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AN L^p-VERSION OF A THEOREM OF D. A. RAIKOV

by Gero FENDLER(*)

1. Introduction.

Let G be a locally compact group, for $p \in (1, \infty)$, let $Pf_p(G)$ denote the closure of $L^1(G)$ in the convolution operator norm on $L^p(G)$. Denote by $W_p(G)$ the dual of $Pf_p(G)$ which is contained in the space of pointwise multipliers of the Figa-Talamanca Herz space $A_p(G)$. (See [5], [8], [9] for all this.)

It is shown in these notes that on the unit sphere of $W_p(G)$ the weak * (i.e. the $\sigma(W_p, Pf_p)$ topology and the A_p-multiplier topology coincide ($u_\beta \longrightarrow u$ in the latter if $\|(u_\beta - u)v\| \longrightarrow 0$ for each $v \in A_p(G)$).

If $p = 2$ and G is amenable then $W_2(G)$ is just the Fourier-Stieltjes algebra of G, denoted B(G), and $A_2(G)$ is the Fourier algebra of G. From this point of view the above enunciation is an L^p-version of a theorem of D.A. Raikov, which asserts that on the positive face of the unit sphere of B(G) the weak * topology coincides with the topology of uniform convergence on compact sets (since $A_p(G)$ always contains functions which take the value one on a given compact set the latter topology is clearly weaker than the $A_p(G)$ multiplier topology; and on norm bounded sets obviously stronger than the weak * topology).

The proof is based on a technique of G.C. Rota [10], first used in harmonic analysis by E.M. Stein ; our application is close to the work of M. Cowling [3]. On the other hand this paper continues the line of studies taken up by E.E. Granirer and M. Leinert in [7].

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2. An estimate for the L^p-operator norm of the sum of two "spectrally disjoint" operators.

If R and S are two commuting normal (of course bounded) operators on an Hilbert space H then, via the Gelfand transform, R and S correspond to some continuous functions on a locally compact space X; further R, S are spectrally disjoint, if the supports of those functions are disjoint. It then follows easily that $\|R+S\| = \max\{\|R\|, \|S\|\}$; we remark that there exists an orthogonal projection P with $PR = R = RP$ and $(1-P)S = S = S(1-P)$.

From this :

$$
\begin{aligned}
\|(R+S)\xi\| = \|(R+S)(P+1-P)\xi\| &= \|PRP\xi + (1-P)S(1-P)\xi\| \\
&= (\|PRP\xi\|^2 + \|(1-P)S(1-P)\xi\|^2)^{1/2} \\
&\leqslant (\|R\|^2\,\|P\xi\|^2 + \|S\|^2\,\|(1-P)\xi\|^2)^{1/2} \\
&\leqslant \max\{\|R\|, \|S\|\}\,(\|P\xi\|^2 + \|(1-P)\xi\|^2)^{1/2} \\
&\leqslant \max\{\|R\|, \|S\|\}\,\|\xi\| \quad \text{for all } \xi \in H.
\end{aligned}
$$

Now let (X, μ) be a σ-finite measure space ; an operator T acting on all L^p-spaces will be called special if :

i) $Tf \geqslant 0$ if $f \geqslant 0$

ii) $\|Tf\|_p \leqslant \|f\|_p \quad f \in L^p(X, \mu),\ 1 \leqslant p \leqslant \infty$

iii) $T1_X = 1_X$

iv) $\int_X Tf(x)\,\overline{g(x)}\,d\mu(x) = \int_X f(x)\,\overline{Tg(x)}\,d\mu(x) \quad f, g \in L^2(X, \mu)$.

Those operators will serve as a substitute for orthogonal projections, since by a method due to G.C. Rota they may be seen as conditional expectations on a certain measure space.

We begin with the following observation :

PROPOSITION. – *Let* $(Y, \mathfrak{F}, \nu)$ *be a* σ*-finite measure space,* $\mathfrak{F}_1 \subset \mathfrak{F}$ *a sub-*σ*-algebra of* $\mathfrak{F}$ *such that* $(Y, \mathfrak{F}_1, \nu)$ *is again* σ*-finite, (which ensures the existence of a conditional expectation operator* E_1 *with respect to* $\mathfrak{F}_1$*).*

Then we have for $\xi, \eta \in L^p(Y, \mathfrak{F}, \nu)$:

$$\|E_1 \xi + (1 - E_1)\eta\|_p \leqslant (\|\xi\|_p^r + 2\|\eta\|_p^r)^{1/r}$$

where $r = p$ if $1 \leqslant p \leqslant 2$ *and* $r = p'$, *the index conjugate to* p, *if* $2 \leqslant p \leqslant \infty$.

Proof. – Clearly

1) $\|E_1 \xi + (1 - E_1)\eta\|_1 \leqslant \|\xi\|_1 + 2\|\eta\|_1$

2) $\|E_1 \xi + (1 - E_1)\eta\|_2^2 \leqslant \|\xi\|_2^2 + \|\eta\|_2^2 \leqslant \|\xi\|_2^2 + 2\|\eta\|_2^2$

3) $\|E_1 \xi + (1 - E_1)\eta\|_\infty \leqslant \|\xi\|_\infty + 2\|\eta\|_\infty$

and the assertion follows from interpolation between 1) and 2) (resp. 2) and 3)) on mixed $l^p(L^q)$-spaces (see [1]).

Let (X, μ) and T be as above. Define $Y = X \times X$ and endow Y with the usual product σ-algebra denoted $\mathfrak{F}$. We define a measure ν on Y by requiring that

$$\nu(S_0 \times S_1) = \int_X \chi_{S_0}(x)\, T\chi_{S_1}(x)\, d\mu(x)$$

(whenever S_0, S_1 are measurable subsets of X).

Denote by $\mathfrak{F}_1$ and $\mathfrak{F}_0$ the σ-algebras of sets $X \times S (S \subseteq X$ measurable$)$, respectively of sets $S \times X$ ($S \subseteq X$ measurable), further denote by E_1, E_0 the corresponding conditional expectation operators. For a measurable function ξ on X we define for $x = (x_0, x_1) \in Y$

$$\xi^i(x_0, x_1) = \xi(x_i) \quad i = 0, 1.$$

Then $\xi \longrightarrow \xi^0$ gives rise to an isometric isomorphism between $L^p(X, \mu)$ and the subspace of $\mathfrak{F}_0$-measurable elements of $L^p(Y, \nu)$; whereas $\xi \longrightarrow \xi^1$, from $L^p(X, \mu)$ to $L^p(Y, \nu)$, does not increase norms.

Further :

$$E_0(\xi^i) = \begin{cases} (T\xi)^0 & \text{if } i = 1 \\ \xi^0 & \text{if } i = 0 \end{cases}$$

$$E_1(\xi^0) = (T\xi)^1 .$$

For a proof of these facts we refer the reader to the book of E.M. Stein [11].

PROPOSITION. – *Let* (X, μ) *be a* σ*-finite measure space,* T *a special operator and* $1 < p < \infty$. *Then for* $\xi_1, \xi_2 \in L^p(X, \mu)$: $\|T^2 \xi_1 + (1 - T^2) \xi_2\| \leqslant (\|\xi_1\|_p^r + 2 \|\xi_2\|_p^r)^{1/r}$, *with* $r = \min\{p, p'\}$.

Proof. – We apply the above procedure to T, then

$$\begin{aligned} \|T^2 \xi_1 + (1 - T^2) \xi_2\| &= \| E_0((T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1)\| \\ &\leqslant \|(T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1\| \\ &= \|E_1(\xi_1^0) + (1 - E_1)(\xi_2^0)\| \\ &\leqslant (\|\xi_1^0\|^r + 2 \|\xi_2^0\|^r)^{1/r}. \end{aligned}$$

COROLLARY. – *Let* R, S *be bounded operators on* $L^p(X, \mu)$, *then we have*

$$\|T^2 R + (1 - T^2) S\| \leqslant (\|R\|^r + 2 \|S\|^r)^{1/r} .$$

3. The weak* topology on the unit sphere of $W_p(G)$.

Let G be a locally compact group, with a fixed left Haar measure dg and modular function Δ. Let $L^p(G)$, $1 \leqslant p \leqslant \infty$, denote the usual Lebesgue spaces with respect to dg and for functions f, h on G let be defined $f * h(x) = \int_G f(g) h(g^{-1} x)\, dg$, $f^\sim(g) = f(g^{-1}) \Delta(g^{-1})$, $f^* = \bar{f}^\sim$, $f^\vee(g) = f(g^{-1})$.

For this section let now $p \in (1, \infty)$ be fixed and let $A_p(G)$ (as in [8]) be the algebra of functions u on G which can be represented as $u = \sum_{n=1}^{\infty} v_n * w_n^\vee$, where

$$\sum_n \|v_n\|_p \cdot \|w_n\|_{p'} < \infty , \quad \frac{1}{p} + \frac{1}{p'} = 1 .$$

The norm on A_p is defined as the inf $\Sigma \|v_n\|_{p'} \|w_n\|_p$ taken over all such representations of u.

If f is an element of $L^1(G)$ then on one hand $w \longmapsto f * w$ defines a convolution operator on $L^p(G)$ and on the other $u \longmapsto \int_G f(g)\, u(g)\, dg$ a continuous linear functional on $A_p(G)$. From $\langle f, v * w^\vee \rangle = \langle f * w , v \rangle$ it follows that the corresponding norms of f coincide.

Let $Pf_p(G)$ denote the closure of $L^1(G)$ in the algebra of convolution operators on $L^p(G)$ and $W_p(G)$ the dual space of $Pf_p(G)$, which is contained in $L^\infty(G)$, and in which $A_p(G)$ is norm non-increasingly embedded.

If t is a nonnegative (almost everywhere) function with $\|t\|_1 = 1$ then $t * t^\sim$, as a convolution operator, is almost a special operator, except that (G, dg) might not be σ-finite.

Let U_α be an open relatively compact neighborhood base at the identity e of G. If $V_\alpha = V_\alpha^{-1}$ are open neighborhoods of e such that $V_\alpha^2 \subset U_\alpha$ then $\tau_\alpha = \lambda(V_\alpha)^{-1} \chi_{V_\alpha}$, where $\lambda(V)$ denotes the Haar measure of V and χ_V its characteristic function, $t_\alpha = \tau_\alpha * \tau_\alpha^\sim$ and $e_\alpha = t_\alpha * t_\alpha$ are approximate identities for $L^1(G)$, e_α being the square of a "special" operator. This last fact we seem really to need in the proof of the following

LEMMA. – *Let* $e_\alpha = t_\alpha * t_\alpha$ *be as above, if* u_β *is a net in* $W_p(G)$ *such that* $u_\beta \longrightarrow u_0$ *in the weak* topology of* $W_p(G)$ *and if* $\|u_\beta\|_{W_p} \longrightarrow \|u_0\|_{W_p}$, *then for* $\epsilon > 0$ *there exist* β_0, α_0 *such that*

i) $\|e_{\alpha_0} * u_\beta - u_\beta\|_{W_p} \leqslant \epsilon$ *for all* $\beta \geqslant \beta_0$

and

ii) $\|e_{\alpha_0} * u_0 - u_0\|_{W_p} \leqslant \epsilon$.

Proof. – Clearly ii) is a consequence of i), so it is enough to prove i) and we may assume that $\|u_0\| = 1$. We suppose now that there is a net u_β which converges to u_0 as described in the lemma and an $\epsilon > 0$ such that for all α_0, β_0 there exists $\beta > \beta_0$ with

$$\|e_{\alpha_0} * u_\beta - u_\beta\| > \epsilon .$$

We shall derive a contradiction.

Let $0 < \eta < \epsilon/2$, to be specified later, and choose $f \in L^1(G)$ with

$$\|f\|_{Pf_p} = 1, \langle f, u_0 \rangle \geqslant 1 - \eta,$$

then choose α_0 with

$$\|e_{\alpha_0} * f - f\|_{Pf_p} \leqslant \eta$$

and β_0 with

$$|\langle u_\beta, e_{\alpha_0} * f \rangle - \langle u_0, e_{\alpha_0} * f \rangle| \leqslant \eta,$$

$$\|u_\beta\| \leqslant 1 + \eta \text{ for all } \beta \geqslant \beta_0.$$

We may now fix $\beta > \beta_0$ with

$$\|e_{\alpha_0} * u_\beta - u_\beta\|_{w_p} > \epsilon$$

and find $g \in L^1(G)$, $\|g\|_{Pf_p} = 1$, with

$$\langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle > \epsilon - \eta$$

i.e. $\langle u_\beta, (e_{\alpha_0} - 1) * g \rangle = \langle u_\beta, (1 - e_{\alpha_0}) * (-g) \rangle > \epsilon - \eta$.

Now, the supports of t_{α_0}, f, g are contained in a σ-finite open subgroup G_0 of G. Since for an $L^1(G)$ function h with support in G_0 : $\|h\|_{Pf_p(G_0)} = \|h\|_{Pf_p(G)}$, we may apply the estimation of the corollary of the last section to $e_{\alpha_0} * f - \lambda g + \lambda e_{\alpha_0} * g$, where $\lambda > 0$:

$$\|e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g)\| \leqslant (\|f\|^r + 2\|-\lambda g\|^r)^{1/r} = (1 + 2\lambda^r)^{1/r}.$$

So on one hand

$$\begin{aligned}\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle &\leqslant \|u_\beta\| (1 + 2\lambda^r)^{1/r} \\ &\leqslant (1 + \eta)(1 + 2\lambda^r)^{1/r},\end{aligned}$$

and on the other

$$\begin{aligned}|\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle| &= |\langle u_0, f \rangle + \langle u_0, e_{\alpha_0} * f - f \rangle \\ &+ \langle u_\beta - u_0, e_{\alpha_0} * f \rangle + \lambda \langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle| \geqslant 1 - 3\eta + \lambda\epsilon/2.\end{aligned}$$

But $1 - 3\eta + \lambda\epsilon/2 \leqslant (1 + \eta)(1 + 2\lambda^r)^{1/r}$ cannot hold for all $\eta \in (\epsilon/2, 0), \lambda > 0$.

We thank the referee for pointing out to us the following implication of the lemma (due to M. Cowling, theorem 3 of [3] ; see [4] for a different proof).

COROLLARY. — *Translations act continuously on* $W_p(G)$.

Proof. — For $h \in G$ let ${}_h u(g) = u(h^{-1} g)$ and $u_h(g) = u(gh)$, $g \in G$.

We first consider left translations, if u is in $W_p(G)$, $\epsilon > 0$ then we find, by the lemma, an element e of $L^1(G)$ with

$$\|e * u - u\|_{W_p} \leqslant \epsilon .$$

Then

$$\begin{aligned}\|{}_h u - u\|_{W_p} &\leqslant \|{}_h u - {}_h(e * u)\|_{W_p} + \|{}_h(e * u) - e * u\|_{W_p} \\ &\qquad + \|e * u - u\|_{W_p} \\ &\leqslant \|u - e * u\|_{W_p} + \|{}_h e - e\|_1 \, \|u\|_{W_p} + \|e * u - u\|_{W_p} \\ &\leqslant 3\epsilon \quad \text{if } h \text{ is in a neighborhood V of the identity,}\end{aligned}$$

choosen such that $\|{}_h e - e\|_1 \leqslant \epsilon \|u\|_{W_p}^{-1}$ for all $h \in V$.

From $\|f\|_{Pf_p} = \|f^{\sim}\|_{Pf_{p'}}$, for $f \in L^1(G)$, we infer that $\|u\|_{W_p} = \|u^{\vee}\|_{W_{p'}}$ for $u \in W_p(G)$, and hence the continuity of right translations, on W_p, follows from that of left translations on $W_{p'}$.

It has been proved by Herz [8], that for $v \in A_p(G)$ and $u \in W_p(G)$ the pointwise product $u \cdot v$ is in $A_p(G)$ and $\|u \cdot v\|_{A_p} \leqslant \|u\|_{W_p} \|v\|_{A_p}$.

We say that a net $u_\beta \in W_p(G)$ converges to $u \in W_p$ in the A_p-multiplier topology, if, for all $v \in A_p$, $u_\beta v \longrightarrow uv$ in A_p norm.

THEOREM. — *On the unit sphere* $S = \{u \in W_p / \|u\|_{W_p} = 1\}$ *of* $W_p(G)$ *the weak* and the* A_p*-multiplier topology coincide.*

Proof. — Let $u_\beta, u \in S$ be such that $u_\beta \longrightarrow u$ in the weak $*$ topology. Let $e_\alpha = t_\alpha * t_\alpha$ be as in the lemma. Then for $v \in A_p(G)$

$$\begin{aligned}\|u_\beta v - uv\| &\leqslant \|(u_\beta - e_{\alpha_0} * u_\beta) v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| \\ &\qquad + \|(e_{\alpha_0} * u - u) v\| \\ &\leqslant \epsilon \|v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| + \epsilon \|v\| ,\end{aligned}$$

when $\beta \geqslant \beta_0$, where α_0, β_0 are choosen according to the lemma.

Since $t_{\alpha_0} \in L^1(G) \cap L^\infty(G)$ has compact support we may apply lemma 6 of [7] and find $\beta_1 \geqslant \beta_0$ such that for $\beta \geqslant \beta_1$ $\|[e_{\alpha_0} * (u_\beta - u)]\, v\| \leqslant \epsilon$.

For the converse it is sufficient to note that $u_\beta \longrightarrow u$ uniformly on compact sets, whenever $u_\beta \longrightarrow u$ in the A_p-multiplier topology and $\|u\|_{W_p}$ is bounded. So, for a compact set K, let $v \in A_p(G)$ be a function which takes the value one on K (e.g. take $v = \lambda(U)^{-1}\, \chi_U * \chi_{K^{-1}U}$, where U is open, relatively compact) then

$$\sup_{g \in K} |(u_\beta - u)(g)| \leqslant \|(u_\beta - u)\, v\|_\infty \leqslant \|(u_\beta - u)\, v\|_{A_p} \longrightarrow 0 .$$

The following corollary is of interest with respect to the problems considered in [6]. To state it, let, for a compact set $K \subset G$, $A_K^p(G) = \{v \in A_p(G)/\operatorname{supp} v \subset K\}$. This space we consider as a subspace of $W_p(G)$.

COROLLARY. – On the unit sphere of $(A_K^p(G), \|\cdot\|_{W_p})$ the weak $*$ and the norm topology coincide.

Proof. – Let $u_\beta, u \in A_K^p(G)$ be such that $u_\beta \longrightarrow u$ in the weak $*$ topology and $\|u_\beta\|_{W_p} = 1 = \|u\|_{W_p}$. Then, for $v \in A_k^p(G)$ which is constant one on K,

$$\|u_\beta - u\|_{W_p} = \|(u_\beta - u)\, v\|_{W_p} \leqslant \|(u_\beta - u)\, v\|_{A_p} \longrightarrow 0$$

by our theorem. The converse is evident.

4. Addendum.

When the paper was already finished we realized that, by our method, we can improve a theorem of E.E. Granirer, theorem 3 of [6], which we think to be central in the cited paper.

Let $MA_p(G)$ be the algebra of (continuous, bounded) functions on G which pointwise multiply $A_p(G)$ into itself and let for $u \in MA_p(G)$ $\|u\|_{MA_p} = \sup\{\|uv\|_{A_p} / \|v\|_{A_p} = 1\}$.

THEOREM. – *Let $u \in MA_p(G)$ be such that $u(g) = \|u\|_{MA_p}$ for an $g \in G$. If u_β is a net in $MA_p(G)$ such that*

$$\|u_\beta\|_{MA_p} \longrightarrow \|u\|_{MA_p}$$

and $u_\beta \longrightarrow u$ *in the* $\sigma(\mathrm{MA}_p(\mathrm{G}), \mathrm{L}^1(\mathrm{G}))$*-topology then* $u_\beta \longrightarrow u$ *in the* A_p*-multiplier topology.*

To prove this theorem we need an auxillary result for whose proof we use that we admit complex scalars for our linear spaces.

PROPOSITION. – *The linear span of* $\{v \in \mathrm{A}_p(\mathrm{G})/v(e) = \|v\|_{\mathrm{A}_p},$ *v has compact support$\}$ is norm dense in* $\mathrm{A}_p(\mathrm{G})$.

Proof. – The dual space of $\mathrm{A}_p(\mathrm{G})$ is the ultra weak operator topology closure of $\mathrm{P}f_p(\mathrm{G})$ in the space of bounded operators on $\mathrm{L}^p(\mathrm{G})$, the duality is given by

$$\langle \mathrm{T}, u \rangle = \sum_{n=1}^{\infty} \int_{\mathrm{G}} \mathrm{T}w_n(g)\, v_n(g)\, dg$$

when $u = \sum_{n=1}^{\infty} v_n * w_n^\vee \in \mathrm{A}_p(\mathrm{G}), \mathrm{T} \in \mathrm{A}_p(\mathrm{G})^*$ (see [9]).

By theorem 4.1 and theorem 9.4 of [2] we have

$$e^{-1}\, \|\mathrm{T}\| \leqslant \sup \{ \langle \mathrm{T}f, f^{\#} \rangle / f \in \mathrm{L}^p(\mathrm{G}),\ \|f\|_p = 1 \},$$

where $f^{\#} = |f|^{p-1} \exp(-i \ \arg(f(\,.\,)))$ is the unique element of $\mathrm{L}^{p'}(\mathrm{G})$ with $\langle f, f^{\#} \rangle = 1$ and norm one.

If we approximate $f \in \mathrm{L}^p(\mathrm{G})$ by $f.\chi_{\mathrm{K}}$, where $\mathrm{K} \subset \mathrm{G}$ is a suitable compact set, in the L^p-norm, then $(f\chi_{\mathrm{K}})^{\#} = f^{\#}\chi_{\mathrm{K}}$ approximates $f^{\#}$ in $\mathrm{L}^{p'}$-norm. This is why we can restrict the supremum to be taken over the elements $f \in \mathrm{L}^p(\mathrm{G})$ with compact support and norm one.

If $f \in \mathrm{L}^p(\mathrm{G})$ has compact support then $v = f^{\#} * f^{\vee}$ will have compact support too, and if $\|f\|_p = 1$ then,

$$1 = \|f\|_p\, \|f^{\#}\|_{p'} \geqslant \|v\|_{\mathrm{A}_p} \geqslant \|v\|_\infty = f^{\#} * f^{\vee}(e) = \|f\|_p^p = 1\,.$$

Hence for any $\mathrm{T} \in \mathrm{A}_p(\mathrm{G})^*$:

$$e^{-1}\, \|\mathrm{T}\| \leqslant \sup \{\langle \mathrm{T}, v \rangle / \, v(e) = \|v\|_{\mathrm{A}_p},\ v \text{ has compact support}\},$$

and the proposition follows by an application of the Hahn-Banach theorem.

Proof of the theorem. – We may assume $\|u\|_{MA_p} = 1$ and, since translations are isometries of $MA_p(G)$, we may further assume $u(e) = \|u\|_{MA_p} = 1$.

Since there exists β_0 such that $\sup\{\|u_\beta\|_{MA_p}/\beta \geq \beta_0\} < \infty$ it suffices, by the above proposition, to show $u_\beta\, v \longrightarrow uv$ when v has compact support, say K, and $v(e) = \|v\|_{A_p} = 1$. Now, the $u_\beta\, v$ and uv are elements of $A^p_K(G)$, and on this space the W_p-norm is equivalent to the A_p-norm (this follows from proposition 1 of [6] and proposition 3 of [8]). Thus we must only show $\|u_\beta\, v - uv\|_{W_p} \longrightarrow 0$.

Clearly, $u_\beta\, v \longrightarrow uv$ in the weak* topology of $A^p_K(G)$, and, if we can show that $\lim\|u_\beta\, v\|_{W_p} = \|uv\|_{W_p}$, then the corollary of the last section finishes the proof.

But,

$$1 = u(e)\,v(e) \leq \|uv\|_{W_p} \leq \liminf \|u_\beta\, v\|_{W_p}$$

and

$$1 = u(e)\,v(e) = \|u\|_{MA_p}\,\|v\|_{A_p} = \lim \|u_\beta\|_{MA_p}\,\|v\|_{A_p}$$

$$\geq \limsup \|u_\beta\, v\|_{A_p} \geq \limsup \|u_\beta\, v\|_{W_p}$$

from which $\lim \|u_\beta\, v\|_{W_p} = 1 = \|uv\|_{W_p}$ follows.

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