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RATIONAL FIBRATIONS, HOMOGENEOUS SPACES WITH POSITIVE EULER CHARACTERISTICS AND JACOBIANS

by H. SHIGA and M. TEZUKA

1. Introduction.

Since D. Sullivan described his differential graded algebra model for the classifying space for fibrations with a given fiber, it has become possible to study rational homotopy theory for fibrations by algebraic methods. The purpose of the present paper is to study fibrations whose fiber has the homotopy type of G/U, where G is a connected compact Lie group and U is its closed subgroup of maximal rank. Then our main result is :

THEOREM A. – *Let* G *be a connected compact Lie group and* U *be a closed subgroup such that* rank $U = \operatorname{rank} G$. *Then every orientable fibration*

$$G/U \xrightarrow{i} E \longrightarrow B$$

with a fiber G/U *is totally non homologous to zero, that is, the induced map*

$$i^* : H^*(E, \mathbf{Q}) \longrightarrow H^*(G/U, \mathbf{Q})$$

is surjective.

Due to the theories of the Sullivan classifying spaces and of the Serre spectral sequences of fibrations, Theorem A is equivalent to the following algebraic result ([11], [13], [15]) :

THEOREM A′. – *Let* G *and* U *be as in Theorem* A. *Then we have*

Key-words : (Rational) Fibration – Homogeneous space – Invariant polynomial (under Weyl group) – Derivation – Jacobian.

$$D_i(H^*(G/U, \mathbf{C}) = 0, \qquad i > 0.$$

Here $D_i(A)$ *denotes the* $\mathbf{Q}$*-vector space of* $\mathbf{Q}$*-derivations of a graded commutative algebra* A *over* $\mathbf{Q}$ *which decreases the degree by* i.

An outline of the proof of Theorem A′ goes as follows. Let T be a common maximal torus of G and U. We fix generators of the cohomology ring of the classifying space BT and so

$$H^*(BT, \mathbf{C}) = \mathbf{C}[X_1, \ldots, X_n].$$

The cohomology ring of BG can be written as

$$H^*(BG, \mathbf{C}) = \mathbf{C}[I_1, \ldots, I_n]$$

where $I_1, \ldots, I_n$ are polynomials of $X_1, \ldots, X_n$ which are invariant under the action of the Weyl group. In our proof, the Jacobian $\det(\partial I_i/\partial X_j)$ plays the key role. Using the works of H. Coxeter and B. Kostant et al. in 1950's, we can prove that the Jacobian is not contained in any prime ideal of the ideal generated by $I_1, \ldots, I_{n-1}$ where we choose $I_1, \ldots, I_n$ so that $\deg I_1 \leqslant \ldots \leqslant \deg I_n$. From this and the fact that $H^*(BT, \mathbf{C})$ is faithfully flat over $H^*(BG, \mathbf{C})$, we can deduce Theorem A′. In the case G/U is a complex flag manifold, the Jacobian becomes the Vandermond's determinant and Theorem A′ can be proved more elementally. Partial results in this case were obtained by W. Meier [9].

As examples of G/U in Theorem A we have complex flag manifolds

$$W(n_1, \ldots, n_k) = U(n)/U(n_1) \times \ldots \times U(n_k) \quad (n = n_1 + \ldots + n_k)$$

and $So(2n)/U(n)$. A list of such homogeneous spaces are given in [2] when G is a simple Lie group.

Theorem A is a special case of Halperin's conjecture.

Related to the proof of Theorem A we show the following result:

THEOREM B. – *Let* K *be a field with characteristic zero and* $P_1, \ldots, P_n$ *be a homogeneous (a quasi homogeneous)* $\mathfrak{M}$ *-regular sequence of elements of the polynomial ring* $K[X_1, \ldots, X_n]$, *where* $\mathfrak{M}$ *is the maximal ideal* $(X_1, \ldots, X_n)$. *Then the Jacobian*

$$\det(\partial P_i/\partial X_j)$$

does not lie in the ideal $(P_1, \ldots, P_n)$.

If $K[X_1, \ldots, X_n]$ is positively graded and $(P_1, \ldots, P_n)$ is a homogeneous ideal of decomposable (i.e. having no linear term) elements, Halperin [6] shows that the quotient ring

$$K[X_1, \ldots, X_n]/(P_1, \ldots, P_n)$$

satisfies the Poincaré duality. Theorem B implies that the Jacobian $\det(\partial P_i/\partial X_j)$ is a fundamental class. Theorem B is obtained by constructing a chain map between the Koszul complex for $(P_1, \ldots, P_n)$ and that for $(X_1, \ldots, X_n)$.

We also study the case of mod p coefficients. We show that Theorem A is valid for mod p coefficient if the prime p does not divide the order of the Weyl group of G.

After we wrote the first version of the manuscript, we obtained following related results:

1) The ideal of $H^*(BT, \mathbf{C})$ generated by $I_1, \ldots, I_{n-1}$ is an intersection of prime ideals on which $W(G)$ acts transitively ([16]).

2) If G is a simple group of type A_n, then the mod p version of Theorem A is true if and only if the prime p does not divide the order of $W(G)$ ([17]).

3) Let G be a simple group of other type than D_n. Then we can describe the graded algebra automorphism group of $H^*(G/U, \mathbf{C})$ in terms of the Weyl group of G and U ([18]).

This paper is organized as follows. In § 2 we review Sullivan's theory of classifying spaces for later use, in § 3 we study the property of the invariant polynomials and their Jacobians, in § 4 the proof of Theorem A is given, in § 5 the proof of Theorem B is given, and in § 6 the case of mod p coefficients is treated.

We would like to take this opportunity to thank Professors S. Halperin and J.C. Thomas for valuable communications and also to Professors A. Kono, K. Shibata and N. Yagita for careful reading of the manuscript and giving many advices.

2. Sullivan's classifying space.

Let $$F \longrightarrow E \longrightarrow B$$

be an orientable fibration over a connected space B such that the fiber F is connected and simply connected. Let $m(\mathrm{F})$ be a minimal model of F and A(B) be a differential graded commutative algebra (abbreviated D.G.A.) representing B. Then there is a corresponding D.G.A. model

$$(m^*(\mathrm{F}), d_{\mathrm{F}}) \longrightarrow (m^*(\mathrm{F}) \otimes \mathrm{A}^*(\mathrm{B}), d) \longrightarrow (\mathrm{A}^*(\mathrm{B}), d_{\mathrm{B}}).$$

The differential of the total space has the following form :

$$d(1 \otimes b) = 1 \otimes d_{\mathrm{B}}(b), \qquad b \in \mathrm{A}^*(\mathrm{B}),$$

$$d(x \otimes 1) = d_{\mathrm{F}}\, x \otimes 1 + \sum_{i \geqslant 0} (-1)^{(i+1)|x|} \sum_{\nu \geqslant 1} \phi^{(\nu)}(x) \otimes b^{(\nu)}_{i+1}, \quad (2.1)$$

$$x \in m^*(\mathrm{F}),$$

where $\phi^{(\nu)}$ is a derivation of $m^*(\mathrm{F})$ decreasing degree by i, that is,

$$\phi^{(\nu)}_i(xy) = \phi^{(\nu)}_i(x)y + (-1)^{i|x|} x\phi^{(\nu)}_i(y)$$

and $\{b_{i+1} ; \nu = 1, 2, \ldots\}$ is a basis of $\mathrm{A}^{i+1}(\mathrm{B})$.

Let $\mathrm{D}_i(\mathrm{F})$ be a set of $\mathbf{Q}$-derivations of $m(\mathrm{F})$ decreasing degree by i for $i > 0$, $\mathrm{D}_0(\mathrm{F}) = \{d_{\mathrm{F}}\phi - \phi d_{\mathrm{F}} \mid \phi \in \mathrm{D}_1(\mathrm{F})\}$. Then $\mathrm{D}_*(\mathrm{F}) = \bigoplus_{i \geqslant 0} \mathrm{D}_i(\mathrm{F})$ forms a graded differential Lie algebra by the rule

1) $[\phi_1, \phi_2] = \phi_1\phi_2 - (-1)^{ij}\phi_2\phi_1$, where $\phi_1 \in \mathrm{D}_i(\mathrm{F})$ and $\phi_2 \in \mathrm{D}_j(\mathrm{F})$,

2) $\delta\phi = d_{\mathrm{F}}\phi - (-1)^{|\phi|}\phi d_{\mathrm{F}}$.

Let $\Lambda(\# s\mathrm{D}_*(\mathrm{F}))$ be the Koszul cochain complex of D.G.L. $\mathrm{D}_*(\mathrm{F})$, that is, the free D.G.A. on the dual vector space $\mathrm{D}_*(\mathrm{F})$ with the degree being shifted up by one, with the differential defined as the dual of δ + the dual of the bracket. From the d-formula of the fibration (2.1) we have a family of linear maps

$$\# \mathrm{A}^{i+1}(\mathrm{B}) \longrightarrow \mathrm{D}_i(\mathrm{F}) \qquad (i = 0, 1, \ldots)$$

$$\# b \longmapsto \sum_{i \geqslant 0} (-1)^{i+1} \sum_{\nu \geqslant 1} \phi^{(\nu)}_i \langle b_{i+1}, \# b\rangle.$$

Sullivan and Schlessinger-Stasheff [11] showed that the set of homotopy class of D.G.A. maps $[\Lambda(\# s\, D_* (F)), A^* (B)]$ corresponds bijectively to the set of equivalence class of rational F-fibrations over B via the above procedure. So we may regard $\Lambda(\# s D_* (F))$ as a D.G.A. representing the rational homotopy type of $BAut_S F$, the classifying space for orientable fibrations with fiber F, and hence it implies

$$H_i (D_* (F), \delta) = \pi_{i+1} (BAut_S F).$$

Let F be a simply connected space such that $H_*(F, \mathbf{Q})$ and $\pi_* (F) \otimes \mathbf{Q}$ are finite dimensional vector spaces and $\chi_\pi (F) = 0$ where

$$\chi_\pi (F) = \sum_{i \geq 2} (-1)^i \dim \pi_i (F) \otimes \mathbf{Q}.$$

Then Halperin [6] showed that the minimal model of F can be written as

$$m(F) = (X_1, \ldots, X_n, \theta_1, \ldots, \theta_n)$$

with $|X_i|$ = even in the degree and

$$dX_i = 0, \; d\theta_i = P_i \; (i = 1, \ldots, n),$$

where P_i are polynomials of $X_i's$ and the sequence $(P_1, \ldots, P_n)$ is a regular sequence. Immediately the cohomology ring $H^* (F)$ is isomorphic to $\mathbf{Q}[X_1, \ldots, X_n]/(P_1, \ldots, P_n)$.

LEMMA 2.1 ([15]). – *Let* F *be as above. Then we have* $H_{even} (D_* (F), \delta) = D_{even} (H^* (F))$.

Proof. – Since every derivation of $m(F)$ can be uniquely determined by its values on the generators, a derivation of degree $2k$ can be written as

$$\phi = \sum_{i=1}^{n} (X_i, R_i) + \sum_{i=1}^{n} (\theta_i, Q_i),$$

where R_i and Q_i are elements of $m^* (F)$ and (a, b) denotes the derivation taking a to b and annihilating the other generators. The boundary operator is given by

$$\delta\,(X_i, R_i) = (X_i, dR_i) + (-1)^{|\phi|} \sum_{j=1}^{n} (\theta_j, \partial P_j/\partial X_i \cdot R_i)$$

$$\delta\,(\theta_i, Q_i) = (\theta_i, dQ_i).$$

Hence ϕ is closed if and only if

$$dR_i = 0, \qquad (i = 1, \ldots, n),$$

$$\left(\sum_{i=1}^{n} \partial P_j/\partial X_i \cdot R_i\right) + dQ_j = 0, \qquad (i = 1, \ldots, n). \tag{1}$$

Suppose R_i and $\overline{R}_i$ are cohomologous and let

$$\overline{\phi} = \Sigma\,(X_i, \overline{R}_i) + \Sigma\,(\theta_i, \overline{Q}_i)$$

be closed. Then we prove that ϕ and $\overline{\phi}$ are homologous. We can choose elements S such that $dS_i = R_i - \overline{R}_i\,(i = 1, 2, \ldots)$. Set $T_i = Q_i - \overline{Q}_i$. Then we have

$$\phi - \overline{\phi} = \Sigma\,(X_i, dS_i) + \Sigma\,(\theta_i, T_i)$$

and

$$\delta\,(X_i, S_i) = (X_i, dS_i) = \sum_k (\theta_k, \partial P_k/\partial X_i \cdot S_i).$$

Hence we have

$$\phi - \overline{\phi} = \delta\left(\sum (X_i, S_i)\right) - \sum_i \sum_k (\theta_k, \partial P_k/\partial X_i \cdot S_i) + \sum_k (\theta_k, T_k)$$

$$= \delta\left(\sum (X_i, S_i)\right) - \sum_k \left(\theta_k, \sum_i \partial P_k/\partial X_i \cdot S_i\right) + \sum_k (\theta_k, T_k). \tag{2}$$

From (1) we see

$$d\left(-\sum_i \partial P_k/\partial X_i \cdot S_i + T_k\right) = -\sum_i \partial P_k/\partial X_i \cdot dS_i + T_k = 0.$$

Hence $-\sum_i \partial P_k/\partial X_i \cdot S_i + T_k$ is a closed element of the ideal $(\theta_1, \ldots, \theta_n)$. Since the cohomology ring $H^*(F)$ is generated by even degree generators, any closed element of odd degree is exact. Hence there are $G_k \in m^*(F)$ $(k = 1, \ldots, n)$ such that

$$dG_k = -\sum_i \partial P_k/\partial X_i \cdot S_i + T_k .$$

By (1) and (2) we get

$$\phi - \bar{\phi} = \delta\,(\Sigma\,(X_i, S_i) - \Sigma\,(\theta_i, G_i)).$$

Thus ϕ and $\bar{\phi}$ are homologous. For closed elements R_i ($i = 1, \ldots, n$), we can take R_i from $\mathbf{Q}[X_1, \ldots, X_n]$ so that $\bar{R}_i$ and R_i are cohomologous. Then by (1)

$$\sum_i (\partial P_j/\partial X_i) \cdot \bar{R}_i \in (P_1, \ldots, P_n) \subset \mathbf{Q}[X_1, \ldots, X_n].$$

Therefore we have the induced derivation $\sum_i (X_i, \bar{R}_i)$ on $H^*(F)$.

For $D \in D_{\text{even}}(H^*(F))$, take R_i to be a representative in $\mathbf{Q}[X_1, \ldots, X_n]$ of

$$D(X_i) \in H^*(F) = \mathbf{Q}[X_1, \ldots, X_n]/(P_1, \ldots, P_n).$$

Since $\sum_{i=1}^{n} \left(\frac{\partial P_j}{\partial X_i}\right) R_i = D(P_j) \in (P_1, \ldots, P_n) \subset \mathbf{Q}[X_1, \ldots, X_n]$, we can take $Q_i \in m(F)$ that satisfies condition (1).

q.e.d.

3. Weyl groups and regular elements.

Let $\mathfrak{G}$ be a finite dimensional simple Lie algebra over $\mathbf{C}$ and $\mathfrak{J}$ be its Cartan subalgebra. We denote by Δ the set of roots with respect to $\mathfrak{J}$. For each root $\alpha \in \Delta$, we define a reflection of the dual space $\mathfrak{J}^*$

$$R_\alpha(X) = X - 2\,(\alpha, X)/(\alpha, \alpha)\,\alpha$$

where (,) is an inner product induced from the Cartan-Killing form restricted to $\mathfrak{J}^*$. The group $W(\mathfrak{J})$ generated by $\{R_\alpha\}_{\alpha \in \Delta}$ is called the Weyl group of $\mathfrak{J}$. Let $\{\alpha_i\}_{1 \leqslant i \leqslant n}$ be the set of the simple roots. Then the transformation of $\mathfrak{J}$ defined by

$$R = R_{\alpha_1} \cdots R_{\alpha_n}$$

is called a Coxeter-Killing transformation. We call an element z of $\mathfrak{J}$ regular if $\alpha(z) \neq 0$ for each $\alpha \in \Delta$. We quote some results due to Coxeter *et al.*

PROPOSITION 3.1. – *Let h be the order of a Coxeter-Killing transformation. Then there exists an eigen vector z with eigen value $e^{2\pi i/h}$ such that z is a regular element of $\mathfrak{J}$.*

For the proof we see [3] and [12].

Let G be an adjoint group of $\mathfrak{G}$. Then the Poincaré polynomial $P_G(t)$ has the form

$$P_G(t) = (1 + t^{2m_1+1})(1 + t^{2m_2+1}) \dots (1 + t^{2m_n+1}).$$

The integers m_i are called the exponents of $W(\mathfrak{G})$ and n is the rank of G. We arrange the exponents so that $m_1 \leqslant \dots \leqslant m_n$.

PROPOSITION 3.2. – $m_i + m_{n-i+1} = h$, $m_1 = 1$.

For the proof, see [3] and [12].

For a complex vector space V we denote by $S(V)$ the symmetric algebra generated by V. If we fix a basis of $\mathfrak{J}^*$, we have an isomorphism $S(\mathfrak{J}^*) = \mathbf{C}[X_1, \dots, X_n]$.

PROPOSITION 3.3 (Borel [1], Chevalley [4]). –

$$H^*(BG, \mathbf{C}) = S(\mathfrak{J}^*)^W = \mathbf{C}[I_1, \dots, I_n],$$

where I_k are homogeneous polynomial in the variables $X_1, \dots, X_n$ and $I_1, \dots, I_n$ are algebraically independent over $\mathbf{C}$. Moreover the sequence is a regular sequence in $S(\mathfrak{J}^*)$ and $\deg I_k = m_k + 1$.

Let $V(I_1, \dots, I_k)$ be an algebraic set in $\mathbf{C}^n$ defined by the equations $I_1 = \dots = I_k = 0$.

LEMMA 3.4. – *The irreducible decomposition of $V(I_1, \dots, I_{n-1})$ is given by*

$$V(I_1, \dots, I_{n-1}) = \cup L_i$$

where L_i is a line through the origin.

Proof. – Since $I_1, \ldots, I_{n-1}$ is a regular sequence, each irreducible component has dimension one. If $(x_1, \ldots, x_n) \in \mathbf{C}^n$ is a point on the algebraic set, the point $(\lambda x_1, \ldots, \lambda x_n)$ $(\lambda \in \mathbf{C})$ is also a point on it. Hence each irreducible component is a line through the origin.

q.e.d.

LEMMA 3.5. – $V(I_1, \ldots, I_n) = 0$.

Proof. – Since $I_1, \ldots, I_n$ is a regular sequence, we have $\dim V(I_1, \ldots, I_n) = 0$. If there is a non zero point x on $V(I_1, \ldots, I_n)$, then a line $\lambda x \in \mathbf{C}^n$ is also contained in it. This contradicts to the fact $V(I_1, \ldots, I_n) = 0$.

q.e.d.

The next proposition is essentially due to B. Kostant [7].

PROPOSITION 3.6. – *Let* $\mathfrak{G}$ *be a finite simple Lie algebra over* $\mathbf{C}$. *Then each non zero point on* $V(I_1, \ldots, I_{n-1})$ *is a regular element of* $\mathfrak{F}$.

Proof. – Since $\mathfrak{G}$ is a simple Lie algebra over $\mathbf{C}$, from Proposition 3.2 we have

$$m_1 = 1 \leqslant \ldots \leqslant m_{n-1} \leqslant m_n < h.$$

According to Proposition 3.1, we take an eigen vector z of a Coxeter-Killing transformation R with eigen value $e^{2\pi i/h}$. Then we have

$$I_k(z) = (R^{-1} I_k)(z) = I_k(Rz) = I_k(e^{2\pi i/h} z) = e^{2\pi i(m_k + 1)/h} I_k(z).$$

Since $e^{2\pi i(m_k + 1)/h} \neq 1$ for $1 \leqslant k \leqslant n-1$ by Proposition 3.2, z is a point on $V(V_1, \ldots, I_{n-1})$. Let x be any non zero point on $V(I_1, \ldots, I_{n-1})$. Then we can take a constant $c \in \mathbf{C}$ that satisfies $I_n(z) = I_n(cx)$ from Lemma 3.5. Now we will prove that z and $v = cx$ are conjugate under the action of $W(\mathfrak{G})$.

Let $A = \{z^{(1)} = z, z^{(2)}, \ldots, z^{(s)}\}$ be the set of all elements of which are conjugate to z, and $B = \{v^{(1)} = v, \ldots, v^{(t)}\}$ be a set of those which are conjugate to v. We assume $A \cap B = \emptyset$, and will deduce a contradiction. We can obviously choose constants a_i^j $(1 \leqslant i \leqslant n, 2 \leqslant j \leqslant s)$ to define

$$F_j(X) = \sum_{i=1} a_i^j (X_i - z_i^{(j)}) \ (2 \leqslant j \leqslant s)$$

such that $F_j(z) \neq 0$, where $z_i^{(j)}$ denote the $i-th$ component of $z^{(j)}$ with respect to the basis $X_1, \ldots, X_n$. Similary we can choose b_i^k $(1 \leqslant i \leqslant n, 1 \leqslant k \leqslant t)$ to define

$$G_k(X) = \sum_{i=1}^{n} b_i^k (X_i - v_i^{(k)}) \ (1 \leqslant k \leqslant t)$$

such that $G_k(z) \neq 0$. We set

$$F(X) = \prod_{2 \leqslant j \leqslant s,\ 1 \leqslant k \leqslant t} F_j(X) G_k(X)$$

and consider the average of F

$$\widetilde{F} = (1/|W|) \sum_{w \in W} wF.$$

Then we have $\widetilde{F}(z) \neq 0$ and $\widetilde{F}(v) = 0$. But $\widetilde{F}$ is an invariant polynomial, and so is a polynomial of $I_1, \ldots, I_n$. Hence we get $F(z) = F(v)$. This is a contradiction. Therefore $z = w(v)$ for some $w \in W$. Then, for each $\alpha \in \Delta$, we have

$$\alpha(x) = c^{-1} \alpha(v) = c^{-1} \alpha(w^{-1} z) = c^{-1} (w\alpha)(z) \neq 0$$

since z is a regular element.

q.e.d.

PROPOSITION 3.7. – *Let* $I_1, \ldots, I_n$ *be a basis system of the invariant algebra* $S(\mathfrak{J}^*)^{W(\mathfrak{G})}$. *Then*

$$\det(\partial I_j / \partial X_i) = c \prod_{\alpha \in \Delta^+} \alpha$$

where Δ^+ *denotes the set of positive roots and* c *is a non zero complex number.*

We refer to [3] for the proof. From Proposition 3.6 and 3.7, we have

PROPOSITION 3.8. – *The polynomial function* $\det(\partial I_j/\partial X_l)$ *is not identically zero on any irreducible component of* $V(I_1, \ldots, I_{n-1})$.

Now we show that Proposition 3.8 is also valid when G is not simple.

Let G be a compact connected one connected Lie group. Then G has a decomposition into the product of simple groups $G_1 \times \ldots \times G_k$ and a maximal torus T of G into the product $T_1 \times \ldots \times T_k$, where $T_i = G_i \cap T$. Using Künneth formula, we have

$$H^*(BG, \mathbf{C}) = \bigotimes_{i=1}^{k} H^*(BG_i, \mathbf{C})$$

and we have

$$H^*(BG_i, \mathbf{C}) = H^*(BT_i, \mathbf{C})^{W(G_i)}.$$

We fix notations as follows: $H^*(BT_i, \mathbf{C}) = \mathbf{C}[X_1^i, \ldots, X_{n_i}^i]$, and $H^*(BG_i, \mathbf{C}) = \mathbf{C}[I_1^i, \ldots, I_{n_i}^i]$, $\deg I_1^i \leqslant \ldots \leqslant \deg I_{n_i}^i$. $I(i)$ denotes the ideal of $H^*(BT, \mathbf{C})$ generated by all the I_j^k except $I_{n_i}^i$. Let $\overline{f_i}$ and f_i be the Jacobian, $\det(\partial I_m^i/\partial X_j^i)$ $1 \leqslant m, j \leqslant n_i$, considered as the element of $H(BT_i, \mathbf{C})/(I_1^i, \ldots, I_{n_i-1}^i)$ and $H(BT, \mathbf{C})/I(i)$ respectively.

PROPOSITION 3.9. – *The polynomial function* f_i *is not identically zero on any irreducible component of* $V(I(i))$.

Proof. – Let $A(i)$ be an ideal of $H^*(BT, \mathbf{C})$ generated by

$$X_1^1, \ldots, X_{n_i-1}^{i-1}, I_1^i, \ldots, I_{n_i-1}^i, X_1^{i+1}, \ldots, X_{n_h}^k.$$

Then $V(I_1^i, \ldots, I_{n_i-1}^i)$ in $\mathbf{C}^{n_i}$ will be identified with $V(A(i))$ in $\mathbf{C}^{n_1 + \cdots + n_k}$. Since $A(i) \supset I(i)$, we have

$$V(A(i)) \subset V(I(i)).$$

Let P be a point of $V(I(i))$. Then

$$X_1^j(P) = \ldots = X_{n_j}^j(P) = 0 \quad (j \neq i)$$

by Lemma 3.5. So P is a point on $V(A(i))$ and we have $V(A(i)) = V(I(i))$. By Proposition 3.8 $\overline{f_i}$ is not identically zero on any irreducible component of $V(I_1^l, \ldots, I_{n_l-1}^l)$ and it clearly implies that f_l is not identically zero on any component of $V(A(i)) = V(I(i))$.

q.e.d.

Proposition 3.9 implies that f_l is not contained in any associated prime ideal of $I(i)$. Hence the sequence

$$I_1^1, \ldots, I_1^l, \ldots, I_{n_l-1}^l, f_l, I_1^{l+1}, \ldots, I_1^k, \ldots, I_{n_k}^k$$

is a regular sequence of $H^*(BT, \mathbf{C}) = \mathbf{C}[X_1^1, \ldots, X_{n_k}^k]$.

4. Proof of Theorem A.

Let G be a compact connected Lie group and L(G) be its Lie algebra. Then the complexifications $\mathfrak{g}$ and $\mathfrak{u}$ of L(G) and L(U) respectively have a common Cartan subalgebra $\mathfrak{I}$. Let U be a closed subgroup with the maximal rank and L(U) its Lie algebra. Then their complexification $\mathfrak{g}$ and $\mathfrak{u}$ have a common Cartan subalgebra $\mathfrak{I}$. Then we recall Borel's result.

PROPOSITION 4.1 ([1]). – *Let G be a compact connected Lie group and U be a closed subgroup with the same rank. Then we have*

$$H^*(G/U, \mathbf{C}) = S(\mathfrak{I}^*)^{W(\mathfrak{u})}/S^+(\mathfrak{I}^*)^{W(\mathfrak{g})}$$

where $S^+(\mathfrak{I}^*)$ denotes the ideal generated by the positive elements of $S(\mathfrak{I}^*)^{W(\mathfrak{g})}$.

Let G be a compact connected Lie group. Then it is known that G has a presentation $T \times G_1/K$, where G_1 is a compact connected one connected semisimple Lie group, T is a torus group and K is a finite group contained in the product of the center of G and T. A closed subgroup of maximal rank has a presentation $T \times U_1/K$, where U_1 is a closed subgroup of G_1 of maximal rank. Let $\mathfrak{g}_1$ and $\mathfrak{u}_1$ be complexifications of corresponding Lie algebra of G_1 and U_1 respectively and $\mathfrak{I}_1$ be their common Cartan subalgebra. Then Proposition 4.1 takes the form

PROPOSITION 4.2. –

$$H^*(G/U, \mathbf{C}) = S(\mathfrak{T}_1^*)^{W(\mathfrak{u}_1)}/S^+(\mathfrak{T}_1^*)^{W(\mathfrak{g}_1)} = H^*(G_1/U_1, \mathbf{C}).$$

For the present we prepare the next Lemma that is a slight modification of [8] (20.D. p. 150).

LEMMA 4.3. – *Let* G *be a connected compact Lie group and* T *be a maximal torus. Then the inclusion map*

$$H^*(BG, K) \longrightarrow H^*(BT, K)$$

is faithfully flat if K *is a field with characteristic zero. Moreover for an ideal* $I < H^*(BG, K)$ *we have* $I \cdot H^*(BT, K) \cap H^*(BG, K) = I$.

Proof. – The cohomology ring $H^*(BT, K)$ is isomorphic to $K[X_1, \ldots, X_n]$ and $H^*(BG, K)$ to $K[X_1, \ldots, X_n]^{W(G)}$ where n is the rank of G and W(G) the Weyl group of G. Then we see $K[X_1, \ldots, X_n]^{W(G)} = K[P_1, \ldots, P_n]$ and $P_1, \ldots, P_n$ is a regular sequence [1] (proposition 26.1 and Lemma 26.2). Then it is known that $K[X_1, \ldots, X_n]$ is a free $K[P_1, \ldots, P_n]$ module. Hence it is faithfully flat over $K[P_1, \ldots, P_n]$. The second assertion follows from [8] (4.c, p. 27-28). q.e.d.

LEMMA 4.4. – $S^+(\mathfrak{T}^*)^{W(\mathfrak{g})} \cdot S(\mathfrak{T}^*) \cap S(\mathfrak{T}^*)^{W(\mathfrak{g})} = S^+(\mathfrak{T}^*)^{W(\mathfrak{g})}$

Proof. – We apply Lemma 4.3. q.e.d.

Now we prove Theorem A′. To prove Theorem A′ it is enough to show that

$$D_i(H^*(G/U, \mathbf{C})) = 0, \qquad i > 0.$$

We may assume that G is a compact connected one connected semi-simple Lie group by Proposition 4.2. Let $G = G_1 \times \ldots \times G_k$ be a decomposition into simple factors. Let $U_i = U \cap G_i$ and $J_1^i, \ldots, J_{n_i}^i$ be generators of $H^*(BU_i, \mathbf{C})$. We use the same notations as in the proof of Proposition 3.9, then we have by Proposition 4.1

$$H^*(G_i/U_i, \mathbf{C}) = \mathbf{C}[J_1^i, \ldots, J_{n_i}^i]/(I_1^i, \ldots, I_{n_i}^i),$$

$$H^*(G/U, \mathbf{C}) = \bigotimes_{i=1}^{k} H^*(G_i/U_i, \mathbf{C}).$$

Let $D : H^*(G/U, \mathbf{C}) \longrightarrow H^*(G/U, \mathbf{C})$ be a negative derivation. Then D can be lifted to a $\mathbf{C}$-derivation

$$\tilde{D} : \mathbf{C}[J_1^1, \ldots, J_{n_1}^1, J_1^2, \ldots, J_1^k, \ldots, J_{n_k}^k] \longrightarrow \mathbf{C}[J_1^1, \ldots, J_{n_1}^1, J_1^2, \ldots, J_{n_k}^k]$$

such that $\tilde{D}(I_m^j) \in (I_1^1, \ldots, I_{n_1}^1, I_1^2, \ldots, I_1^k, \ldots, I_{n_k}^k)$ for $m = 1, \ldots, n_j$, $j = 1, \ldots, k$. Since D is a negative derivation, we have

$$\tilde{D}(I_m^i) \in I(i), \quad m = 1, \ldots, n_i, \quad i = 1, \ldots, k.$$

From Leibnitz's law we have

$$\tilde{D}(I_m^i) = \sum_{j=1}^{n_i} \frac{\partial I_m^i}{\partial J_j^i} \cdot \tilde{D}(J_j^i), \quad m = 1, \ldots, n_i, \quad i = 1, \ldots, k.$$

Then by Cramer's formula

$$\det \left(\frac{\partial I_m^i}{\partial J_\ell^i} \right)_{1 \leq m, \ell \leq n_i} \cdot \tilde{D}(J_j^i) \in I(i), \quad i = 1, \ldots, k. \tag{1}$$

We embed $H^*(BU, \mathbf{C})$ into $H^*(BT, \mathbf{C})$ and let $I(i)^c$ be an ideal generated by all the I_j^k except $I_{n_i}^i$ in $H^*(BT, \mathbf{C})$. Then we have

$$\det \left(\frac{\partial I_m^i}{\partial X_s^i} \right) \cdot \tilde{D}(J_j^i) = \det \left(\frac{\partial I_m^i}{\partial J_\ell^i} \right) \det \left(\frac{\partial J_\ell^i}{\partial X_s^i} \right) \cdot \tilde{D}(J_j^i) \in I(i)^c$$

by (1). Then by Proposition 3.9 and the account below it, $\det \left(\frac{\partial I_m^i}{\partial X_s^i} \right)$ is not a zero divisor of $\mathbf{C}[X_1^1, \ldots, X_{n_k}^k]/I(i)^c$. Hence we have

$$\tilde{D}(J_j^i) \in I(i)^c, \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, k.$$

Applying Lemma 4.2, we obtain

$$\tilde{D}(J_j^i) \in I(i)^c \cap H^*(BU) = I(i).$$

This implies that D is zero derivation. q.e.d.

Now we prove Theorem A (compare [9] and [15]). Combining Theorem A and Lemma 2.1 we have

$$\pi_{\mathrm{odd}}\,(\mathrm{BAut_S\ F}) \otimes \mathbf{Q} = 0 .$$

Therefore the generators of the minimal model of $\mathrm{BAut_S\ F}$ possess even degree. Hence we have $\mathrm{H^{odd}}\,(\mathrm{BAut_S\ F}, \mathbf{Q}) = 0$. Taking account of $\mathrm{H^{odd}}\,(\mathrm{F}, \mathbf{Q}) = 0$, the Serre spectral sequence of the universal fibration

$$\mathrm{F} \longrightarrow \widetilde{\mathrm{E}} \longrightarrow \mathrm{BAut_S\ F}$$

collapses at the E_2-level, which implies that this fibration is totally non homologous to zero. Since every fibration is induced from the universal one, we have Theorem A.

5. Koszul complex and Jacobian.

In this section we prove Theorem B. First we consider the case when $\mathrm{P}_1, \dots, \mathrm{P}_n$ are homogeneous polynomials.

Let $(\mathrm{P}_1, \dots, \mathrm{P}_n)$ be a homogeneous $\mathfrak{M}$-regular sequence in $\mathrm{K}[\mathrm{X}_1, \dots, \mathrm{X}_n]$. We denote $\mathrm{K}[\mathrm{X}_1, \dots, \mathrm{X}_n]$ and $(\mathrm{P}_1, \dots, \mathrm{P}_n)$ by A and I respectively.

LEMMA 5.1. – *The natural surjection*

$$\mathrm{A/I} \longrightarrow \mathrm{A}/\mathfrak{M}$$

induces the map

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\mathrm{A}}\,(\mathrm{A}/\mathfrak{M}, \mathrm{A}) & \longrightarrow & \mathrm{Ext}^n_{\mathrm{A}}\,(\mathrm{A/I}, \mathrm{A}) \\
\wr\| & & \wr\| \\
\mathrm{K} & & \mathrm{A/I} \\
\Psi & & \Psi \\
1 & \longrightarrow & c\,\det\,(\partial \mathrm{P}_i/\partial \mathrm{X}_j)
\end{array}$$

where c *is a non-zero constant defined by* $c = \left(\prod_{i=1}^{n} \deg(\mathrm{P}_i)\right)$.

Proof. – Let

$$\mathrm{C} = \bigotimes_{s=0}^{n} \mathrm{A} \otimes \Lambda^s (e_1 , \ldots , e_n) , \; d(e_i) = \mathrm{P}_i \quad (i = 1 , \ldots , n)$$

be the Koszul complex associated with the regular sequence

$$(\mathrm{P}_1 , \ldots , \mathrm{P}_n)$$

and

$$\mathrm{C} = \bigoplus_{s=0}^{n} \mathrm{A} \otimes \Lambda^s (f_1 , \ldots , f_n) , \quad d(f_i) = \mathrm{X}_i$$

be the Koszul complex with the regular sequence $(\mathrm{X}_1 , \ldots , \mathrm{X}_n)$. Let $\overline{\theta} : \mathrm{A} \otimes \Lambda^1 (e_1 , \ldots , e_n) \longrightarrow \mathrm{A} \otimes \Lambda^1 (f_1 , \ldots , f_n)$ be a A-linear map defined by

$$\overline{\theta}(e_i) = n_i^{-1} \sum_j \partial \mathrm{P}_i / \partial \mathrm{X}_j \, . \, f_j$$

and n_i denote the degree of P_i. We can extend θ to $\overline{\theta} : \mathrm{C} \longrightarrow \overline{\mathrm{C}}$ by

$$\theta (e_{i_1} \ldots e_{i_k}) = \overline{\theta}(e_{i_1}) \wedge \ldots \wedge \theta (e_{i_k}) .$$

For the polynomials

$$\mathrm{P}_i = \sum_{\alpha_1 + \ldots + \alpha_n = n_i} a_{\alpha_1 \ldots \alpha_n} \mathrm{X}_1^{\alpha_1} \ldots \mathrm{X}_n^{\alpha_n} ,$$

we get

$$\theta (e_i) = n_i^{-1} \sum_{\alpha_1 + \ldots + \alpha_n = n_i} \sum_{j=1}^{n} a_{\alpha_1 \ldots \alpha_n} \alpha_j \mathrm{X}_1^{\alpha_1} \ldots \mathrm{X}_j^{\alpha_j - 1} \ldots \mathrm{X}_n^{\alpha_n} f_j$$

and

$$d\theta (e_i) = n_i^{-1} \sum_{j=1}^{n} a_{\alpha_1 \ldots \alpha_n} \alpha_j \mathrm{X}_1^{\alpha_1} \ldots \mathrm{X}_j^{\alpha_j} \ldots \mathrm{X}_n^{\alpha_n}$$
$$= \mathrm{P}_i .$$

Therefore the following diagram commutes

$$\begin{tikzcd}
A\, e_1 \oplus \ldots \oplus A\, e_n \arrow[r, "d"] \arrow[d, "\theta"] & A \arrow[r] \arrow[d] & A/I \arrow[d] \\
A\, f_1 + \ldots + A\, f_n \arrow[r, "d"] & A \arrow[r] & A/\mathfrak{M}\,.
\end{tikzcd}$$

Next we have

$$d\,(e_i \wedge e_j) = \theta\,(P_i\, e_j - P_j\, e_i)$$

$$= P_i\,\Big(n_j^{-1} \sum_k \partial P_i/\partial X_k \cdot f_k\Big) - P_j\,\Big(n_i^{-1} \sum_k \partial P_i/\partial X_k \cdot f_k\Big)$$

and

$$d\theta\,(e_i \wedge e_j) = d\,\Big(\Big(n_i^{-1} \sum_k \partial P_i/\partial X_k \cdot f_k\Big) \wedge \Big(n_j^{-1} \sum_k \partial P_j/\partial X_k \cdot f_k\Big)\Big)$$

$$= P_i\,\Big(n_j^{-1} \sum_k \partial P_j/\partial X_k \cdot f_k\Big) - P_j\,\Big(n_i^{-1} \sum_k \partial P_i/\partial X_k \cdot f_k\Big)\,.$$

In this way we see that

$$\theta : C \longrightarrow \bar{C}$$

is a chain map.

In dimension n, the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A\, e_1 \wedge \ldots \wedge e_n \arrow[r] \arrow[d, "\theta"] & C_{n-1} \arrow[r] \arrow[d, "\theta"] & {} \\
0 \arrow[r] & A\, f_1 \wedge \ldots \wedge f_n \arrow[r] & C_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Then θ is expressed

$$\theta\,(e_1 \wedge \ldots \wedge e_n) = \theta\,(e_1) \wedge \ldots \wedge \theta\,(e_n)$$

$$= \Big(n_1^{-1} \sum_k \partial P_1/\partial X_k \cdot f_k\Big) \wedge \ldots \wedge \Big(n_n^{-1} \sum_k \partial P_n/\partial X_k \cdot f_k\Big)$$

$$= \det \begin{bmatrix} n_1^{-1}\, \partial P_1/\partial X_1 & & n_n^{-1}\, \partial P_n/\partial X_1 \\ & & \\ n_1^{-1}\, \partial P_1/\partial X_n & & n_n^{-1}\, \partial P_n/\partial X_n \end{bmatrix} f_1 \wedge \dots \wedge f_n$$

$$= \Big(\prod_{i=1} n_i\Big)^{-1} \det(\partial P_i/\partial X_j)\, f_1 \wedge \dots \wedge f_n\,.$$

Hence θ induces a homomorphism

$$\begin{array}{ccc} \theta : \mathrm{Ext}^n_A(A/\mathfrak{M}, A) & \longrightarrow & \mathrm{Ext}^n_A(A/I, A) \\ \wr\| & & \wr\| \\ \mathbf{Q} & & A/I \end{array}$$

such that $\theta(1) = (\Pi\, n_i)^{-1} \det(\partial P_i/\partial X_j) \mod I$.

q.e.d.

Now we show

THEOREM B′. – *Let* $(P_1, \dots, P_n)$ *be a homogeneous regular sequence. Then we have*

$$\det(\partial P_i/\partial X_j) \notin (P_1, \dots, P_n)\,.$$

Proof. – Since $\dim_K A/I < \infty$, there exists a positive integer r such that $\mathfrak{M}^r \subset I$. Especially $(X_1^r, \dots, X_n^r) \subset I$. We consider the commutative diagram of natural surjections

$$\begin{tikzcd} A/I & A/(X_1^r, \dots, X_n^r) \arrow[l, "i_3"'] \arrow[dl, "i_2"] \\ A/\mathfrak{M} \arrow[from=u, "i_1"'] & \end{tikzcd}$$

It induces the commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^n_A(A/I, A) \arrow[r, "i_3^*"] & \mathrm{Ext}^n_A(A/(X_1^r, \dots, X_n^r), A) \\ \mathrm{Ext}^n_A(A/\mathfrak{M}, A) \arrow[u, "i_1^*"] \arrow[ur, "i_2^*"'] & \end{tikzcd}$$

It is easy to see that $i_2^* \neq 0$. Therefore from Lemma 4.1, we obtain

$$\det(\partial P_i/\partial X_j) \notin (P_1, \dots, P_n).$$

q.e.d.

We call $P(X_1, \dots, X_n)$ is a quasi homogeneous polynomial if it is obtained from a homogeneous polynomial in the variables $Y_1, \dots, Y_n$ by polynomial transformations $X_i = X_i(Y_1, \dots, Y_n)$ where $X_1, \dots, X_n$ is a regular sequence of $K[Y_1, \dots, Y_n]$.

Finally, we prove the general case.

Let $B = A[Y_1, \dots, Y_n]$ and $(P_1, \dots, P_n)$ be an ideal generated by $P_1(Y), \dots, P_n(Y)$ in B. Then $- \otimes B$ is exact by the argument in the proof of Lemma 6.3. Hence $P_1(Y), \dots, P_n(Y)$ is a regular homogeneous sequence of B and we obtain by Theorem B′

$$\det(\partial P_i/\partial Y_j) = \det(\partial P_i/\partial X_k)\det(\partial X_k/\partial Y_j) \notin (P_1, \dots, P_n)^c.$$

Therefore

$$\det(\partial P_i/\partial X_j) \notin (P_1, \dots, P_n)^c.$$

Then we have

$$\det(\partial P_i/\partial X_j) \notin (P_1, \dots, P_n).$$

This completes the proof of Theorem B.

6. The case of mod p coefficient.

In this section we discuss Theorem A in case of mod p coefficient. We check our previous argument for the case of characteristic p. The purpose of this section is to prove the following result:

THEOREM C. – *Let* G *be a compact connected Lie group and* U *be a closed subgroup of* G *such that* rank G = rank U. *Let*

$$G/U \xrightarrow{i} E \longrightarrow B$$

be any orientable fibration with G/U *as a fiber. Then the induced map*

$$i^* : H^*(E, F_p) \longrightarrow H^*(G/U, F_p)$$

is surjective if a prime p *does not divide the order of the Weyl group* $W(G)$ *of* G, *where* F_p *denotes a prime field of characteristic* p.

The proof of Theorem C is very similar to that of Theorem A. So it is sufficient to write the detailed proof at where we cannot derive from previous one soon. Let $D_i(A)$ denote a F_p vector space of F_p-derivations of F_p algebra A decreasing the degree by i.

LEMMA 6.1. – *Let* $F \xrightarrow{i} E \longrightarrow B$ *be an orientable fibration. Then the induced map* $i^* : H^*(E, F_p) \longrightarrow H^*(F, F_p)$ *is surjective if* $D_i(H^*(F, F_p)) = 0$ *for* $i > 0$.

Proof. – Suppose that $d_2 = \ldots = d_{r-1} = 0$. Then we can write $E_r^{0,*} = H^*(F)$ and $E_r^{r,*-r+1} = H^r(B) \otimes H^{*-r+1}(F)$ and the differential is considered as a derivation

$$d_r : H^*(F) \longrightarrow H^r(B) \otimes H^{*-r+1}(F).$$

For $b \in H^r(B)$ evaluating on the dual of b we have a map

$$p_b : H^r(B) \otimes H^{*-r+1}(F) \longrightarrow H^{*-r+1}(F).$$

Then the composition $p_b \circ d_r$ is a F_p derivation on $H^*(F)$ decreasing the degree by $r - 1$. By assumption we have $p_b \circ d_r = 0$ for any b. This implies $d_r = 0$. Hence the Serre spectral sequence collapses at E_2 level.

q.e.d.

From this Lemma we can reduce Theorem C to the following:

THEOREM C. – *Let* p *be a prime such that* p *does not divide the order of the Weyl group* $W(G)$. *Then* $D_i(H^*(G/U, k)) = 0$ *for* $i > 0$ *where* k *is an algebraic closed field of characteristic* p.

Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{G}$ and Δ^+ be the set of positive root. Since the Cartan Killing form is non degenerate on $\mathfrak{H}$ we can choose a unique element $h'_\alpha \in \mathfrak{H}$ satisfying $(h, h'_\alpha) = \alpha(h)$ for all $h \in \mathfrak{H}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental root system. Then we define $h_\alpha = \dfrac{2}{(\alpha, \alpha)} h'_\alpha$ and set

$h_i = h_{\alpha_i}$ $(i = 1, \ldots, n)$. We denote by $W(\mathfrak{G})$ the Weyl group generated by reflections R_{α_i}.

LEMMA 6.2. – *Let* $\mathfrak{I}_{\mathbf{Z}} = \bigoplus_{i=1}^{n} \mathbf{Z} h_i$. *Then* $W(\mathfrak{G})$ *act on* $\mathfrak{I}_{\mathbf{Z}}$ *and* h_α *is contained in* $\mathfrak{I}_{\mathbf{Z}}$ *for each root* α.

For the proof see [12] (Lemma 1 p. 2).

Throughout this section k denotes an algebraic closed field of characteristic p such that $(p, |W(G)|) = 1$ and $Z_{(p)}$ denotes the localization of $\mathbf{Z}$ at p. Let V be a module over a commutative ring A and B be a A algebra. Then we write $V_B = V \underset{A}{\otimes} B$ and $V_B^* = \mathrm{Hom}_A(V, B) = V^* \underset{A}{\otimes} B$.

LEMMA 6.3. – *Let* p *be a prime so that* $(p, |W(G)|) = 1$. *Then* $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ *is a basis for* $\mathfrak{I}^*_{\mathbf{Z}(p)}$.

Proof. – Let Λ_i $1 \leqslant i \leqslant n$ be the element of $\mathfrak{I}^*_{\mathbf{Z}(p)}$ defined by $\Lambda_i(h_j) = \delta_{ij}$. Then $\{\Lambda_i\}$ $1 \leqslant i \leqslant n$ is a basis for $\mathfrak{I}^*_{\mathbf{Z}(p)}$. Then we can write

$$\alpha_i = \sum_{j=1}^{n} a_{ij} \Lambda_j \qquad 1 \leqslant i \leqslant n \text{ and } a_{ij} \in \mathbf{Z}_{(p)}.$$

Taking the value on $h_j \in \mathfrak{I}_Z$, we have

$$\alpha_i(h_j) = a_{ij} \text{ and } \alpha_i(h_j) = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j).$$

This implies that a matrix (a_{ij}) is a Cartan matrix. From [3] (p. 45) det (a_{ij}) is given as follows:

type	det (a_{ij})
A_n	$n + 1$
B_n	2
C_n	2
D_n	4
E_6	3
E_7	2
E_8	1
F_4	1
G_2	1

We see that det (a_{ij}) is a unit of $\mathbf{Z}_{(p)}$. By Cramer's formula $\{\alpha_i\}$ $1 \leqslant i \leqslant n$ is a basis for $\mathfrak{I}^*_{\mathbf{Z}_{(p)}}$. q.e.d.

LEMMA 6.4. – *Let $\overline{\Delta}^+$ be the image of Δ^+ under the reduction map $\mathfrak{I}^*_{\mathrm{Z}_{(p)}} \longrightarrow \mathfrak{I}^*_{\mathrm{F}_p}$. Then the cardinality of Δ^+ and $\overline{\Delta}^+$ are equal.*

Proof. – Note that if $\alpha = \sum_{i=1}^{n} m_i \alpha_i \in \Delta^+$ then m_i is non negative integer not greater than the corresponding of the dominant root. Then we have the Lemma from the list of dominant root.

type	dominant root
A_n	$\alpha_1 + \ldots + \alpha_n$
B_n	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n$
C_n	$2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n$
D_n	$\alpha_1 + 2\alpha_2 + \ldots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$

Let V be a $\mathbf{Z}_{(p)}$ module and V_k be $\mathrm{V}_{\mathbf{Z}} \underset{(p)}{\otimes} k$. Then we denote by $\overline{x}$ the element $x \otimes 1 \in \mathrm{V}_k$. Since R_i is an isomorphism on $\mathfrak{I}^*_{\mathbf{Z}_{(p)}}$, we can define a Coxeter Killing transformation $\overline{\mathrm{R}}$ of $\mathfrak{I}^*_k$ with respect to $\overline{\Pi} = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$ by

$$\overline{\mathrm{R}} = \overline{\mathrm{R}}_1 \circ \cdots \circ \overline{\mathrm{R}}_n .$$

We denote by μ_m a primitive m-th root of unity.

PROPOSITION 3.1′. – *Let $\overline{h}$ be the order of $\overline{\mathrm{R}}$. Then we have $\overline{h} = h$. Moreover there exists an eigen vector $\overline{z} \in \mathfrak{I}_k$ with an eigen value μ_h such that $\overline{\alpha}(\overline{z}) \neq 0$ for all $\overline{\alpha} \in \overline{\Delta}$.*

Proof. – Since h (the order of a Coxeter Killing transformation over R) divides the order of $\mathrm{W}(\mathfrak{G})$, p does not divide h. Let $e^{2\pi i/h}$ be an eigen value of R and $\mathrm{F} = \mathbf{Q}(e^{2\pi i/h})$ be a field extension of a rational field $\mathbf{Q}$ by $e^{2\pi i/h}$. Let W_h be a group generated multiplicatively by $e^{2\pi i/h}$ and A be the set of algebraic integers in F. Let $\mathfrak{P}$ be a prime ideal of A such that $\mathfrak{P} \cap \mathbf{Z} = (p)$. Then we have

$$X^{h-1} + \ldots + X + 1 = \prod_{\substack{\xi \in W_h \\ \xi \neq 1}} (X - \xi).$$

By setting $X = 1$ we have $h = \prod_{\substack{\xi \in W_h \\ \xi \neq 1}} (1 - \xi)$. Since $p \nmid h$, this implies that $1 - \xi \in \mathfrak{P}$ if $\xi \neq 1$. Therefore $\mu = \overline{e^{2\pi i/h}} \in k$ is a primitive $h-th$ root of unity. Let $\bar{z} \in \mathfrak{I}_k$ be an eigen vector with eigen value μ. Suppose that $\bar{\alpha}(\bar{z}) = 0$ for $\bar{\alpha} \in \bar{\Delta}$. Since $w(\Pi)$ is also a fundamental root system, we may take $\bar{R} = \bar{R}_{\beta_1} \ldots \bar{R}_{\beta_{n-1}} \bar{R}_{\alpha}$ as Coxeter Killing transformation where $w(\Pi) = \{\beta_1, \ldots, \beta_{n-1}, \alpha_n\}$. Set $\bar{R}' = \bar{R}_{\beta_1} \ldots \bar{R}_{\beta_{n-1}}$. Then we have $\bar{R}'(\bar{z}) = \mu\bar{z}$ since $\bar{R}_{\alpha}(z) = \bar{z}$. Therefore $\bar{R}'$ has an eigen value $e^{2\pi i/h}$. This contradicts to the table 2 in Coxeter [4]. q.e.d.

PROPOSITION 3.3'. – *The invariant subalgebra* $S(\mathfrak{I}_k^*)^{W(\mathfrak{g})}$ *is generated by* $\bar{I}_1, \ldots, \bar{I}_n$ *over* k *and* $\deg \bar{I}_j = \deg I_j$ $1 \leqslant j \leqslant n$. *Moreover the sequence* $\bar{I}_1, \ldots, \bar{I}_n$ *is a regular sequence in* $S(\mathfrak{I}_k^*)$.

Compare [1].

Let $V(\bar{I}_1, \ldots, \bar{I}_k)$ be an algebraic set in k^n defined by the equation $\bar{I}_1 = \ldots = \bar{I}_k = 0$. We call $\bar{z} \in \mathfrak{I}_k$ a regular element if $\bar{\alpha}(\bar{z}) \neq 0$ for all $\bar{\alpha} \in \bar{\Delta}$. By Proposition 3.1 and Proposition 3.3, the proof of Proposition 3.6 goes well in this case. Thus we have

PROPOSITION 3.6'. – *Each non zero point on* $V(\bar{I}_1, \ldots, \bar{I}_{n-1})$ *is a regular element of* $\mathfrak{I}_k$.

The following result is obtained in parallel with the proof of Theorem B :

THEOREM B'. – *Let* $P_1, \ldots, P_n$ *be a homogeneous* $\mathfrak{M}$*-regular sequence in* $k[X_1, \ldots, X_n]$ *such that* $(\deg P_i, \mathrm{ch}(k)) = 1$, $1 \leqslant i \leqslant n$. *Then we have* $\det(\partial P_i/\partial X_j) \notin (P_1, \ldots, P_n)$.

PROPOSITION 3.7'. – *Let* $\bar{I}_1, \ldots, \bar{I}_n$ *be a basic system of the invariant algebra* $S(\mathfrak{I}_k^*)^{W(\mathfrak{g})}$. *Then* $\det(\partial \bar{I}_i/\partial \alpha_j) = c \prod_{\bar{\alpha} \in \bar{\Delta}^+} \bar{\alpha}$

where c *is a non zero element of* k.

Proof. – Since

$$|W(\mathfrak{G})| = \prod_{k=1}^{n} (m_k + 1) \text{ and } (\mathrm{ch}(k), |W(\mathfrak{G})|) = 1,$$

we have $\det(\partial \bar{I}_i/\partial \alpha_j) \neq 0$ by Theorem B. Consider a formal de Rham complex

$$C = k[\bar{\alpha}_1, \ldots, \bar{\alpha}_n] \otimes \Lambda(d\bar{\alpha}_1, \ldots, d\bar{\alpha}_n).$$

Then $W(\mathfrak{G})$ act on C by

$$R_i(\bar{\alpha}_j) = \bar{\alpha}_j - 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)\,\bar{\alpha}_i$$
$$R_i(d\bar{\alpha}_j) = d\bar{\alpha}_j - 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)\,d\bar{\alpha}_i.$$

It is clear that this action is compatible with d. Hence we have

$$\begin{aligned} d\bar{I}_1 \wedge \ldots \wedge d\bar{I}_n &= d(\bar{R}_k \bar{I}_1) \wedge \ldots \wedge d(\bar{R}_k \bar{I}_n) \\ &= \bar{R}_k(d\bar{I}_1) \wedge \ldots \wedge \bar{R}_k(d\bar{I}_n) \\ &= \bar{R}_k(d\bar{I}_1 \wedge \ldots \wedge d\bar{I}_n) = \bar{R}_k(\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)\, d\bar{\alpha}_1 \wedge \ldots \wedge d\bar{\alpha}_n) \\ &= -\bar{R}_k(\det(\partial \bar{I}_i/\partial \bar{\alpha}_j))\, d\bar{\alpha}_1 \wedge \ldots \wedge d\bar{\alpha}_n. \end{aligned}$$

Therefore we get $\bar{R}_\alpha(\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)) = \det(R_\alpha)\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)$ for $\bar{\alpha} \in \bar{\Delta}^+$. Let v be a point on a Hyperplane $\bar{H}_\alpha$ defined by $\bar{\alpha}(\bar{H}_\alpha) = 0$. Then we have

$$\begin{aligned} -\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)(v) &= \bar{R}_k(\det(\partial \bar{I}_i/\partial \bar{\alpha}_j))(v) \\ &= \det(\partial \bar{I}_i/\partial \bar{\alpha}_j)(\bar{R}_k v) \\ &= \det(\partial \bar{I}_i/\partial \bar{\alpha}_j)(v). \end{aligned}$$

This implies $\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)(v) = 0$. For $w \in H_\alpha, \bar{\alpha} \in \bar{\Delta}^+$, we can take $\bar{R}_\alpha, \bar{H}_i$ so that $\bar{H}_\alpha = \bar{R}_\alpha \bar{H}_i$ and we have $\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)(w) = 0$ by similar way. Therefore $\prod_{\bar{\alpha} \in \bar{\Delta}^+} \alpha$ divide $\det(\partial \bar{I}_i/\partial \bar{\alpha}_j)$. Comparing the degree we have the Proposition.

q.e.d.

LEMMA 4.3′. – *Let* U *be a closed subgroup of compact connected Lie group* G *with* rank U = rank G *and* T *be a common* *maximal torus. Then* $H^*(BT, k)$ *is a finite free module over* $H^*(BU, k)$ *via the natural inclusion* $H^*(BU, k) \longrightarrow H^*(BT, k)$. *In particular for an ideal* $I \subset H^*(BU, k)$ *we have*

$$I \cdot H^*(BT, k) \cap H^*(BU, k) = I.$$

Proof. – Noting that $p \nmid |W(G)|$ implies $p \nmid |W(U)|$, the proof is done in the same way as the proof of Lemma 4.3.

q.e.d.

For other part our previous argument goes well with minor change. Thus we have proved Theorem C.

Remark. – The condition that a prime p does not divide the order of the Weyl group need not be best possible.

BIBLIOGRAPHY

[1] A. BOREL, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math.*, 57 (1953), 115-207.

[2] A. BOREL and J.D. SIEBENTHAL, Les sous-groupes fermés de rang maximal de Lie clos, *Comm. Math. Helv.*, 23 (1949), 200-221.

[3] R.W. CATER, *Simple groups of Lie type*, John Wilely and Sons, London, 1972.

[4] C. CHEVALLEY, Invariants of finite groups generated by reflections, *Amer. J. Math.*, 77 (1955), 778-782.

[5] H. COXETER, The product of the generators of a finite group generated by reflections, *Duke Math. J.*, 18 (1951), 391-441.

[6] S. HALPERIN, Finitess in the minimal models of Sullivan, *Trans. A.M.S.*, 230 (1977), 173-199.

[7] B. KOSTANT, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.*, 81 (1959), 997-1032.

[8] H. MATSUMURA, *Commutative Algebra*, second edition, Benjamin (1980).

[9] W. MEIER, Rational Universal fibration and Flag manifolds, *Math. Ann.*, 258 (1982), 329-340.

[10] D. QUILLEN, The mod 2 cohomology rings of extra-special 2-groups and the Spinor groups, *Math. Ann.*, 194 (1971), 197-212.

[11] M. SCHLESSINGER and J. STASCHEFF, Deformation theory and rational homotopy type, (to appear).

[12] R. STEINBERG, *Lectures on Chevalley groups*, Yale Univ. (1967).

[13] D. SULLIVAN, Infinitesimal computations in Topology, *Publ. I.H.E.S.*, 47 (1977), 269-332.

[14] J.C. THOMAS, Homotopie rationnelle des fibrations de Serre, *Ann. Inst. Fourier*, 31-3 (1981), 71-90.

[15] J.C. THOMAS, Quelques questions commentées sur la fibre d'Eilengerg-Moore d'une fibration de Serre, *Publ. Lille*, 3, n° 6 (1981).

[16] H. SHIGA, Classifying maps and homogeneous spaces, (preprint).

[17] A. KONO, H. SHIGA and M. TEZUKA, A note on the cohomology of a fiber space whose fiber is a homogeneous space, (preprint).

[18] H. SHIGA and M. TEZUKA, Cohomology automorphisms of some Homogeneous spaces, to appear in *Topology* and its applications (Singapore conference volume).

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Added in proof: J. Aguadé pointed out that Theorem C and C′ was obtained by L. Smith and H.G. Munkholm (A Note on realization of graded complete intersection algebras by the cohomology of a space, Quat. J., 33 (1982) 379-384).