^{0,\beta}$ at a. By Whitney's extension theorem [27, I.4.1] and Hestenes's lemma [37, IV.4.3], there exists $g_i \in \mathscr{I}(N; X_{k+1})$ such that $G_{i,a} = \hat{g}_{i,a}$, for all $a \in X_k$, as claimed.

To finish the proof, we must express f in terms of the columns φ^j of A. By (3) and (10.3), $\theta^i_a(y) = \sum_{j=1}^{q} \xi_{ij,a}(y)\varphi^j_a(y)$, $i = 1, \dots, t$, where $\varphi^j_a(y) = \varphi^j(a+y)$, $\xi_{ij,a} \in \mathcal{O}_a$, and the coefficients $\xi_{ij,\beta}(a)$ of $\xi_{ij,a}(y) = \sum_\beta \xi_{ij,\beta}(a) y^\beta$ are quotients of analytic functions by products of powers of the $\psi^l_{\beta_l,j_l}(a)$. Put $\xi_{i,a} = (\xi_{i1,a}, \dots, \xi_{iq,a})$. By the formal division algorithm and Remark 6.7, there exist unique $\eta_{i,a}(y) \in \mathcal{O}^q_a$ such that $\xi_{i,a} - \eta_{i,a} \in \mathscr{B}_a$ and $\operatorname{supp} \eta_{i,a} \cap \mathfrak{N}_k(\mathscr{B}) = \varnothing$. Write $\eta_{i,a} = (\eta_{i1,a}, \dots, \eta_{iq,a})$ and $\eta_{ij,a}(y) = \sum_\beta \eta_{ij,\beta}(a) y^\beta$, $j = 1, \dots, q$. By (4), the $\eta_{ij,\beta}(a)$ extend to X_k as quotients of analytic functions. By the uniqueness of formal division, $\eta_{ij,a}(b-a+y) = \eta_{ij,b}(y)$, for b in some neighborhood of a in $X_k - X_{k+1}$ (cf. the proof of Corollary 7.7 (3)). Thus the $\eta_{ij,a}$ are the formal power series expansions at a of analytic functions η_{ij} defined in a neighborhood of $X_k - X_{k+1}$.

If $a \in X_k - X_{k+1}$, then $\hat{f}_a = \sum_i G_{i,a}\theta_a^i = \sum_{i,j} \eta_{ij,a}G_{i,a}\varphi_a^j$. Put $H_{j,a} = \sum_i \eta_{ij,a}G_{i,a}$ if $a \in X_k - X_{k+1}$, and $H_{j,a} = 0$ if $a \in X_{k+1}$, $j = 1, \ldots, q$. Then there exist $h_j \in \mathscr{I}(N;X_{k+1})$ such that $H_{j,a} = \hat{h}_{j,a}$ for all $a \in X_k$, $j = 1, \ldots, q$. Thus, $f - A \cdot h \in \mathscr{I}(N;X_k)^p$, where $h = (h_1, \ldots, h_q)$. □

11. Modules over a ring of composite differentiable functions.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let M and N denote analytic manifolds (over $\mathbf{K}$), and let $\varphi : M \to N$ be an analytic mapping. Let A and B be $p \times q$ and $p \times r$ matrices of analytic functions on M, respectively. We use the notation of 8.2. If $a \in M$, let $\mathscr{R}_a = \{G \in \hat{\mathcal{O}}^q_{\varphi(a)} : \hat{\Phi}_a(G) \in \operatorname{Im} \hat{B}_a\}$.

Let $\mathscr{B} \subset \mathcal{O}^p_M$ denote the sheaf of $\mathcal{O}_M$-modules generated by the columns of B. Let U be a coordinate neighborhood of some point in M, with coordinates $x_1, \ldots, x_m$, say. By Theorem 7.4, the diagram of initial exponents $\mathfrak{N}(\mathscr{B}_a) \subset \mathbf{N}^m \times \{1, \ldots, p\}$ is Zariski semicontinuous on U. Thus, after perhaps shrinking U, there is a filtration by closed analytic subsets, $U = X_0 \supset X_1 \supset \ldots \supset X_{t+1} = \varnothing$, such that $\mathfrak{N}(\mathscr{B}_a)$ is constant on each $X_\lambda - X_{\lambda+1}$. Let $b \in N$. The following proposition shows that $\mathscr{R}_a$ is constant on every connected component of $(X_\lambda - X_{\lambda+1}) \cap \varphi^{-1}(b)$, $\lambda = 0, \ldots, t$.

Proposition 11.1. — *Let* U *be a local coordinate chart in* M. *Let* $b \in N$ *and let* S *be a locally closed semianalytic subset of* U *such that* $S \subset \varphi^{-1}(b)$. *Suppose that* $\mathfrak{N}(\mathscr{B}_a)$ *is constant on* S. *Let* $f \in \mathcal{O}(U)^p$ *and let* $G \in \hat{\mathcal{O}}^q_b$. *Then*

$$\mathscr{H} = \{a \in S : \hat{f}_a - \hat{\Phi}_a(G) \in \operatorname{Im} \hat{B}_a\}$$

is open and closed in S.

Proof. — We can assume that U (respectively, N) is an open neighborhood of the origin in $\mathbf{K}^m$ (respectively, $\mathbf{K}^n$), and that $\varphi(0) = 0$ and $b = 0$. We identify (the components of) φ and f and (the entries of) A and B with their convergent power series expansions at 0. If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, then

$$\begin{aligned} f(x+y) &- \mathrm{A}(x+y)\cdot \mathrm{G}(\varphi(x+y)-\varphi(x)) \\ &= \sum_{\alpha\in\mathbf{N}^m} \frac{\mathrm{D}^\alpha f(x)}{\alpha!} y^\alpha - \mathrm{A}(x+y)\cdot \sum_{\beta\in\mathbf{N}^n} \frac{\mathrm{D}^\beta \mathrm{G}(0)}{\beta!}\left(\sum_{\alpha>0} \frac{\mathrm{D}^\alpha\varphi(x)}{\alpha!} y^\alpha\right)^\beta, \end{aligned}$$

where α (respectively, β) denotes a multiindex in $\mathbf{N}^m$ (respectively, $\mathbf{N}^n$). Thus

$$f(x+y) - \mathrm{A}(x+y)\cdot \mathrm{G}(\varphi(x+y)-\varphi(x)) = \sum_{\alpha\in\mathbf{N}^m} \frac{\mathrm{H}_\alpha(x)}{\alpha!} y^\alpha,$$

where the H_α converge in a common neighborhood of 0 (which we can take to be U). (For all $\alpha \in \mathbf{N}^m$, each component of $\mathrm{H}_\alpha(x) - \mathrm{D}^\alpha f(x)$ is a finite linear combination of certain products of derivatives of the components of φ times derivatives of the entries of A.)

Let $\mathfrak{N} = \mathfrak{N}(\mathscr{B}_a)$, $a \in \mathrm{S}$, and let (α_i, j_i), $i = 1, \ldots, k$, denote the vertices of $\mathfrak{N}$. For each $a \in \mathrm{S}$, let $g_a^i(y) \in \hat{\mathscr{O}}_a^p = \mathbf{K}[[y]]^p$, $i = 1, \ldots, k$, denote the standard basis of $\hat{\mathscr{B}}_a$, where $\operatorname{in} g_a^i = y^{\alpha_i, j_i}$. Then each $g_a^i(y) = \sum_{\alpha,j} g_{\alpha,j}^i(a) y^{\alpha,j}$ is convergent, and each $g_{\alpha,j}^i(a)$ is analytic on S (Corollary 6.8).

Let $a \in \mathrm{S}$ and let $h_a(y) = \sum_\alpha \mathrm{H}_\alpha(a) y^\alpha/\alpha!$. By Theorem 6.2, there exist unique $q_{i,a}(y) \in \hat{\mathscr{O}}_a$ and $r_a(y) \in \hat{\mathscr{O}}_a^p$ such that $(\alpha_i, j_i) + \operatorname{supp} q_{i,a} \subset \Delta_i$, $\operatorname{supp} r_a \subset \Delta$ (where Δ_i, Δ are as in § 6), and

$$h_a(y) = \sum_{i=1}^k q_{i,a}(y) g_a^i(y) + r_a(y). \tag{11.2}$$

Write $r_a(y) = \sum_{\alpha,j} r_{\alpha,j}(a) y^{\alpha,j}$. Then each $r_{\alpha,j}(a)$ is analytic on S (cf. Remark 6.5). By (11.2), $h_a \in \operatorname{Im} \hat{\mathrm{B}}_a$ if and only if each $r_{\alpha,j}(a) = 0$; i.e., $\mathscr{H}$ is closed.

Since $f(y) - \mathrm{A}(y)\cdot \mathrm{G}(\varphi(y)) \in \hat{\mathscr{B}}_0 \subset \mathbf{K}[[y]]^p$, there exist unique $q_i(y) \in \hat{\mathscr{O}}_0$ such that $(\alpha_i, j_i) + \operatorname{supp} q_i \subset \Delta_i$ and $f(y) - \mathrm{A}(y)\cdot\mathrm{G}(\varphi(y)) = \sum_{i=1}^k q_i(y) g_0^i(y)$. Consider the identity

$$f(x+y) - \mathrm{A}(x+y)\cdot \mathrm{G}(\varphi(x+y)) = \sum_{i=1}^k q_i(x+y) g_0^i(x+y). \tag{11.3}$$

Suppose that $0 \in S$. Let $\mathscr{I} \subset \mathcal{O}_0 = \mathbf{K}\{x\}$ denote the ideal of germs of analytic functions at 0 which vanish on S. Write $\mathcal{O}_{S,0} = \mathcal{O}_0/\mathscr{I}$ and $\hat{\mathcal{O}}_{S,0} = \hat{\mathcal{O}}_0/\mathscr{I} \cdot \hat{\mathcal{O}}_0$. We expand each term of (11.3) as a power series in y with coefficients in $\hat{\mathcal{O}}_0 = \mathbf{K}[[x]]$, and take the induced power series in y with coefficients in $\hat{\mathcal{O}}_{S,0}$. Since each component of φ vanishes on S, the left-hand side of (11.3) gives the same result as reducing the coefficients of $\sum H_\alpha(x)y^\alpha/\alpha!$ modulo $\mathscr{I}$; write $h_x(y)$ for the resulting element of $\mathcal{O}_{S,0}[[y]]^p$. Likewise, write $q_{i,x}(y)$ and $g^i_x(y)$ for the elements of $\hat{\mathcal{O}}_{S,0}[[y]]$ and $\hat{\mathcal{O}}_{S,0}[[y]]^p$ induced by $q_i(x+y)$ and $g^i_0(x+y)$, respectively. Thus,

$$h_x(y) = \sum_{i=1}^{k} q_{i,x}(y) g^i_x(y). \tag{11.4}$$

Since $(\alpha_i, j_i) + \operatorname{supp} q_i \subset \Delta_i$, then $(\alpha_i, j_i) + \operatorname{supp} q_{i,x} \subset \Delta_i$. Clearly, in $g^i_x(y) = y^{\alpha_i, j_i}$.

On the other hand, by the formal division algorithm, there are unique $Q_{i,x}(y) \in \hat{\mathcal{O}}_{S,0}[[y]]$ and $R_x(y) \in \hat{\mathcal{O}}_{S,0}[[y]]^p$ such that $(\alpha_i, j_i) + \operatorname{supp} Q_{i,x} \subset \Delta_i$, $\operatorname{supp} R_x \subset \Delta$, and

$$h_x(y) = \sum_{i=1}^{k} Q_{i,x}(y) g^i_x(y) + R_x(y). \tag{11.5}$$

Since the coefficients of $h_x(y)$ belong to $\mathcal{O}_{S,0}$, so do those of $Q_{i,x}(y)$ and $R_x(y)$ (cf. Remark 6.5); moreover, all coefficients can be evaluated in a common neighborhood of 0 in S.

Comparing (11.4) and (11.5), we get $R_x(y) = 0$. But from (11.2) and (11.5), $R_a(y) = r_a(y)$ for $a \in S$ sufficiently close to 0. Therefore, all $r_{\alpha,j}(a)$ vanish on S near 0; i.e., $\mathscr{H}$ is open. □

COROLLARY 11.6. — *If φ is proper, then (locally in* N*), there is a bound s on the number of distinct submodules $\mathscr{R}_a$ of $\hat{\mathcal{O}}^q_b$, where $a \in \varphi^{-1}(b)$.*

Proof. — Let $U, X_0, \ldots, X_{t+1}$ be as above. Suppose that U is relatively compact and each X_λ is semianalytic in M. Then, for each $\lambda = 0, \ldots, t$, there is a bound on the number of connected components of $(X_\lambda - X_{\lambda+1}) \cap \varphi^{-1}(b)$ [11], [12], [20, Thm. 2.5]. The result follows from Proposition 11.1. □

Remark 11.7. — Suppose φ is proper. Then (locally in N), there is a bound s' on the number of connected components of a fiber $\varphi^{-1}(b)$. If $B = 0$, then Corollary 11.6 is satisfied with $s = s'$.

In the remainder of this section, we assume that $\mathbf{K} = \mathbf{R}$. Let $\varphi^* : \mathscr{C}^\infty(\mathrm{N}) \to \mathscr{C}^\infty(\mathrm{M})$ denote the ring homomorphism induced by φ, and let $\Phi : \mathscr{C}^\infty(\mathrm{N})^q \to \mathscr{C}^\infty(\mathrm{M})^p$ denote the module homomorphism over φ^* defined by $\Phi(g) = \mathrm{A}\cdot(g\circ\varphi)$, where $g \in \mathscr{C}^\infty(\mathrm{N})^q$. Let $\mathrm{B}\cdot : \mathscr{C}^\infty(\mathrm{M})^r \to \mathscr{C}^\infty(\mathrm{M})^p$ denote the $\mathscr{C}^\infty(\mathrm{M})$-homomorphism induced by multiplication by the matrix B.

Let $(\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r)\hat{} = \{f \in \mathscr{C}^\infty(\mathrm{M})^p :$ for all $b \in \varphi(\mathrm{M})$, there exists $\mathrm{G}_b \in \hat{\mathcal{O}}_b^q$ such that $\hat{f}_a - \hat{\Phi}_a(\mathrm{G}_b) \in \operatorname{Im} \hat{\mathrm{B}}_a$, for all $a \in \varphi^{-1}(b)\}$.

THEOREM 11.8. — *Suppose that* φ *is proper. Then each of the equivalent conditions of Theorem 8.2.5 implies that*

$$\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r = (\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r)\hat{}\,.$$

Remark 11.9. — Let Z be a closed subanalytic subset of N. Our proof of Theorem 11.8 will show that each of the equivalent conditions of Theorem 8.2.5 implies the following stronger result: If $f \in (\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r)\hat{}$ and $\hat{f}_a \in \operatorname{Im} \hat{\mathrm{B}}_a$ for all $a \in \varphi^{-1}(\mathrm{Z})$, then there exists $g \in \mathscr{I}(\mathrm{N};\mathrm{Z})^q$ and $h \in \mathscr{C}^\infty(\mathrm{M})^r$ such that $f = \Phi(g) + \mathrm{B}\cdot h$.

Remark 11.10. — In the case that $\mathrm{A} = \mathrm{I}$ and $\mathrm{B} = 0$, it is enough to assume that φ is semiproper [5, Rmk. 3.5]. The following example shows that « semiproper » is not sufficient in general: Let $\mathrm{M} = \mathrm{M}_1 \cup \mathrm{M}_2$ be the disjoint union of $\mathrm{M}_1 = \mathbf{R}^2$ and $\mathrm{M}_2 = \mathbf{R}^2$. Let $\mathrm{N} = \mathbf{R}^2$. Define $\varphi : \mathrm{M} \to \mathrm{N}$ by $\varphi(x,y) = (x,y)$ if $(x,y) \in \mathrm{M}_1$, $\varphi(x,y) = (x,xy)$ if $(x,y) \in \mathrm{M}_2$. Let $p = q = 1$ and let $\mathrm{A}(x,y) = 0$ on M_1, $\mathrm{A}(x,y) = 1$ on M_2. Take $\mathrm{B} = 0$. Define $f \in \mathscr{C}^\infty(\mathrm{M})$ by $f(x,y) = 0$ on M_1 and $f(x,y) = ye^{-1/x^2y^2}$ on M_2. Let (u,v) denote the coordinates of N. Then f is flat on $\varphi^{-1}(\{u=0\})$, and outside $\varphi^{-1}(\{u=0\})$, $f = \Phi(g)$, where $g(u,v) = (v/u)e^{-1/v^2}$. Hence $f \in (\Phi\mathscr{C}^\infty(\mathrm{N}))\hat{}$. Clearly, $f \notin \Phi\mathscr{C}^\infty(\mathrm{N})$. This example satisfies the conditions of Theorem 8.2.5 because $\varphi|\mathrm{M}_2$ is generically a submersion (cf. § 13).

Remark 11.11. — The assertion that $\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r = (\Phi\mathscr{C}^\infty(\mathrm{N})^q + \mathrm{B}\cdot\mathscr{C}^\infty(\mathrm{M})^r)\hat{}$ is local in N. Hence we can assume that N is an open subset of $\mathbf{R}^n$ and, by Corollary 11.6, that there is a bound s on the number of distinct submodules $\mathscr{R}_a \subset \hat{\mathcal{O}}_b^q$, where $a \in \varphi^{-1}(b)$, $b \in \mathrm{N}$. We will prove Theorem 11.8 using the conditions of Theorem 8.2.5 with this s.

We will also use the following:

Remark 11.12. — Let X be a germ at the origin of a closed analytic subset of $\mathbf{R}^m$. Let X^C denote the complexification of X, and let Sing X^C denote (the germ of) the singular points of X^C. The real part Σ of Sing X^C is (a germ of) a proper analytic subset of X. There exist $f_i(x) \in \mathbf{R}\{x\} = \mathbf{R}\{x_1, \ldots, x_m\}$, $1 \leqslant i \leqslant k$, such that the complexifications $f_i(z)$ of the $f_i(x)$ generate the ideal in $\mathbf{C}\{z\} = \mathbf{C}\{z_1, \ldots, z_m\}$ of convergent power series which vanish on X^C. Then, for all $a \in X - \Sigma$, $\mathscr{I}_{X,a}$ is generated by the f_i (where we have used the same symbol for a germ at the origin and a representative of the germ in a suitable neighborhood, and where $\mathscr{I}_X$ denotes the sheaf of germs of real analytic functions vanishing on X).

Proof of Theorem 11.8. — We make the assumptions of Remark 11.11. If $b \in \varphi(M)$, then there exist $a^1, \ldots, a^s \in \varphi^{-1}(b)$ such that $\bigcap_{a \in \varphi^{-1}(b)} \mathscr{R}_a = \bigcap_{i=1}^{s} \mathscr{R}_{a^i}$. If $\mathbf{a} \in M^s_\varphi$, $\mathbf{a} = (a^1, \ldots, a^s)$, we put $\mathscr{R}_\mathbf{a} = \bigcap_{i=1}^{s} \mathscr{R}_{a^i}$. Since the diagram of initial exponents $\mathfrak{N}_\mathbf{a} = \mathfrak{N}(\mathscr{R}_\mathbf{a})$ is Zariski semicontinuous on M^s_φ (8.2.5(4)), there is a locally finite filtration of M^s_φ by closed analytic subsets, $M^s_\varphi = Z_0 \supset Z_1 \supset \ldots Z_\nu \supset Z_{\nu+1} \supset \ldots$, such that, for all $\nu \in \mathbf{N}$, $\mathfrak{N}_\mathbf{a}$ is constant on $Z_\nu - Z_{\nu+1}$ and, for all $\mathbf{a} \in Z_\nu - \varphi^{-1}(\varphi(Z_{\nu+1}))$, $\mathscr{R}_\mathbf{a} = \bigcap_{a \in \varphi^{-1}(\varphi(\mathbf{a}))} \mathscr{R}_a$.

It follows that there is a locally finite partition $\{X_\mu\}_{\mu \in \mathbf{N}}$ of M^s_φ such that, for each μ:

(1) X_μ is a relatively compact connected smooth semianalytic subset of M^s_φ, and $\bar{X}_\mu$ lies in a product coordinate chart U_μ in M^s.

(2) $\bar{X}_\mu - X_\mu \subset \cup_{\lambda<\mu} X_\lambda$.

(3) $\mathfrak{N}_\mathbf{a}$ is constant, say $\mathfrak{N}_\mathbf{a} = \mathfrak{N}_\mu$, on X_μ.

(4) Let $Y_\mu = \varphi(\cup_{\lambda<\mu} X_\lambda)$. Then, for all $\mathbf{a} \in X_\mu - \varphi^{-1}(Y_\mu)$, $\mathscr{R}_\mathbf{a} = \bigcap_{a \in \varphi^{-1}(\varphi(\mathbf{a}))} \mathscr{R}_a$.

(5) (By Remark 11.12.) There exist finitely many elements $\theta_{\mu i}$ of $\mathscr{O}(U_\mu)$ such that, if $W_\mu = \{\mathbf{x} \in U_\mu : \theta_{\mu i}(\mathbf{x}) = 0$ for all $i\}$, then $\dim X_\mu = \dim W_\mu$ and, for all $\mathbf{a} \in X_\mu$, $\mathscr{I}_{X_\mu, \mathbf{a}} = \mathscr{I}_{W_\mu, \mathbf{a}} =$ the ideal generated by the $\theta_{\mu i}$ at $\mathbf{a}$ (where $\mathscr{I}_{X_\mu, \mathbf{a}}$ denotes the germs of real analytic functions vanishing on X_μ at $\mathbf{a}$). In particular, X_μ is an open subset of the smooth part of W_μ.

Let $f \in (\Phi\mathscr{C}^\infty(N)^q + B\cdot\mathscr{C}^\infty(M)^r)\hat{}$. It is enough to prove that, for each μ, there exist $g \in \mathscr{C}^\infty(N)^q$ and $h \in \mathscr{C}^\infty(M)^r$ such that $f - \Phi(g) - B\cdot h$ is flat on $\varphi^{-1}(Y_{\mu+1})$. By induction, we can assume that f is flat on $\varphi^{-1}(Y_\mu)$.

Let $X = X_\mu - \varphi^{-1}(Y_\mu)$. If $X = \emptyset$, we can take $g = 0$ and $h = 0$. Suppose $X \neq \emptyset$. Then $\varphi|X : X \to N - Y_\mu$ is proper. Let $\mathbf{a} \in X$, $\mathbf{a} = (a^1, \ldots, a^s)$, and let $b = \varphi(\mathbf{a})$. By (3) and the formal division algorithm (Theorem 6.2), there is a unique $G_b \in \hat{\mathcal{O}}_b^q$ such that

$$\text{(11.13)} \qquad \operatorname{supp} G_b \cap \mathfrak{N}_\mu = \emptyset,$$

and $\hat{f}_{a^i} - \hat{\Phi}_{a^i}(G_b) \in \operatorname{Im} \hat{B}_{a^i}, i = 1, \ldots, s$. Then, by (4), for all $a \in \varphi^{-1}(b)$, $\hat{f}_a - \hat{\Phi}_a(G_b) \in \operatorname{Im} \hat{B}_a$.

Write $G_b = (G_{1,b}, \ldots, G_{q,b})$, $G_{j,b} = \sum_{\beta \in \mathbf{N}^n} G_{j,b}^\beta y^\beta \in \hat{\mathcal{O}}_b = \mathbf{R}[[y]]$, where $y = (y_1, \ldots, y_n)$. Then (11.13) is equivalent to: $D^\beta G_{j,b} = 0$ for all $(\beta, j) \in \mathfrak{N}_\mu$.

LEMMA 11.14. – *For each* $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, *there exists* $g_j^\beta \in \mathscr{C}^\infty(X)$ *such that:*

(i) g_j^β *extends continuously to zero on* $\bar{X} - X$.

(ii) *For all* $\mathbf{a} \in X$, $g_{j,\mathbf{a}}^\beta = \hat{\iota}_{\mathbf{a}}^* \circ \hat{\varphi}_{\mathbf{a}}^* (D^\beta G_{j,\varphi(\mathbf{a})})$, *where* $\hat{\iota}_{\mathbf{a}}^* : \hat{\mathcal{O}}_{M_\varphi^s, \mathbf{a}} \to \hat{\mathcal{O}}_{X, \mathbf{a}}$ *is induced by the inclusion* $\iota : X \to M_\varphi^s$.

It follows from (ii) and an estimate of Glaeser [16, §§ 4, 5] (or [37, pp. 180-181]) that, for each $j = 1, \ldots, q$, there exists $g_j' \in \mathscr{C}^\infty(N - Y_\mu)$ such that $\hat{g}_{j,b}' = G_{j,b}$ for all $b \in \varphi(X) = Y_{\mu+1} - Y_\mu$. By (i), for all $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, $D^\beta g_j' | \varphi(X)$ extends continuously to zero on Y_μ. Since $Y_{\mu+1}$ is subanalytic, it follows that there exist $g_j \in \mathscr{C}^\infty(N)$ such that g_j is flat on Y_μ and $\hat{g}_{j,b} = G_{j,b}$, for all $b \in \varphi(X)$. Put $g = (g_1, \ldots, g_q)$. Then $(f - \Phi(g))\hat{}_a \in \operatorname{Im} \hat{B}_a$, for all $a \in \varphi^{-1}(Y_{\mu+1})$. By Theorem 10.1 (and Remark 10.2), there exists $h \in \mathscr{C}^\infty(M)^r$ such that $f - \Phi(g) - B \cdot h$ is flat on $\varphi^{-1}(Y_{\mu+1})$, as required. □

Proof of Lemma 11.14. – If $(\beta, j) \in \mathfrak{N}_\mu$, then $D^\beta G_{j,b} = 0$, for all $b \in \varphi(X)$. Hence it is enough to prove the assertion for $(\beta, j) \notin \mathfrak{N}_\mu$. Let $\mathbf{a} \in X$, $\mathbf{a} = (a^1, \ldots, a^s)$. We have $\hat{f}_{a^i} - \hat{A}_{a^i} \cdot (G_{\varphi(\mathbf{a})} \circ \hat{\varphi}_{a^i}) \in \operatorname{Im} \hat{B}_{a^i}$, $i = 1, \ldots, s$; i.e., $(\hat{f}_{a^i})_{1 \leq i \leq s} - \hat{\mathbf{\Phi}}_{\mathbf{a}}(G_{\varphi(\mathbf{a})}) \in \operatorname{Im} \hat{\mathbf{B}}_{\mathbf{a}}$.

For each $\ell \in \mathbf{N}$, let ${}^\ell F_{\mathbf{a}}$ (respectively, ${}^\ell G_{\mathbf{a}}$) denote the image of $(\hat{f}_{a^i})_{1 \leq i \leq s}$ (respectively, of $G_{\varphi(\mathbf{a})}$) by the lower (respectively, upper) horizontal arrow in the completion of the left-hand diagram (8.2.6); thus,

$$\text{(11.15)} \qquad {}^\ell F_{\mathbf{a}} - \hat{A}_{\ell,a} \cdot {}^\ell G_{\mathbf{a}} \in \operatorname{Im} \hat{B}_{\ell,\mathbf{a}}.$$

Recall that ${}^{\ell}G_{\mathbf{a}}$ is the element of $\oplus_{|\beta|\leqslant\ell}\hat{\mathcal{O}}^q_{X,\mathbf{a}}$ induced by $(D^\beta G_{\varphi(\mathbf{a})}\circ\hat{\varphi}_{\mathbf{a}})_{|\beta|\leqslant\ell}$. Write ${}^{\ell}G_{\mathbf{a}} = (G^\beta_{\mathbf{a}})_{|\beta|\leqslant\ell} = (G^\beta_{j,\mathbf{a}})_{|\beta|\leqslant\ell,1\leqslant j\leqslant q}$, where each $G^\beta_{j,\mathbf{a}}\in\hat{\mathcal{O}}_{X,\mathbf{a}}$ and $G^\beta_{\mathbf{a}} = (G^\beta_{j,\mathbf{a}})_{1\leqslant j\leqslant q}$. Then $G^\beta_{j,\mathbf{a}} = 0$ for all $(\beta,j)\in\mathfrak{N}_\mu$.

We use the notation of 8.2, 8.3. Let $k\in\mathbf{N}$. According to Theorem 8.2.5. (1), there exists $\ell = \ell(k)\in\mathbf{N}$ such that $\ell(k,\mathbf{a})\leqslant\ell$ for all $\mathbf{a}\in X$. Let $\rho_{\ell,k}(X) = \max_{\mathbf{a}\in X}\rho_{\ell,k}(\mathbf{a})$ and let $\sigma_{\ell,k}(X) = \max_{\mathbf{a}\in X}\sigma^X_{\ell,k}(\mathbf{a})$. Put $Y_{\ell,k} = \{\mathbf{a}\in X:\ \rho_{\ell,k}(\mathbf{a})<\rho_{\ell,k}(X)\}$ and $Z_{\ell,k} = \{\mathbf{a}\in X:\ \sigma^X_{\ell,k}(\mathbf{a})<\sigma_{\ell,k}(X)\}$. Then $Y_{\ell,k}$ and $Z_{\ell,k}$ are proper analytic subsets of X. Let $\mathbf{a}\in X$. Define $T^X_{\ell,k}(\mathbf{a})$ and $\hat{T}_{\ell,k,\mathbf{a}}$ as in 8.3. From (11.15):

$$\mathrm{ad}^{\sigma_{\ell,k}(X)}\hat{S}_{\ell,k,\mathbf{a}}\circ\mathrm{Ad}^{\rho_{\ell,k}(X)}\hat{D}_{\ell,k,\mathbf{a}}\cdot{}^{\ell}F_{\mathbf{a}} = \hat{T}_{\ell,k,\mathbf{a}}\cdot{}^{k}G_{\mathbf{a}},$$

where $\hat{S}_{\ell,k,\mathbf{a}} = \mathrm{Ad}^{\rho_{\ell,k}(X)}\hat{D}_{\ell,k,\mathbf{a}}\circ\hat{B}_{\ell,\mathbf{a}}$.

Let $e(k)$ denote the number of exponents $(\beta,j)\in\mathbf{N}^n\times\{1,\dots,q\}$ such that $(\beta,j)\notin\mathfrak{N}_\mu$ and $|\beta|\leqslant k$. Suppose $\mathbf{a}\in X-(Y_{\ell,k}\cup Z_{\ell,k})$. By the formal division algorithm (Theorem 6.2) and Remarks 8.2.4 and 8.3.1, $\operatorname{rank}T^X_{\ell,k}(\mathbf{a}) = e(k)$; moreover, if $V_{\mathbf{a}}(k)$ denotes the subspace

$$\{G = (G^\beta_j)_{|\beta|\leqslant k,1\leqslant j\leqslant q}\in\oplus_{|\beta|\leqslant k}(\hat{\mathcal{O}}_{X,\mathbf{a}}/\mathfrak{m}_{X,\mathbf{a}}\cdot\hat{\mathcal{O}}_{X,\mathbf{a}})^q:\quad G^\beta_j = 0 \text{ if } (\beta,j)\in\mathfrak{N}_\mu\},$$

then $\operatorname{rank}T^X_{\ell,k}(\mathbf{a})|V_{\mathbf{a}}(k) = e(k)$.

By the induction hypothesis and Cramer's rule, there is a minor $\delta = \delta_k$ of order $e(k)$ of $T^X_{\ell,k}$ such that δ is not identically zero on X and such that, for all $\mathbf{a}\in X$ and $(\beta,j)\notin\mathfrak{N}_\mu$, $|\beta|\leqslant k$,

$$\hat{\delta}_{\mathbf{a}}\cdot G^\beta_{j,\mathbf{a}} = (\xi^\beta_j)\hat{}_{\mathbf{a}}, \tag{11.16}$$

where $\xi^\beta_j\in\mathscr{C}^\infty(X)$ is the restriction to $X = X_\mu-\varphi^{-1}(Y_\mu)$ of a $\mathscr{C}^\infty$ function on U_μ which is flat on $\varphi^{-1}(Y_\mu)$. The minor δ is the restriction to X of an analytic function defined on U_μ (which we also denote δ).

Suppose $(\beta,j)\notin\mathfrak{N}_\mu$, $|\beta|\leqslant k$. By Whitney's extension theorem [27, I.4.1], there exists $\eta^\beta_j\in\mathscr{C}^\infty(U_\mu)$ such that η^β_j is flat on $W_\mu-X$ and $\eta^\beta_j|X = \xi^\beta_j$. Then, by (11.16) and (5) above, for all $\mathbf{a}\in U_\mu$, $(\eta^\beta_j)\hat{}_{\mathbf{a}}$ belongs to the ideal in $\hat{\mathcal{O}}_{U_\mu,\mathbf{a}}$ generated by $\hat{\delta}_{\mathbf{a}}$ and the $\hat{\theta}_{\mu i,\mathbf{a}}$. By Theorem 10.1, there exists $h^\beta_j\in\mathscr{C}^\infty(U_\mu)$ such that $\eta^\beta_j-\delta\cdot h^\beta_j$ belongs to the ideal generated by the $\theta_{\mu i}$ in $\mathscr{C}^\infty(U_\mu)$. Then h^β_j vanishes on $\bar{X}-X$ and, if $g^\beta_j = h^\beta_j|X$, then $\hat{g}^\beta_{j,\mathbf{a}} = G^\beta_{j,\mathbf{a}}$ for all $\mathbf{a}\in X$, as required.

□

CHAPTER III

SEMICONTINUITY RESULTS

12. Algebraic morphisms.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathbf{K}[x]$ (respectively, $\mathbf{K}[[x]]$) denote the ring of polynomials (respectively, formal power series) in $x = (x_1, \ldots, x_m)$.

DEFINITION 12.1. — *Let* U *be an open subset of* $\mathbf{K}^m$. *An analytic function* $f \in \mathcal{O}(\mathrm{U})$ *is* Nash *if it is algebraic over the ring* $\mathbf{K}[x]$ *of polynomials in the coordinates* $x = (x_1, \ldots, x_m)$ *of* $\mathbf{K}^m$; *i.e., there is a nonzero polynomial* $\mathrm{P}(x,y) \in \mathbf{K}[x,y]$ *such that* $\mathrm{P}(x,f(x)) = 0$ *for all* $x \in \mathrm{U}$. *Let* N(U) *denote the ring of Nash functions on* U.

We can define a category of *Nash manifolds* and *Nash mappings* using, as local models, open subsets U of $\mathbf{K}^m$, $m \in \mathbf{N}$, together with the rings N(U).

THEOREM 12.2. — *Let* M *and* N *denote Nash manifolds, and let* $\varphi : \mathrm{M} \to \mathrm{N}$ *be a Nash mapping. Let* A *and* B *be* $p \times q$ *and* $p \times r$ *matrices, respectively, whose entries are Nash functions on* M. *We use the notation of* 8.2, 8.4. *Let* $s \in \mathbf{N}$. *Assume that* N *is an open subset of* $\mathbf{K}^n$. *Then the diagram of initial exponents* $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}(\mathscr{R}_{\mathbf{a}})$ *is Zariski semicontinuous on* M^s_φ.

Remarks 12.3. — (1) Our proof of Theorem 12.2 together with Proposition 9.6 in fact establishes 12.2 under the following more general hypothesis: Let M and N denote analytic manifolds. Let $\varphi : \mathrm{M} \to \mathrm{N}$ be an analytic mapping, and A, B matrices of analytic functions on M, satisfying the following condition: For every $a \in \mathrm{M}$, there are (analytic) coordinate neighborhoods U of a in M and V of $\varphi(a)$ in N, such that $\varphi(\mathrm{U}) \subset \mathrm{V}$ and both the components of $\varphi|\mathrm{U}$ and the entries of $\mathrm{A}|\mathrm{U}$ and $\mathrm{B}|\mathrm{U}$ belong to N(U).

(2) In the special case that M and N are algebraic manifolds, φ is a regular (rational) mapping, and A, B are matrices of regular functions on M, our proofs actually show that $\mathfrak{N}_{\mathbf{a}}$ is Zariski semicontinuous in the algebraic sense; i.e., for each $\mathbf{a} \in \mathrm{M}^s_\varphi$, $\{\mathbf{x} \in \mathrm{M}^s_\varphi : \mathfrak{N}_{\mathbf{x}} \geqslant \mathfrak{N}_{\mathbf{a}}\}$ is a closed algebraic subset of M^s_φ.

To prove Theorem 12.2, we will use a version of « Artin approximation with respect to nested subrings » (cf. [2], [3], [33]):

DEFINITION 12.4. — *A formal power series* $f(x) \in \mathbf{K}[[x]]$ *is* algebraic *if it is algebraic over* $\mathbf{K}[x]$. *The algebraic elements of* $\mathbf{K}[[x]]$ *form a subring which we denote* $\mathbf{K}\langle x\rangle$.

Clearly, $\mathbf{K}\langle x\rangle \subset \mathbf{K}\{x\}$, the ring of convergent power series. Let $(x) = (x_1, \ldots, x_m)$ denote the ideal in $\mathbf{K}[[x]]$ generated by $x_1, \ldots x_m$.

Remark 12.5 [3]. — Let $f_1(x) \in \mathbf{K}[[x]]$. Then $f_1(x)$ is algebraic if and only if there exist $r \in \mathbf{N}$, $f_i(x) \in \mathbf{K}[[x]]$, $i = 2, \ldots, r$, and $F_j(x,y) \in \mathbf{K}[x,y]$, $j = 1, \ldots, r$ where $y = (y_1, \ldots, y_r)$, such that:

(1) $F(x,f(x)) = 0$, where $f = (f_1, \ldots, f_r)$ and $F = (F_1, \ldots, F_r)$;

(2) $\det\left(\dfrac{\partial F}{\partial y}\right)(0,f(0)) \neq 0$.

THEOREM 12.6. — *Let*

$$f(x,y,u,v) = 0 \tag{12.7}$$

be a system of equations in $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $u = (u_1, \ldots, u_p)$ *and* $v = (v_1, \ldots, v_q)$, *where* $f = (f_1, \ldots, f_r)$ *and each* $f_j \in \mathbf{K}\langle x,y,u,v\rangle$. *Assume that* f *is linear with respect to* v; *i.e.*,

$$f(x,y,u,v) = \sum_{i=0}^{q} v_i g_i(x,y,u),$$

where $v_0 = 1$ *and each* $g_i \in \mathbf{K}\langle x,y,u\rangle^r$. *Suppose that* (12.7) *admits a solution* $u = \hat{u}(x) \in \mathbf{K}[[x]]^p$, $v = \hat{v}(x,y) \in \mathbf{K}[[x,y]]^q$, *where* $\hat{u}(0) = 0$. *Then, for all* $t \in \mathbf{N}$, (12.7) *has a solution* $u = u(x) \in \mathbf{K}\langle x\rangle^p$, $v = v(x,y) \in \mathbf{K}\langle x,y\rangle^q$ *such that* $u(x) - \hat{u}(x) \in (x)^t \cdot \mathbf{K}[[x]]^p$ *and* $v(x,y) - \hat{v}(x,y) \in (x,y)^t \cdot \mathbf{K}[[x,y]]^q$.

Remark 12.8. — The analogue of Theorem 12.6 for convergent power series is *false*: Let $f(x) = f(x_1, x_2)$ and $\varphi_i(x)$, $i = 1, 2, 3$, be as in Example 2.8. Then the equation $f(x) - g(y) = \sum_{i=1}^{3} h_i(x,y)(y_i - \varphi_i(x))$ admits a formal solution $g(y)$, $h_i(x,y)$, $i = 1, 2, 3$, but no such convergent solution.

LEMMA 12.9. — *Theorem* 12.6 *holds under the stronger assumption that each* $f_j(x,y,u,v) \in \mathbf{K}[x,y,u,v]$. *(In this case, it is unnecessary to assume* $\hat{u}(0) = 0$.*)*

Proof. — For convenience, we make the following change of notation: v will mean $(v_0, v_1, \ldots, v_q)$, where $v_0 = 1$. We also put $\hat{v}(x,y) = (\hat{v}_0(x,y), \ldots, \hat{v}_q(x,y))$, where $\hat{v}_0(x,y) = 1$. Let A denote the localization of the ring $\mathbf{K}[[x]][y]$ at the ideal generated by x and y. Let $\hat{\mathrm{A}}$ denote the completion of A ; of course, $\hat{\mathrm{A}} = \mathbf{K}[[x,y]]$.

Each $g_i(x,y,\hat{u}(x)) \in \mathrm{A}$. Since $v = \hat{v}(x,y)$ is a solution of the system $\sum_{i=0}^{q} v_i g_i(x,y,\hat{u}(x)) = 0$, then, by Krull's theorem, there is a solution $v = \bar{v}(x,y)$, where $\bar{v}_0 = 1$ and each $\bar{v}_i(x,y) \in \mathrm{A}$. Clearly, $\bar{v}$ can be chosen to approximate $\hat{v}$ to any given order.

We can write $\bar{v}(x,y) = \bar{w}(x,y)/\bar{w}_0(x,y)$, where $\bar{w} = (\bar{w}_0, \ldots, \bar{w}_q)$, each $\bar{w}_i \in \mathbf{K}[[x]][y]$ and $\bar{w}_0(0,0) \neq 0$. Then $\sum_i \bar{w}_i(x,y) g_i(x,y,\hat{u}(x)) = 0$. Write each $\bar{w}_i$ and g_i as a polynomial in $y_1, \ldots, y_n$: $\bar{w}_i(x,y) = \sum_\alpha \hat{w}_{i\alpha}(x) y^\alpha \in \mathbf{K}[[x]][y]$, $g_i(x,y,u) = \sum_\alpha g_{i\alpha}(x,u) y^\alpha \in \mathbf{K}[x,u][y]^r$, where α denotes a multiindex in $\mathbf{N}^n$. Then $u = \hat{u}(x)$, $w_{i\alpha} = \hat{w}_{i\alpha}(x)$ is a formal solution of the system of polynomial equations

$$\sum_{i=0}^{q} \sum_{\alpha+\beta=\gamma} w_{i\alpha} g_{i\beta}(x,u) = 0, \qquad \gamma \in \mathbf{N}^n.$$

By Artin's theorem [2, Thm. 1.10], there is an algebraic solution $u = u(x)$, $w_{i\alpha} = w_{i\alpha}(x)$ which approximates the given formal solution to any specified order.

Put $w_i(x,y) = \sum_\alpha w_{i\alpha}(x) y^\alpha$ and $v(x,y) = w(x,y)/w_0(x,y)$, where $w = (w_0, \ldots, w_q)$. Then $u = u(x)$, $v = v(x,y)$ is an algebraic solution of (12.7). Clearly, the solution can be chosen to approximate $\hat{u}(x)$, $\hat{v}(x,y)$ to any specified order. □

Proof of Theorem 12.6. — We make the same notational changes as in Lemma 12.9 : v will mean $v = (v_0, v_1, \ldots, v_q)$, where $v_0 = 1$, etc. Write $g_i = (g_{i1}, \ldots, g_{ir})$, $i = 0, \ldots, q$, where each $g_{ij} \in \mathbf{K}\langle x,y,u \rangle$. By Remark 12.5, there exist $s \in \mathbf{N}$, $s > q$, as well as $g_{ij}(x,y,u) \in \mathbf{K}\langle x,y,u \rangle$, $i = q+1, \ldots, s$, $j = 1, \ldots, r$, and $\mathrm{G}_{k\ell}(x,y,u,z) \in \mathbf{K}[x,y,u,z]$,

$k = 0, \ldots, s$, $\ell = 1, \ldots, r$, where $z = (z_{ij})$, $i = 0, \ldots, s$, $j = 1, \ldots, r$, such that:

(1) $G(x,y,u,g(x,y,u)) = 0$, where $g = (g_{ij})$, $G = (G_{k\ell})$;

(2) $\det\left(\frac{\partial G}{\partial z}\right)(0,g(0)) \neq 0$.

By the implicit function theorem,

$$z - g(x,y,u) + g(0) = c(x,y,u,z)\cdot G(x,y,u,g(0) + z),$$

where $c(x,y,u,z) = (c_{ijk\ell}(x,y,u,z))$ is a matrix whose rows are indexed by (i,j) and whose columns are indexed by (k,ℓ). Each entry $c_{ijk\ell}(x,y,u,z) \in \mathbf{K}\langle x,y,u,z\rangle$. Then, for each $j = 1, \ldots, r$,

$$\sum_{i=0}^{q} v_i g_{ij}(x,y,u) = \sum_{i=0}^{q} v_i\cdot(g_{ij}(0) + z_{ij}) - \sum_{i=0}^{q}\sum_{k,\ell} v_i c_{ijk\ell}(x,y,u,z)G_{k\ell}(x,y,u,g(0) + z).$$

Consider the system of polynomial equations

$$(12.10)\qquad \sum_{i=0}^{q} v_i\cdot(g_{ij}(0) + z_{ij}) = \sum_{k,\ell} w_{jk\ell}G_{k\ell}(x,y,u,g(0) + z),$$

$j = 1, \ldots, r$, where u, v and $w = (w_{jk\ell})$ are the unknowns. Then (12.10) admits a formal solution $u = \hat{u}(x)$, $v = \hat{v}(x,y)$ and $w_{jk\ell} = \hat{w}_{jk\ell}(x,y,z) = \sum_{i=0}^{q} \hat{v}_i(x,y)c_{ijk\ell}(x,y,\hat{u}(x),z)$. Let $t \in \mathbf{N}$. By Lemma 12.9, there exist $u = u(x) \in \mathbf{K}\langle x\rangle^p$, $v = v'(x,y,z) \in \mathbf{K}\langle x,y,z\rangle^{q+1}$ and $w_{jk\ell} = w_{jk\ell}(x,y,z) \in \mathbf{K}\langle x,y,z\rangle$ such that $v_0'(x,y,z) = 1$, $u(x) - \hat{u}(x) \in (x)^t\cdot\mathbf{K}[[x]]^p$, $v'(x,y,z) - \hat{v}(x,y) \in (x,y,z)^t\cdot\mathbf{K}[[x,y,z]]^{q+1}$, and

$$(12.11)\qquad \sum_{i=0}^{q} v_i'(x,y,z)\cdot(g_{ij}(0) + z_{ij}) = \sum_{k,\ell} w_{ij\ell}(x,y,z)G_{k\ell}(x,y,u(x),g(0) + z),$$

$j = 1, \ldots, r$. Substitute $z_{ij} = g_{ij}(x,y,u(x)) - g_{ij}(0)$ into (12.11), to get

$$\sum_{i=0}^{q} v_i(x,y)g_i(x,y,u(x)) = 0,$$

where $v_i(x,y) = v_i'(x,y,g(x,y,u(x)) - g(0))$, $i = 0, \ldots, q$. □

Remark 12.12. – Let $f_1(x) \in \mathbf{C}\langle x\rangle = \mathbf{C}\langle x_1, \ldots, x_m\rangle$. Let $f_i(x)$, $i = 2, \ldots, r$, and $F_j(x,y)$, $j = 1, \ldots, r$, $y = (y_1, \ldots, y_r)$, be as in Remark 12.5. Put $Z = \{(x,y) \in \mathbf{C}^{m+r} : F(x,y) = 0\}$. We can assume that the projection $\pi(x,y) = x$ of Z onto $\mathbf{C}^m$ is finite. The smooth points of Z which are not critical points of π project onto the complement of a proper algebraic subset V of $\mathbf{C}^m$. Clearly, f_1 extends to $\mathbf{C}^m - V$ as a *multivalued* holomorphic function, whose various determinations are algebraic at every point of $\mathbf{C}^m - V$. By differentiating the system of equations $F(x,f(x)) = 0$ with respect to x_j, we can see that the partial derivative $\partial f_1/\partial x_j$ also extends to $\mathbf{C}^m - V$ as a multivalued holomorphic function whose various determinations are algebraic at every point.

Proof of Theorem 12.2. – By Lemma 9.5, we can assume that M is connected. Let $\mathbf{a}_0 \in M^s_\varphi \subset M^s$. There is a product coordinate neighborhood $U = \prod_{i=1}^{s} U^i$ of $\mathbf{a}_0$ in M^s such that the components of φ and the entries of A and B all restrict to Nash functions on each U^i. Let $x = (x_1, \ldots, x_m)$ (respectively, $y = (y_1, \ldots, y_n)$) denote the coordinates of each U^i (respectively, of N). The notation of this paragraph will be fixed throughout the remainder of the section.

LEMMA 12.13. – *Let* $\mathbf{a} \in M^s_\varphi \cap U$, $\mathbf{a} = (a^1, \ldots, a^s)$. *Let* $\mathbf{\Phi_a}$: $\mathcal{O}^q_{\varphi(\mathbf{a})} \to \bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i}$ *and* $\mathbf{B_a} : \bigoplus_{i=1}^{s} \mathcal{O}^r_{a^i} \to \bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i}$, *as well as* $\hat{\mathbf{\Phi}}_\mathbf{a}$ *and* $\hat{\mathbf{B}}_\mathbf{a}$, *be as in* 8.2. *Let* $G \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$ *and* $H \in \bigoplus_{i=1}^{s} \mathcal{O}^r_{a^i}$. *Put* $f = \hat{\mathbf{\Phi}}_\mathbf{a}(G) + \hat{\mathbf{B}}_\mathbf{a}(H) \in \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^p_{a^i}$, $f = (f^1, \ldots, f^s)$. *Suppose each* $f^i \in \hat{\mathcal{O}}^p_{a^i} = \mathbf{K}[[x]]^p$ *is algebraic. Let* $t \in \mathbf{N}$. *Then there exist* $g \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$ *and* $h \in \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^r_{a^i}$ *such that g and h are algebraic,* $f = \mathbf{\Phi_a}(g) + \mathbf{B_a}(h)$, *and* $g - G \in \mathfrak{m}^t_{\varphi(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$, $h - H \in \bigoplus_{i=1}^{s} \mathfrak{m}^t_{a^i} \cdot \hat{\mathcal{O}}^r_{a^i}$.

Proof. – Write $H = (H^1, \ldots, H^s)$. Then

$$f^i(x) = \hat{A}_{a^i}(x) \cdot G(\hat{\varphi}_{a^i}(x) - \varphi(a^i)) + \hat{B}_{a^i}(x) \cdot H^i(x), \tag{12.14}$$

$i = 1, \ldots, s$. In other words, for each $i = 1, \ldots, s$, there is a $p \times n$ matrix $Q^i(x,y)$ with entries in $\mathbf{K}[[x,y]]$ such that

$$\begin{aligned} f^i(x) - \hat{A}_{a^i}(x) \cdot G(y) - \hat{B}_{a^i}(x) \cdot H^i(x) \\ = Q^i(x,y) \cdot (y - \hat{\varphi}_{a^i}(x) + \varphi(a^i)). \end{aligned} \tag{12.15}$$

In this system of equations, $G(y)$ and the $H^i(x)$, $Q^i(x,y)$ are the « unknowns ». Since A, B and φ are algebraic, then, by Theorem 12.6, there is an algebraic solution $g(y)$, $h_1^i(x,y)$, $q^i(x,y)$ of (12.15); i.e.,

$$(12.16)\quad f^i(x) - \hat{A}_{a^i}(x)\cdot g(y) - \hat{B}_{a^i}(x)\cdot h_1^i(x,y) = q^i(x,y)\cdot(y - \hat{\varphi}_{a^i}(x) + \varphi(a^i)),$$

$i = 1, \ldots, s$, such that $g(y) - G(y) \in (y)^t\cdot \mathbf{K}[[y]]^q$ and each $h_1^i(x,y) - H^i(x) \in (x,y)^t\cdot \mathbf{K}[[x,y]]^r$. Substitute $y = \hat{\varphi}_{a^i}(x) - \varphi(a^i)$ back into (12.16), for each i, to see that $g(y)$, $h^i(x) = h_1^i(x, \hat{\varphi}_{a^i}(x) - \varphi(a^i))$ is a solution of (12.14); clearly $h^i(x) - H^i(x) \in (x)^t\cdot\mathbf{K}[[x,y]]^r$.

□

Corollary 12.17. — *$\mathscr{R}_{\mathbf{a}} = \{G \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})} : \hat{\mathbf{\Phi}}_{\mathbf{a}}(G) \in \operatorname{Im} \hat{\mathbf{B}}_{\mathbf{a}}\}$ is generated by algebraic elements.*

Proof. — Let (β,j) be a vertex of $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}(\mathscr{R}_{\mathbf{a}})$. By Lemma 12.13, there exists $g \in \mathscr{R}_{\mathbf{a}}$ such that g is algebraic and $\operatorname{in} g = y^{\beta,j}$.

□

We now complete the proof of Theorem 12.2. We can assume that $\mathbf{K} = \mathbf{C}$. Let X denote an irreducible germ at $\mathbf{a}_0$ of a closed analytic subset of M^s_φ. We can assume that X is a closed analytic subset of U and that its smooth points are connected. Let $\mathfrak{N}_X$ denote the generic diagram of initial exponents (Definition 8.4.3). By Proposition 8.4.6(1), it suffices to find a proper closed analytic subset W of X such that $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}_X$ for all $\mathbf{a} \in X - W$.

Let (β_ℓ, k_ℓ), $\ell = 1, \ldots, t$, denote the vertices of $\mathfrak{N}_X$. Let $k = k(X)$ as in Definition 8.4.1, so that each $|\beta_\ell| \leqslant k$. Let D_k be as in (8.3.2) and let $Z \subset X$ be as in Remark 8.4.4. By Lemma 8.4.5, $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}_X$ for all $\mathbf{a} \in D_k \cap (X - Z)$.

Let $\mathbf{a}_1 \in D_k \cap (X - Z)$, $\mathbf{a}_1 = (a_1^1, \ldots, a_1^s)$. Put $b_1 = \varphi(\mathbf{a}_1)$. Let $G^\ell(y) = y^{\beta_\ell, k_\ell} - r^\ell(y)$, $\ell = 1, \ldots, t$, denote the standard basis of $\mathscr{R}_{\mathbf{a}_1}$, so that $\operatorname{supp} r^\ell \cap \mathfrak{N}_X = \varnothing$; for each ℓ. By Corollaries 6.8 and 12.17, each $G^\ell(y)$ is convergent. Thus, for b in some neighborhood of b_1, we can substitute $b - b_1 + y$ into G^ℓ, and expand in powers of y:

$$\begin{aligned} G^\ell(b - b_1 + y) &= (b - b_1 + y)^{\beta_\ell, k_\ell} - r^\ell(b - b_1 + y) \\ &= y^{\beta_\ell, k_\ell} - \tilde{r}^\ell_b(y), \end{aligned}$$

where supp $\tilde{r}_b^\ell(y) \cap \mathfrak{N}_X = \varnothing$. For $\mathbf{a}$ in a sufficiently small neighborhood of $\mathbf{a}_1$ in M_φ^s, put $G_\mathbf{a}^\ell(y) = G^\ell(\varphi(\mathbf{a}) - b_1 + y)$. Then $G_\mathbf{a}^\ell(y) = y^{\beta_\ell, k_\ell} - r_\mathbf{a}^\ell(y)$, where $r_\mathbf{a}^\ell = \tilde{r}_{\varphi(\mathbf{a})}^\ell$. Clearly, $G_\mathbf{a}^\ell \in \mathscr{R}_\mathbf{a}$. If $\mathbf{a} \in X - Z$, then $\mathfrak{N}_\mathbf{a} \subset \mathfrak{N}_X$ by Proposition 8.4.6.(2), and it follows that $\operatorname{in} G_\mathbf{a}^\ell = y^{\beta_\ell, k_\ell}$. In particular, $\mathfrak{N}_\mathbf{a} = \mathfrak{N}_X$ in a neighborhood of $\mathbf{a}_1$ in X.

By Lemma 12.13, for each $\ell = 1, \ldots, t$, there exist $g^\ell \in \hat{\mathcal{O}}_{\varphi(\mathbf{a}_1)}^q$, $h_\ell \in \bigoplus_{i=1}^s \hat{\mathcal{O}}_{a_1^i}^r$, $h_\ell = (h_\ell^1, \ldots, h_\ell^s)$, such that g^ℓ and each h_ℓ^i are algebraic, $\operatorname{in} g^\ell = y^{\beta_\ell, k_\ell}$, and $\mathbf{\Phi}_{\mathbf{a}_1}(g^\ell) = \mathbf{B}_{\mathbf{a}_1}(h_\ell)$. In particular, $g^\ell \in R_{\mathbf{a}_1}$. For each $\ell = 1, \ldots, t$, put

$$G^\ell(v\,;y) = \sum_{\beta \in \mathbf{N}^n} (D^\beta g^\ell)(v) \frac{y^\beta}{\beta!} \in \hat{\mathcal{O}}_{b_1}[[y]]^q,$$

$$H_\ell^i(u\,;x) = \sum_{\alpha \in \mathbf{N}^m} (D^\alpha h_\ell^i)(u) \frac{x^\alpha}{\alpha!} \in \hat{\mathcal{O}}_{a_1^i}[[x]]^r, \qquad i = 1, \ldots, s,$$

where $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$. By the formal division algorithm (cf. Remark 6.5),

$$y^{\beta_\ell, k_\ell} = \sum_{j=1}^t Q_j(v;y) G^j(v;y) + R^\ell(v;y), \tag{12.18}$$

$\ell = 1, \ldots, t$, where, for each ℓ,

$$Q_\ell(v;y) \in \hat{\mathcal{O}}_{b_1}[[y]], \qquad R^\ell(v;y) \in \hat{\mathcal{O}}_{b_1}[[y]]^q, \qquad \operatorname{supp} R^\ell(v;y) \cap \mathfrak{N}_X = \varnothing,$$

and the coefficients of Q_ℓ and R^ℓ (as elements of $\hat{\mathcal{O}}_{b_1}$) are algebraic. (They are linear combinations of the coefficients of the $G^\ell(v;y)$ divided by products of powers of the $D^{\beta_\ell} g_{k_\ell}^\ell(v)$, where $g^\ell = (g_1^\ell, \ldots, g_q^\ell)$.)

For each $\ell = 1, \ldots, t$, write

$$R^\ell(v;y) = \sum_{(\beta,j) \notin \mathfrak{N}_X} \hat{R}_{\beta,j}^\ell(v) y^{\beta,j}.$$

It follows from Remark 12.12 that there exist:

(1) A proper algebraic subset V of N such that $b_1 \notin V$, and, for each $i = 1, \ldots, s$, a proper algebraic subset W^i of U^i such that $a_1^i \notin W^i$.

(2) For each $\ell = 1, \ldots, t$ and $(\beta,j) \notin \mathfrak{N}_X$, an (*a priori*, multivalued) analytic function $\rho^\ell_{\beta,j}$ defined on $N - V$, such that $\hat{R}^\ell_{\beta,j}(v)$ is the formal Taylor expansion $(R^\ell_{\beta,j})\hat{}_{b_1}(v)$ of some branch $R^\ell_{\beta,j}$ of $\rho^\ell_{\beta,j}$ at b_1. Likewise, for each $\ell = 1, \ldots, t$, multivalued analytic functions defined on $N - V$ (respectively, multivalued analytic functions defined on $U^i - W^i$, $i = 1,\ldots,s$) which extend the coefficients of Q_ℓ (respectively, the coefficients of H^i_ℓ, $i = 1, \ldots, s$).

For each $\ell = 1, \ldots, t$, write $r^\ell_{\mathbf{a}}(y) = \sum_{(\beta,j)\notin \mathfrak{N}_X} r^\ell_{\beta,j}(\mathbf{a})y^{\beta,j}$. We claim that, for $\mathbf{a}$ in a sufficiently small neighborhood of $\mathbf{a}_1$ in $X - Z$,

$$r^\ell_{\beta,j}(\mathbf{a}) = R^\ell_{\beta,j}(\varphi(\mathbf{a})), \tag{12.19}$$

for all ℓ, β, j. Indeed, if $\mathbf{a}$ belongs to a suitable neighborhood of $\mathbf{a}_1$, then $R^\ell_{\beta,j}(\varphi(\mathbf{a})) = \hat{R}^\ell_{\beta,j}(\varphi(\mathbf{a})-b_1)$ and

$$G^\ell(\varphi(\mathbf{a})-b_1;y) = g^\ell(\varphi(\mathbf{a})-b_1+y) \in \mathscr{R}_{\mathbf{a}}.$$

Thus $y^{\beta_\ell,k_\ell} - R^\ell(\varphi(\mathbf{a})-b_1;y) \in \mathscr{R}_{\mathbf{a}}$. Moreover,

$$\operatorname{supp} R^\ell(\varphi(\mathbf{a})-b_1;y) \cap \mathfrak{N}_X = \varnothing.$$

For $\mathbf{a}$ close enough to $\mathbf{a}_1$ in $X - Z$, $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}_X$, so that

$$G^\ell_{\mathbf{a}}(y) = y^{\beta_\ell,k_\ell} - R^\ell(\varphi(\mathbf{a})-b_1;y),$$

by uniqueness of the standard basis; hence (12.19).

Let $W = X \cap (\varphi^{-1}(V) \cup \bigcup_{i=1}^{s} (\mu^i)^{-1}(W^i))$, where $\mu^i : M^s_\varphi \to M$ denotes the projection $\mu^i(\mathbf{x}) = x^i$, $\mathbf{x} = (x^1,\ldots,x^s)$. Then W is a closed analytic subset of X, and $\mathbf{a}_1 \notin W$. By (12.19) and (2) above, the coefficients $r^\ell_{\beta,j}(\mathbf{a})$ of each $G^\ell_{\mathbf{a}}(y) = y^{\beta_\ell,k_\ell} - r^\ell_{\mathbf{a}}(y)$, as well as the coefficients of the Q_ℓ composed with φ, and the coefficients of the H^i_ℓ, can be analytically continued (as multivalued functions) throughout $X - W$. By continuity and (12.18), if $\mathbf{a} \in W$, then any analytic continuation of (the coefficients of) $G^\ell_\bullet(y)$ to $\mathbf{a}$ results in an element of $\mathscr{R}_{\mathbf{a}}$. If $\mathbf{a} \in X - (Z \cup W)$, then $\mathfrak{N}_{\mathbf{a}} \subset \mathfrak{N}_X$; it follows from uniqueness of the standard basis that any analytic continuation of $G^\ell_\bullet(y)$ to $\mathbf{a}$ gives the same result, and that $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}_X$. □

13. Regular mappings.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

THEOREM 13.1. – *Let* M *and* N *be analytic manifolds* (*over* $\mathbf{K}$) *and let* $\varphi : \mathrm{M} \to \mathrm{N}$ *be an analytic mapping. Suppose that* φ *is regular* (*as in* 2.7). *Let* $s \in \mathbf{N}$. *For each* $\mathbf{a} \in \mathrm{M}^s_\varphi$, *let* $\mathrm{H}_\mathbf{a}$ *denote the Hilbert-Samuel function of the ring* $\hat{\mathcal{O}}_{\varphi(\mathbf{a})}/\mathcal{R}_\mathbf{a}$, *where* $\mathcal{R}_\mathbf{a} = \bigcap_{i=1}^{s} \operatorname{Ker} \hat{\varphi}^*_{a^i}$, $\mathbf{a} = (a^1, \ldots, a^s)$. *Then* $\mathrm{H}_\mathbf{a}$ *is Zariski semicontinuous on* M^s_φ.

Remark 13.2 (Tougeron). – If $s = 1$, the uniform Chevalley estimate (8.2.5(1)) can be proved using results of [39].

Remark 13.3. – Let V be an analytic manifold, and let Z be a closed analytic subset of V. We denote by $\mathcal{I}_\mathrm{Z}$ the subsheaf of ideals of $\mathcal{O}_\mathrm{V}$ of germs of analytic functions which vanish on Z. Suppose that $\dim \mathrm{V} = n$ and that Z has pure dimension $n - 1$. Let $b \in \mathrm{V}$. Then $\mathcal{I}_{\mathrm{Z},b}$ is a principal ideal. Let μ be as in Remark 6.10(2); we call $\mu_\mathrm{Z}(b) = \mu$ the *multiplicity* of Z at b. Thus $\mu_\mathrm{Z}(b)$ is the largest $\mu \in \mathbf{N}$ such that $\mathcal{I}_{\mathrm{Z},b} \subset \mathfrak{m}_b^\mu$, where $\mathfrak{m}_b$ is the maximal ideal of $\mathcal{O}_{\mathrm{V},b}$.

Proof of Theorem 13.1. – By Lemma 9.5, we can assume that the generic rank $r_1(a)$ of φ near a is constant on M; say $r_1(a) = n - k$, $a \in \mathrm{M}$. Let $\mathbf{a}_0 \in \mathrm{M}^s_\varphi$, $\mathbf{a}_0 = (a_0^1, \ldots, a_0^s)$. Put $b_0 = \varphi(\mathbf{a}_0)$. We can assume that N is an open subset of $\mathbf{K}^n$ and $b_0 = 0$. Since φ is regular, then, after replacing M and N by suitable neighborhoods of $\{a_0^1, \ldots, a_0^s\}$ and b_0 (respectively) if necessary, there is a closed analytic subset Z of N of dimension $n = k$, such that $\varphi(\mathrm{M}) \subset \mathrm{Z}$ and $\mathcal{I}_{\mathrm{Z},0} = \bigcap_{i=1}^{s} \operatorname{Ker} \varphi^*_{a_0^i}$.

The result is trivial if $k = 0$. Suppose that $k = 1$. We can assume that $\mathbf{K} = \mathbf{C}$ and that Z has pure dimension $n - 1$. Since Z is coherent, the multiplicity of Z is Zariski semicontinuous, by Theorem 7.4 and Remark 6.10. Let $\eta : \mathrm{Z}' \to \mathrm{Z}$ denote the normalization of Z. Since η is finite, it follows that (after shrinking N if necessary) there is a filtration of Z by closed analytic subsets,

$$\mathrm{Z} = \mathrm{Z}_0 \supset \mathrm{Z}_1 \supset \ldots \supset \mathrm{Z}_{t+1} = \varnothing,$$

such that, for each $i = 0, \ldots, t$:

(1) $Z_i - Z_{i+1}$ is smooth and connected.

(2) Let $Z'_i = \eta^{-1}(Z_i)$. Then $\eta \mid (Z'_i - Z'_{i+1}) : Z'_i - Z'_{i+1} \to Z_i - Z_{i+1}$ is a smooth covering projection.

(3) The multiplicity of Z is constant on $Z_i - Z_{i+1}$.

It follows from (2) that, for each i, there are finitely many analytic sets Z_{ij} defined in a neighborhood of $Z_i - Z_{i+1}$, such that, for all $b \in Z_i - Z_{i+1}$, the germs $Z_{ij,b}$ of the Z_{ij} at b are the distinct irreducible components of Z_b. Then, by (3), for each i and j, the multiplicity of $Z_{ij,b}$ is constant on $Z_i - Z_{i+1}$.

Let $X_i = \varphi^{-1}(Z_i)$, $i=0,\ldots,t$. Suppose that $\mathbf{a}=(a^1,\ldots,a^s) \in X_i - X_{i+1}$. Then, for each $\ell = 1, \ldots, s$, there is a j such that $\operatorname{Ker} \varphi^*_{a^\ell} = \mathscr{I}_{Z_{ij},\varphi(\mathbf{a})}$. It follows that $\operatorname{Ker} \varphi^*_{x^\ell} = \mathscr{I}_{Z_{ij},\varphi(\mathbf{x})}$ for $\mathbf{x}=(x^1,\ldots,x^s)$ in some neighborhood of $\mathbf{a}$ in $X_i - X_{i+1}$. Therefore, by Remark 6.10, the Hilbert-Samuel function $H_\mathbf{a}$ is constant on each connected component of $X_i - X_{i+1}$. By Proposition 8.3.7, $H_\mathbf{a}$ is Zariski semicontinuous on M^s_φ. This completes the proof in the case $k = 1$.

In general, by the representation theorem for germs of analytic sets [32, Ch. III], we can assume:

(1) There is a neighborhood V' of O in $\mathbf{K}^{n-k}$ such that $N = V' \times \mathbf{K}^k \subset \mathbf{K}^{n-k} \times \mathbf{K}^k$.

(2) Let $y = (y_1,\ldots,y_n)$ denote the coordinates in $\mathbf{K}^n$. Then, for each $i = 1, \ldots, k$, there is a monic polynomial $P_i \in \mathcal{O}(V')[Y_{n-i+1}]$ such that P_i vanishes on Z.

(3) Let $d_i = \text{degree } P_i$, $i = 1, \ldots, k$. Put $P = P_k$ and $d = d_k$. Let $\Delta(y_1,\ldots,y_{n-k})$ denote the discriminant of P. Then Δ is not identically zero and, for all $j = 1, \ldots, d$ and all $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbf{N}^k$ with $0 \leqslant \alpha_i < d_i$, $i = 1, \ldots, k$, there exists $v_{\alpha j} \in \mathcal{O}(V')$ such that

$$Q_\alpha = \Delta \cdot y_{n-k+1}^{\alpha_k} \cdots y_n^{\alpha_1} - \sum_{j=1}^{d} v_{\alpha j} \cdot y_{n-k+1}^{d-j}$$

vanishes on Z.

Suppose $\mathbf{a} = (a^1,\ldots,a^s) \in M^s_\varphi$ and $b = \varphi(\mathbf{a})$, $b = (b_1,\ldots,b_n)$. Set $b' = (b_1,\ldots,b_{n-k})$. Suppose $G \in \hat{\mathcal{O}}_b = \mathbf{K}[[y]]$. Then, by the formal Weierstrass division theorem, there exist $G_\alpha \in \hat{\mathcal{O}}_{b'}$, $0 \leqslant \alpha_i < d_i$,

$i = 1, \ldots, k$, such that

$$G - \sum_{0 \leq \alpha_i < d_i} G_\alpha \cdot y_{n-k+1}^{\alpha_k} \cdots y_n^{\alpha_1} \in (P_i) \cdot \hat{\mathcal{O}}_b,$$

where (P_i) denotes the ideal of $\mathcal{O}_b$ generated by the P_i. By (3), there exist $H_j \in \hat{\mathcal{O}}_{b'}$, $j = 1, \ldots, d$, such that

$$\hat{\Delta}_{b'} \cdot G - \sum_{j=1}^{d} H_j \cdot y_{n-k+1}^{d-j} \in (P_i, Q_\alpha) \cdot \hat{\mathcal{O}}_b .$$

Let $\pi : N \to V = V' \times \mathbf{K}$ denote the projection $\pi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-k+1})$. Put $\psi = \pi \circ \varphi$. Then ψ is regular and has generic rank $n - k$. If $G \in \bigcap^{s} \operatorname{Ker} \hat{\varphi}^*_{a^\ell}$, then $H = \sum_{j=1}^{d} H_j \cdot y_{n-k+1}^{d-j} \in \bigcap_{\ell=1}^{s} \operatorname{Ker} \hat{\Psi}^*_{a^\ell}$. It follows from the case $k = 1$ and Theorems 8.2.5 and 9.1, that there is a neighborhood U' of $\mathbf{a}_0$ in M_φ^s and a filtration of U' by closed analytic sets, $U' = Y_0 \supset Y_1 \supset \ldots \supset Y_{t+1} = \varnothing$, such that, for each $\lambda = 0, \ldots, t$, there exist finitely many $h_{\lambda\mu} \in \mathcal{M}(Y_\lambda ; Y_{\lambda+1})[[y_1, \ldots, y_{n-k+1}]]$ such that the $h_{\lambda\mu}(\mathbf{a}; y_1, \ldots, y_{n-k+1})$ generate $\bigcap_{\ell=1}^{s} \operatorname{Ker} \hat{\psi}^*_{a^\ell}$, $\mathbf{a} = (a^1, \ldots, a^s) \in Y_\lambda - Y_{\lambda+1}$. Then by Proposition 9.4, there is a neighborhood U of $\mathbf{a}_0$ in M_φ^s and a filtration of U by closed analytic sets, $U = X_0 \supset X_1 \supset \ldots \supset X_{r+1} = \varnothing$, such that, for each $\lambda = 0, \ldots, r$, there exist finitely many elements $g_{\lambda\mu} \in \mathcal{M}(X_\lambda ; X_{\lambda+1})[[y]]$ such that the $g_{\lambda\mu}(\mathbf{a}; y)$ generate $\bigcap_{\ell=1}^{s} \operatorname{Ker} \hat{\varphi}^*_{a^\ell}$, for all $\mathbf{a} = (a^1, \ldots, a^s) \in X_\lambda - X_{\lambda+1}$. Therefore, by Lemma 7.2 (2) and Proposition 8.3.7, the Hilbert-Samuel function $H_{\mathbf{a}}$ is Zariski semi-continuous on M_φ^s. □

14. The finite case.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let M and N denote analytic manifolds (over $\mathbf{K}$) and let $\varphi : M \to N$ be an analytic mapping. If $a \in M$, then $\mathcal{O}_a$ is an $\mathcal{O}_{\varphi(a)}$-module via the homomorphism $\varphi_a^* : \mathcal{O}_{\varphi(a)} \to \mathcal{O}_a$.

DEFINITION 14.1. — *We say that* φ *is* locally finite *if, for every* $a \in \mathrm{M}$, $\mathcal{O}_a$ *is a finitely generated* $\mathcal{O}_{\varphi(a)}$*-module. (This definition extends to morphisms of (possibly singular) analytic spaces.)*

THEOREM 14.2. — *Let* M *and* N *be analytic manifolds, and let* $\varphi : \mathrm{M} \to \mathrm{N}$ *be a locally finite analytic mapping. Let* A *and* B *be* $p \times q$ *and* $p \times r$ *matrices of analytic functions on* M, *respectively. We use the notation of* 8.2. *Let* $s \in \mathbf{N}$. *Then there is a uniform Chevalley estimate* (8.2.5(1)) *on* M^s_φ.

Theorem 14.2 extends to the case that M is a (possibly singular) analytic space which is Cohen-Macauley: see Remark 14.13 after the proof.

Proof of Theorem 14.2. — We can assume that $\mathbf{K} = \mathbf{C}$ and that N is an open neighborhood of 0 in $\mathbf{C}^n$. By Lemma 9.5, we can assume that M has pure dimension m. Let $\mathbf{a}_0 = (a_0^1, \dots, a_0^s) \in \mathrm{M}^s_\varphi$. Shrinking N and replacing M by an appropriate neighborhood of $\{a_0^1, \dots, a_0^s\}$, we can assume that φ is proper and that $\mathrm{Z} = \varphi(\mathrm{M})$ is a closed analytic subset of N, each irreducible component of which contains $\boldsymbol{\varphi}(\mathbf{a}_0)$.

Suppose that $\boldsymbol{\varphi}(\mathbf{a}_0) = 0$ in $\mathrm{N} \subset \mathbf{C}^n$. Since $\dim \mathrm{Z} = m$, we can assume that $\mathrm{N} = \mathrm{N}' \times \mathrm{N}'' \subset \mathbf{C}^m \times \mathbf{C}^{n-m}$ and that the projection $\pi : \mathrm{N} \to \mathrm{N}'$ induces a *finite* (i.e., proper and locally finite) mapping of Z onto N'. Let $\theta = \pi \circ \varphi$, $\theta = (\theta_1, \dots, \theta_m)$. Let $a \in \mathrm{M}$ and let $\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$ denote the ideal in $\mathcal{O}_a$ generated by $\mathfrak{m}_{\theta(a)}$ (via the homomorphism θ_a^*). Since θ is finite, $\dim_{\mathbf{C}} \mathcal{O}_a / \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a < \infty$.

LEMMA 14.3. — *Let* $\ell = \dim_{\mathbf{C}} \mathcal{O}_a / \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$. *Then* $\mathfrak{m}_a^{\ell+1} \subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$.

Proof. — If $j \geqslant 1$ and $\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^j = \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^{j+1}$, then, by Nakayama's lemma, $\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a = \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^j$, so that $\mathfrak{m}_a^j \subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$. Suppose $\mathfrak{m}_a^{\ell+1} \not\subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$. Then, for all $j \leqslant \ell + 1$,

$$\dim_{\mathbf{C}} \mathcal{O}_a / (\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^{j+1}) > \dim_{\mathbf{C}} \mathcal{O}_a / (\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^j).$$

Therefore, $\dim_{\mathbf{C}} \mathcal{O}_a / \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a \geqslant \dim_{\mathbf{C}} \mathcal{O}_a / (\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a + \mathfrak{m}_a^{\ell+2}) > \ell$; a contradiction. □

Remark 14.4. — We define the *multiplicity* $\mathrm{mult}_a \theta$ of θ at a by

$$\mathrm{mult}_a \theta = \dim_{\mathbf{K}_{\theta(a)}} \mathcal{O}_a \underset{\mathcal{O}_{\theta(a)}}{\otimes} \mathbf{K}_{\theta(a)},$$

where $\mathbf{K}_{\theta(a)}$ denotes the field of fractions of $\mathcal{O}_{\theta(a)}$. Then $\text{mult}_a \theta = \dim_{\mathbf{C}} \mathcal{O}_a / \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$ (by [31, Ch. 6, Thm. A.10] and [40, App. 6, Thm. 3]). Let d denote the number of points in a generic fiber of θ. Then, for all $b \in \mathrm{N}'$, $\sum_{a \in \theta^{-1}(b)} \text{mult}_a \theta = d$ (Weil's formula [31, Ch. 6, (A.8)]).

COROLLARY 14.5. — *For all* $a \in \mathrm{M}$, $\mathfrak{m}_a^{d+1} \subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$.

Let X be an irreducible germ at $\mathbf{a}_0$ of a closed analytic subset of M_φ^s. In order to prove Theorem 14.2, it suffices to find (a germ at $\mathbf{a}_0$ of) a proper closed analytic subset Y of X, and a function $\ell = \ell(k)$ from $\mathbf{N}$ to itself, such that, for $\mathbf{a} \in \mathrm{X} - \mathrm{Y}$ in some neighborhood of $\mathbf{a}_0$, $\ell(k, \mathbf{a}) \leqslant \ell(k)$ for all $k \in \mathbf{N}$. (We use the same symbol for a germ at $\mathbf{a}_0$ and a suitable representative of the germ in some neighborhood.)

Put $\boldsymbol{\theta} = \pi \circ \boldsymbol{\varphi} : \mathrm{M}_\varphi^s \to \mathrm{N}'$. (Clearly, $\mathrm{M}_\varphi^s \subset \mathrm{M}_\theta^s \subset \mathrm{M}^s$; $\boldsymbol{\theta}$ is the restriction to M_φ^s of the mapping $\mathrm{M}_\theta^s \to \mathrm{N}'$ induced by θ.) Then $\boldsymbol{\theta}$ is finite.

LEMMA 14.6. — *There exists (a germ at* $\mathbf{a}_0$ *of) a proper analytic subset* Y′ *of* X *and, for all* $i = 1, \ldots, s$, *a positive integer* d_i, *such that :*

(1) $\mathrm{Y}' = \mathrm{X} \cap \boldsymbol{\theta}^{-1}(\boldsymbol{\theta}(\mathrm{Y}'))$;

(2) $\text{mult}_{a^i} \theta = d_i$ *for all* $\mathbf{a} = (a^1, \ldots, a^s) \in \mathrm{X} - \mathrm{Y}'$.

Proof. — Let $a \in \mathrm{M}$. By Remark 14.4 and Corollary 14.5, $\text{mult}_a \theta = \dim_{\mathbf{C}} \mathcal{O}_a / \mathfrak{m}_a^{d+1} - \dim_{\mathbf{C}} \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1}$. With respect to local coordinates $x = (x_1, \ldots, x_m)$ in M, the vector space $\mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1}$ is generated by the equivalence classes modulo $\mathfrak{m}_a^{d+1}$ of $(x-a)^\alpha \cdot (\theta_j(x) - \theta_j(a))$, where $j = 1, \ldots, m$ and $\alpha \in \mathbf{N}^m$, $|\alpha| \leqslant d$. Thus $\dim_{\mathbf{C}} \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1}$ is the rank of a matrix whose entries are analytic functions in a. (Its columns are the partial derivatives through order d of the $(x-a)^\alpha \cdot (\theta_j(x) - \theta_j(a))$ with respect to x, evaluated at $x = a$.) Therefore, $\text{mult}_a \theta$ is (analytic) Zariski (upper-) semicontinuous. The result follows since $\boldsymbol{\theta}$ is finite. □

Remark 14.7. — Let $\mathbf{a}_1 = (a_1^1, \ldots, a_1^s) \in \mathrm{M}_\varphi^s$. Suppose that $\{a_1^1, \ldots, a_1^s\}$ contains r distinct elements $c^1, \ldots, c^r$, where c^j is repeated μ^j times, $j = 1, \ldots, r$, and $\sum \mu^j = s$. Choose connected open neighborhoods U^j of c^j in M, $j = 1, \ldots, r$, and V of $\boldsymbol{\theta}(\mathbf{a}_1)$ in N′, such that the U^j are mutually disjoint and $\theta(\mathrm{U}^j) = \mathrm{V}$ for each j. Put $\mathrm{U} = \cup\, \mathrm{U}^j$.

Then :

(1) Since $\theta|U$ is finite, $\sum_{a \in U \cap \theta^{-1}(b)} \operatorname{mult}_a \theta$ is constant on V.

(2) If $\mathbf{a} = (a^1, \ldots, a^s)$ is sufficiently close to $\mathbf{a}_1$ in $\overset{s}{M_\varphi}$, then $\{a^1, \ldots, a^s\}$ contains μ^j elements of U^j, for each j.

COROLLARY 14.8. — *Let* Y′ *be as in Lemma* 14.6. *There exists* $r \leqslant s$ *and a surjection* σ *of* $\{1, \ldots, s\}$ *onto* $\{1, \ldots, r\}$ *satisfying the following conditions : Let* $M^r_\varphi \to M^s_\varphi$ *denote the embedding given by* $(a^1, \ldots, a^r) \to (a^{\sigma(1)}, \ldots, a^{\sigma(s)})$. *Then :*

(1) $X \subset M^r_\varphi$.

(2) *If* $\mathbf{a} = (a^1, \ldots, a^r) \in X - Y'$ *and* $i \neq j$, *then* $a^i \neq a^j$.

Proof. — It follows from Lemma 14.6 and Remark 14.7 that, for each i and j, $\{\mathbf{a} = (a^1, \ldots, a^s) \in X - Y' : a^i = a^j\}$ is open in $X - Y'$. Clearly, it is closed. Since $X - Y'$ is connected, the result follows. □

Let Y′ be as in Lemma 14.6. According to Corollary 14.8, we can assume, in our proof of Theorem 14.2, that if $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y'$ and $i \neq j$, then $a^i \neq a^j$.

For each $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y'$, put $\mathscr{F}_{\mathbf{a}} = \bigoplus_{i=1}^{s} \mathcal{O}_{a^i}$ and $E_{\mathbf{a}} = \bigoplus_{i=1}^{s} \mathcal{O}_{a^i}/\mathfrak{m}_{\theta(a^i)} \cdot \mathcal{O}_{a^i}$. Then $\mathscr{F}_{\mathbf{a}}$ is an $\mathcal{O}_{\theta(\mathbf{a})}$-module via the homomorphism $(\theta^*_{a^i})_{1 \leqslant i \leqslant s} : \mathcal{O}_{\theta(\mathbf{a})} \to \bigoplus_{i=1}^{s} \mathcal{O}_{a^i}$, and $E_{\mathbf{a}}$ is a vector space over $\mathbf{C}$. Clearly, $E_{\mathbf{a}}$ identifies with $\mathscr{F}_{\mathbf{a}}/\mathfrak{m}_{\theta(\mathbf{a})} \cdot \mathscr{F}_{\mathbf{a}}$.

Replacing M, if necessary, by a smaller neighborhood of $\{a_0^1, \ldots, a_0^s\}$, we can assume there exist $\eta_1, \ldots, \eta_\sigma \in \mathcal{O}(M)$ and $\mathbf{a}_1 \in X - Y'$ such that the η_j induce a basis of $E_{\mathbf{a}_1}$. (We can, for example, choose $\eta_1, \ldots, \eta_\sigma$ to be polynomial with respect to local coordinates in a neighborhood of each a_0^i.) By Lemma 14.6, $\dim_{\mathbf{C}} E_{\mathbf{a}} = \sum_{i=1}^{s} d_i$ is constant on $X - Y'$. Thus there is (a germ at $\mathbf{a}_0$ of) a proper analytic subset Y of X such that $Y' \subset Y$ and the η_j induce a basis of $E_{\mathbf{a}}$, for all $\mathbf{a} \in X - Y$. Since $\boldsymbol{\theta}$ is finite, we can assume that $Y = X \cap \boldsymbol{\theta}^{-1}(\boldsymbol{\theta}(Y))$.

LEMMA 14.9. — *For each* $\mathbf{a} \in X - Y$, $\eta_1, \ldots, \eta_\sigma$ *induce a free set of generators of the module* $\mathscr{F}_{\mathbf{a}}$ *over* $\mathcal{O}_{\theta(\mathbf{a})}$.

Proof. — Let $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y$. By Nakayama's lemma, $\eta_1, \ldots, \eta_\sigma$ induce a set of generators of $\mathscr{F}_{\mathbf{a}}$ over $\mathcal{O}_{\theta(\mathbf{a})}$. By Remark 14.4, $\sigma = \dim_{\mathbf{C}} E_{\mathbf{a}} = \sum_{i=1}^{s} \operatorname{mult}_{a^i} \theta = \sum_{i=1}^{s} \dim_{\mathbf{K}_{\theta(\mathbf{a})}} \mathcal{O}_{a^i} \otimes_{\mathcal{O}_{\theta(\mathbf{a})}} \mathbf{K}_{\theta(\mathbf{a})}$, where $\mathbf{K}_{\theta(\mathbf{a})}$ is the field of fractions of $\mathcal{O}_{\theta(\mathbf{a})}$. Thus $\sigma = \dim_{\mathbf{K}_{\theta(\mathbf{a})}} \mathscr{F}_{\mathbf{a}} \otimes_{\mathcal{O}_{\theta(\mathbf{a})}} \mathbf{K}_{\theta(\mathbf{a})}$, as required. □

COROLLARY 14.10. — *Put $\ell_1(k) = (d+1)(k+1) - 1$, where $k \in \mathbf{N}$. Let $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y$ and let $H_j \in \hat{\mathcal{O}}_{\theta(\mathbf{a})}$, $j = 1, \ldots, \sigma$. If $\sum_{j=1}^{\sigma} \hat{\theta}^*_{a^i}(H_j) \cdot \hat{\eta}_{j,a^i} \in \mathfrak{m}_{a^i}^{\ell_1(k)+1} \cdot \hat{\mathcal{O}}_{a^i}$, $i = 1, \ldots, s$, then each $H_j \in \mathfrak{m}_{\theta(\mathbf{a})}^{k+1} \cdot \hat{\mathcal{O}}_{\theta(\mathbf{a})}$.*

Proof. — If $a \in M$, then, by Corollary 14.5, $\mathfrak{m}_a^{d+1} \subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a$. Therefore, for all $\mathbf{a} = (a^1, \ldots, a^s) \in M^s_\varphi$, $\bigoplus_{i=1}^{s} \mathfrak{m}_{a^i}^{(d+1)k} \cdot \hat{\mathcal{O}}_{a^i} \subset \mathfrak{m}^k_{\theta(\mathbf{a})} \cdot \hat{\mathscr{F}}_{\mathbf{a}}$, where $\hat{\mathscr{F}}_{\mathbf{a}} = \bigoplus_{i=1}^{s} \hat{\mathcal{O}}_{a^i}$. The result follows from Lemma 14.9. □

LEMMA 14.11. — *Let $f \in \mathcal{O}(M)$. Then :*

(1) *If $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y$, there exist unique $h_{j,\mathbf{a}} \in \mathcal{O}_{\theta(\mathbf{a})}$, $j = 1, \ldots, \sigma$, such that, for each $i = 1, \ldots, s$, $\hat{f}_{a^i} = \sum_{j=1}^{\sigma} \theta^*_{a^i}(h_{j,\mathbf{a}}) \cdot \hat{\eta}_{j,a^i}$.*

(2) *For each $j = 1, \ldots, \sigma$ and $\beta \in \mathbf{N}^m$, let $h^\beta_j(\mathbf{a}) = D^\beta h_{j,\mathbf{a}}(\theta(\mathbf{a}))$, where $\mathbf{a} \in X - Y$. Then $h^\beta_j \in \mathscr{M}(X; Y)$.*

Proof. — (1) By Lemma 14.9.

(2) If $a \in M$, let $\Theta_a : \mathcal{O}^\sigma_{\theta(a)} \to \mathcal{O}_a$ denote the module homomorphism over θ^*_a defined by $\Theta_a(g) = \sum_{j=1}^{\sigma} \theta^*_a(g_j) \cdot \hat{\eta}_{j,a}$, where $g = (g_1, \ldots, g_\sigma) \in \mathcal{O}^\sigma_{\theta(a)}$. If $\mathbf{a} = (a^1, \ldots, a^s) \in M^s_\varphi \subset M^s_\theta$, let $\mathbf{\Theta}_{\mathbf{a}} : \mathcal{O}^\sigma_{\theta(\mathbf{a})} \to \bigoplus_{i=1}^{s} \mathcal{O}_{a^i}$ denote the composition of $\bigoplus_{i=1}^{s} \Theta_{a^i}$ with the diagonal injection $\mathcal{O}^\sigma_{\theta(\mathbf{a})} \to \bigoplus_{i=1}^{s} \mathcal{O}^\sigma_{\theta(\mathbf{a})}$.

Suppose that $\mathbf{a} \in X - Y$. Acording to (1), $(\hat{f}_{a^i})_{1 \leq i \leq s} = \mathbf{\Theta}_{\mathbf{a}}(h_{\mathbf{a}})$, where $h_{\mathbf{a}} = (h_{1,\mathbf{a}}, \ldots, h_{\sigma,\mathbf{a}})$. We use the formalism of 8.2 and 8.3, where $p = 1$, $q = \sigma$, $B = 0$, $\mathbf{\Phi}_{\mathbf{a}}$ is replaced by $\mathbf{\Theta}_{\mathbf{a}}$, etc. For each $\ell \in \mathbf{N}$, let ${}^\ell F_{\mathbf{a}}$ (respectively, ${}^\ell H_{\mathbf{a}}$) denote the image of $(\hat{f}_{a^i})_{1 \leq i \leq s}$ (respectively, of $h_{\mathbf{a}}$) by the lower (respectively, upper) horizontal arrow in the left-hand diagram of (8.2.6); thus, ${}^{\ell}F_{\mathbf{a}} = A_{\ell,\mathbf{a}} \cdot {}^{\ell}H_{\mathbf{a}}$. Recall that ${}^{\ell}H_{\mathbf{a}}$ is the element of $\bigoplus_{|\beta|\leq\ell}\mathcal{O}^{\sigma}_{X,\mathbf{a}}$ induced by $(D^{\beta}h_{\mathbf{a}}\circ\hat{\theta}_{\mathbf{a}})_{|\beta|\leq\ell}$. Write ${}^{\ell}H_{\mathbf{a}} = (H_{\beta,j,\mathbf{a}})_{|\beta|\leq\ell, 1\leq j\leq\sigma}$, where each $H_{\beta,j,\mathbf{a}} \in \mathcal{O}_{X,\mathbf{a}}$.

Let $k \in N$ and let $\ell = \ell_1(k)$. Then

$$\mathrm{Ad}^{\rho_{\ell,k}(X)}D_{\ell,k,\mathbf{a}} \cdot {}^{\ell}F_{\mathbf{a}} = C_{\ell,k,\mathbf{a}} \cdot {}^{k}H_{\mathbf{a}}.$$

Let $e(k)$ denote the number of pairs $(\beta,j) \in N^m \times \{1,\ldots,\sigma\}$ such that $|\beta| \leq k$ ($e(k)$ is the number of columns of $C_{\ell,k,\mathbf{a}}$). By Corollary 14.10 and Lemma 8.1.1 (2), rank $C^{X}_{\ell,k}(\mathbf{a}) = e(k)$. Then, by Cramer's rule, for all $(\beta,j) \in N^m \times \{1,\ldots,\sigma\}$, $|\beta| \leq k$, we obtain $\zeta_{\beta,j}$, $\omega_{\beta,j} \in \mathcal{O}(U)$ (U is a product coordinate neighborhood of $\mathbf{a}_0$ in M^s) such that, if $\mathbf{a} \in X - Y$, then $\omega_{\beta,j}(\mathbf{a}) \neq 0$ and $H_{\beta,j,\mathbf{a}} = \zeta_{\beta,j,\mathbf{a}}/\hat{\omega}_{\beta,j,\mathbf{a}}$, as required. □

We can now complete the proof of Theorem 14.2. Since the projection of Z onto N′ is finite, then, by the finite coherence theorem of Grauert and Remmert [32, Ch. IV, Thm. 7], we can assume there exist $\xi_1,\ldots,\xi_\rho \in \mathcal{O}(N)$ satisfying the following condition: For all $b \in Z$ and $G \in \hat{\mathcal{O}}_b$, there exist $G_1,\ldots,G_\rho \in \hat{\mathcal{O}}_{\pi(b)}$ such that $G - \sum_{h=1}^{\rho} \hat{\pi}^*_b(G_h)\cdot\hat{\xi}_{h,b} \in \mathcal{I}_{Z,b}\cdot\hat{\mathcal{O}}_b$, where $\mathcal{I}_Z$ denotes the sheaf of germs of analytic functions which vanish on Z.

Let $\mathbf{a} \in X - Y$, $\mathbf{a} = (a^1,\ldots,a^s)$. By Lemma 14.11 (1), there exist unique $p \times q$ matrices $C_{hj,\mathbf{a}}$, $h = 1,\ldots,\rho$, $j = 1,\ldots,\sigma$, and unique $p \times r$ matrices $D_{\ell j,\mathbf{a}}$, ℓ, $j = 1,\ldots,\sigma$, all with entries in $\mathcal{O}_{\theta(\mathbf{a})}$, such that, for all $i = 1,\ldots,s$,

$$(\hat{\xi}_{h,\varphi(a^i)}\circ\hat{\varphi}_{a^i})\cdot A_{a^i} = \sum_{j=1}^{\sigma} \hat{\eta}_{j,a^i}\cdot(C_{hj,\mathbf{a}}\circ\hat{\theta}_{a^i}),$$

$$\hat{\eta}_{\ell,a^i}\cdot B_{a^i} = \sum_{j=1}^{\sigma} \hat{\eta}_{j,a^i}\cdot(D_{\ell j,\mathbf{a}}\circ\hat{\theta}_{a^i}).$$

By Lemmas 14.11 (2) and 7.2 (3) and Remark 7.6, there exists $\lambda \in N$ satisfying the following condition: Let $\mathbf{a} \in X - Y$. Suppose that $G_h \in \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$, $h = 1,\ldots,\rho$, $H_\ell \in \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$, $\ell = 1,\ldots,\sigma$, and $\sum_{h=1}^{\rho} C_{hj,\mathbf{a}}\cdot G_h + \sum_{\ell=1}^{\sigma} D_{\ell j,\mathbf{a}}\cdot H_\ell \in \mathfrak{m}^{k+\lambda}_{\theta(\mathbf{a})}\cdot\hat{\mathcal{O}}^p_{\theta(\mathbf{a})}$, $j = 1,\ldots,\sigma$. Then there exist $G'_h \in \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$ and

$H'_\ell \in \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$ such that $\sum_h C_{hj,\mathbf{a}} \cdot G'_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot H'_\ell = 0$, $j = 1, \ldots, \sigma$, and $G_h - G'_h \in \mathfrak{m}^k_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$, $H_\ell - H'_\ell \in \mathfrak{m}^k_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$.

Let $\ell(k) = \ell_1(k+\lambda)$, $k \in \mathbf{N}$. We claim that $\ell(k,\mathbf{a}) \leqslant \ell(k)$ for all $\mathbf{a} \in X - Y$ and $k \in \mathbf{N}$: Let $\mathbf{a} \in X - Y$ and let $G \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$. Suppose that $A_{a^i} \cdot (G \circ \hat{\varphi}_{a^i}) + B_{a^i} \cdot H^i \in \mathfrak{m}^{\ell(k)+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}$, where $H^i \in \hat{\mathcal{O}}^r_{a^i}$, $i = 1, \ldots, s$. There exist $G_1, \ldots, G_\rho \in \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$ such that $G - \sum_h \xi_{h,\varphi(\mathbf{a})} \cdot (G_h \circ \hat{\pi}_{\varphi(\mathbf{a})}) \in \mathscr{I}_{Z,\varphi(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$. Also, there exist unique $H_1, \ldots, H_\sigma \in \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$ such that $H^i = \sum_\ell \hat{\eta}_{\ell,a^i} \cdot (H_\ell \circ \hat{\theta}_{a^i})$, $i = 1, \ldots s$. Thus, for each $i = 1, \ldots, s$,

$$A_{a^i} \cdot (G \circ \hat{\varphi}_{a^i}) + B_{a^i} \cdot H^i = \sum_{j=1}^{\sigma} \hat{\eta}_{j,a^i} \cdot \left(\left(\sum_{h=1}^{\rho} C_{hj,\mathbf{a}} \cdot G_h + \sum_{\ell=1}^{\sigma} D_{\ell j,\mathbf{a}} \cdot H_\ell \right) \circ \hat{\theta}_{a^i} \right).$$

By Corollary 14.10, $\sum_h C_{hj,\mathbf{a}} \cdot G_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot H_\ell \in \mathfrak{m}^{k+\lambda+1}_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^p_{\theta(\mathbf{a})}$, $j = 1, \ldots, \sigma$. Thus there exist $G'_1, \ldots, G'_\rho \in \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$ and $H'_1, \ldots, H'_\sigma \in \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$ such that $\sum_h C_{hj,\mathbf{a}} \cdot G'_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot H'_\ell = 0$, $j = 1, \ldots, \sigma$, and each $G_h - G'_h \in \mathfrak{m}^{k+1}_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$. Put $G' = \sum_{h=1}^{\rho} \xi_{h,\varphi(\mathbf{a})} \cdot (G'_h \circ \hat{\pi}_{\varphi(\mathbf{a})})$. Then $A_{a^i} \cdot (G' \circ \hat{\varphi}_{a^i}) \in \operatorname{Im} \hat{B}_{a^i}$, $i = 1, \ldots, s$, and $G - G' \in \mathfrak{m}^{k+1}_{\varphi(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$, as claimed. This completes the proof of Theorem 14.2. □

Remark 14.12. — (1) Let $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y$. Let $G \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$ and let $H \in \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^r_{a^i}$, $H = (H^1, \ldots, H^s)$. Let $f = \hat{\mathbf{\Phi}}_{\mathbf{a}}(G) + \hat{\mathbf{B}}_{\mathbf{a}}(H) \in \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^p_{a^i}$; i.e., $f = (f^1, \ldots, f^s)$, where each $f^i = A_{a^i} \cdot (G \circ \hat{\varphi}_{a^i}) + B_{a^i} \cdot H^i$. Suppose that $f^i \in \mathcal{O}^p_{a^i}$, $i = 1, \ldots, s$. Then, for all $k \in \mathbf{N}$, there exists $g \in \mathcal{O}^q_{\varphi(\mathbf{a})}$ and $h \in \bigoplus_{i=1}^{s} \mathcal{O}^r_{a^i}$ such that $f = \mathbf{\Phi}_{\mathbf{a}}(g) + \mathbf{B}_{\mathbf{a}}(h)$, $g - G \in \mathfrak{m}^k_{\varphi(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$, and $h - H \in \bigoplus_{i=1}^{s} \mathfrak{m}^k_{a^i} \cdot \hat{\mathcal{O}}^r_{a^i}$: We use the notation introduced above. Let $G_1, \ldots, G_\rho \in \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$ such that $G - \sum_h \xi_{h,\varphi(\mathbf{a})} \cdot (G_h \circ \hat{\pi}_{\varphi(\mathbf{a})}) \in \mathscr{I}_{Z,\varphi(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$, and let $H_1, \ldots, H_\sigma \in \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$ such that $H^i = \sum_\ell \hat{\eta}_{\ell,a^i} \cdot (H_\ell \circ \hat{\theta}_{a^i})$, $i = 1, \ldots, s$. By

Lemma 14.9, $\sum_h C_{hj,\mathbf{a}} \cdot G_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot H_\ell \in \mathcal{O}^p_{\theta(\mathbf{a})}$, $j = 1, \ldots, \sigma$. By Krull's theorem, there exist $g_1, \ldots, g_\rho \in \mathcal{O}^q_{\theta(\mathbf{a})}$ and $h_1, \ldots, h_\sigma \in \mathcal{O}^r_{\theta(\mathbf{a})}$ such that

$$\sum_h C_{hj,\mathbf{a}} \cdot g_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot h_\ell = \sum_h C_{hj,\mathbf{a}} \cdot G_h + \sum_\ell D_{\ell j,\mathbf{a}} \cdot H_\ell,$$

$j = 1, \ldots, \sigma$, and each $g_h - G_h \in \mathfrak{m}^k_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^q_{\theta(\mathbf{a})}$, $h_\ell - H_\ell \in \mathfrak{m}^k_{\theta(\mathbf{a})} \cdot \hat{\mathcal{O}}^r_{\theta(\mathbf{a})}$. Put $g = \sum_h \hat{\xi}_{h,\varphi(\mathbf{a})} \cdot (g_h \circ \hat{\pi}_{\varphi(\mathbf{a})})$, $h^i = \sum_\ell \hat{\eta}_{\ell,a^i} \cdot (h_\ell \circ \hat{\theta}_{a^i})$, $i = 1, \ldots, s$, and $h = (h^1, \ldots, h^s)$.

(2) Let $\mathbf{a} = (a^1, \ldots, a^s) \in X - Y$. Then $\mathcal{R}_\mathbf{a} = \{G \in \hat{\mathcal{O}}^q_{\varphi(\mathbf{a})} : \hat{\Phi}_\mathbf{a}(G) \in \operatorname{Im} \hat{\mathbf{B}}_\mathbf{a}\}$ is generated by $\mathcal{R}_\mathbf{a} \cap \mathcal{O}^q_{\varphi(\mathbf{a})}$ (cf. Corollary 12.17).

Remark 14.13. — Let X be an analytic space over **K**. It follows from theorems of Buchsbaum and Eisenbud [9, Thms. 1.2, 2.1] and [37, I.5.1] that $\{x \in X : \mathcal{O}_{X,x}$ is Cohen-Macauley$\}$ is open in X. (We are grateful to David Eisenbud for the reference.) We say that X is *Cohen-Macauley* if, for all $x \in X$, $\mathcal{O}_{X,x}$ is a Cohen-Macauley ring. Thus, a Cohen-Macauley real analytic space admits a Cohen-Macauley complexification.

Our proof of Theorem 14.2 extends to the case that M is a Cohen-Macauley analytic space with essentially no change : We can assume that $\mathbf{K} = \mathbf{C}$. The equalities of Remark 14.4 remain valid. In Lemma 14.11, we can assume that M is embedded in an open subspace W of $\mathbf{C}^m$, and that $\mathcal{O}_M = \mathcal{O}_W / L \cdot \mathcal{O}^r_W$, where L is a $1 \times r$ matrix with entries in $\mathcal{O}(W)$; the same proof goes through using the formalism of 8.2, 8.3 with $B = L$ rather than $B = 0$.

N.B. : Bibliography published in the first issue of volume **37** (1987).

Manuscrit reçu le 16 décembre 1985.

E. Bierstone and P. D. Milman,
University of Toronto
Dept. of Mathematics
Toronto, Canada M5S 1A1.