ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 35, nº 4 (1985), p. 127-150
<http://www.numdam.org/item?id=AIF_1985__35_4_127_0>

Ann. Inst. Fourier, Grenoble
35, 4 (1985), 127-150.

A VALUE-DISTRIBUTION CRITERION FOR THE CLASS L LOG L, AND SOME RELATED QUESTIONS

by **M. ESSÉN** (1), **D. F. SHEA** (2) and **C. S. STANTON**

1. Introduction.

Let F belong to the Nevanlinna class of functions analytic in the unit disk U, so that

$$T(1,F) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta < \infty .$$

In particular, $\lim_{r\to 1} F(re^{i\theta}) = F(e^{i\theta})$ exists a.e.. We shall say that $\operatorname{Re} F \in L \log L$ if

$$\sup_{0<r<1} \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| \log^+ |\operatorname{Re} F(re^{i\theta})|\, d\theta = \int_0^{2\pi} |\operatorname{Re} F(e^{i\theta})| \log^+ |\operatorname{Re} F(e^{i\theta})|\, d\theta < \infty .$$

The class L log L is closely related to the Hardy space $H^1(U)$, as is shown by the following classical results of Zygmund [22]:

THEOREM A. — *If* $\operatorname{Re} F \in L \log L$, *then* $F \in H^1$.

(1) Research supported by the Swedish Natural Science Research Council.
(2) Research supported by the U.S. National Science Foundation.

Key-words: Hardy space - Harmonic measures - Riesz measure - Conjugate function - Nevanlinna theory.

THEOREM B. — *If* $F \in H^1$ *and* $\operatorname{Re} F > 0$, *then* $\operatorname{Re} F \in L \log L$.

In this paper we prove some refinements of Theorem B. We state our basic results in terms of the usual Nevanlinna counting function

$$N(r,w) = N(r,w;F) = \int_0^r n(t,w)\, dt/t$$

where $n(t,w) = \Sigma_{|z_\nu| \leq t}\ 1$ and $\{z_\nu\} = f^{-1}(w)$. Our main result is

THEOREM 1. — *Let* $F \in H^1(U)$. *The following are equivalent :*

(1.1 *a*) $\displaystyle\int_{-\infty}^{\infty} N(1, iv) \log^+ |v|\, dv < \infty$.

(1.1 *b*) $\operatorname{Re} F \in L \log L$.

(1.1 *c*) $\displaystyle\int_0^{2\pi} |\operatorname{Re} F(e^{i\theta})| \log^+ |F(e^{i\theta})|\, d\theta < \infty$.

Remark 1. — We note that if $\operatorname{Re} F > 0$, then $N(1,iv) = 0$ for all $v \in \mathbf{R}$. Hence Theorem B follows from the equivalence of (1.1 *a*) and (1.1 *b*).

Remark 2. — We could replace the integration over the imaginary axis in (1.1 *a*) by integration over any vertical line, i.e., for any real u (1.1 *a*) is equivalent to

(1.1 *a'*) $$\int_{-\infty}^{\infty} N(1,u+iv) \log^+ |v|\, dv < \infty .$$

Once the Theorem has been proved, this follows immediately since $N(1,u+iv;F) = N(1,iv;F-u)$ and $\operatorname{Re}(F-u) \in L \log L$ if and only if $\operatorname{Re} F \in L \log L$.

In Sections 3 through 6, we give some further refinements of Theorem B: from a geometrical condition on the range of F, we can deduce that $\operatorname{Re} F \in L \log L$. To apply Theorem 1, we need a criterion to decide whether $F \in H^1(U)$. In this context, our main tool is a more general result which may have independent interest : in terms of harmonic measure, it gives a necessary and sufficient condition for $F \in H^p(U)$, $0 < p < \infty$ (cf. Theorem 7 in Section 5)). We also consider cases when the hypothesis $F \in H^1(U)$ in Theorem 1 is omitted. The material in Sections 5 and 6 overlaps with certain work of Burkholder in [4], [5].

The starting-point of our work was a study of the relation between the classical criterion of Zygmund and the following result of A. Baernstein. Let $f \in L^1(T)$ be a given real-valued function and consider

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \, d\varphi . \tag{1.2}$$

We have $F(e^{i\theta}) = f(e^{i\theta}) + i\tilde{f}(e^{i\theta})$ a.e. on T. Let g be the symmetric decreasing rearrangement of f and $\tilde{g}$ the conjugate function of g with mean value zero. In particular, we have

$$\|g\|_1 = \|f\|_1 , \int_T |g| \log^+ |g| = \int_T |f| \log^+ |f| .$$

In [2], Baernstein proved that

$$\|\tilde{f}\|_1 \leqslant \|\tilde{g}\|_1 . \tag{1.3}$$

Thus, when $f \in L^1(T)$ is given, (1.3) implies a sufficient condition for F to be in H^1, namely: $F \in H^1$ if $\|\tilde{g}\|_1 < \infty$. However, this consequence of (1.3) does not actually yield a new criterion for $F \in L^1$, in view of the following consequence of Theorem 1:

COROLLARY 1. – $\tilde{g} \in L^1(T) \Leftrightarrow g \in L \log L$.

Proof. – Assume that $\tilde{g} \in L^1(T)$. From the discussion in Section 6 in Baernstein [2], we see that the analytic function G associated to g by (1.2) maps U univalently onto a Steiner symmetric domain, i.e., a domain with the property that for all $u \in \mathbf{R}$,

$$G(U) \cap \{\mathrm{Re}\, w = u\} = \{w = u + iv : |v| < b(u)\} .$$

If there exists u_0 such that $b(u_0) < \infty$, condition (1.1 a') will hold for $u = u_0$, and Theorem 1 implies that $g \in L \log L$ which is equivalent to $f \in L \log L$. But such a u_0 exists because if $b(u) = \infty$ for all $u \in \mathbf{R}$, G would map U onto the whole complex plane.

The converse assertion is simply Zygmund's Theorem A; Corollary 1 is proved.

Remark. – In a private discussion, Lennart Carleson has shown us a simple real-variable proof of Corollary 1.

In Section 2, we deduce Theorem 1 from a general identity:

THEOREM 2. – *Let* F *be analytic in* U *and let* Φ *be subharmonic in* **C** *with* $\Phi(F(0))$ *finite. Then*

$$(1.4)\quad (2\pi)^{-1}\int_0^{2\pi} \Phi(F(re^{i\theta}))\,d\theta = \int_C N(r,w)\,d\mu(w) + \Phi(F(0)),\qquad 0 < r < 1,$$

where μ *is the Riesz measure of* Φ.

For each choice of Φ in Theorem 2, we get a formula connecting a mean of F over a circle in U with an integral of $N(r,.)$ over the range of F. Examples of such formulas will be given in Section 2. The present proof of Theorem 1 avoids some lengthy estimates in the original proof of Essén and Shea, as announced in [8]; this simplification is made possible because of Stanton's proof of (1.4) in [19]. We apply (1.4) here with $\Phi(w) = |\mathrm{Re}\, w| \log(1+|w|^2)$, cf. (2.5) below.

If μ is a positive finite measure on **C** and $-\Phi$ is the logarithmic potential of μ, then (1.4) is a classical formula of Frostman, cf. [16, p. 177].

We can extend Theorem 2 in a number of ways. For example, it holds when Φ is δ-subharmonic, i.e. the difference of two subharmonic functions, with μ a signed measure. An identity like (1.4) is true for f meromorphic (in the disk or in the plane) provided Φ has sufficiently small growth at infinity. The theorem also can be extended to analytic functions mapping the polydisk or ball of $\mathbf{C}^n$ into **C**. Details of these extensions are given in Section 7.

2. Proofs of Theorems 1 and 2.

We need the following well-known facts on the Nevanlinna counting function (for further information and references, cf. Section 4 in Essén and Shea [7]):

$N(1,w,F) = \lim_{r\to 1^-} N(r,w,F)$ exists and is uniformly bounded except near $F(0)$. The upper regularization $N(w) = N(w,F)$ of $N(1,w,F)$, defined by

$$N(w) = \limsup_{\zeta\to w} N(1,\zeta,F),$$

is subharmonic in $\mathbf{C}\setminus\{F(0)\}$ and coincides with $N(1,w,F)$ off a set of w-values of logarithmic capacity zero. The function $N(w) + \log|w-F(0)|$ can be defined at $F(0)$ to be subharmonic in $\mathbf{C}$.

Throughout the paper, the H^p-norms are defined as in Duren [6], p. 35. The class h^p of harmonic functions in U is defined in [6], p. 2. The set of interior points of a set K is denoted by K^0.

Proof of Theorem 2. – Let r be fixed, $0 < r < 1$. Then there exists a compact set K such that $\operatorname{supp} N(r,.) \subset K^0$ and we have

$$\Phi(\zeta) = \int_K \log|\zeta - w|\, d\mu(w) + h(\zeta),$$

where h is harmonic in K^0. From Jensen's formula

$$(2\pi)^{-1}\int_0^{2\pi} \log|F(re^{i\theta})-w|\, d\theta = N(r,w) + \log|F(0)-w|,$$

we deduce that

$$\begin{aligned}(2\pi)^{-1}&\int_0^{2\pi} \Phi(F(re^{i\theta}))\, d\theta \\ &= \int_K d\mu(w)\int_0^{2\pi}\log|F(re^{i\theta})-w|\, d\theta/(2\pi) + h(F(0)) \\ &= \int_K (N(r,w) + \log|F(0)-w|)\, d\mu(w) + h(F(0)) \\ &= \int_K N(r,w)\, d\mu(w) + \Phi(F(0)).\end{aligned}$$

The theorem is proved.

Next, we give a list of some special subharmonic functions Φ, their associated Riesz measures μ and the formulas which follow from (1.4). Most of these formulas are known; (2.5) is new and is basic to our proof of Theorem 1.

We write $w = u + iv$. δ_u and δ_v are Dirac measures supported by the v- and u-axis, respectively. (Formally, we should write $\delta_{\{u=0\}} \otimes 1$ and $1 \otimes \delta_{\{v=0\}}$). We put $k(t) = (2+t)(1+t)^{-2}$ and $C(t) = 2\pi t \log(1+t)$.

Φ	$\lvert u\rvert$	$\lvert v\rvert$	$\lvert w\rvert$	$\lvert u\rvert \log(1+\lvert u\rvert)$	$\lvert u\rvert \log(1+\lvert w\rvert^2)$
$\Delta\Phi = 2\pi\, d\mu$	$2\,\delta_u$	$2\,\delta_v$	$\lvert w\rvert^{-1}$	$k(\lvert u\rvert)$	$2\,\delta_u \log(1+v^2) + 4\lvert u\rvert k(\lvert w\rvert^2)$

Applying (1.4), we obtain

$$\int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|\, d\theta = 2\int_{-\infty}^{\infty} N(r,iv)\, dv + 2\pi|\operatorname{Re} F(0)|, \tag{2.1}$$

$$\int_0^{2\pi} |\operatorname{Im} F(re^{i\theta})|\, d\theta = 2\int_{-\infty}^{\infty} N(r,u)\, du + 2\pi|\operatorname{Im} F(0)|, \tag{2.2}$$

$$\int_0^{2\pi} |F(re^{i\theta})|\, d\theta = \iint_C N(r,w)\, du\, dv/|w| + 2\pi|F(0)|, \tag{2.3}$$

$$\int_0^{2\pi} (|\operatorname{Re} F| \log(1+|\operatorname{Re} F|))(re^{i\theta})\, d\theta = \iint_C N(r,w)k(|u|)\, du\, dv + C(|\operatorname{Re} F(0)|), \tag{2.4}$$

$$\begin{aligned}\int_0^{2\pi} (|\operatorname{Re} F| \log(1+|F|^2))(re^{i\theta})\, d\theta &= 4\iint_C N(r,w)|u|k(|w|^2)\, du\, dv \\ &+ 2\int_{-\infty}^{\infty} N(r,iv)\log(1+v^2)\, dv + 2\pi|\operatorname{Re} F(0)|\log(1+|F(0)|^2).\end{aligned} \tag{2.5}$$

The equation $\Delta\Phi = 2\pi\, d\mu$ is interpreted in the distributional sense. This means that for all $\psi \in C_0^\infty(\mathbf{R}^2)$, we have

$$\iint_C \Phi\, \Delta\psi = 2\pi \iint_C \psi\, d\mu, \tag{2.6}$$

(cf. Lemmas 3.6 and 3.8 in Hayman and Kennedy [12]).

We illustrate the computation of these formulas by deriving the one needed for (2.5). We choose $\Phi(w) = |u|\log(1+|w|^2)$. Then if $u \neq 0$, $\Phi \in C^\infty$ near $u + iv$ and $\Delta\Phi = 4|u|k(|w|^2)$.

Let $\psi \in C_0^\infty(\mathbf{R}^2)$. From Green's theorem, we deduce that

$$\iint_{\{u>0\}} \Phi\, \Delta\psi = \iint_{\{u>0\}} \psi(w)4uk(|w|^2)\, du\, dv + \int_{-\infty}^{\infty} \psi(iv)\log(1+v^2)\, dv,$$

$$\iint_{\{u<0\}} \Phi\, \Delta\psi = \iint_{\{u<0\}} \psi(w)4|u|k(|w|^2)\, du\, dv + \int_{-\infty}^{\infty} \psi(iv)\log(1+v^2)\, dv.$$

Adding these two formulas, and using (2.6), we obtain

$$2\pi \iint_{\mathbf{C}} \psi \, d\mu = \iint_{\mathbf{C}} \psi(w) 4|u| k(|w|^2) \, du \, dv + 2 \int_{-\infty}^{\infty} \psi(iv) \log(1+v^2) \, dv,$$

which is the fifth formula in the table above.

We rewrite (2.5) in the following way:

$$I_1(r) = 4I_2(r) + 2I_3(r) + 2\pi |\operatorname{Re} F(0)| \log(1+|F(0)|^2).$$

Let $I_j = \sup I_j(r), 0 < r < 1, j = 1, 2, 3$.

LEMMA 1. — *Let* F *be analytic in* U. *Then* $F \in H^1(U)$ *if and only if* $\operatorname{Re} F \in h^1$ *and* I_2 *is finite.*

Proof. — Let $F \in H^1(U)$. From the inequality $k(t) < t^{-1}, t > 0$, it follows that $|u| k(|w|^2) \leqslant |u| |w|^{-2} \leqslant |w|^{-1}$ and I_2 must be finite since we have (2.3). Trivially, we have $\operatorname{Re} F \in h^1$.

Conversely, if I_2 is finite, we use the subharmonicity of $N(r,w)$ in $\mathbf{C} \backslash \{F(0)\}$ to deduce that

$$\begin{aligned}\int_{|u|>2|F(0)|} N(r,u) \, du &\leqslant 4 \int_{-\infty}^{\infty} (\pi u^2)^{-1} \, du \iint_{|\zeta - u| < |u|/2} N(r,\zeta) \, d\xi \, d\eta \\ &\leqslant (4/\pi) \iint_{D_+} N(r,\zeta)(\xi^2 - 3\eta^2)^{1/2} \, d\xi \, d\eta / |\zeta|^2 \\ &\leqslant (4/\pi) \iint_{\mathbf{C}} N(r,\zeta) |\xi| \, d\xi \, d\eta / |\zeta|^2 = I_4(r),\end{aligned}$$

where $D_+ = \{\zeta = \xi + i\eta : \xi^2 \geqslant 3\eta^2\}$.

If I_2 is finite, $\sup\limits_{0<r<1} I_4(r)$ will also be finite. It is now clear from (2.2) that $\operatorname{Im} F \in h^1$. Since $\operatorname{Re} F \in h^1$, we must have $F \in H^1(U)$ and the lemma is proved.

Proof of Theorem 1. — The proof will show that

$$(1.1\,a) \rightarrow (1.1\,b) \rightarrow (1.1\,c) \rightarrow (1.1\,a).$$

To prove $(a) \rightarrow (b)$, we first note that it follows from (a) that I_3 is finite and from Lemma 1 that I_2 is finite. Thus, by (2.5), I_1 is finite and we have proved (b).

To prove $(b) \rightarrow (c)$, we consider the following simple chain of inequalities:

$$|u| \log(1+u^2+v^2) \leqslant 2|u| \log(1+|u|+|v|) \leqslant 2(|u| \log(1+|u|)+|v|).$$

If we now choose $u = \operatorname{Re} F$, $v = \operatorname{Im} F$ and integrate with respect to θ, we obtain

$$I_1 \leqslant 2 \int_0^{2\pi} (|\operatorname{Re} F| \log(1+|\operatorname{Re} F|) + |\operatorname{Im} F|)(e^{i\theta})\, d\theta < \infty,$$

and have proved (c).

The implication $(c) \rightarrow (a)$ is an immediate consequence of (2.5).

Remark. — It is easy to prove Zygmund's Theorem A that (1.1 b) implies $F \in H^1(U)$, using these methods. This is immediate from (2.3), (2.4) and the inequality $k(|u|) \geqslant (|u|+1)^{-1} \geqslant (2|w|)^{-1}$ (valid for $|w| \geqslant 1$), which yield

$$\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta \leqslant \frac{1}{\pi} \int_0^{2\pi} |\operatorname{Re} F| \log(1+|\operatorname{Re} F|)(re^{i\theta})\, d\theta + C_1$$

for $C_1 = T(1,F) + |F(0)| + 1$.

In the opposite direction, (2.3) and (2.5) together with $|u| k(|w|^2) \leqslant |w|^{-1}$ imply

$$\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F| \log(1+|F|^2)^{1/2}(re^{i\theta})\, d\theta \leqslant \frac{1}{\pi} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta + C_2,$$

$$C_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} N(1,iv) \log(1+v^2)^{1/2}\, dv + |\operatorname{Re} F(0)| \log(1+|F(0)|).$$

With heavy restrictions on $F(U)$, such as $\operatorname{Re} F > 0$, inequalities of this type are classical (cf. [22], p. 256).

Let us finally give some further examples of formulas which are immediate consequences of Theorem 2. Successively choosing $\Phi(w)$ as $\log^+ |w|$, $\log(1+|w|^2)$, $|w|^p$ with $p > 0$, $|u|^p$ with $p > 1$ and as $|w| A(\arg w)$ where

$$A(\varphi) = \begin{cases} (1/2)\varphi \sin\varphi, & |\varphi| \leqslant \pi/2, \\ (1/2)(\pi-\varphi)\sin\varphi - \cos\varphi, & \pi/2 \leqslant \varphi \leqslant 3\pi/2, \end{cases}$$

we obtain

$$(2.7)\quad \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta = \int_0^{2\pi} N(r,e^{i\varphi})\, d\varphi + 2\pi \log^+ |F(0)|,$$

$$(2.8)\quad \int_0^{2\pi} \log(1+|F(re^{i\theta})|^2)\, d\theta$$
$$= 4 \iint_C N(r,w)(1+|w|^2)^{-2}\, du\, dv + 2\pi \log(1+|F(0)|^2)$$

$$(2.9)\quad \int_0^{2\pi} |F(re^{i\theta})|^p\, d\theta = p^2 \iint_C N(r,w)|w|^{p-2}\, du\, dv$$
$$+ 2\pi |F(0)|^p,\ p > 0.$$

$$(2.10)\quad \int_0^{2\pi} |\mathrm{Re}\, F(re^{i\theta})|^p\, d\theta$$
$$= p(p-1) \iint_C N(r,w)|u|^{p-2}\, du\, dv + 2\pi |\mathrm{Re}\, F(0)|^p,\ p>1,$$

$$(2.11)\quad \int_0^{2\pi} (|F|A(\arg F))(re^{i\theta})\, d\theta$$
$$= \iint_C N(r,w)|u|\, |w|^{-2}\, du\, dv + 2\pi |F(0)| A(\arg F(0)).$$

Remarks. – Equation (2.7) is Cartan's identity (see Hayman [11], p. 8). Equation (2.8) is a version of a classical formula for the Ahlfors characteristic (see (3.1), p. 173, in Nevanlinna [16]). Equations (2.9), (2.1) and (2.2) are classical (see e.g. Lehto [15], pp. 12, 14). Baernstein derives (2.1), (2.2), (2.9) and (2.10) from Cartan's identity in [2].

3. The class L log L and estimates of harmonic measure.

What more can we say about the connection between the closed set E on which $N(z) = N(z,F)$ vanishes, and the integrability condition (1.1 *b*)? From now on, we assume that $F(0) = 0$.

We need an idea of M. Benedicks [3], developed to study positive harmonic functions vanishing on the boundary in sets of the form $C \setminus E_0$, where E_0 is a subset of the imaginary axis. Our set E is not necessarily restricted in this way.

Following Benedicks, we introduce a function β_E which measures how « thin » the set E is at infinity near the imaginary axis. If $z \neq 0$, let K_z be the open square in the plane with centre at z, sides parallel to the axis and with side length $|z|$. Let $\Omega_z = K_z \backslash E$. In Ω_z, we consider the harmonic function V^z which has boundary values 1 on ∂K_z and 0 on $E \cap K_z$. We define $\beta_E(z) = V^z(z)$.

THEOREM 3. — *Let* $F \in H^1(U)$ *and assume that* $F(0) = 0$. *A sufficient condition for* $\operatorname{Re} F$ *to be in* $L \log L$ *is that*

$$\int_{|y|>1} \beta_E(iy) \log |y| \, dy/y < \infty . \tag{3.1}$$

In Section 4, we shall give examples of conditions on the omitted set E which ensure that (3.1) holds.

In the proof of Theorem 3, we need

LEMMA 2. — *Assume that* $F \in H^p(U)$ *for some* $p > 0$ *and that* $F(0) = 0$. *Then*

$$N(z,F) \leqslant C_p \|F\|_p^p |z|^{-p}, \; z \neq 0, \tag{3.2}$$

where $C_p = p^{-1}$, $0 < p \leqslant 1$, *and* $C_p \leqslant 4$, $p > 1$.

Proof. — For any F in the Nevanlinna class with $F(0) = 0$, it follows from Jensen's theorem that we have

$$N(w,F) \leqslant (2\pi)^{-1} \int_0^{2\pi} \log (1 + |F(e^{i\theta})| \, |w|^{-1}) \, d\theta . \tag{3.3}$$

When $0 < p \leqslant 1$, (3.2) is an immediate consequence of (3.3) and the inequality

$$\log (1+u) \leqslant u^p/p, \; 0 < p \leqslant 1, \; u > 0 .$$

When $p > 2$, we use the fact that $N(z)$ is subharmonic in $\mathbf{C}\backslash\{0\}$ to deduce that, if $\rho = 2/p$,

$$\begin{aligned} N(z) &\leqslant (\pi\rho^2|z|^2)^{-1} \iint_{|w-z|<\rho|z|} N(w) \, du \, dv \\ &\leqslant |z|^{-p}\rho^{-2}(1-\rho)^{2-p} \iint_{\mathbf{C}} N(w)|w|^{p-2} \, du \, dv/\pi \\ &\leqslant C_p \|F\|_p^p |z|^{-p}, \qquad C_p = (1-2/p)^{2-p}/2 . \end{aligned}$$

In the last step, we used (2.9). The argument is similar when $1 < p \leqslant 2$, with $\rho = 1$ and $C_p = (2/p^2)2^{2-p}$.

Proof of Theorem 3. — Using the maximum principle, we deduce from Lemma 2 that

$$N(\zeta) \leqslant 2\|F\|_1 V^{iy}(\zeta)/|y|,\ \zeta \in \Omega_{iy},$$
$$N(iy) \leqslant 2\|F\|_1 \beta_E(iy)/|y|,\ y \neq 0.$$

Hence, Theorem 3 is an immediate consequence of Theorem 1.

In the study of the function $\beta_E(z)$, we need two lemmas of Hayman and Pommerenke [13].

LEMMA A. — *Let* E_1 *be a compact subset of* $\{z:|z|\leqslant R/2\}$ *and let* ω_{E_1} *be the harmonic measure of* E_1 *in* $\{z:|z|<R\}\backslash E_1$. *Then*

$$\omega_{E_1}(z) \geqslant \alpha(R,E_1),\ |z| \leqslant R/2, \tag{3.4}$$

where $\alpha(R,E_1) = \log(5/4)/\log(5R/4\,\mathrm{cap}\,E_1)$.

Lemma A is proved in Section 3 in [13].

LEMMA B. — *Let* E *be a given closed set in the plane and let* $E_1 = E \cap \{z:|z-it|\leqslant R/2\}$. *Let* $\rho > R$, *and let* ω *be the harmonic measure of the outer circle in* $\{z:|z-it|<\rho\}\backslash E$. *We define* $B(r) = \max\limits_{|z-it|=r} \omega(z)$. *Then*

$$B(R/2) \leqslant (1-\alpha(R,E_1))B(R). \tag{3.5}$$

Proof (Adapted from the first part of the proof of Theorem 1 in [13].). — We define $\omega(z) = 0$, $z \in E \cap \{z:|z-it|<\rho\}$. Let ω_1 be the harmonic measure of E_1 in $\{z:|z-it|<R\}\backslash E_1$. If $h(z) = \omega(z) - B(R)(1-\omega_1(z))$, it is easy to check that $h(z)$ is non-positive in $\{z:|z-it|<R\}\backslash E$. Applying Lemma A, we obtain (3.5).

4. Applications of the estimates in Section 3.

We say that a closed set $E \subset \mathbf{C}$ satisfies condition (K_1) if there exist positive numbers δ and a in the interval (0,1) such that for all real t with $|t|$ sufficiently large, we have

$$\mathrm{cap}\,\{E \cap \{z:|z-it|\leqslant R\}\} \geqslant \delta R,\ |t|^a \leqslant R \leqslant |t|. \tag{4.1}$$

THEOREM 4. — *Let* F *be in the Nevanlinna class and assume that* $F(0) = 0$. *If the set* $E = \{z : N(z,F) = 0\}$ *satisfies condition* (K_1), *then* $\operatorname{Re} F \in h^1$, *i.e.,*

$$\sup_{0<r<1} \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| \, d\theta < \infty .$$

Proof. — From (3.3) we see that $N(z,F)$ is uniformly bounded when $|z| \geqslant 1$. From (2.1) we see that it is sufficient to prove that $\int_{-\infty}^{\infty} N(it) < \infty$.

Let ω be that harmonic measure of the outer circle in $\{z : |z-it| < |t|/2\} \backslash E$. Applying Lemma B with $\rho = |t|/2$, we see that for some $b > 0$, we have

$$B(R/2) \leqslant (1-\alpha(R,E_1))B(R), \, b < R < |t|/2 .$$

It follows from condition (K_1) that for all sufficiently large $|t|$, we have

$$\alpha(R,E_1) \geqslant \gamma > 0, \quad 2|t|^a < R \leqslant |t|/2 .$$

Putting $R_0 = 2|t|^a$, we obtain

$$B(R_0) \leqslant (1-\gamma)^p B(2^p R_0) \leqslant (1-\gamma)^p ,$$

where we can take $2^{p+1}|t|^a \approx |t|$, i.e., $p \approx (1-a) \log |t| / \log 2$. Thus, if $|t|$ is large, we have

$$\beta_E(it) \leqslant \omega(it) \leqslant \text{Const.} \, |t|^{-c} , \tag{4.2}$$

where $c = (1-a)\gamma/\log 2$.

Since $N(z)$ is bounded when $|z| > 1$, it follows from the maximum principle that

$$N(it) \leqslant \text{Const.} \, \beta_E(it) \leqslant \text{Const.} \, |t|^{-c}, \, |t| \geqslant 1 .$$

The Poisson integrals of N in $\{\operatorname{Re} z > 0\}$ and $\{\operatorname{Re} z < 0\}$ are majorants of $N(z)$ in the respective halfplanes. We conclude that

$$N(z) \leqslant \text{Const.} \, |z|^{-c}, \, |z| \geqslant 1 ,$$

provided that $0 < c < 1$.

Repeating the previous argument, we see that

$$N(it) \leqslant \text{Const.}\,|t|^{-c}\beta_E(it) \leqslant \text{Const.}\,|t|^{-2c}, \ |t| \geqslant 1.$$

Continuing in this way, we obtain

$$N(it) \leqslant \text{Const.}\,|t|^{-qc}, \ |t| \geqslant 1,$$

where q is the integer determined by $qc > 1$ and $(q-1)c<1$. (If $qc=1$, we can decrease c slightly so that $qc < 1$, $(q+1)c>1$). Thus, we have $\int_{-\infty}^{\infty} N(it)\,dt < \infty$, and Theorem 4 is proved.

As a second application of our ideas, we consider the class L log L. We say that a closed set E in the complex plane satisfies condition (K_2) if there exist positive numbers $\delta \in (0,1)$ and q such that for all sufficiently large $|t|$, we have

$$\text{(4.3)} \quad \text{cap}\,(E\cap\{z:|z-it|\leqslant R\}) \geqslant \delta R, |t|\,(\log|t|)^{-q} \leqslant R \leqslant |t|/2.$$

In the same way as in the proof of Theorem 4, we define for all sufficiently large $|t|$

$$\gamma = \inf \alpha(R,E_1),\ 2|t|(\log|t|)^{-q} \leqslant R \leqslant |t|/2.$$

THEOREM 5. – *Let* $F \in H^1(U)$ *and assume that* $F(0) = 0$. *If the set* $E = \{z:N(z;F)=0\}$ *satisfies condition* (K_2) *with* $q\gamma > 2\log 2$, *then* $\operatorname{Re} F \in L \log L$.

Proof. – Arguing in the same way as in the proof of Theorem 4 and choosing $R_0 = 2|t|\,(\log|t|)^{-q}$, we have

$$B(R_0) \leqslant (1-\gamma)^p B(2^p R_0) \leqslant (1-\gamma)^p,$$

where we can take $2^p R_0 \approx |t|$, i.e., $p \approx (q/\log 2)\log\log|t|$. Thus, for $|t|$ large,

$$\beta_E(it) \leqslant \omega(it) \leqslant (1-\gamma)^p \leqslant e^{-\gamma p} \approx (\log|t|)^{-q/\log 2} = (\log|t|)^{-(2+\varepsilon)},$$

where $\varepsilon > 0$. Theorem 5 now follows from Theorem 3.

We now point out an immediate consequence of Theorem 3 and some sharp estimates of Benedicks [3].

THEOREM 6. — *Let* $p \geqslant 1$ *be a real number and put*

$$E = \bigcup_{m \neq 0} [\operatorname{sign}(m)|m|^p - d_m, \operatorname{sign}(m)|m|^p + d_m],$$

where $\{d_m\}_{-\infty}^{\infty}$, $0 < d_m < 1/2$, *is a sequence of positive numbers such that*

$$\log d_m \approx \log d_k, \ k \approx m,$$

k, $m \to \infty$ *and* k, $m \to -\infty$. *If* $F \in H^1(U)$ *and* $N(w,F) = 0$, $w \in E$, *a sufficient condition for* $\operatorname{Re} F \in L \log L$ *is that*

$$\Sigma \log (1/d_m) \log m/m^2 < \infty . \tag{4.4}$$

Remark. — It is clear that the set E can be chosen to be a very small subset of the imaginary axis.

Proof. — At the end of the proof of Theorem 5 in [3], Benedicks gives the estimate

$$\beta_E(it) \leqslant \text{Const.} (\log p + (p-1) \log m + \log (1/d_m) + 1)/m,$$
$$m^p \leqslant t \leqslant (m+1)^p, m = 1, 2, \ldots .$$

This gives the convergence of $\int_1^\infty \beta_E(iy) \log y \, dy/y$ provided that (4.4) holds. The argument as $t \to -\infty$ is similar. Thus, Theorem 6 follows from Theorem 3.

5. H^p-classes and harmonic measure.

To apply Theorem 1, we need a geometric criterion on the range of an analytic function F to decide whether $F \in H^1(U)$. Our main tool is the following observation which we state as

THEOREM 7. — *Let* $F : U \to F(U)$ *be analytic with* $F(0) = 0$, *and assume that* $\mathbf{C} \backslash F(U)$ *has positive capacity. Let* ω_R *be the harmonic measure of the outer circle in that component* D_R *of* $\{(z,F(z)) : z \in U, |F(z)| < R\}$ *which contains* $(0,0) = 0$. *Then, for* $0 < p < \infty$, $F \in H^p(U)$ *if and only if*

$$\int_0^\infty R^{p-1} \omega_R(0) \, dR < \infty . \tag{5.1}$$

Remark 1. – Here we understand the range of F to lie on a Riemann surface $\mathscr{R}$, and ω_R to be harmonic measure on $\mathscr{R}$. If F is univalent, it is not necessary to use this terminology: ω_R will be the harmonic measure of the circle $\{w:|w|=R\}$ in that component of $F(U) \cap \{w:|w|<R\}$ which contains the origin. The rest of Theorem 7 will remain unchanged.

Remark 2. – As a corollary, we obtain the following result of Hayman and Weitsman [14]: Let ω'_R be the harmonic measure of the outer circle in $F(U) \cap \{w:|w|<R\}$. Then $F \in H^p(U)$ if

$$\int_0^\infty R^{p-1}\omega'_R(0)\,dR < \infty. \tag{5.1'}$$

This is immediate from Theorem 7 since we have $\omega_R(0) \leqslant \omega'_R(0)$.

Added in proof. – Conversely, if $F \in {}^{\theta p}(U)$, then (5.1′) holds. An argument proving this when F is the universal covering map of U onto V, V such that $\mathbf{C}\backslash V$ has positive capacity, is given in Section 6 of [8*a*]. The general case follows via subordination.

Remark 3. – Theorem 7 is equivalent to a result of Burkholder (Theorem 2.2, p. 189 in [4]). In Section 6, we shall use Theorem 7 to discuss another result of Burkholder (cf. [5], p. 115-116).

Proof of Theorem 7. – Assume that (5.1) holds. We define $F_\rho(z) = F(\rho z)$, $0 < \rho < 1$. Let $R > 0$ be given and let $h_\rho = h_{\rho,R}$ be the harmonic function on U which is 1 on $\{e^{i\theta}:|F_\rho(e^{i\theta})|>R\}$ and 0 on $\{e^{i\theta}:|F_\rho(e^{i\theta})|\leqslant R\}$. Let $\omega_{\rho,R}$ be the harmonic measure of the outer circle in that component $D_{\rho,R}$ of $\{(z,F_\rho(z)):z \in U,|F_\rho(z)|<R\}$ which contains $(0,0) = 0$.

We claim that for $(z,F_\rho(z)) \in D_{\rho,R}$, we have

$$h_\rho(z) \leqslant \omega_{\rho,R}(F_\rho(z)). \tag{5.2}$$

To prove this, we consider

$$E_{\rho,R} = \{z \in U:|F_\rho(z)|<R\}.$$

If $z \in \partial E_{\rho,R} \cap U$, we have $|F_\rho(z)| = R$ and

$$\omega_{\rho,R}(F_\rho(z)) = 1 \geqslant h_\rho(z).$$

If $z \in \partial E_{\rho,R} \cap T$, we have $|F_\rho(z)| \leqslant R$ and

$$h_\rho(z) = 0 \leqslant \omega_{\rho,R}(F_\rho(z)).$$

Hence (5.2) follows from the maximum principle. Since we have $D_{\rho,R} \subset D_R$, we conclude that

$$h_\rho(0) \leqslant \omega_{\rho,R}(0) \leqslant \omega_R(0).$$

We have assumed that the complement of $F(U)$ has positive capacity and thus F is in the Nevanlinna class (cf. R. Nevanlinna [16], p. 209). For almost all R, we have

$$(2\pi)^{-1} m\{e^{i\theta} : |F(e^{i\theta})| > R\} = \lim_{\rho \to 1_-} h_\rho(0) \leqslant \omega_R(0).$$

Since we have (5.1), it is now clear that $F \in H^p(U)$ because

$$\|F\|_p^p = \int_0^\infty (2\pi)^{-1} m\{e^{i\theta} : |F(e^{i\theta})| > R\}\, dR^p \leqslant p \int_0^\infty \omega_R(0) R^{p-1}\, dR < \infty.$$

This concludes the first part of the proof.

Conversely, let us assume that $F \in H^p(U)$ for some $p > 0$. We shall also assume that F is continuous on $U \cup T$. If this is not the case, we argue as in the first part of the proof. Let NF be the nontangential maximal function of F (let the opening angle of the associated Stolz domain be $2\pi/3$ (cf. Petersen [17], p. 8)). Let $H = H_R$ be the harmonic function on U which is 1 on $\{e^{i\theta} : NF(e^{i\theta}) \geqslant R\}$ and 0 on $\{e^{i\theta} : NF(e^{i\theta}) < R\}$. If $|F(z_0)| = R$, where $z_0 = r_0 e^{i\alpha} = (1-\delta)e^{i\alpha}$ with $\delta \in (0,1)$, we have

$$NF(e^{i\theta}) \geqslant R, \ |\theta - \alpha| < \delta,$$

and it follows that

$$H(z_0) \geqslant (2\pi)^{-1} \int_{|\varphi-\alpha|<\delta} (1-r_0^2)(1+r_0^2-2r_0\cos(\varphi-\alpha))^{-1}\, d\varphi$$

$$\geqslant \pi^{-1} \int_0^\delta \delta(\delta^2+t^2)^{-1}\, dt = 1/4.$$

Let $E_R = \{z \in U : |F(z)| < R\}$. We claim that

$$\omega_R(z, F(z)) \leqslant 4H(z), \quad z \in E_R. \tag{5.3}$$

Again, we use the maximum principle. If $z \in \partial E_R \cap U$, we have $|F(z)| = R$ and $4H(z) \geqslant 1$. Thus, (5.3) holds in this case. If $z \in \partial E_R \cap T$, we have either $NF(z) \geqslant R$ and $H(z) = 1$ or $|F(z)| \leqslant NF(z) < R$ and consequently $\omega_R(z,F(z)) = 0 \leqslant 4H(z)$. In both cases, (5.3) is true. In a standard way, we conclude that

$$\omega_R(0) \leqslant (2/\pi)m\{e^{i\theta}:NF(e^{i\theta})\geqslant R\},$$

$$\int_0^\infty \omega_R(0)\, dR^p \leqslant (2/\pi)\|NF\|_p^p \leqslant \text{Const.}\ \|F\|_{H^p}^p.$$

In the last step, we used a result of Hardy and Littlewood (cf. Theorem IV.40, p. 186 in Tsuji [21]). This concludes the proof of Theorem 7.

6. Examples.

All examples F_Φ discussed below satisfy condition (1.1 a), while F_Φ may or may not be in $H^1(U)$. In case $F_\Phi \in H^1(U)$, these examples may be considered to yield variants of Zygmund's Theorem B, mentioned in the Introduction, by means of an obvious subordination argument.

A simple first example is $F_0(z) = 2z(1-z^2)^{-1}$ which maps U onto $\mathbf{C}\backslash\{w=iv:|v|\geqslant 1\}$. Consequently, (1.1 a) is true for F_0. On the other hand, F_0 is not in $H^1(U)$.

We proceed to construct a class of univalent functions $F = F_\Phi$ which are such that $F(U)$ avoids a neighborhood of the imaginary axis near infinity. The function F will be or will not be in $H^1(U)$ depending on the size of this neighborhood. Let

$$D = D(\Phi) = \{z=re^{i\theta}:||\theta|-\pi/2|\leqslant\Phi(r), r\geqslant 2\},$$

where the function Φ will be in one of the following two classes : We say that $\Phi:[2,\infty) \to [0,\pi/3]$ is in Q_1 if Φ is continuous, $\Phi(r) \to 0$, $r \to \infty$, and $\Phi(2) = 0$.

We say that $\Phi:[2,\infty) \to [0,\pi/3]$ is in Q_2 if $\Phi \in Q_1$ and Φ is differentiable with $\Phi' \in L^\infty$ and with $\int_2^\infty r\Phi'(r)^2\, dr < \infty$.

Let $F = F_\Phi$ map U onto $\mathbf{C}\backslash D$ in such a way that $F(0) = 0$. We also introduce $J = J(\Phi) = \int_2^\infty \Phi(r)\, dr/r$.

PROPOSITION. — *If* $\Phi \in Q_2$ *and* $J(\Phi)$ *is finite,* F *will not be in* $H^1(U)$. *If* $\Phi \in Q_1$ *and* $J(\Phi)$ *is infinite, with*

$$\int_2^\infty \left\{\exp\left(-\frac{2}{\pi}\int_2^R \Phi(t)\frac{dt}{t}\right)\right\}\frac{dR}{R} < \infty, \tag{6.1}$$

then $F \in H^1(U)$.

To prove the Proposition, we consider the harmonic measure, ω_R, of the outer circle in $F(U) \cap \{z:|z|<R\}$. From Haliste ([9], formulas (2.1) and (2.3)), we see that if $\Phi \in Q_1$ and R is large enough, we have

$$\begin{aligned}\omega_R(0) &\leqslant (4/\pi)\exp\left(4\pi - \pi\int_2^R (\pi - 2\Phi(t))^{-1}\,dt/t\right)\\ &\leqslant (C_0/R)\exp\left(-\frac{2}{\pi}\int_2^R \Phi(t)\,dt/t\right), \quad C_0 = 8e^{4\pi}.\end{aligned}$$

Now, (6.1) implies $\int_0^\infty \omega_R(0)\,dR < \infty$ and thus $F \in H^1(U)$, by Theorem 7.

From Theorem 2.1 in Haliste [9], we see that if $\Phi \in Q_2$ and R is large enough, we have

$$\omega_R(0) \geqslant C_1 \exp\left(-\pi\int_2^R (\pi - 2\Phi(t))^{-1}\,dt/t \right. \\ \left. - \pi\int_2^R \{t\Phi'(t)^2/(\pi - 2\Phi(t))\}\,dt/3\right), \tag{6.2}$$

where $C_1 = (1/9)\exp(-8\pi(1+4\|\Phi'\|_\infty^2/3))$.

It follows that if $\Phi \in Q_2$ and J is finite, we have

$$\omega_R(0) \approx \exp\left(-\int_2^R (1-2\Phi(t)/\pi)^{-1}\,dt/t\right) \approx \exp(-2J/\pi)/R.$$

Thus, we see that $\int_2^\infty \omega_R(0)\,dR = \infty$. Applying Theorem 7, we see that $F \notin H^1(U)$, and we have proved the first part of the Proposition.

Let us in particular take $\Phi(r) = (\log r)^{-a}$, when $r \geqslant 3$. The associated domain is essentially of the form

$$\{z = x+iy : |x| \leqslant |y|\,(\log|y|)^{-a}, |y| \geqslant 3\}.$$

When $a > 1$, the argument above applies and F_Φ is not in $H^1(U)$. On the other hand, (1.1 a) is clearly true.

When $0 < a < 1$, (6.1) holds and consequently $F_\Phi \in H^1(U)$. It follows from Theorem 1 that $\operatorname{Re} F_\Phi \in L \log L$.

If $\Phi(r) = C(\log r)^{-1}$, $r \geqslant 3$, we have $\omega_R(0) \approx R^{-1}(\log R)^{-2C/\pi}$ when R is large and it follows that

$$(6.3) \qquad F_\Phi \notin H^1(U),\ C \leqslant \pi/2,\ F_\Phi \in H^1(U),\ C > \pi/2.$$

This illustrates the second part of the Proposition. In particular, it follows from Theorem 1 that $\operatorname{Re} F_\Phi \in L \log L$ if $C > \pi/2$.

This last example is related to a problem considered by Burkholder (cf. [5], p. 115-116). Let $S_\delta = \{x+iy : x>1, |y|<\delta x \log x\}$ and let F_δ be a univalent analytic function mapping U onto S_δ. Burkholder uses his theorem on "generalized subordination" to prove that

$$(6.4) \qquad F_\delta \in H^1(U),\ \delta < 2/\pi,\ F_\delta \notin H^1(U),\ \delta > 2/\pi.$$

Using our notation with $D(\Phi) \cap \{\operatorname{Re} z > 0\} = S_\delta$, we have

$$\Phi(r) = (\delta \log r)^{-1} + O(\log\log r/(\log r)^2),\ r \to \infty,$$

and it follows from (6.3) that

$$F_\delta \in H^1(U),\ \delta < 2/\pi,\ F_\delta \notin H^1(U),\ \delta \geqslant 2/\pi.$$

Thus, we obtain Burkholder's result (6.4), as well as the boundary case $\delta = 2/\pi$.

Remark. — Using estimates of harmonic measure in "strip domains", K. Haliste has in [10] given still another method to treat Burkholder's problem, including the boundary case.

The following observation is due to Haliste. Let $T_\delta = \{re^{i\theta} : r>1, |\theta| > p^{-1} \arctan(\delta p \log r)\}$ and let G_δ be a univalent analytic function mapping U onto T. Then

$$G_\delta \in H^p(U), \qquad \delta < 2/\pi, \qquad G_\delta \notin H^p(U), \qquad \delta \geqslant 2/\pi.$$

This result also follows in a simple way from our Theorem 7.

Let us now return to the more general regions $D(\Phi)$ considered earlier. If a function F is such that

$$(6.5) \qquad F(U) \subset \mathbf{C} - D(\Phi)$$

with $J(\Phi)$ finite, then we cannot expect $F \in H^1(U)$. If however we require (6.5) to hold with $D(\Phi)$ replaced by a somewhat larger set, we can achieve $F \in H^1(U)$ and thus will be able to apply Theorem 1. Our last example is of this type.

Let Φ be in Q_1 with $J(\Phi)$ finite and let Ω be a collection of intervals contained in $(-\infty,-2] \cup [2,\infty)$ which is such that for all sufficiently large R and for a constant $c > 1$, we have

$$\int_{\Omega(R)} dt/t \geqslant (c\pi/2)\log\log R\,, \qquad \int_{\Omega(-R)} dt/t \geqslant (c\pi/2)\log\log R\,.$$

Here $\Omega(R) = \Omega \cap [2,R]$ and $\Omega(-R) = \Omega \cap [-R,-2]$.

Let $\Omega_0(R)$ be the one of the two sets $\Omega(R)$ and $\Omega(-R)$ which has the smallest logarithmic length. Let F map U univalently onto the infinite covering surface over $C\backslash(D(\Phi)\cup\Omega)$ in such a way that $F(0) = 0$. From standard estimates of harmonic measure (cf. Tsuji [21], p. 116), we see that

$$\omega_R(0) \leqslant \omega'_R(0) \leqslant \text{Const.}\exp\left(-\left(\int_2^{R/2} + \int_{\Omega_0(R/2)}\right)(1-2\Phi(t)/\pi)^{-1}\,dt/t\right).$$

Thus, for R large, we have

$$\omega_R(0) \leqslant \text{Const.}\, R^{-1}(\log R)^{-c},$$

and consequently $\int_2^{\infty} \omega_R(0)\,dR < \infty$. From Theorem 7, we see that $F \in H^1(U)$. Applying Theorem 1, we conclude that $\operatorname{Re} F \in L\log L$.

Finally, we observe that the function $G(z) = iw/\log^2(1+w)$ with $w = (1+z)/(1-z)$ is in $H^1(U)$, but $\operatorname{Re} G \notin L\log L$. Thus by Theorem 1 the integral (1.1 a) diverges, and in fact $N(1,iv) > (v\log^2 v)^{-1}$ is easy to see, for large v.

7. Extensions of Theorem 2.

Theorem 2 can be extended to meromorphic functions f, provided the subharmonic function Φ is not very large at infinity. We put $M(r,\Phi) = \sup_\theta \Phi(re^{i\theta})$. We have

THEOREM 8. — *Suppose f is meromorphic in $\{z:|z|<\mathrm{R}\}$, where $0 < \mathrm{R} \leqslant \infty$, and that f does not have a pole at the origin. Let Φ be subharmonic in $\mathbf{C}$, with $\Phi(f(0))$ finite, and suppose that for some $\tau \in (0,1)$*

$$\Phi(w) \leqslant \mathrm{O}(|w|^{\tau}), \quad w \to \infty. \tag{7.1}$$

Then, for each r such that f does not have a pole on the circle $\{z:|z|=r\}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi(f(re^{i\theta}))\, d\theta = \int_{\mathbf{C}} (\mathrm{N}(r,w)-\mathrm{N}(r,\infty))\, d\mu(w) + \Phi(f(0)), \tag{7.2}$$

where μ is the Riesz measure of Φ and $0 < r < \mathrm{R}$.

Here the main case of interest is that of Φ small at ∞, in the sense that (7.1) holds for all $\tau > 0$; in this case (7.2) is finite for every $r < \mathrm{R}$. (Compare the Φ in (2.7)-(2.9).)

Proof. — Our assumption (7.1) implies that the following representation for Φ holds on the entire plane (cf. Hayman and Kennedy [12], pp. 141, 146):

$$\Phi(z) = \int_{\{|w|<1\}} \log|z-w|\, d\mu(w) + \int_{\{|w|\geqslant 1\}} \log|zw^{-1}-1|\, d\mu(w) + c, \tag{7.3}$$

where c is a real constant, and $\int_{|w|\geqslant 1} d\mu(w)/|w| < \infty$. Put $z = f(re^{i\theta})$ in (7.3) and integrate $d\theta$, as in the proof of Theorem 2, using Jensen's theorem on $f - w$ or $w^{-1}f - 1$ according as $|w| < 1$ or $|w| \geqslant 1$. Using (7.3) again, with $z = f(0)$, to evaluate c, we obtain (7.2).

We can also extend our results to functions mapping the polydisk or ball of $\mathbf{C}^n$ to $\mathbf{C}$. Let U^n be the unit polydisk in $\mathbf{C}$:

$$\mathrm{U}^n = \{z \in \mathbf{C}^n : |z_j| < 1, j=1,\ldots,n\}.$$

U^n has distinguished boundary

$$\mathrm{T}^n = \{z \in \mathbf{C}^n : |z_1| = \cdots = |z_n| = 1\}.$$

For an n-tuple $\varphi = (\varphi_1,\ldots,\varphi_n)$, $\varphi_j \in [0,2\pi]$, we define a function f_φ on the unit disc by

$$f_\varphi(\zeta) = f(\zeta e^{i\varphi_1},\ldots,\zeta e^{i\varphi_n}).$$

We define a counting function for $w \in \mathbf{C}$ by

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{T^n} N(r,w;f_\varphi)\, d\varphi_1 \ldots d\varphi_n .$$

Here $N(r,w;f_\varphi)$ is the usual one-dimensional counting function for the function f_φ. Jensen's formula is ([18], p. 326):

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{T^n} \log |f(re^{i\varphi_1},\ldots,re^{i\varphi_n}) - w|\, d\varphi_1 \ldots d\varphi_n - \log|f(0)-w| .$$

Now consider the unit ball B^n in $\mathbf{C}^n$:

$$B^n = \{z \in \mathbf{C}^n : \Sigma |z_j|^2 < 1\} .$$

The boundary of B^n is the unit sphere S^{2n-1}. For $z \in S^{2n-1}$ we define a function f_z on the unit disc by $f_z(\zeta) = f(\zeta z)$. For the ball, the counting function is

$$N_f(r,w) = \frac{1}{C_n} \int_{S^{2n-1}} N(r,w;f_z)\, d\sigma(z) .$$

Here the volume element $d\sigma$ is Lebesgue measure on S^{2n-1} and C_n is the volume of S^{2n-1}, i.e. $C_n = \dfrac{2\pi^n}{(n-1)!}$.

In this setting, Jensen's formula is ([20], p. 404):

$$N_f(r,w) = \frac{1}{C_n} \int_{S^{2n-1}} \log|f(rz)-w|\, d\sigma(z) - \log|f(0)-w| .$$

Using these versions of Jensen's formula as in the proof of Theorem 2, we get

THEOREM 9. — *Suppose Φ is subharmonic in the complex plane with Riesz measure μ. If f is holomorphic in the unit polydisk U^n, then*

$$\left(\frac{1}{2\pi}\right)^n \int_{T^n} \Phi(f(re^{i\varphi_1},\ldots,re^{i\varphi_n})\, d\varphi_1 \ldots d\varphi_n = \int_{\mathbf{C}} N(r,w)\, d\mu(w) + \Phi(f(0)) .$$

If f is holomorphic in the unit ball B^n, then

$$\frac{1}{C_n} \int_{S^{2n-1}} \Phi(f(rz))\, d\sigma(z) = \int_{\mathbf{C}} N(r,w)\, d\mu(w) + \Phi(f(0)) .$$

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Manuscrit reçu le 17 févirer 1984.

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