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NEARLY FINITE CHACON TRANSFORMATION

TRANSFORMATION DE CHACON PRESQUE FINIE

Abstract. — We construct an infinite measure preserving version of Chacon transformation, and prove that it has a property similar to Minimal Self-Joinings in finite measure: its Cartesian powers have as few invariant Radon measures as possible.

Résumé. — Nous construisons une version de la transformation de Chacon en mesure infinie, et prouvons qu'elle satisfait une propriété similaire aux autocouplages minimaux en mesure finie : ses puissances cartésiennes ont aussi peu de mesures de Radon invariantes que possible.

1. Introduction

1.1. Motivations

The purpose of this work is to continue the study, started in [JRdlR18] and [Dan18], of what the Minimal Self-Joinings (MSJ) property could be in the setting of infinite-measure preserving transformations. We want here to construct an infinite measure

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preserving transformation whose Cartesian powers have as few invariant measures as possible. As in the aforementioned papers, we restrict ourselves to Radon measures (giving finite mass to compact sets), since in general there are excessively many infinite invariant measures for a given transformation (think of the sum of Dirac masses along an orbit).

A first attempt in this direction was to consider the so-called *infinite Chacon transformation* introduced in [AFS97]. Indeed, the construction of this infinite measure preserving rank-one transformation is strongly inspired by the classical finite measure preserving Chacon transformation, which enjoys the MSJ property [dJRS80]. The identification of invariant measures for Cartesian powers of the infinite Chacon transformation was the object of our previous work [JRdlR18]. In addition to the products of graph measures arising from powers of the transformation (see the beginning of Section 3.3 for details), we found in the case of infinite Chacon some kind of unexpected invariant measures, the so-called *weird measures*. These weird measures have marginals which are singular with respect to the original invariant measure, but it is shown in [Dan18, Example 5.4] that an appropriate convex combination of weird measures can have absolutely continuous marginals.

We propose here another rank-one transformation, which we call the *nearly finite Chacon transformation*, hereafter denoted by T. Although it preserves an infinite measure μ, its construction is designed to mimic as much as possible the behaviour of the classical Chacon transformation, so that the phenomenon of weird measures disappears. Our main result, Theorem 3.10, is the following: there exists a μ-conull set X_∞ such that, for each $d \geqslant 1$, the ergodic $T^{\times d}$-invariant Radon measures on X_∞^d are the product measure $\mu^{\otimes d}$ and products of graph measures arising from powers of T. Corollary 3.11 then identifies all $T^{\times d}$-invariant Radon measures whose marginals are absolutely continuous with respect to μ as sums of countably many ergodic components which are of the form given in the theorem.

Beyond the question of the MSJ property in the infinite measure world, the example presented in this paper is also of crucial importance in the study of Poisson suspensions. A Poisson suspension is a finite measure preserving dynamical system constructed from an infinite measure preserving system: a state of the space is a realization of a Poisson point process whose intensity is the infinite invariant measure, and each random point evolves according to the dynamics of the infinite measure preserving transformation (we refer to [Roy07] for a complete presentation of Poisson suspensions). Although of different nature, Poisson suspensions share surprising properties with another category of finite measure preserving dynamical systems of probabilistic origin: Gaussian dynamical systems, which are constructed from finite measures on the circle. A beautiful theory has been developed in [LPT00], concerning a special class of Gaussian systems called GAGs (a French acronym for *Gaussian systems with Gaussian self-joinings*). The keystone for the construction of a GAG system is a striking theorem due to Foiaş and Strătilă [FS67]: if a measure supported on a Kronecker subset of the circle appears as the spectral measure of some ergodic stationary process, then this process is Gaussian. The Poisson counterpart of GAG, called $\mathcal{P}a\mathcal{P}$ (Poisson suspension with Poisson self-joinings) is presented in [JRdlR17], where the construction of a $\mathcal{P}a\mathcal{P}$ example relies on a theorem *à la*

Foiaş-Strătilă (see [JRdlR17, Theorem 3.4]). Roughly speaking, according to this theorem, if some ergodic point process evolves under a dynamics directed by an infinite measure preserving transformation *with special properties*, then this point process is Poissonian. The special properties needed here are precisely those given by Corollary 3.11. Therefore, systems enjoying those properties play in the theory of Poisson suspensions the same role as measures supported on Kronecker subset in the setting of Gaussian systems.

For some applications in the study of Poisson suspensions developed in [JRdlR17], we also need an additional property which is the existence of a measurable law of large numbers. Proposition 8.4 shows that the nearly finite Chacon transformation satisfies a stronger property called rational ergodicity.

1.2. Roadmap of the paper

Section 2 is devoted to the construction of the nearly finite Chacon transformation, and to first elementary results. For pedagogical reasons, we start in Section 2.1 by defining the nearly finite Chacon transformation with the cutting-and-stacking method on $\mathbb{R}_+$ equipped with the Lebesgue measure, as it is easier to visualize the structure of the Rokhlin towers in this setting. Most steps of the construction are identical to construction of the classical Chacon transformation. There is just a fast increasing sequence (n_ℓ) of integers such that each n_ℓ-th step of the construction differs from classical Chacon, which ensures that the invariant measure has infinite mass. Then we turn in Section 2.2 to a more convenient (but isomorphic) model for our purposes, which is a transformation T on a set X of sequences on a countable alphabet. In Section 2.3, we describe basic properties of a typical point with respect to the invariant measure μ, and define the conull set X_∞ referred to in Theorem 3.10.

Section 3 contains the main results concerning Radon measures on X^d which are $T^{\times d}$-invariant. Section 3.1 first states some basic facts about Radon measures on X^d. We give a criterion for such a measure to be $T^{\times d}$-invariant (Lemma 3.2). We also define a notion of convergence of Radon measures (Definition 3.3), which is specially adapted to the formulation of Hopf's ratio ergodic theorem, and give useful lemmas concerning this convergence. In Section 3.2, we treat the easy case of totally dissipative measures: Proposition 3.9 eliminates the possibility of a totally dissipative $T^{\times d}$-invariant Radon measure supported on X_∞^d. In Section 3.3, we state our main result (Theorem 3.10) and establish the bases of a proof by induction on d. At the end of Section 3.3, we fix once and for all a $T^{\times d}$-invariant Radon measure σ supported on X_∞^d, for some $d \geqslant 2$. The remainder of the paper is completely devoted to proving that either σ is a graph measure arising from powers of T, or it can be decomposed as a product of two measures on which we can apply the induction hypothesis. In Section 3.4, we choose once and for all a σ-typical point $x \in X_\infty^d$, on the orbit of which we estimate the properties of σ. We also introduce in Definition 3.12 the central notion of *n-crossings*, which are finite subintervals of $\mathbb{Z}$ depending on the position of the orbit of the typical point x with respect to the n-th Rokhlin tower of the rank-one construction. The analysis of the structure of those n-crossings constitutes the core of our proof. In Section 3.5, we provide a criterion for σ to be a graph measure arising from powers of T, stated in terms of n-crossings (Proposition 3.17).

Section 4 is devoted to the proof of Proposition 4.1, which is a central result in the analysis of the structure of n-crossings. Section 4.1 describes a hierarchy of abstract subsets of $\mathbb{Z}$ and provides a lemma on the combinatorics of subsets in this hierarchy (Lemma 4.2). Then Section 4.2 explains how to apply this lemma to the structure of n-crossings. Section 4.3 provides a useful corollary of Proposition 4.1 in terms of the measure σ.

Section 5 is devoted to the study of the convergence of *empirical measures*, which are finite sums of Dirac masses on points on the orbit of x, corresponding to finite subsets of $\mathbb{Z}$. We provide two criteria, Proposition 5.7 and Proposition 5.9, for a sequence of such empirical measures to converge to σ.

In Section 6 we present the main tool used to decompose σ as a product measure. We introduce the notion of *twisting transformation* (Definition 6.1), which is simply a transformation of X^d acting as T on some coordinates, and as Id on others. Based on a theorem from [JRdlR18], Proposition 6.2 shows that, if σ is invariant by such a twisting transformation, then σ decomposes as a product measure to which we can apply the induction hypothesis. Then Proposition 6.3 provides a criterion for σ to be invariant by some twisting transformation.

All the preceding tools are used in Section 7, where the proof of Theorem 3.10 is completed. If the criterion given by Proposition 3.17 for σ to be a graph measure fails, then for infinitely many integers n there exists some n-crossing, not too far from 0, with some special property. We treat several cases according to the positions of these integers with respect to the sequence (n_ℓ). With the help of Propositions 6.3 and 6.2, we show that in all cases σ decomposes as a product of two measures to which the induction hypothesis applies.

The last (short) section is devoted to some additional properties of the nearly finite Chacon transformation. We give some corollaries of the main result concerning the commutant and the factors of the transformation, and we also deal with rational ergodicity and the existence of a measurable law of large numbers.

2. Construction of the nearly finite Chacon transformation

2.1. Cutting-and-stacking construction on $\mathbb{R}_+$

As previously explained, this transformation is designed to mimic the classical finite measure preserving Chacon transformation as much as possible, yet it must preserve an infinite measure. The construction will make use of two predefined increasing sequences of integers: $1 \ll n_1 \ll n_2 \ll \cdots \ll n_\ell \ll \cdots$ and $\ell_0 := 1 \ll \ell_1 \ll \ell_2 \ll \cdots \ll \ell_k \ll \cdots$, satisfying some growth conditions to be specified later (see below conditions (2.1) and (2.2)). For each $\ell \geqslant 1$, there exists a unique integer $k \geqslant 0$ such that $\ell_k \leqslant \ell < \ell_{k+1}$, and we denote this integer by $k(\ell)$.

In the first step we consider the interval $[0, 1)$, which is cut into three subintervals of equal length. We take the extra interval $[1, 4/3)$ and stack it above the middle piece. Then we stack all these intervals left under right, getting a tower of height

$h_1 := 4$. The transformation T maps each point to the point exactly above it in the tower. At this step T is yet undefined on the top of the tower.

After step n we have a tower of height h_n, called tower n, the levels of which are intervals of length $1/3^n$. These intervals are closed to the left and open to the right. At step $(n+1)$, tower n is cut into three subcolumns of equal width. If $n \notin \{n_\ell : \ell \geqslant 1\}$, we do as in the standard finite measure preserving Chacon transformation: we add an extra interval of length $1/3^{n+1}$ above the middle subcolumn, and we stack the three subcolumns left under right to get tower $n+1$ of height $h_{n+1} = 3h_n + 1$. If $n = n_\ell$ for some ℓ, we add $h_{n-k(\ell)}$ extra intervals above each of the three subcolumns, and a further extra interval above the second subcolumn. Then we stack the three subcolumns left under right and get tower $n+1$ of height $h_{n+1} = 3h_n + 3h_{n-k(\ell)} + 1$. (See Figure 2.1.)

At each step, we pick the extra intervals successively by taking the leftmost interval of desired length in the unused part of $\mathbb{R}_+$. Extra intervals used at step $n+1$ are called $(n+1)$-*spacers*.

We want the Lebesgue measure of tower n to increase to infinity, which is easily satisfied provided the sequence ℓ_k grows sufficiently fast. Indeed, for each $n \geqslant 1$ we have $h_{n+1} \leqslant 6h_n + 1 \leqslant 7h_n$, whence $h_n/h_{n+1} \geqslant 1/7$. It follows that for each $k \geqslant 0$ and each $\ell_k \leqslant \ell < \ell_{k+1}$,

$$\mathrm{Leb}(\text{tower } n_\ell + 1) \geqslant \mathrm{Leb}(\text{tower } n_\ell) \left(1 + \frac{h_{n_\ell - k}}{h_{n_\ell}}\right) \geqslant \left(1 + 7^{-k}\right).$$

Therefore it is enough for example to assume that for each $k \geqslant 0$,

$$\left(1 + 7^{-k}\right)^{\ell_{k+1} - \ell_k} \geqslant 2. \tag{2.1}$$

Under this assumption, we get at the end a rank-one transformation defined on $\mathbb{R}_+$ which preserves the Lebesgue measure.

We will also assume that for each ℓ,

$$n_\ell > n_{(\ell-1)} + 2\ell. \tag{2.2}$$

2.2. Construction on a set of sequences

For technical reasons, it will be more convenient to consider a model of the nearly finite Chacon transformation in which the ambient space is a totally disconnected non compact metric space, and each level of each tower is a compact clopen set.

Consider the countable alphabet $\mathbb{A} := \{*\} \cup \mathbb{N}$. To each $t \in \mathbb{R}_+$, we associate the sequence $\varphi(t) = \left(j_n(t)\right)_{n \geqslant 0} \in \mathbb{A}^{\mathbb{N}}$ defined by

$$j_n(t) := \begin{cases} * & \text{if } t \notin \text{tower } n, \\ j & \text{if } t \text{ is in level } j \text{ of tower } n \ (0 \leqslant j < h_n). \end{cases}$$

By condition (2.1), $\mathbb{R}_+ = \bigcup_n$ tower n, and for each n, tower $n \subset$ tower $n+1$. Hence for each $t \in \mathbb{R}_+$,

$$\exists\, n_0 \geqslant 0 : \forall\, n < n_0,\ j_n(t) = *,\ \text{and}\ \ \forall\, n \geqslant n_0, j_n(t) \in \{0, \dots, h_n - 1\}.$$

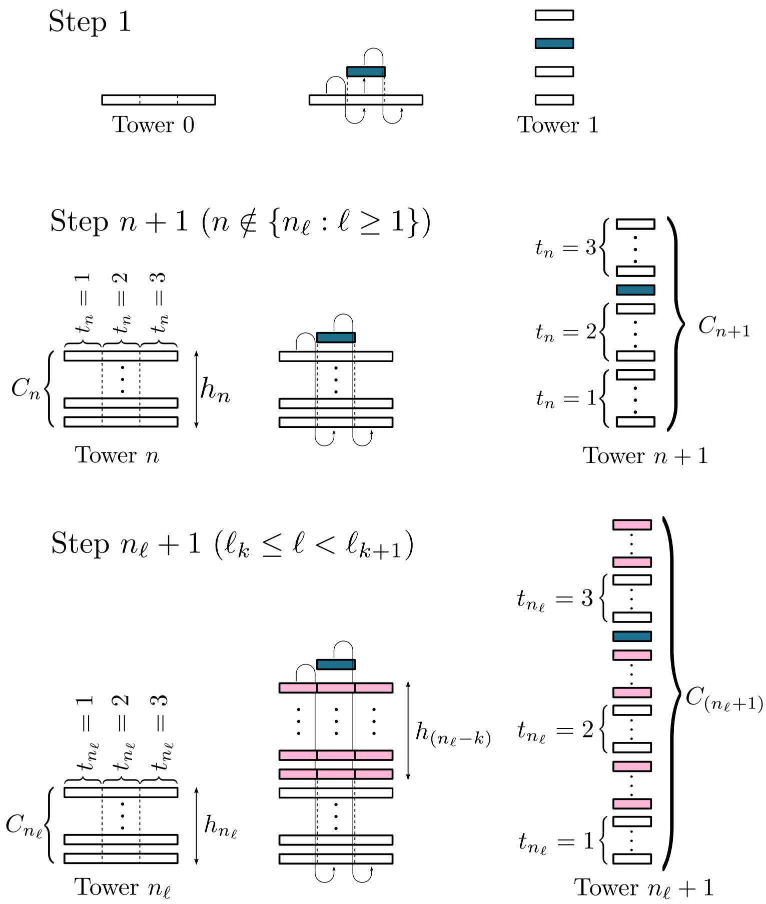

Figure 2.1. Construction of the nearly finite Chacon transformation by cutting and stacking.

Moreover, each level of tower $n+1$ is either completely outside tower n or completely inside a single level of tower n. Let us introduce, for each $n \geqslant 1$, the map p_n : $\{0, \dots, h_{n+1} - 1, *\} \to \{0, \dots, h_n - 1, *\}$ defined by

- $p_n(*) := *$,
- $\forall\ j \in \{0, \dots, h_{n+1} - 1\}$, $p_n(j) := *$ if level j of tower $n+1$ is completely outside tower n, and $p_n(j) := j' \in \{0, \dots, h_n - 1\}$ if level j of tower $n+1$ is completely inside level j' of tower n.

Then the sequence $\big(j_n(t)\big)_{n \geqslant 0}$ satisfies the following compatibility condition.

$$\forall\ n \geqslant 0,\ j_n(t) = p_n\Big(j_{n+1}(t)\Big).$$

In particular, $j_n(t)$ completely determines $j_m(t)$ for each $0 \leqslant m \leqslant n$. We also observe that p_n satisfies the following property:

(2.3) If $p_n(j) \in \{0, \dots, h_n - 2\}$, then $j \in \{0, \dots, h_{n+1} - 2\}$ and $p_n(j+1) = p_n(j) + 1$.

Now we can define our space X, to which belongs $\varphi(t)$ for each $t \in \mathbb{R}_+$:

$$X := \Big\{(j_n)_{n \geqslant 0} \in \mathbb{A}^{\mathbb{N}} : \forall\ n \geqslant 0,\ j_n = p_n(j_{n+1}) \text{ and } \exists\ n_0,\ j_{n_0} \neq *\Big\}.$$

X inherits its topology from the product topology of $\mathbb{A}^{\mathbb{N}}$. In particular it is a totally disconnected metrizable space, but it is not compact (in fact X is not closed in $\mathbb{A}^{\mathbb{N}}$, as the infinite sequence $(*, *, \ldots)$ is in $\overline{X} \setminus X$).

For each $n \geqslant 0$ and each $x \in X$, we denote by $j_n(x)$ the n-th coordinate of x. For each $j \in \{0, \ldots, h_n - 1\}$, we define the subset of X

$$L_n^j := \{x \in X : j_n(x) = j\}.$$

Then L_n^j is compact and clopen in X. Moreover the family of sets (L_n^j) form a basis of the topology on X.

To define the transformation T on X, we need the following easy lemma.

LEMMA 2.1. — *For each $x = (j_n)_{n \geqslant 0} \in X$, there exists $\overline{n}$ such that, for each $n \geqslant \overline{n}$, $j_n \in \{0, \ldots, h_n - 2\}$.*

Proof. — Remember that at each step $n_\ell + 1$, some spacers are added on the last subcolumn of tower n_ℓ. Hence, $j_{n_\ell+1} = h_{n_\ell+1} - 1$ implies $j_{n_\ell} = *$. Now take ℓ large enough so that $j_{n_\ell} \neq *$. Then $j_{n_\ell+1} < h_{n_\ell+1} - 1$, and (2.3) shows by an immediate induction that $j_m < h_m - 1$ for each $m \geqslant n_\ell + 1$. □

We define the measurable transformation $T : X \to X$ as follows: for $x = (j_n)_{n \geqslant 0} \in X$, we consider the smallest integer $\overline{n}$ satisfying the property given in Lemma 2.1. Then we set $T(x) := (j'_n)_{n \geqslant 0}$, where $j'_n := j_n + 1$ if $n \geqslant \overline{n}$, and the finite sequence $(j'_1, j'_2, \ldots, j'_{\overline{n}-1})$ is determined by the value of $j'_{\overline{n}}$ and the compatibility conditions $j'_n = p_n(j'_{n+1})$, $1 \leqslant n < \overline{n}$. Note that T is one-to-one, and $T(X) = X \setminus \{\mathbf{0}\}$, where $\mathbf{0} := (0, 0, \ldots)$.

For each $n \geqslant 1$ and each $0 \leqslant j < h_n - 1$, $T(L_n^j) = L_n^{j+1}$, hence $(L_n^0, \ldots, L_n^{h_n-1})$ is a Rokhlin tower for T. By construction, the family of Rokhlin towers we get in this way has the same structure as the family of Rokhlin towers we constructed by cutting-and-stacking on $\mathbb{R}_+$. From now on, "tower n" will rather designate the Rokhlin tower $(L_n^0, \ldots, L_n^{h_n-1})$ in X. The main advantage that we get compared to the construction on $\mathbb{R}_+$ is the following elementary fact.

Remark 2.2. — If $(L_n^{j_n})_{n \geqslant \overline{n}}$ is a sequence of levels in the successive Rokhlin towers, such that $L_{n+1}^{j_{n+1}}$ is always included in $L_n^{j_n}$ (equivalently, $j_n = p_n(j_{n+1})$), then $\bigcap_n L_n^{j_n}$ is always a singleton

(Note that such an intersection can be empty in the construction on $\mathbb{R}_+$.)

Let μ be the pushforward of the Lebesgue measure on $\mathbb{R}_+$ by φ. Then μ is an infinite, σ-finite and T-invariant measure on X, and it satisfies

$$\forall\, n \geqslant 0,\ \forall\, j \in \{0, \ldots, h_n - 1\},\ \mu(L_n^j) = 3^{-n}.$$

Additional notations

For each $n \geqslant 0$, we denote by C_n the subset of X formed by the union of all the levels of tower n. Note that for each $n \geqslant 0$, $C_n \subset C_{n+1}$, and that $X = \bigcup_{n \geqslant 0} C_n$. For $x \in C_n$, note that $j_n(x)$ indicates the level of tower n to which x belongs.

We also define a function t_n on C_n, taking values in $\{1,2,3\}$, which indicates for each point whether it belongs to the first, the second, or the third subcolumn of tower n. We thus have for $x \in C_n$ and $n \notin \{n_\ell : \ell \geqslant 1\}$

$$j_{n+1}(x) = \begin{cases} j_n(x) & \text{if } t_n(x) = 1, \\ j_n(x) + h_n & \text{if } t_n(x) = 2, \\ j_n(x) + 2h_n + 1 & \text{if } t_n(x) = 3. \end{cases} \tag{2.4}$$

In the case where $n = n_\ell$ for some $\ell \geqslant 1$, we have to replace h_n by $h_{n_\ell} + h_{n_{\ell - k(\ell)}}$ in the above formula. In particular, we always have

$$j_{n+1}(x) \geqslant j_n(x). \tag{2.5}$$

Consider two integers $0 \leqslant n < m$. By construction, tower n is subdivided into 3^{m-n} subcolumns which appear as bundles of h_n consecutive levels in tower m: we call them *occurrences of tower n inside tower m*. These occurrences are naturally ordered, from bottom to top of tower m. For a point x in tower n, the precise occurrence of tower n inside tower m to which x belongs is determined by the sequence $t_n(x), t_{n+1}(x), \dots, t_{m-1}(x)$. For example, x belongs to the last occurrence of tower n inside tower m if and only if $t_n(x) = t_{n+1}(x) = \cdots = t_{m-1}(x) = 3$.

Remark 2.3. — Observe that for each $\ell \geqslant 2$ and each $n_{(\ell-1)} + 1 \leqslant n \leqslant n_\ell - 1$, there is 0 or 1 spacer between two consecutive occurrences of tower n inside tower n_ℓ.

2.3. Behaviour of μ-typical points

Lemma 2.4. — *There exists a μ-conull subset X_∞ of X satisfying: for each $x \in X_\infty$, there exists an integer $\ell(x)$ such that, for all $\ell \geqslant \ell(x)$, for each $n_{(\ell-1)} \leqslant n \leqslant n_\ell - \ell$, $x \in C_n$ but x is neither in the first hundred nor in the last hundred occurrences of tower n inside tower n_ℓ.*

Proof. — If we consider x as a random point chosen according to the normalized μ-measure on C_n, then the random variables $t_n(x), t_{n+1}(x), \dots, t_{m-1}(x)$ are i.i.d. and uniformly distributed in $\{1,2,3\}$. Hence the probability that x belongs to some specified occurrence of tower n inside tower m is $1/3^{m-n}$.

Since the series $\sum 1/3^\ell$ converges, by Borel Cantelli there exists a subset X_n of full μ-measure inside C_n such that, for each $x \in X_n$, there is only a finite number of integers ℓ such that x belongs to the first hundred or to the last hundred occurrences of tower $(n_\ell - \ell)$ inside tower n_ℓ.

Setting

$$X_\infty := X \setminus \bigcup_n (C_n \setminus X_n),$$

we get a conull subset of X, and for each $x \in X_\infty$, there exists an integer $\ell(x)$ such that, for all $\ell \geqslant \ell(x)$, $x \in C_{n_{(\ell-1)}} \subset C_{n_\ell - \ell}$, but x is neither in the first hundred nor in the last hundred occurrence of tower $(n_\ell - \ell)$ inside tower n_ℓ. If $n_{(\ell-1)} \leqslant n \leqslant n_\ell - \ell$, the first (respectively last) hundred occurrences of tower n inside tower n_ℓ are included in the first (respectively last) hundred occurrences of tower $(n_\ell - \ell)$ inside tower n_ℓ, and this concludes the proof. ☐

Remark 2.5. — In particular, for each $x \in X_\infty$, if $n > n_{\ell(x)}$, then x does not belong to the first level of tower n. And since $\mathbf{0}$ is in the first level of tower n for each n, we have $\mathbf{0} \notin X_\infty$.

Remark 2.6. — As $n_{(\ell-1)} + \ell < n_\ell - \ell$ by (2.2), we may also assume that for each $x \in X_\infty$ and each $\ell \geqslant \ell(x)$, x is neither in the first hundred nor in the last hundred occurrences of tower $n_{(\ell-1)}$ inside tower $n_\ell - \ell$.

3. Invariant Radon measures for Cartesian powers of the nearly finite Chacon transformation

We fix a natural integer $d \geqslant 1$, and we study the action of the Cartesian power $T^{\times d}$ on X^d. Recall that a measure σ on X^d is a *Radon measure* if it is finite on each compact subset of X^d (equivalently, if $\sigma(C_n^d) < \infty$ for each n). In particular, a Radon measure on X^d is σ-finite (but the converse is not true).

Our purpose is to describe all Radon measures on X^d which are $T^{\times d}$-invariant and whose marginals are absolutely continuous with respect to μ.

3.1. Basic facts about Radon measures on X^d

We call *n-box* a subset of C_n^d which is a Cartesian product $L_n^{j_1} \times \cdots \times L_n^{j_d}$, where each $L_n^{j_i}$ is a level of the Rokhlin tower C_n. A *box* is a subset which is an n-box for some $n \geqslant 0$. The family of all boxes form a basis of compact clopen sets of the topology of X^d.

We consider the following ring of subsets of X^d

$$\mathscr{R} := \{B \subset X^d : \exists\, n \geqslant 0,\ B \text{ is a finite union of } n\text{-boxes}\}.$$

Proposition 3.1. — *Any finitely additive functional $\sigma : \mathscr{R} \to \mathbb{R}_+$ can be extended to a unique measure on the Borel σ-algebra $\mathscr{B}(X^d)$, which is Radon.*

Proof. — Using Theorems F p. 39 and A p. 54 (Caratheodory's extension theorem) in [Hal50], we only have to prove that, if $(R_k)_{k\geqslant 1}$ is a decreasing sequence in $\mathscr{R}$ such that $\lim_{k\to\infty} \downarrow \sigma(R_k) > 0$, then $\bigcap_k R_k \neq \emptyset$. But this is obvious since, under this assumption, each R_k is a compact nonempty set. □

In particular, to define a Radon measure σ on X^d, we only have to define $\sigma(B)$ for each box B, with the compatibility condition that, if B is an n-box for some $n \geqslant 0$, then $\sigma(B) = \sum_{B' \subset B} \sigma(B')$, where the sum ranges over the 3^d $(n+1)$-boxes which are contained in B.

We call *n-diagonal* a Rokhlin tower for $T^{\times d}$ which is of the form

$$\Big(B, T^{\times d}(B), \ldots, (T^{\times d})^{r-1}(B)\Big),$$

where each $(T^{\times d})^j(B)$ is an n-box, and which is maximal in the following sense: B has one projection which is the bottom level L_n^0 of tower n, $(T^{\times d})^{r-1}B$ has one projection which is the top level $L_n^{h_n-1}$ of tower n, and the projections of each $(T^{\times d})^j B$, $1 \leqslant j \leqslant r-2$ are neither the bottom level nor the top level of tower n. (See Figure 3.1)

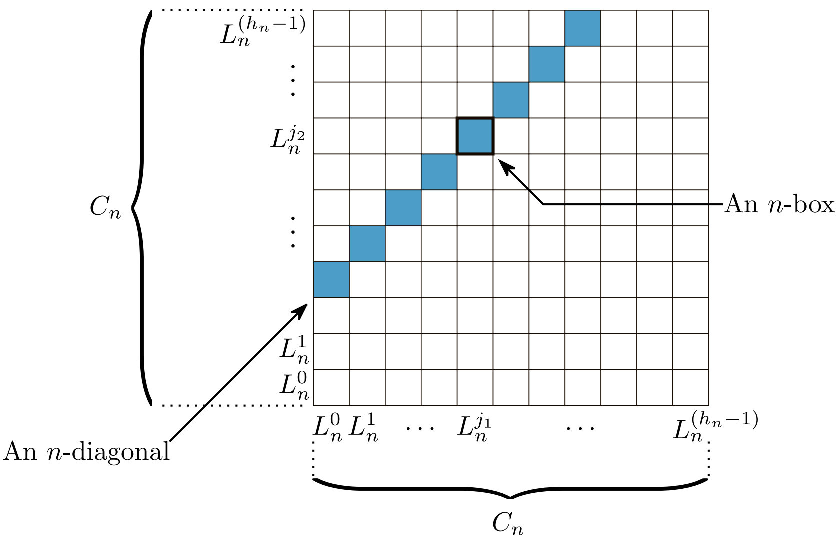

Figure 3.1. An n-diagonal inside C_n^d (here $d = 2$)

Lemma 3.2. — *Set $X_{\mathbf{0}}^d := \{x = (x_1, \ldots, x_d) \in X^d : \exists\ i, x_i = \mathbf{0}\}$. Let σ be a Radon measure on X^d. Then σ is $T^{\times d}$-invariant if and only if the following two conditions hold:*

(1) $\sigma\left(X_{\mathbf{0}}^d\right) = 0$.

(2) *for each n, all the n-boxes lying on an n-diagonal always have the same measure.*

Proof. — Assume first that σ is a $T^{\times d}$-invariant Radon measure on X^d. Recalling that $\mathbf{0}$ has no preimage by T, we see that $(T^{\times d})^{-1}(X_{\mathbf{0}}^d) = \emptyset$, whence $\sigma\left(X_{\mathbf{0}}^d\right) = 0$. Moreover, since n-boxes on an n-diagonal are levels of a $T^{\times d}$-Rokhlin tower, the second condition obviously holds. Reciprocally, assume that the two conditions given in the statement of the lemma hold. For each n, let Ω_n be the subset of C_n^d constituted of all n-boxes of the form $L_n^{j_1} \times \cdots \times L_n^{j_d}$, where for each i $j_i \neq 0$. Then the second condition implies that σ and $(T^{\times d})_*(\sigma)$ coincide on Ω_n for each n. But

$$\bigcup_{n \geqslant 0} \Omega_n = X \setminus X_{\mathbf{0}}^d.$$

On the other hand, we have $(T^{\times d})_*(\sigma)(X_{\mathbf{0}}^d) = \sigma\Big((T^{\times d})^{-1}(X_{\mathbf{0}}^d)\Big) = \sigma(\emptyset) = 0$. With the first condition we see that σ and $(T^{\times d})_*(\sigma)$ also coincide on $X_{\mathbf{0}}^d$, hence they are equal. □

Definition 3.3 (Convergence of Radon measures on X^d). — *We say that a sequence (σ_k) of Radon measures on X^d converges to the nonzero Radon measure σ if, for each n large enough so that $\sigma(C_n^d) > 0$, we have*

- *$\sigma_k(C_n^d) > 0$ for all large enough k,*
- *for each n-box B,*

$$\frac{\sigma_k(B)}{\sigma_k(C_n^d)} \xrightarrow[k\to\infty]{} \frac{\sigma(B)}{\sigma(C_n^d)}.$$

Observe that, when a sequence of Radon measures converges in the above sense, then its limit is unique up to a multiplicative positive constant. Observe also that the convergence is unchanged if we multiply each measure σ_k by a positive real number (which may vary with k).

Remark 3.4. — If the sequence (σ_k) of Radon measures on X^d converges to the nonzero Radon measure σ, then for n such that $\sigma(C_n^d) > 0$ and for each $m \geqslant n$, for each m-box $B \subset C_n^d$, we also have

$$\frac{\sigma_k(B)}{\sigma_k(C_n^d)} \xrightarrow[k\to\infty]{} \frac{\sigma(B)}{\sigma(C_n^d)}.$$

Consequently, the above holds also when $B \in \mathscr{R}$ is included in C_n^d.

Indeed, as C_n^d is a finite union of m-boxes, we have

$$\frac{\sigma_k(C_n^d)}{\sigma_k(C_m^d)} \xrightarrow[k\to\infty]{} \frac{\sigma(C_n^d)}{\sigma(C_m^d)}.$$

Then we can write, for an m-box $B \subset C_n^d$,

$$\frac{\sigma_k(B)}{\sigma_k(C_n^d)} = \frac{\sigma_k(B)}{\sigma_k(C_m^d)}\frac{\sigma_k(C_m^d)}{\sigma_k(C_n^d)} \xrightarrow[k\to\infty]{} \frac{\sigma(B)}{\sigma(C_m^d)}\frac{\sigma(C_m^d)}{\sigma(C_n^d)} = \frac{\sigma(B)}{\sigma(C_n^d)}.$$

Proposition 3.5. — *Let (σ_k) be a sequence of Radon measures on X^d, and assume that there exists some $\underline{n} \geqslant 0$ satisfying*

- *$\sigma_k(C_{\underline{n}}^d) > 0$ for all large enough k,*
- *for each $n > \underline{n}$, the sequence $\left(\sigma_k(C_n^d)/\sigma_k(C_{\underline{n}}^d)\right)_k$ is bounded.*

Then there is a subsequence (k_j) and a nonzero Radon measure σ on X^d such that (σ_{k_j}) converges to σ.

Proof. — Multiplying each σ_k by a positive real number if necessary, we may assume that for all large enough k, $\sigma_k(C_{\underline{n}}^d) = 1$. Then the second assumption ensures that for each box B, the sequence $(\sigma_k(B))_k$ is bounded. By a standard diagonal procedure, we can find a subsequence (k_j) such that for each box B, $\sigma_{k_j}(B)$ has a limit which we denote by $\sigma(B)$. Then σ defines a finitely additive functional on the ring $\mathscr{R}$ of finite unions of boxes, with values in $\mathbb{R}_+$. By Proposition 3.1, σ can be extended to a Radon measure on $\mathscr{B}(X^d)$, which is nonzero since $\sigma(C_{\underline{n}}^d) = 1$. And we obviously have the convergence of (σ_{k_j}) to σ. □

Proposition 3.6. — *Let (σ_k) and (γ_k) be two sequences of Radon measures on X^d, and assume there exist two nonzero Radon measures σ and γ, an integer $\underline{n} \geqslant 1$ and a real number $\theta > 0$ such that*

- *$\sigma_k \xrightarrow[k\to\infty]{} \sigma$,*
- *$\gamma_k \xrightarrow[k\to\infty]{} \gamma$,*
- *$\forall\, k$, $\gamma_k \leqslant \sigma_k$*
- *$\forall\, n \geqslant \underline{n}$, for all large enough k (depending on n), $\gamma_k(C_n^d) \geqslant \theta\sigma_k(C_n^d)$.*

Then $\gamma \ll \sigma$.

Proof. — Let $m \geqslant n \geqslant \underline{n}$, and let B be an m-box. For all large enough k, we have by assumption
$$\frac{\gamma_k(B)}{\gamma_k(C_n^d)} \leqslant \frac{\sigma_k(B)}{\theta\sigma_k(C_n^d)}.$$
But by Remark 3.4, we have
$$\frac{\gamma_k(B)}{\gamma_k(C_n^d)} \xrightarrow[k\to\infty]{} \frac{\gamma(B)}{\gamma(C_n^d)}, \text{ and } \frac{\sigma_k(B)}{\sigma_k(C_n^d)} \xrightarrow[k\to\infty]{} \frac{\sigma(B)}{\sigma(C_n^d)}.$$
It follows that
$$\frac{\gamma(B)}{\gamma(C_n^d)} \leqslant \frac{\sigma(B)}{\theta\sigma(C_n^d)}.$$
The above inequality extends to each $B \in \mathscr{R}$ contained in C_n^d, and then to each $B \in \mathscr{B}(X)$ contained in C_n^d. In particular, if $B \subset C_n^d$ is Borel measurable and satisfies $\sigma(B) = 0$, then we also have $\gamma(B) = 0$. And since $X = \bigcup_n C_n^d$, this concludes the proof. □

Remark 3.7. — For each $\underline{\ell} \geqslant 1$, the definition of the ring $\mathscr{R}$ is unchanged if we consider only the finite unions of n_ℓ-boxes, for some $\ell \geqslant \underline{\ell}$. Hence in Propositions 3.5 and 3.6, it is enough for the conclusions to hold that the assumptions be verified only when $n \in \{n_\ell : \ell \geqslant \underline{\ell}\}$.

3.2. Dissipative case

Lemma 3.8. — *For each $x = (x_1, \dots, x_d) \in X_\infty^d$, for $\ell > \max\{\ell(x_i) : i = 1, \dots, d\}$, we have*
$$\#\{j \geqslant 0 : (T^{\times d})^j(x) \in C_{n_\ell+1}^d\} = \infty.$$

Proof. — If $\ell > \max\{\ell(x_i) : i = 1, \dots, d\}$, we know by Lemma 2.4 that each coordinate x_i is in $C_{n_\ell+1}$, but is not in the last occurrence of tower $(n_\ell + 1)$ inside tower $n_{(\ell+1)}$. Moreover by Remark 2.5, x_i is not in the first level of tower $(n_\ell + 1)$. The next occurrence of tower $(n_\ell + 1)$ inside tower $n_{(\ell+1)}$ appears after 0 or 1 spacer by Remark 2.3. As the height of tower $(n_\ell + 1)$ is $h_{(n_\ell+1)}$, $T^{h_{(n_\ell+1)}}(x_i)$ is either in the same level of tower $(n_\ell + 1)$ as x_i, or in the level immediately below. Thus $T^{h_{(n_\ell+1)}}(x_i) \in C_{n_\ell+1}$. But the same applies to any $\ell' \geqslant \ell$, and we get that $T^{h_{(n_{\ell'}+1)}}(x_i)$ is either in the same level of tower $(n_{\ell'} + 1)$ as x_i, or in the level immediately below. Since these two levels are both included in $C_{n_\ell+1}$ we get that $T^{h_{(n_{\ell'}+1)}}(x_i) \in C_{n_\ell+1}$. □

Proposition 3.9. — *There is no Radon, $T^{\times d}$-invariant and totally dissipative measure for which X_∞^d is a conull set. In particular, there is no Radon, $T^{\times d}$-invariant and totally dissipative measure whose marginals are absolutely continuous with respect to μ.*

Proof. — Suppose that σ is such a measure. Let W be a wandering set for σ, with
$$\sigma\left(X^d \setminus \bigcup_{j\in\mathbb{Z}} (T^{\times d})^j W\right) = 0.$$

As X^d_∞ is a conull set, we may assume that $W \subset X^d_\infty$. By the previous lemma, $W = \bigcup_n W_n$, where

$$W_n := \left\{x \in W : \#\{j \geqslant 0 : (T^{\times d})^j(x) \in C^d_n\} = \infty\right\}.$$

Hence there exists some n with $\sigma(W_n) > 0$. The ergodic decomposition of σ writes

$$\sigma = \int_W \left(\sum_{j\in\mathbb{Z}} \delta_{(T^{\times d})^j(x)}\right) \mathrm{d}\sigma(x),$$

so we get $\sigma(C^d_n) = \infty$, which contradicts the fact that σ is Radon. □

3.3. Main result

An obvious example of a $T^{\times d}$-invariant Radon measure on X^d is the product measure $\mu^{\otimes d}$. Another example is what we call a *graph measure arising from powers of T*: this is a measure σ of the form

$$\sigma(A_1 \times \cdots \times A_d) = \alpha\mu(A_1 \cap T^{-e_2}A_2 \cap \cdots \cap T^{-e_d}A_d), \tag{3.1}$$

for some integers $e_2, \ldots, e_d$ and some fixed positive real number α. Such a measure is concentrated on the subset

$$\left\{(x_1, \ldots, x_d) \in X^d : x_i = T^{e_i}x_1 \text{ for all } i = 2, \ldots, d\right\}.$$

Theorem 3.10. — *For each $d \geqslant 1$, the infinite measure preserving dynamical system $(X^d, \mu^{\otimes d}, T^{\times d})$ is conservative ergodic.*

Moreover, if σ is a nonzero, Radon, $T^{\times d}$-invariant and ergodic measure on X^d, such that

$$\sigma\left(X^d \setminus X^d_\infty\right) = 0, \tag{3.2}$$

then there exists a partition of $\{1, \ldots, d\}$ into r subsets $D_1, \ldots, D_r$, such that $\sigma = \sigma^{D_1} \otimes \cdots \otimes \sigma^{D_r}$, where σ^{D_j} is a graph measure on X^{D_j} arising from powers of T.

Corollary 3.11. — *If σ is a nonzero, Radon, $T^{\times d}$-invariant measure on X^d, whose marginals are absolutely continuous with respect to μ, then σ decomposes as a sum of countably many ergodic components, which are all of the form described in Theorem 3.10.*

To prove Theorem 3.10 in the case $d = 1$, we even do not need assumption (3.2) as we can show that μ is, up to a multiplicative constant, the only T-invariant, Radon measure on X (the proof is the same as for the Chacon infinite transformation, see [JRdlR18, Proposition 2.4]).

We also note that, if we have proved the second part of the theorem for some $d \geqslant 2$, then the first one follows immediately. Indeed, if $\mu^{\otimes d}$ were not ergodic, then almost all its ergodic components would satisfy (3.2), hence would be a product of graph measures different from $\mu^{\otimes d}$. But this would mean that for $\mu^{\otimes d}$-almost all $x \in X^d$, there exist at least two coordinates of x lying on the same T-orbit, which of course is absurd. Hence $\mu^{\otimes d}$ is ergodic, and by Proposition 3.9, it is conservative.

The remainder of this paper is devoted to the proof by induction of the second part of Theorem 3.10. So we now assume that for some $d \geqslant 2$, the statement is true up to $d-1$. We consider a nonzero, Radon, $T^{\times d}$-invariant and ergodic measure σ on X^d, satisfying (3.2).

We will show that either σ is a graph measure arising from powers of T, or it can be decomposed into a product of two measures $\sigma_1 \times \sigma_2$, σ_i being a $T^{\times d_i}$-invariant Radon measure on X^{d_i} for some $1 \leqslant d_i < d$, $d_1 + d_2 = d$. In this latter case we can apply the induction hypothesis to each σ_i, which yields the announced result.

3.4. Choice of a σ-typical point

By Proposition 3.9, the system $(X^d, \sigma, T^{\times d})$ is conservative ergodic. By Hopf's ergodic theorem, if $B \subset C \subset X^d$ with $0 < \sigma(C) < \infty$, we have for σ-almost every point $x = (x_1, \dots, x_d) \in X^d$

$$\frac{\sum_{j\in J} \mathbb{1}_B((T^{\times d})^j x)}{\sum_{j\in J} \mathbb{1}_C((T^{\times d})^j x)} \xrightarrow[|J|\to\infty]{} \frac{\sigma(B)}{\sigma(C)}, \tag{3.3}$$

where the sums in the above expression range over a set J of consecutive integers containing 0.

We say that $x \in X^d$ is *typical* if, for all n large enough so that $\sigma(C_n^d) > 0$, Property (3.3) holds whenever B is an n-box and C is C_n^d. We know that σ-almost every $x \in X^d$ is typical. Therefore, there exists a point $x = (x_1, \dots, x_d)$ such that

(3.4) For each $j \in \mathbb{Z}$, $(T^{\times d})^j x$ is typical.

Since there are only countably many boxes, we may also assume that

(3.5) For each box B, $x \in B \Longrightarrow \sigma(B) > 0$.

Moreover, by (3.2), we can further assume that

(3.6) $\forall\, i = 1, \dots, d,\ \ x_i \in X_\infty$.

From now on, we fix a point $x = (x_1, \dots, x_d)$ satisfying the above assumptions (3.4), (3.5) and (3.6). We will derive properties of σ from the observations made on the orbit of this point x.

By an *interval*, we mean in this paper a finite set of consecutive integers. We will need the following key notion in our argument.

Definition 3.12. — *We call* n-crossing *a maximal interval $J \subset \mathbb{Z}$ with the following properties:*

- $(T^{\times d})^j x \in C_n^d$ *for each* $j \in J$,
- *for each* $1 \leqslant i \leqslant d$, $j \mapsto t_n(T^j x_i)$ *is constant on* J.

An n-crossing is said to be synchronized *if* $t_n(T^j x_1) = \cdots = t_n(T^j x_d)$ *for each j in this n-crossing.*

Note that an n-crossing has at most h_n elements. If j is the smallest (respectively the largest) element of an n-crossing, then there exists $1 \leqslant i \leqslant d$ such that $T^j x_i$ is in the first (respectively the last) level of tower n. Observe also that when j runs over an n-crossing, $(T^{\times d})^j x$ successively passes through each n-box of some n-diagonal.

3.5. Characterizations of graph measures arising from powers of T

Lemma 3.13. — *The following assertions are equivalent:*

(1) *σ is a graph measure arising from powers of T;*
(2) *$\exists\, e_2,\dots,e_d\in\mathbb{Z}$: $x_i=T^{e_i}x_1$ for each $i=2,\dots,d$;*
(3) *$\exists\, \underline{n}:\forall\, n\geqslant \underline{n}$, $t_n(x_1)=\cdots=t_n(x_d)$;*
(4) *$\exists\, j,\exists\, \underline{n}:\forall\, n\geqslant \underline{n}$, $t_n(T^jx_1)=\cdots=t_n(T^jx_d)$.*

Proof. — Let us first prove that (1) $\Longrightarrow$ (2). If σ is a graph measure arising from powers of T, then there exist a positive real number α and integers $e_2,\dots,e_d$ such that for all measurable subsets $A_1,\dots,A_d$ of X, (3.1) holds. Observe that, if ℓ is large enough so that $h_{n_\ell-k(\ell)}>\max\{|e_2|,\dots,|e_d|\}$, then for each $i=2,\dots,d$ and each $j,j'\in\{0,\dots,h_{n_\ell}-1\}$,

$$L^j_{n_\ell}\cap T^{-e_i}L^{j'}_{n_\ell}=\begin{cases}L^j_{n_\ell} & \text{if } j'=j+e_i,\\ \emptyset & \text{otherwise.}\end{cases}$$

It follows that the only n_ℓ-boxes that may be charged by σ are of the form $L^{j_1}_{n_\ell}\times L^{j_1+e_2}_{n_\ell}\times\cdots\times L^{j_1+e_d}_{n_\ell}$ for some j_1. By assumption (3.5), it follows that for each $i=2,\dots,d$, $j_{n_\ell}(x_i)=j_{n_\ell}(x_1)+e_i$. Since this is true for all large enough ℓ, this in turn implies that for each $i=2,\dots,d$, $x_i=T^{e_i}x_1$.

Conversely, if (2) holds, the same argument shows that if ℓ is large enough so that $h_{n_\ell-k(\ell)}>\max\{|e_2|,\dots,|e_d|\}$, then the only n_ℓ-boxes that can contain x are the n_ℓ-boxes of the form $L^{j_1}_{n_\ell}\times L^{j_1+e_2}_{n_\ell}\times\cdots\times L^{j_1+e_d}_{n_\ell}$ for some j_1. Note that the n_ℓ-boxes of this form constitute an n_ℓ-diagonal, which we denote by D. But (2) is also valid for each $(T^{\times d})^jx$, $j\in\mathbb{Z}$ hence the argument also applies to each $(T^{\times d})^jx$. Thus, if B is an n_ℓ-box which is not on D, then $(T^{\times d})^jx\notin B$ for each $j\in\mathbb{Z}$. Now, remembering that x is typical for σ, we have for each n_ℓ-box $B=L^{j_1}_{n_\ell}\times\cdots\times L^{j_d}_{n_\ell}$

$$\frac{\sigma(B)}{\sigma(C^d_{n_\ell})}=\lim_{k\to\infty}\frac{\sum_{-k\leqslant j\leqslant k}\mathbb{1}_B((T^{\times d})^jx)}{\sum_{-k\leqslant j\leqslant k}\mathbb{1}_{C^d_{n_\ell}}((T^{\times d})^jx)}.$$

The above limit is 0 if B is not on D. Moreover, note that each time the orbit of x passes through $C^d_{n_\ell}$, $(T^{\times d})^jx$ successively passes through each n_ℓ-box on D. Hence if B is on D, the limit is equal to the inverse of the number of n_ℓ boxes on D. In particular the limit is proportional to $\mu\left(L^{j_1}_{n_\ell}\cap T^{-e_2}L^{j_1}_{n_\ell}\cap\cdots\cap T^{-e_d}L^{j_d}_{n_\ell}\right)$. The coefficient of proportionality depends a priori on ℓ, but since each n_ℓ-box is a union of disjoint $n_{\ell+1}$-boxes, we see that in fact this coefficient does not depend on ℓ. Finally, this gives (3.1) in the case of an n_ℓ-box for each large enough ℓ, and this is enough to conclude that (3.1) holds for each measurable set of the form $A_1\times\cdots\times A_d$. We have so far proved the equivalence of (1) and (2).

Now let us turn to the proof of (2) $\Longrightarrow$ (3). Since $x_1\in X_\infty$, we have $j_n(x_1)\to\infty$ and $h_n-j_n(x_1)\to\infty$ as $n\to\infty$. If (2) holds, we then have $j_n(x_i)=j_n(x_1)+e_i$ for each $i=1,\dots,d$ and each n large enough so that $\min\{j_n(x_1),h_n-j_n(x_1)\}>\max\{|e_2|,\dots,|e_d|\}$. But then for such an n we also have $t_n(x_i)=t_n(x_1)$ for each $i=1,\dots,d$.

The implication (3) $\Longrightarrow$ (4) is obvious.

Assume now that (4) holds with $j = 0$ (i.e. that, in fact, (3) holds). For $i = 2, \ldots, d$, we then have by an easy induction that $j_n(x_i) - j_n(x_1) = j_{\underline{n}}(x_i) - j_{\underline{n}}(x_1)$ for each $n \geqslant \underline{n}$. Setting $e_i := j_{\underline{n}}(x_i) - j_{\underline{n}}(x_1)$ for $i = 2, \ldots, d$, we get that $x_i = T^{e_i} x_1$ and we have (2). Now if (4) holds with some $j \in \mathbb{Z}$, we get (2) for $(T^{\times d})^j x$, which is clearly equivalent to (2) for x. Thus we have proved that (4) $\Longrightarrow$ (2) and this concludes the proof of the lemma. □

For the remainder of the paper, we also fix a real number $0 < \eta < 1$, small enough so that $\eta < \frac{1}{100d}$. In particular we will need the inequality $(1 - \eta)^2 > 1/2$.

Definition 3.14. — *For each n, let $\boldsymbol{I}_n := \{-\lfloor h_n/2 \rfloor, \ldots, -\lfloor h_n/2 \rfloor + h_n - 1\}$ be the interval of length h_n and centered at 0. For each $n \geqslant 0$, we call* substantial n-crossing *any n-crossing whose intersection with $\boldsymbol{I}_n$ contains at least ηh_n elements.*

Lemma 3.15. — *If $n = n_{(\ell-1)}$ for some large enough ℓ, then substantial n-crossings cover a proportion at least $(1 - (d + 2)\eta)$ of $\boldsymbol{I}_n$. In particular, there exists at least one substantial n-crossing. Moreover, if all substantial n-crossings are synchronized, then each substantial n-crossing is of size at least $(1 - (d + 2)\eta)h_n$, and there are at most two of them.*

Proof. — Let us start by considering the case of an integer n which is of the form $n = n_{(\ell-1)}$ for some $\ell > \max_i \ell(x_i)$. We also assume that ℓ is large enough so that

$$\frac{1}{3^{k(\ell-1)}} < \frac{\eta}{2d}.$$

We set $n' := n_{(\ell-1)} - k(\ell - 1)$, and we observe that the above assumption ensures that

$$\frac{h_{n'} + 1}{h_n} < \frac{\eta}{d}. \tag{3.7}$$

We know by Lemma 2.4 that $x \in C^d_{n_{(\ell-1)}} = C^d_n$, and that the interval

$$\{-100h_{(n_\ell - \ell)}, \ldots, 100h_{(n_\ell - \ell)}\}$$

is contained in a single n_ℓ-crossing. A fortiori, $\boldsymbol{I}_n$ is contained in a single n_ℓ-crossing. Therefore, if a coordinate $T^j x_i$ reaches the top of tower n and comes back to C_n on the interval $\boldsymbol{I}_n$, then the two passages in C_n are separated by at most $h_{n'} + 1$. Moreover, this can happen at most once on the interval $\boldsymbol{I}_n$ for each i. It follows that the set of integers $j \in \boldsymbol{I}_n$ such that $(T^{\times d})^j x \notin C^d_n$ is constituted of at most d pieces, and its cardinality is bounded above by ηh_n by (3.7). Then there exist at most $(d+1)$ n-crossings intersecting $\boldsymbol{I}_n$, and they cover a proportion at least $(1 - \eta)$ of $\boldsymbol{I}_n$. Now the proportion of $\boldsymbol{I}_n$ covered by n-crossings which are not substantial is less than $(d + 1)\eta$, hence the proportion of $\boldsymbol{I}_n$ covered by substantial n-crossings is at least $(1 - (d + 2)\eta)$. This proves the first part of the lemma

Let us assume now that all substantial n-crossings are synchronized. If we have only one substantial n-crossing, then this n-crossing is of size at least $(1-(d+2)\eta)h_n$, and we have for j in this n-crossing

$$\left| j_n(T^j x_{i_1}) - j_n(T^j x_{i_2}) \right| \leqslant (d + 2)\eta h_n. \tag{3.8}$$

If we have at least two substantial n-crossings, note that between two of them, there is at least one coordinate passing through the top of tower n, and for which t_n has increased by 1 mod 3. Since the $t_n(T^j x_i)$, $i = 1, \dots, d$ are supposed to be equal on each substantial n-crossing, we deduce that *each* coordinate passes through the top of tower n between two substantial n-crossings. As this happens at most once for each coordinate on $\boldsymbol{I}_n$, we see that there are at most two substantial n-crossings. Finally, from the first part of the lemma it follows that two consecutive substantial n-crossings are separated by at most $(d+2)\eta h_n$ points. We deduce that, on any substantial n-crossing, (3.8) holds, hence each substantial n-crossing is of size at least $(1-(d+2)\eta)h_n$. ☐

Remark 3.16. — The preceding lemma extends easily to the case when $n_{(\ell-1)} \leqslant n \leqslant n_\ell - \ell$. Indeed, when $n_{(\ell-1)} + 1 \leqslant n \leqslant n_\ell - \ell$ the proof is even simpler, as two successive passages in C_n are now separated by at most one.

Proposition 3.17. — *The measure σ is a graph measure arising from powers of T if and only if for each large enough n, all substantial n-crossings are synchronized.*

Proof. — First assume that σ is a graph measure arising from powers of T. Then by Lemma 3.13, we know that there exists $e_2, \dots, e_d \in \mathbb{Z}$ such that $x_i = T^{e_i} x_1$ for each $i = 2, \dots, d$. Take n large enough so that $\max\{|e_2|, \dots, |e_d|\} < \eta h_n$. Let J be a substantial n-crossing. In particular the size of J is at least ηh_n. Hence there exists $\bar{j} \in J$ such that $\eta h_n \leqslant j_n(T^{\bar{j}} x_1) \leqslant (1-\eta)h_n$. We deduce that $j_n(T^{\bar{j}} x_i) = j_n(T^{\bar{j}} x_1) + e_i$ for each $i = 2, \dots, d$. But we also have $\eta h_n \leqslant j_{n+1}(T^{\bar{j}} x_1) \leqslant (1-\eta)h_n$ and this ensures that $j_{n+1}(T^{\bar{j}} x_i) = j_{n+1}(T^{\bar{j}} x_1) + e_i$. By (2.4), the equality $j_{n+1}(T^{\bar{j}} x_i) - j_{n+1}(T^{\bar{j}} x_1) = j_n(T^{\bar{j}} x_i) - j_n(T^{\bar{j}} x_1)$ implies $t_n(T^{\bar{j}} x_i) = t_n(T^{\bar{j}} x_1)$. Finally, as $j \mapsto t_n(T^j x_i)$ is constant on the n-crossing J, we see that J is synchronized.

Conversely, assume that there exists $\underline{n}$ such that for each $n \geqslant \underline{n}$, all substantial n-crossings are synchronized. Without loss of generality, we may assume that $\underline{n}$ is of the form $n_{(\ell-1)}$, for some ℓ large enough to apply Lemma 3.15. Then we know that there exists at least one substantial $\underline{n}$-crossing $J_{\underline{n}}$, of size at least $(1-(d+2)\eta)h_{\underline{n}}$. For $j \in J_{\underline{n}}$ and for each $i = 2, \dots, d$, $|j_{\underline{n}}(T^j x_i) - j_{\underline{n}}(T^j x_1)| \leqslant (d+2)\eta h_{\underline{n}}$. Let us prove by induction that for each $n \geqslant \underline{n}$, there exists a substantial n-crossing J_n, of size at least $(1-(d+2)\eta)h_n$, and containing $J_{\underline{n}}$. We already know that this property is true for $\underline{n}$. Assume it is true up to n for some $n \geqslant \underline{n}$. Then, the n-crossing J_n extends to a unique $(n+1)$-crossing J_{n+1}. As J_n intersects $\boldsymbol{I}_n$ and is of size at most h_n, $J_n \subset \boldsymbol{I}_{n+1}$. It follows that

$$|J_{n+1} \cap \boldsymbol{I}_{n+1}| \geqslant |J_n| \geqslant (1-(d+2)\eta)h_n \geqslant \eta h_{n+1},$$

which proves that J_{n+1} is a substantial $(n+1)$-crossing. Moreover, since the size of J_n is at least $(1-(d+2)\eta)h_n$, we have for $j \in J_n$ and each $i = 2, \dots, d$ $|j_n(T^j x_i) - j_n(T^j x_1)| \leqslant (d+2)\eta h_n$. But J_n is synchronized, hence by (2.4), we have for $j \in J_n$

$$|j_{n+1}(T^j x_i) - j_{n+1}(T^j x_1)| = |j_n(T^j x_i) - j_n(T^j x_1)| \leqslant (d+2)\eta h_n \leqslant (d+2)\eta h_{n+1}.$$

This equality extends to $j \in J_{n+1}$ since the difference is constant on an $(n+1)$-crossing. This proves that the size of J_{n+1} is at least $(1-(d+2)\eta)h_{n+1}$. Now if we take any $j \in J_{\underline{n}}$, we have $j \in J_n$ for each $n \geqslant \underline{n}$, and since we assumed that each substantial n-crossing is synchronized, we have $t_n(T^j x_1) = \cdots = t_n(T^j x_d)$, i.e. we have (4) of Lemma 3.13. This proves that σ is a graph measure arising from powers of T. $\square$

Remark 3.18. — In the preceding proof, the induction provides in fact a stronger inequality for the sizes of the substantial n-crossings (J_n): $|J_n| \geqslant h_n - (d+2)\eta h_{\underline{n}}$.

4. Combinatorics of some sets of integers

The purpose of this section is to establish Proposition 4.1 on the combinatorics of the set of integers j such that $(T^{\times d})^j x \in C_n^d$ for a given large n.

Proposition 4.1. — *There exist constants $K_1 > 0$ and $K_2 > 0$ such that, for any large enough integer $\underline{\ell}$, and any integer $1 \leqslant c \leqslant h_{n_{\underline{\ell}}}$, the following holds: if $I \subset \mathbb{Z}$ is an interval contained in an $n_{(\underline{\ell}+\ell)}$-crossing for some $\ell \geqslant 1$, and if the length of I is at least $\eta h_{n_{(\underline{\ell}+\ell-1)}}$, then*

- *the proportion of integers $j \in I$ such that $(T^{\times d})^j x \in C_{n_{\underline{\ell}}}^d$ is at least $(1-\eta)^{2\ell}$;*
- *among all the integers $j \in I$ such that $(T^{\times d})^j x \in C_{n_{\underline{\ell}}}^d$, the proportion of those belonging to an $n_{\underline{\ell}}$-crossing of size $\leqslant c$ is bounded above by*

$$K_1 \frac{c}{h_{n_{\underline{\ell}}}} + \frac{K_2}{3^{\underline{\ell}}}.$$

For this we will introduce a hierarchy of more and more complex subsets of $\mathbb{Z}$, prove by induction some combinatorial results on abstract sets in this hierarchy, and finally show how to apply these results in the particular case we are interested in.

4.1. A hierarchy of subsets of $\mathbb{Z}$

This part of the argument is completely abstract and independent of the rest of the paper, but we keep the notations d (an integer, $d \geqslant 2$) and η (a positive real number between 0 and 1). We set

$$K_1 := \frac{1 + \frac{2}{\eta}}{1-\eta} d.$$

We fix two sequences of positive integers $(c_\ell)_{\ell \geqslant 1}$ and $(s_\ell)_{\ell \geqslant 1}$, satisfying

$$\forall\, \ell \geqslant 1,\ \frac{s_\ell}{c_\ell} < \frac{1}{d} \frac{\eta}{\eta+1} \eta, \tag{4.1}$$

and

$$\forall\, \ell \geqslant 1,\ \frac{c_\ell}{c_{\ell+1}} < \frac{\eta}{K_1}. \tag{4.2}$$

Let $F \subset \mathbb{Z}$, and let $I \subset \mathbb{Z}$ be an interval. We call *piece* of $F \cap I$ any maximal interval included in $F \cap I$, and we call *hole* of $F \cap I$ any maximal interval included in $I \setminus F$. (Thus, I is the disjoint union of the pieces and the holes of $F \cap I$, which alternate.)

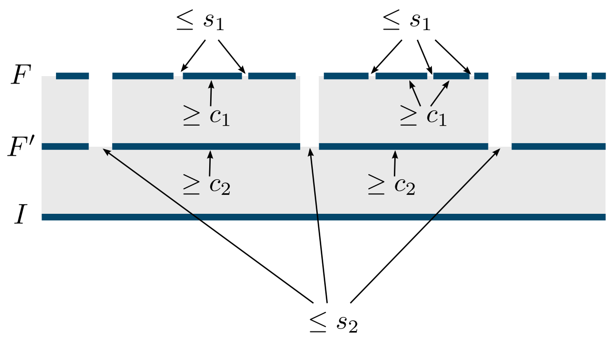

Figure 4.1. A set F of order 2 inside an interval I.

We say that F *is of order 1 inside the interval* I if

- each hole of $F \cap I$ is of size $\leqslant s_1$,
- two consecutive holes of $F \cap I$ are always separated by a piece of size at least c_1.

Recursively, we say that F *is of order* $\ell \geqslant 2$ *inside the interval* I if there exists a subset $F' \subset \mathbb{Z}$ such that

- $F \subset F'$,
- each hole of $F' \cap I$ is of size $\leqslant s_\ell$,
- two consecutive holes of $F' \cap I$ are always separated by a piece of size at least c_ℓ,
- for each piece I' of $F' \cap I$, F is of order $(\ell - 1)$ inside I'.

(See Figure 4.1.) Note that, if F is of order ℓ inside the interval I, then F is of order ℓ inside each subinterval $J \subset I$.

Lemma 4.2. — *Let $F_1, \dots, F_d$ be d subsets of $\mathbb{Z}$, and let $I \subset \mathbb{Z}$ be an interval. Assume that for some $\ell \geqslant 1$, F_i is of order ℓ inside I for each $i = 1, \dots, d$, and that the size of I is at least ηc_ℓ. Set $F := \bigcap_{i=1}^d F_i$. Then*

- *the density of F inside I satisfies*

$$\frac{|F \cap I|}{|I|} \geqslant (1-\eta)^{2\ell}, \tag{4.3}$$

- *for a given integer c, $1 \leqslant c < h_1$, the proportion of integers in $F \cap I$ lying in pieces of $F \cap I$ with size $\leqslant c$ is bounded above by*

$$K_1 \left(\frac{c}{c_1} + \frac{c_1}{c_2} \frac{1}{(1-\eta)^4} + \cdots + \frac{c_{\ell-1}}{c_\ell} \frac{1}{(1-\eta)^{2\ell}} \right)$$

Proof. — Let us first establish the result for $\ell = 1$. We assume that each F_i is of order 1 inside I, and that $|I| \geqslant \eta c_1$. For each $i = 1, \dots, d$, let k_i be the number of holes of $F_i \cap I$. Then by definition of order 1, there are at least $k_i - 1$ pieces of $F_i \cap I$ with size at least c_1, whence $c_1(k_i - 1) \leqslant |I|$, and

$$k_i \leqslant |I|/c_1 + 1.$$

Since each hole of $F_i \cap I$ has size $\leqslant s_1$, we deduce that the cardinality of $I \setminus F_i$ is bounded by $s_1(|I|/c_1 + 1)$. This yields by the inequality $1 \leqslant \frac{1}{\eta}\frac{|I|}{c_1}$ and (4.1):

$$|I \setminus F| \leqslant ds_1(|I|/c_1 + 1) \leqslant d\left(1 + \frac{1}{\eta}\right) s_1|I|/c_1 \leqslant \eta|I|.$$

We thus get $|F \cap I|/|I| \geqslant 1 - \eta \geqslant (1-\eta)^2$, which is the first point. Moreover, the number k of holes of $F \cap I$ satisfies $k \leqslant k_1 + \cdots + k_d \leqslant d|I|/c_1 + d$, whence the number m of pieces of $F \cap I$ satisfies

$$m \leqslant d|I|/c_1 + d + 1 \leqslant d|I|/c_1 + 2d \leqslant d\left(1 + \frac{2}{\eta}\right)|I|/c_1.$$

It follows that the number r of points of $F \cap I$ lying in a piece of size $\leqslant c$ satisfies

$$r \leqslant mc \leqslant d\left(1 + \frac{2}{\eta}\right)|I|\frac{c}{c_1}.$$

As we already know that $|F \cap I| \geqslant (1-\eta)|I|$, we get by definition of K_1

$$\frac{r}{|F \cap I|} \leqslant \frac{\left(1 + \frac{2}{\eta}\right)}{1-\eta} d\frac{c}{c_1} = K_1\frac{c}{c_1},$$

which establishes the second point for $\ell = 1$.

Now we assume by induction that the result is true up to $\ell - 1$ for some $\ell \geqslant 2$ and we consider a family $(F_i)_{1 \leqslant i \leqslant d}$ of subsets of $\mathbb{Z}$, which are of order ℓ inside an interval I satisfying $|I| \geqslant \eta c_\ell$. By definition of order ℓ, for each i there exists a subset $F'_i \subset \mathbb{Z}$ satisfying

- $F_i \subset F'_i$,
- each hole of $F'_i \cap I$ is of size $\leqslant s_\ell$,
- two consecutive holes of $F'_i \cap I$ are always separated by a piece of size at least c_ℓ,
- for each piece I' of $F'_i \cap I$, F'_i is of order $(\ell - 1)$ inside I'.

Since $|I| \geqslant \eta c_\ell$, the argument developed for order 1 applies for $F' := \bigcap_{i=1}^d F'_i$ (with $(c_{\ell-1}, c_\ell, s_\ell)$ in place of (c, c_1, s_1)). We thus get

$$|F' \cap I| \geqslant (1-\eta)|I|, \tag{4.4}$$

and denoting by r' the number of points of $F' \cap I$ lying in pieces of $F' \cap I$ of size $< c_{\ell-1}$, we have (using also (4.2))

$$\frac{r'}{|F' \cap I|} \leqslant K_1\frac{c_{\ell-1}}{c_\ell} < \eta. \tag{4.5}$$

Let G stand for the union of all pieces of $F' \cap I$ of size $\geqslant c_{\ell-1}$. The above inequality can be rewritten as

$$\frac{|G|}{|F' \cap I|} > (1-\eta). \tag{4.6}$$

Let J be an arbitrary piece of G. Since for each i, F_i is of order $\ell-1$ inside J, and by definition of G, $|J|\geqslant c_{\ell-1}\geqslant \eta c_{\ell-1}$, the induction hypothesis gives

$$\frac{|F\cap J|}{|J|}\geqslant (1-\eta)^{2\ell-2}.$$

Summing over all pieces of G we get, using also (4.6) and (4.4)

$$|F\cap I|\geqslant |F\cap G|\geqslant (1-\eta)^{2\ell-2}|G|\geqslant (1-\eta)^{2\ell}|I|, \tag{4.7}$$

which is the first point at order ℓ.

Moreover, if r_J denotes the number of points of $F\cap J$ lying in pieces of $F\cap J$ of size smaller than c, then

$$\frac{r_J}{|F\cap J|}\leqslant K_1\left(\frac{c}{c_1}+\frac{c_1}{c_2}\frac{1}{(1-\eta)^4}+\cdots+\frac{c_{\ell-2}}{c_{\ell-1}}\frac{1}{(1-\eta)^{2\ell-2}}\right).$$

Now let us denote by r the number of points of $F\cap I$ lying in pieces of $F\cap I$ of size smaller than c. The contribution to r of points in G is $\sum_J r_J$ (where the sum ranges over all pieces J of G), and by the previous inequality, it satisfies

$$\begin{aligned}\sum_J r_J &\leqslant K_1\left(\frac{c}{c_1}+\frac{c_1}{c_2}\frac{1}{(1-\eta)^4}+\cdots+\frac{c_{\ell-2}}{c_{\ell-1}}\frac{1}{(1-\eta)^{2\ell-2}}\right)|F\cap G|\\ &\leqslant K_1\left(\frac{c}{c_1}+\frac{c_1}{c_2}\frac{1}{(1-\eta)^4}+\cdots+\frac{c_{\ell-2}}{c_{\ell-1}}\frac{1}{(1-\eta)^{2\ell-2}}\right)|F\cap I|.\end{aligned}$$

The contribution to r of points in $F\setminus G$ is clearly at most $|F\setminus G|$, which can be bounded above as follows

$$\begin{aligned}|(F\cap I)\setminus G| &\leqslant |(F'\cap I)\setminus G| && \text{(because } F\subset F'\text{)}\\ &= r' && \text{(by definition of } G \text{ and } r'\text{)}\\ &\leqslant K_1\frac{c_{\ell-1}}{c_\ell}|F'\cap I| && \text{(by (4.5))}\\ &\leqslant K_1\frac{c_{\ell-1}}{c_\ell}|I| && \\ &\leqslant K_1\frac{c_{\ell-1}}{c_\ell}\frac{|F\cap I|}{(1-\eta)^{2\ell}} && \text{(by (4.7))}\end{aligned}$$

Summing the two contributions and using the above inequalities, we get

$$r\leqslant K_1\left(\frac{c}{c_1}+\frac{c_1}{c_2}\frac{1}{(1-\eta)^4}+\cdots+\frac{c_{\ell-2}}{c_{\ell-1}}\frac{1}{(1-\eta)^{2\ell-2}}+\frac{c_{\ell-1}}{c_\ell}\frac{1}{(1-\eta)^{2\ell}}\right)|F\cap I|,$$

which is the second point at order ℓ. □

4.2. Application to the structure of n-crossings

We want now to apply the preceding lemma in order to obtain some statistical results on long range of successive n-crossings. We fix some integer $\underline{\ell}$, large enough to satisfy some conditions to be specified later, and we set $\underline{k}:=k(\underline{\ell})$ We define the sequences $(c_\ell)_{\ell\geqslant 1}$ and $(s_\ell)_{\ell\geqslant 1}$ as follows.

- $c_1 := h_{n_{\underline{\ell}}}$,
- $s_1 := h_{(n_{\underline{\ell}}-\underline{k})} + 1$,
- in general, $c_\ell := h_{n_{(\underline{\ell}+\ell-1)}}$, and $s_\ell := h_{n_{(\underline{\ell}+\ell-1)}-k(\underline{\ell}+\ell-1)} + 1$.

Using the fact that we always have $h_n/h_{n+1} < 1/3$, we observe that for each $\ell \geqslant 1$, with $k := k(\underline{\ell}+\ell-1)$,

$$\frac{s_\ell}{c_\ell} = \frac{h_{n_{(\underline{\ell}+\ell-1)}-k}+1}{h_{n_{(\underline{\ell}+\ell-1)}}} < 2\frac{h_{n_{(\underline{\ell}+\ell-1)}-k}}{h_{n_{(\underline{\ell}+\ell-1)}}} < \frac{2}{3^k} \leqslant \frac{2}{3^{\underline{k}}}.$$

Hence (4.1) is satisfied if $\underline{\ell}$ is large enough. The fact that (4.2) holds if $\underline{\ell}$ is large enough follows from the following easy consequence of (2.2):

$$\frac{h_{n_\ell}}{h_{n_{\ell+1}}} < \frac{h_{n_{(\ell+1)}} - (\ell+1)}{h_{n_{(\ell+1)}}} < \frac{1}{3^{\ell+1}} \xrightarrow[\ell\to\infty]{} 0.$$

We can therefore assume that $\underline{\ell}$ is large enough so that both (4.1) and (4.2) hold.

We want to apply Lemma 4.2 to the subsets F_i $(i = 1, \dots, d)$ defined by

$$F_i := \left\{j \in \mathbb{Z} : T^j x_i \in C_{n_{\underline{\ell}}}\right\}.$$

Let $I \subset \mathbb{Z}$ be an interval, $n \geqslant 1$ and $i \in \{1, \dots, d\}$. We say that x_i *climbs into tower n along I* if for each $j \in I$, $T^j x_i \in C_n$, and there is no $j \in I$ such that $j+1 \in I$, $T^j x_i \in L_n^{h_n-1}$ and $T^{j+1}x_i \in L_n^0$. Note that I is included in an n-crossing if and only if each coordinate x_i climbs into tower n along I.

Lemma 4.3. — *For each interval $I \subset \mathbb{Z}$ and each $i \in \{1, \dots, d\}$, if x_i climbs into tower $n_{(\underline{\ell}+\ell)}$ along I, then F_i is of order ℓ inside I.*

Proof. — By construction of the Nearly Finite Chacon Transformation, two successive occurrences of tower $n_{\underline{\ell}}$ inside tower $n_{(\underline{\ell}+1)}$ are separated either by $h_{n_{\underline{\ell}}-\underline{k}}$ or by $h_{n_{\underline{\ell}}-\underline{k}}+1$ spacers. Hence, if x_i climbs into tower $n_{(\underline{\ell}+1)}$ along I, F_i is of order 1 inside I. This proves the lemma in the case $\ell = 1$.

Assume that the statement of the lemma is true up to $\ell-1$ for some $\ell \geqslant 2$. We consider

$$F_i' := \left\{j \in \mathbb{Z} : T^j x_i \in C_{n_{(\underline{\ell}+\ell-1)}}\right\}.$$

We clearly have $F_i \subset F_i'$.

Two successive occurrences of tower $n_{(\underline{\ell}+\ell-1)}$ inside tower $n_{(\underline{\ell}+\ell)}$ are separated either by $h_{n_{(\underline{\ell}+\ell-1)}-k}$ or by $h_{n_{(\underline{\ell}+\ell-1)}-k}+1$ spacers, where k is determined by $\ell_k \leqslant \underline{\ell}+\ell-1 < \ell_{k+1}$. Hence, if x_i climbs into tower $n_{(\underline{\ell}+\ell)}$ along I, each hole of $F_i' \cap I$ is of size $\leqslant h_{n_{(\underline{\ell}+\ell-1)}-k}+1 = s_\ell$, and two consecutive holes of $F_i' \cap I$ are separated by a piece of $F_i' \cap I$ of size $h_{n_{(\underline{\ell}+\ell-1)}} = c_\ell$. Moreover, along each piece of $F_i' \cap I$, x_i climbs into tower $n_{(\underline{\ell}+\ell-1)}$. Therefore the property for $\ell-1$ ensures that F_i is of order $\ell-1$ inside each piece of $F_i' \cap I$. It follows that F_i is of order ℓ inside I, and the lemma is proved by induction. ☐

Proof of Proposition 4.1. — With the subsets F_i defined as above, we see that

$$F := \bigcap_{1\leqslant i\leqslant d} F_i = \left\{j \in \mathbb{Z} : (T^{\times d})^j x \in C^d_{n_{\underline{\ell}}}\right\}.$$

Observe that the pieces of F are precisely the $n_{\underline{\ell}}$-crossings.

Assume that the interval $I \subset \mathbb{Z}$ is included in an $n_{(\underline{\ell}+\ell)}$-crossing for some $\ell \geqslant 1$ (remember that this is equivalent to: each coordinate x_i climbs into tower $n_{(\underline{\ell}+\ell)}$ along I). Then, putting together Lemma 4.2 and Lemma 4.3, and provided that the length of I be at least $\eta c_\ell = \eta h_{n_{(\underline{\ell}+\ell-1)}}$, we get:

- the proportion of $j \in I$ such that $(T^{\times d})^j x \in C^d_{n_{\underline{\ell}}}$ is at least $(1-\eta)^{2\ell}$,
- for each $1 \leqslant c \leqslant h_{n_{\underline{\ell}}}$, the proportion of $j \in F \cap I$ belonging to an $n_{\underline{\ell}}$-crossing of size $\leqslant c$ is bounded above by

$$K_1 \left(\frac{c}{h_{n_{\underline{\ell}}}} + \frac{h_{n_{\underline{\ell}}}}{h_{n_{(\underline{\ell}+1)}}} \frac{1}{(1-\eta)^4} + \cdots + \frac{h_{n_{(\underline{\ell}+\ell-2)}}}{h_{n_{(\underline{\ell}+\ell-1)}}} \frac{1}{(1-\eta)^{2\ell}} \right). \tag{4.8}$$

Let us estimate the general term of the above sum, using the inequality $h_{n_\ell}/h_{n_{(\ell+1)}} < 1/3^{\ell+1}$, and the assumption $(1-\eta)^2 > 1/2$.

$$\begin{aligned} \frac{h_{n_{(\underline{\ell}+\ell-2)}}}{h_{n_{(\underline{\ell}+\ell-1)}}} \frac{1}{(1-\eta)^{2\ell}} &< \frac{1}{3^{(\underline{\ell}+\ell-1)}} \frac{1}{(1-\eta)^{2\ell}} \\ &= \frac{1}{3^{(\underline{\ell}-1)}} \frac{1}{(3(1-\eta)^2)^{\ell}} \\ &< \frac{1}{3^{(\underline{\ell}-1)}} \left(\frac{2}{3}\right)^{\ell}. \end{aligned}$$

It follows that there exist a constant K_2 such that (4.8) $\leqslant K_1 \frac{c}{h_{n_{\underline{\ell}}}} + \frac{K_2}{3^{\underline{\ell}}}$. ☐

4.3. Measure of the edge of C^d_n

For each $n \geqslant 0$, we say that an n-box $L^{j_1}_n \times \cdots \times L^{j_d}_n$ is *on the edge of* C^d_n if there exists $i \in \{1, \ldots, d\}$ such that $j_i = 0$ or $j_i = h_n - 1$. We denote by ∂C^d_n the union of all such n-boxes.

As a first application of Proposition 4.1, we have the following result.

Corollary 4.4. —

$$\delta(n) := \frac{\sigma(\partial C^d_n)}{\sigma(C^d_n)} \xrightarrow[n\to\infty]{} 0.$$

sketch of proof. — This is a direct consequence of the following facts:

- Since x is typical for σ, the quotient $\delta(n)$ can be estimated by the ratio
$$\frac{\sum_{j\in I} \mathbb{1}_{\partial C^d_n}(T^{\times d})^j x}{\sum_{j\in I} \mathbb{1}_{C^d_n}(T^{\times d})^j x}$$
for a large interval I containing 0.
- The subset of $j \in \mathbb{Z}$ such that $(T^{\times d})^j x \in C^d_n$ is partitioned into n-crossings, and in each n-crossing J there are exactly two integers j (the minimum and the maximum of J) such that $(T^{\times d})^j x \in \partial C^d_n$.
- By Proposition 4.1, most n-crossings are large if n is large. ☐

5. Convergence of sequences of empirical measures

For each finite subset $J \subset \mathbb{Z}$, we denote by γ_J the *empirical measure*

$$\gamma_J := \sum_{j \in I} \delta_{(T^{\times d})^j x}.$$

The validity of Property (3.3) whenever B is an n-box and C is C_n^d (remember that x has been chosen as a typical point) means that, if (J_n) is a sequence of intervals containing 0, with $|J_n| \xrightarrow[n\to\infty]{} \infty$, then we have the convergence $\gamma_{J_n} \xrightarrow[n\to\infty]{} \sigma$.

Our purpose in this section is to extend this convergence to the case where the intervals J_n do not necessarily contain 0, but are *not too far* from 0. We will also treat the case where the subsets J_n are no longer intervals, but union of intervals with a sufficiently regular structure.

We fix a real number $\varepsilon > 0$, small enough so that $(1-\varepsilon)^2 > 1-\eta$. Then we consider an integer $c \geqslant 1$, large enough so that $\frac{c-1}{c} > 1-\varepsilon$.

In Sections 5.1 and 5.2, we consider a fixed integer $\underline{\ell}$, large enough so that the result of Proposition 4.1 holds. We can also assume that

$$K_1 \frac{c}{h_{n_{\underline{\ell}}}} + \frac{K_2}{3^{\underline{\ell}}} < \varepsilon. \tag{5.1}$$

We are going to estimate the behaviour of empirical measures with respect to $n_{\underline{\ell}}$-boxes. The following lemmas are devoted to the control of

$$\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big) = \sum_{j\in I} \mathbb{1}_{C^d_{n_{\underline{\ell}}}}\Big((T^{\times d})^j(x)\Big)$$

for particular intervals I.

5.1. Consecutive n-intervals

For $n \geqslant 1$, we call *n-interval* any interval $I = \{j, j+1, \dots, j+h_n-1\}$ of length h_n and such that j is a multiple of h_n. (The second condition is completely artificial, it is only useful to define canonically a cutting of any interval into intervals of length h_n.)

Lemma 5.1. — *Let p_1 be the smallest integer such that $3^{p_1} > 2d+1$ and $p_1 > d$. There exists a constant $0 < \theta_1 < 1$ (depending only on η and d) for which the following holds.*

Let $\ell > \underline{\ell}+1$, and let n be such that $n_{(\ell-1)} - k(\ell-1) + p_1 \leqslant n < n_\ell$. Whenever I_1 and I_2 are two consecutive n-intervals, both contained in the same n_ℓ-crossing, we have

$$\theta_1 \gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big) < \gamma_{I_2}\Big(C^d_{n_{\underline{\ell}}}\Big) < \frac{1}{\theta_1}\gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Proof. — We divide the proof into two cases.

Case 1: $n_{(\ell-1)} + 1 \leqslant n \leqslant n_\ell$. — Set $j_1 := \min I_1$ and $j_2 := \min I_2 = j_1 + h_n$. Proposition 4.1 applies to I_1, and this ensures that, among the $\gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big)$ integers j such that $(T^{\times d})^{j_1+j}x \in C^d_{n_{\underline{\ell}}}$, a proportion at least $(1-\varepsilon)$ (by (5.1)) belong to an $n_{\underline{\ell}}$-crossing of size at least c. Then, among those belonging to an $n_{\underline{\ell}}$-crossing of size at least c, a proportion at least $\frac{c-1}{c}$ are not the minimum of their $n_{\underline{\ell}}$-crossing. By the choice of ε and c, we get the partial following result: a proportion at least $1-\eta$ of integers $j \in \{0, \dots, h_n - 1\}$ are such that, for each $i = 1, \dots, d$, $T^{j_1+j}x_i \in C_{n_{\underline{\ell}}}$, but $T^{j_1+j}x_i$ is not in the bottom level of tower $n_{\underline{\ell}}$. Let us consider such an integer j. Observe that, since I_2 is in the same n_ℓ-crossing as I_1, the coordinate $T^{j_1+j}x_i$ cannot be in the last occurrence of tower n inside tower n_ℓ. Hence it will pass through zero or one spacer before coming back to C_n. Then we can use a similar argument as in the proof of Lemma 3.8: according to whether coordinate i sees a spacer or not, $T^{j_2+j}x_i = T^{j_1+j+h_n}x_i$ is either in the same level of tower n as $T^{j_1+j}x_i$, or in the level immediately below. And the same applies if we consider the levels of tower $n_{\underline{\ell}}$. Hence $T^{j_2+j}x_i \in C_{n_{\underline{\ell}}}$.

This proves that $\gamma_{I_2}\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant (1-\eta)\gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big)$. But we can do a similar reasoning starting from I_2 and going backwards, and we get the announced inequalities for any $0 < \theta_1 \leqslant (1-\eta)$.

Case 2: $n_{(\ell-1)} - k(\ell-1) + p_1 \leqslant n \leqslant n_{(\ell-1)}$. — To simplify the notations, we set $n' := n_{(\ell-1)} - k(\ell-1)$. The reason why we cannot do the same reasoning as in the previous case is the following: when for some j the coordinate $T^j x_i$ leaves tower n, it will come back to C_n after 0, 1, $h_{n'}$ or $h_{n'}+1$ spacers. Because of this huge number of spacers that might separate two climbings into tower n, we cannot be sure that $T^{j+h_n}x_i$ will be in $C_{n_{\underline{\ell}}}$. To circumvent this difficulty, we introduce what we call the *fake tower n'*: it is the Rokhlin tower of height $h_{n'}$ whose levels are the $h_{n'}$ spacers placed on top of tower $n_{(\ell-1)}$ in the construction. Let us denote by $\tilde L^0_{n'}, \dots, \tilde L^{h_{n'}-1}_{n'}$ its consecutive levels. We note that this fake tower n' is disjoint from $C_{n_{(\ell-1)}}$, a fortiori it is disjoint from $C_{n_{\underline{\ell}}}$. However we can construct a *fake* $C_{n_{\underline{\ell}}}$ inside the fake tower n' by mimicking the structure of $C_{n_{\underline{\ell}}}$ inside tower n'. More precisely, we set

$$\tilde C_{n_{\underline{\ell}}} := \bigsqcup_{j : L^j_{n'} \subset C_{n_{\underline{\ell}}}} \tilde L^j_{n'}, \qquad \text{and} \qquad \overline{C}_{n_{\underline{\ell}}} := C_{n_{\underline{\ell}}} \sqcup \tilde C_{n_{\underline{\ell}}}.$$

If we consider $\overline{C}_{n_{\underline{\ell}}}$ instead of $C_{n_{\underline{\ell}}}$, then everything happens as if the n-intervals were both contained in a single $n_{(\ell-1)}$-crossing. Hence we can use Case 1 with $(\ell-1)$ in place of ℓ, which yields

$$(1-\eta)\gamma_{I_1}\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) < \gamma_{I_2}\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) < \frac{1}{(1-\eta)}\gamma_{I_1}\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big). \tag{5.2}$$

It remains now to compare $\gamma_I\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big)$ with $\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big)$ for $I = I_1$ or $I = I_2$.

For this we will consider n'-intervals intersecting I. Let J be such an n'-interval. We say that it is *suspect* if there exists $1 \leqslant i \leqslant d$ and $j \in J$ such that $T^j x_i \notin C_{n_{(\ell-1)}}$. Note that, by definition of a suspect interval, if $J \subset I$ is an n'-interval which is not suspect, then $\gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) = \gamma_J\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big)$. Observe also that, when $T^j x_i$ leaves $C_{n_{(\ell-1)}}$, then it comes back after at most $h_{n'}+1$ spacers (remember that everything takes place inside an n_ℓ-crossing, therefore the coordinates do not leave C_{n_ℓ}). Moreover, when it comes back to $C_{n_{(\ell-1)}}$, it stays in $C_{n_{(\ell-1)}}$ for a time $\geqslant h_{n_{(\ell-1)}} \geqslant h_n$. Since $|I| = h_n$, each coordinate $1 \leqslant i \leqslant d$ is responsible for at most 2 suspect n'-intervals intersecting I, and we conclude that there exist at most $2d$ suspect n'-intervals intersecting I. Moreover, since we assumed that $n \geqslant n' + p_1$, we have $h_n/h_{n'} > 3^{p_1} > 2d+1$, and this ensures that there exists at least one n'-interval contained in I which is not suspect.

Now if J' is a suspect interval intersecting I, we can find a chain $J' = J'_0, J'_1, \dots, J'_r = J$ of consecutive n'-intervals, where $J'_0, \dots, J'_{r-1}$ are suspect (hence $r \leqslant 2d$), $J = J'_r$ is not suspect and contained in I. Applying Case 1 $r \leqslant 2d$ times (with n' in place of n and $\ell - 1$ in place of ℓ gives

$$\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) = \gamma_J\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) \geqslant (1-\eta)^{2d}\gamma_{J'}\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) \geqslant (1-\eta)^{2d}\gamma_{J'}\Big(\tilde{C}^d_{n_{\underline{\ell}}}\Big).$$

Since only suspect intervals can contribute to $\gamma_I\Big(\tilde{C}^d_{n_{\underline{\ell}}}\Big)$, and since there are at most $2d$ of them, summing the preceding inequality over all suspect intervals J' intersecting I yields

$$2d\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant (1-\eta)^{2d}\gamma_I\Big(\tilde{C}^d_{n_{\underline{\ell}}}\Big).$$

In other words,

$$\gamma_I\Big(\tilde{C}^d_{n_{\underline{\ell}}}\Big) \leqslant \frac{2d}{(1-\eta)^{2d}}\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Adding $\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big)$ on both sides, we get

$$\gamma_I\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) \leqslant \left(\frac{2d}{(1-\eta)^{2d}} + 1\right)\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Inserting the above inequality for $I = I_1$ in (5.2), we get

$$\gamma_{I_2}\Big(C^d_{n_{\underline{\ell}}}\Big) \leqslant \gamma_{I_2}\Big(\overline{C}^d_{n_{\underline{\ell}}}\Big) < \frac{1}{\theta_1}\gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big)$$

with

$$\theta_1 := (1-\eta)\left(\frac{2d}{(1-\eta)^{2d}} + 1\right)^{-1}.$$

But we can exchange the roles of I_1 and I_2 and this gives the announced result. □

Remark 5.2. — Let ℓ and n be as in Lemma 5.1. Assume that I_1 and I_2 are consecutive n-intervals, but only I_1 is supposed to be contained in some n_ℓ crossing J. Then we get the inequality

$$\gamma_{I_2\cap J}\Big(C^d_{n_{\underline{\ell}}}\Big) < \frac{1}{\theta_1}\gamma_{I_1}\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Indeed, we can always change what happens on $I_2 \setminus J$ to do as if both I_1 and I_2 were included in the same n_ℓ-crossing.

5.2. Contribution of substantial subintervals

Lemma 5.3. — *Let p_2 be the smallest integer such that $\frac{1}{3^{p_2}} < \frac{1}{3}\eta$. For each $M > 0$, there exists a real number $0 < \theta_2(M) < 1$ (depending also on η and d) for which the following holds.*

Let $\ell > \underline{\ell} + 1$, and let n be such that

$$n_{(\ell-1)} - k(\ell-1) + p_1 + p_2 \leqslant n < n_\ell. \tag{5.3}$$

For each interval I of length $|I| \leqslant Mh_n$ contained in an n_ℓ-crossing, for each subinterval $J \subset I$ with $|J| \geqslant \eta h_n$, we have

$$\gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \theta_2(M)\,\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Remark 5.4. — Note that if ℓ is large enough, we have $n_{(\ell-1)} - k(\ell-1) + p_1 + p_2 < n_{(\ell-1)}$, hence the above is valid in particular for $n_{(\ell-1)} \leqslant n < n_\ell$.

Proof. — Under the assumptions of the lemma, we have

$$n_{\underline{\ell}} < n_{(\ell-1)} - k(\ell-1) + p_1 \leqslant n - p_2 < n_\ell.$$

We consider the $(n-p_2)$-intervals included in I, and we will apply Lemma 5.1 to them. Remember that $h_{n-p_2} \geqslant \frac{1}{7^{p_2}} h_n$. Hence the number of $(n-p_2)$-intervals contained in I is at most $7^{p_2}M$. Moreover their length h_{n-p_2} satisfies $h_{n-p_2} < \frac{1}{3^{p_2}} h_n < \frac{1}{3}|J|$. Hence there is at least one $(n-p_2)$-interval included in J. Let us call it J'. Now, if I' is another $(n-p_2)$-interval contained in I, a repeated use of Lemma 5.1 yields

$$\gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \gamma_{J'}\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \theta_1^{7^{p_2}M} \gamma_{I'}\Big(C^d_{n_{\underline{\ell}}}\Big).$$

There might also exist two $(n-p_2)$ intervals intersecting I at its extremities but not contained in I, hence not necessarily contained in the n_ℓ-crossing. If I' is such an interval, we use Remark 5.2 and get the same inequality (with $\gamma_{I'\cap I}$ instead of $\gamma_{I'}$). Summing over all the $(n-p_2)$-intervals intersecting I, we get

$$(7^{p_2}M + 2)\,\gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \theta_1^{7^{p_2}M} \gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big).$$

This gives the announced result, with $\theta_2(M) := \theta_1^{7^{p_2}M} / \,(7^{p_2}M + 2)$. □

5.3. How to apply Proposition 3.5

Here we want to provide some conditions so that Proposition 3.5 applies to a sequence of empirical measures (γ_{J_m}) for some sequence of intervals (J_m). We make the following assumptions.

(5.4) For each m, there exists an integer ℓ_m with $\ell_m \to \infty$ as $m \to \infty$, such that J_m is contained in some n_{ℓ_m}-crossing,

and

(5.5) For each m, there exists an integer $n(m)$ satisfying

- $n_{(\ell_m-1)} - k(\ell_m - 1) + p_1 + 2p_2 \leqslant n(m) < n_{\ell_m}$,
- $\eta h_{n(m)} \leqslant |J_m| \leqslant h_{n(m)}$,

Note that, as soon as $\underline{\ell}$ is large enough so that Proposition 4.1 applies, the first point of this proposition ensures that $\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right) > 0$ for m large enough, which is the first assumption needed to apply Proposition 3.5.

It remains, for some fixed $\underline{\ell}$ and ℓ, to control the ratio $\gamma_{J_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right)/\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right)$, which is the purpose of the following lemma.

Lemma 5.5. — *Let $\underline{\ell}$ be large enough so that Proposition 4.1 applies and* (5.1) *holds. Assume also that*

$$(K_1 + K_2)\frac{1}{3^{\underline{\ell}}} < \eta. \tag{5.6}$$

Let $\ell \geqslant 1$, and let (J_m) be a sequence of intervals satisfying (5.4) *and* (5.5)*. Then for each m large enough*

$$\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right) \geqslant (1-\eta)^{2\ell+1}\theta_2(7)\gamma_{J_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right).$$

Proof. — We first consider the case where $n(m) \geqslant n_{(\ell_m-1)}$. Then we can apply Proposition 4.1 (with $\underline{\ell}+\ell$ in place of $\underline{\ell}$) to show that, if $\ell_m \geqslant \underline{\ell}+\ell+1$, the proportion of integers in $\{j \in J_m : (T^{\times d})^j x \in C^d_{n_{(\underline{\ell}+\ell)}}\}$ belonging to an $n_{(\underline{\ell}+\ell)}$-crossing of size less than $h_{n_{(\underline{\ell}+\ell-1)}}$ is bounded above by

$$K_1\frac{h_{n_{(\underline{\ell}+\ell-1)}}}{h_{n_{(\underline{\ell}+\ell)}}} + \frac{K_2}{3^{\underline{\ell}+\ell}} \leqslant (K_1+K_2)\frac{1}{3^{\underline{\ell}+\ell}} < \eta.$$

Now, if $I \subset J_m$ is an $n_{(\underline{\ell}+\ell)}$-crossing with $|I| \geqslant h_{n_{(\underline{\ell}+\ell-1)}}$, another application of Proposition 4.1 proves that the proportion of integers $j \in I$ such that $(T^{\times d})^j x \in C^d_{n_{\underline{\ell}}}$ is at least $(1-\eta)^{2\ell}$. We finally get in this case

$$\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right) \geqslant (1-\eta)^{2\ell+1}\gamma_{J_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right). \tag{5.7}$$

Now we consider the case where $n_{(\ell_m-1)} - k(\ell_m-1) + p_1 + 2p_2 \leqslant n(m) < n_{(\ell_m-1)}$. Let n be the largest integer, $n \leqslant n(m)$, such that $h_n \leqslant |J_m|$. If $n < n(m)$, then we have $h_{n+1} > |J_m| \geqslant \eta h_{n(m)}$. But on the other hand, $h_{n+1} < h_{n(m)}/3^{n(m)-n-1}$. Taking into account the definition of p_2 (Lemma 5.3), we get that $n(m) - n < p_2$, and finally that n satisfies

$$n_{(\ell_m-1)} - k(\ell_m-1) + p_1 + p_2 < n < n_{(\ell_m-1)}.$$

Since $|J_m| < h_{n_{(\ell_m-1)}}$, we observe that each coordinate can leave $C_{n_{(\ell_m-1)}}$ only once on J_m, and when it does so, it stays outside $C_{n_{(\ell_m-1)}}$ on an interval of length $\leqslant h_{n_{(\ell_m-1)}-k(\ell_m-1)} + 1$. Since there are d coordinates, the set of integers $j \in J_m$ such that $(T^{\times d})^j x \in C^d_{n_{(\ell_m-1)}}$ is cut into at most $(d+1)$ pieces, and its cardinality is at least

$$|J_m| - d\left(h_{n_{(\ell_m-1)}-k(\ell_m-1)} + 1\right) \geqslant |J_m| - \frac{d}{3^{p_1+p_2}} h_n$$
$$\geqslant |J_m| \left(1 - \frac{d}{3^{p_1+p+2}}\right)$$
$$\geqslant |J_m| \left(1 - \frac{\eta}{6}\right) \geqslant \frac{|J_m|}{2}$$

Therefore, there exists at least one subinterval $\tilde{J}_m \subset J_m$, with size

$$|\tilde{J}_m| \geqslant \frac{|J_m|}{2(d+1)} \geqslant \eta h_n,$$

and which is contained in a single $n_{(\ell_m-1)}$-crossing. Since $n_{(\ell_m-2)} < n < n_{(\ell_m-1)}$, the estimation (5.7) is valid for $\tilde{J}_m$ in place of J_m, i.e.

$$\gamma_{\tilde{J}_m}\left(C^d_{n_{\underline{\ell}}}\right) \geqslant (1-\eta)^{2\ell+1} \gamma_{\tilde{J}_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right). \tag{5.8}$$

Note that, if $n < n(m)$, then by definition of n we have $|J_m| < h_{n+1} < 7h_n$. If $n = n(m)$, $|J_m| = h_{n(m)} = h_n$. Hence in all cases we have $\eta h_n \leqslant |\tilde{J}_m| \leqslant |J_m| \leqslant 7h_n$. So we can also apply Lemma 5.3 with $I := J_m$, $J := \tilde{J}_m$, and $\underline{\ell} + \ell$ in place of $\underline{\ell}$. This yields

$$\gamma_{\tilde{J}_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right) \geqslant \theta_2(7) \gamma_{J_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right). \tag{5.9}$$

Combining (5.8) and (5.9), we get

$$\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right) \geqslant \gamma_{\tilde{J}_m}\left(C^d_{n_{\underline{\ell}}}\right)$$
$$\geqslant (1-\eta)^{2\ell+1} \gamma_{\tilde{J}_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right)$$
$$\geqslant (1-\eta)^{2\ell+1} \theta_2(7) \gamma_{J_m}\left(C^d_{n_{(\underline{\ell}+\ell)}}\right). \qquad \square$$

With the above lemma, we see that all the conditions needed to apply Proposition 3.5 are satisfied, and this gives the following result.

Lemma 5.6. — *Let (J_m) be a sequence of intervals satisfying (5.4) and (5.5). Then there is a subsequence $\left(\gamma_{J_{m_j}}\right)$ which converges to some nonzero Radon measure.*

5.4. Convergence of sequences of empirical measures

Proposition 5.7. — *Let (I_m) and (J_m) be two sequences of intervals, with $J_m \subset I_m$. Assume that there exist two sequences of integers (ℓ_m) and $(n(m))$, and a real number $M > 0$ such that*

- *$\ell_m \to \infty$,*
- *$n_{(\ell_m-1)} - k(\ell_m - 1) + p_1 + 2p_2 \leqslant n(m) < n_{\ell_m}$,*
- *I_m is contained in some n_{ℓ_m}-crossing,*
- *$\eta h_{n(m)} \leqslant |J_m| \leqslant h_{n(m)}$,*

- $|I_m| \leqslant M h_{n(m)}$.

If $\gamma_{I_m} \xrightarrow[m\to\infty]{} \sigma$, *we also have* $\gamma_{J_m} \xrightarrow[m\to\infty]{} \sigma$.

Proof. — The assumptions (5.4) and (5.5) are satisfied for the sequence of intervals (J_m), hence Lemma 5.6 applies to the sequence of measures $\left(\gamma_{J_m}\right)$. Therefore, it is enough to prove that, if γ_{J_m} converges to some nonzero Radon measure γ, then $\gamma = \sigma$ up to some multiplicative constant. So, let us assume that $\gamma_{J_m} \to \gamma$. Since $J_m \subset I_m$, we have $\gamma_{J_m} \leqslant \gamma_{I_m}$. We can also apply Lemma 5.3 which shows that, for each large enough integer $\underline{\ell}$, we have as soon as $n(m) > n_{\underline{\ell}} + 1$

$$\gamma_{J_m}\left(C^d_{n_{\underline{\ell}}}\right) \geqslant \theta_2(M)\, \gamma_{I_m}\left(C^d_{n_{\underline{\ell}}}\right).$$

Then Proposition 3.6 ensures that $\gamma \ll \sigma$. Now by ergodicity of $(X^d, \sigma, T^{\times d})$, it only remains to show that γ is $T^{\times d}$-invariant. For this we want to apply Lemma 3.2. Since $X^d_{\mathbf{0}} \subset X^d \setminus X^d_\infty$, we have $\sigma(X^d_{\mathbf{0}}) = 0$ hence $\gamma(X^d_{\mathbf{0}}) = 0$ by absolute continuity. Finally, observe that if for some fixed integer n, B and B' are two n-boxes contained in the same n-diagonal, then for any m, as J_m is an interval,

$$|\gamma_{J_m}(B) - \gamma_{J_m}(B')| \leqslant 1.$$

Indeed, the times j when the orbit of x falls in B alternate with the times when the orbit of x fall in B'. On the other hand the first point of Proposition 4.1 ensures that

$$\gamma_{J_m}\left(C^d_n\right) \xrightarrow[m\to\infty]{} \infty,$$

and it follows that $\gamma(B) = \gamma(B')$. Lemma 3.2 now proves that γ is $T^{\times d}$-invariant. □

Remark 5.8. — Note that the condition $\gamma_{I_m} \xrightarrow[m\to\infty]{} \sigma$ is automatically satisfied if $0 \in I_m$ for each m, since we took x as a typical point for σ.

Now we want to extend Proposition 5.7 to the case where (J_m) is no longer a sequence of intervals, but J_m is a subset of I_m with a sufficiently regular structure.

Proposition 5.9. — *Let* (I_m) *and* (J_m) *be two sequences of finite subsets of* $\mathbb{Z}$, *and let* $M > 0$. *We assume that the following conditions are satisfied.*

- $J_m \subset I_m$ *for each* m.
- *There exists a sequence of integers* (ℓ_m) *with* $\ell_m \to \infty$ *as* $m \to \infty$, *such that for each* m, I_m *is an interval contained in some* n_{ℓ_m}*-crossing.*
- *There exists a sequence of integers* $(n(m))$ *with*

$$n_{(\ell_m - 1)} - k(\ell_m - 1) + p_1 + 2p_2 \leqslant n(m) < n_{\ell_m},$$

such that for each m, J_m *is a disjoint union of intervals of common size* $s(m)$, *where* $\eta h_{n(m)} \leqslant s(m) \leqslant h_{n(m)}$, *and* $I_m \setminus J_m$ *does not contain an interval of size greater than* $M h_{n(m)}$.

If $\gamma_{I_m} \xrightarrow[m\to\infty]{} \sigma$, *then we also have* $\gamma_{J_m} \xrightarrow[m\to\infty]{} \sigma$.

Proof. — We just have to justify that the same arguments as in the proof of Proposition 5.7 apply also in this case. First, we want to prove that the conclusion of Lemma 5.6 holds for (J_m). For this, it is enough to observe that all the pieces of

J_m satisfy assumptions (5.4) and (5.5). Hence the estimation given in Lemma 5.5 is valid for each piece of J_m, and then it is also valid for J_m itself.

Now, let $\underline{\ell}$ be a large enough integer, and take m large enough so that $n(m) > n_{\underline{\ell}}+1$. Let J be any piece of J_m (in particular we have $|J| = s(m) \geqslant \eta h_{n(m)}$), and let I be the interval constituted of J and the two adjacent pieces of $I_m \setminus J_m$. Then we have $|I| \leqslant 40h_{n(m)} + s(m) \leqslant 100h_{n(m)}$, and we can apply Lemma 5.3 to I and J to get

$$\gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \theta_2\gamma_I\Big(C^d_{n_{\underline{\ell}}}\Big).$$

Summing over all pieces J of J_m, we get

$$\gamma_{J_m}\Big(C^d_{n_{\underline{\ell}}}\Big) \geqslant \theta_2\gamma_{I_m}\Big(C^d_{n_{\underline{\ell}}}\Big),$$

which is the second key step in the proof of Proposition 5.7. This ensures that, if $\gamma_{J_m} \to \gamma$, then $\gamma \ll \sigma$.

Finally, we have to see that γ is $T^{\times d}$-invariant, and it is enough for that to show that, if for some fixed n we consider two n-boxes B and B' on the same n-diagonal, then $\gamma(B) = \gamma(B')$. But for each m and each piece J of J_m, we have $|\gamma_J(B) - \gamma_J(B')| \leqslant 1$, whereas by the first point of Proposition 4.1, we know that

$$\min_{J \text{ piece of } J_m} \gamma_J\Big(C^d_{n_{\underline{\ell}}}\Big) \xrightarrow[m\to\infty]{} \infty. \qquad \square$$

6. Twisting transformations and decomposition of σ as a product

The purpose of this section is to provide a criterion ensuring that σ can be decomposed into the product of two measures $\sigma_1 \times \sigma_2$, σ_i being a $T^{\times d_i}$-invariant Radon measure on X^{d_i} for some $1 \leqslant d_i < d$, $d_1 + d_2 = d$. We will need for that to introduce the following type of transformation of X^d.

Definition 6.1. — *The transformation $S : X^d \to X^d$ is said to be a* twisting transformation *if there exists a partition $\{1,\dots,d\} = G_0 \sqcup G_1$ into two nonempty subsets such that for each $(y_1,\dots,y_d) \in X^d$,*

$$S(y_1,\dots,y_d) = (z_1,\dots,z_d), \text{ where } z_i := \begin{cases} Ty_i & \text{if } i \in G_1,\\ y_i & \text{if } i \in G_0.\end{cases}$$

The reason why we introduce those twisting transformations is that, if we are able to prove that σ is invariant by some twisting transformation then σ can be decomposed as a product of two measures. More precisely, by Theorem A.1 in [JRdlR18] we have the following result.

Proposition 6.2. — *Assume that σ is invariant by some twisting transformation S, and let $\{1,\dots,d\} = G_0 \sqcup G_1$ be the partition associated with S. Then there exist Radon measures σ_0 and σ_1 on X^{G_0} and X^{G_1} respectively, such that*

- *$\sigma = \sigma_0 \otimes \sigma_1$;*
- *each σ_s is $T^{\times|G_a|}$-invariant ($a = 0, 1$), and the system $(X^{G_a}, T^{\times|G_a|}, \sigma_a)$ is conservative ergodic.*

Thus, if the assumption of the above proposition is satisfied, we can write σ as the product of two measures which are invariant by some smaller Cartesian power of T, and to which we can apply the induction hypothesis to finish the proof of Theorem 3.10. We want now to give a condition under which we are able to prove that σ is indeed invariant by some twisting transformation.

In the next proposition, we use again the notation Ω_n which was introduced in the proof of Lemma 3.2: recall that Ω_n is the union of all n-boxes of the form $L_n^{j_1} \times \cdots \times L_n^{j_d}$ where for all $i = 1, \dots, d$, $j_i \neq 0$. We observe that, if B is an n-box contained in Ω_n and if S is a twisting transformation, then $S^{-1}B$ is also an n-box (but not necessarily contained in Ω_n). Note also that $\Omega_n \subset \Omega_{n+1}$ for each n.

Proposition 6.3. — *Assume that there exist (σ'_n), (σ_n), two sequences of Radon measures on X^d, and a sequence (S_n) of twisting transformations satisfying*

- $\sigma_n \xrightarrow[n\to\infty]{} \sigma$,
- $\sigma'_n \xrightarrow[n\to\infty]{} \sigma$,
- *For each $m \geqslant 0$, and for n large enough (depending on m), for each m-box $B \subset \Omega_m$, $\sigma'_n(S_n^{-1}B) = \sigma_n(B)$.*

Then there exists a twisting transformation S such that σ is S-invariant. In particular, σ is a product measure as in Proposition 6.2.

Proof. — Note that, d being fixed here, there exist only finitely many twisting transformations. Therefore, considering subsequences if necessary, we may assume that there exist a twisting transformation S such that $S_n = S$ for each n.

Now, let $m \geqslant 0$ be large enough so that $\delta(m) < 1/2$ (see Corollary 4.4). In particular, $\sigma(C_m^d) > 0$. Let $m' \geqslant m$. Then, for each n large enough (depending on m'), if B is an m'-box contained in Ω_m, then $B \subset \Omega_{m'}$ and we know that $\sigma'_n(S^{-1}B) = \sigma_n(B)$. By Remark 3.4, the assumptions of the lemma also yield

$$\frac{\sigma'_n(S^{-1}B)}{\sigma'_n(C_m^d)} \xrightarrow[n\to\infty]{} \frac{\sigma(S^{-1}B)}{\sigma(C_m^d)}.$$

But the left-hand side of the above formula is equal to

$$\frac{\sigma'_n(S^{-1}B)}{\sigma'_n(C_m^d)} = \frac{\sigma_n(B)}{\sigma_n(C_m^d)} \frac{\sigma_n(C_m^d)}{\sigma'_n(C_m^d)},$$

where

$$\frac{\sigma_n(B)}{\sigma_n(C_m^d)} \xrightarrow[n\to\infty]{} \frac{\sigma(B)}{\sigma(C_m^d)}.$$

It remains to control the ratio $\sigma_n(C_m^d)/\sigma'_n(C_m^d)$. For this, we write

$$\sigma'_n(C_m^d) = \sigma'_n(S^{-1}\Omega_m) + \sigma'_n(C_m^d \setminus S^{-1}\Omega_m),$$

and

$$\sigma_n(C_m^d) = \sigma_n(\Omega_m) + \sigma_n(C_m^d \setminus \Omega_m).$$

We observe that the first terms $\sigma'_n(S^{-1}\Omega_m)$ and $\sigma_n(\Omega_m)$ are equal. Moreover, as $C_m^d \setminus \Omega_m$ and $C_m^d \setminus S^{-1}\Omega_m$ are included in ∂C_m^d, we have by Corollary 4.4

$$\frac{\sigma_n(\Omega_m)}{\sigma_n(C_m^d)} \xrightarrow[n\to\infty]{} \frac{\sigma(\Omega_m)}{\sigma(C_m^d)} \geqslant 1 - \delta(m)$$

and

$$\frac{\sigma'_n(S^{-1}\Omega_m)}{\sigma'_n(C^d_m)} \xrightarrow[n\to\infty]{} \frac{\sigma(S^{-1}\Omega_m)}{\sigma(C^d_m)} \geqslant 1-\delta(m),$$

where $\delta(m)\to 0$ as $n\to\infty$. In particular, $\sigma_n(\Omega_m)$ and $\sigma'_n(S^{-1}\Omega_m)$ are positive if $\delta(m)<1/2$ and n is large enough. Hence we can write

$$\frac{\sigma_n(C^d_m)}{\sigma'_n(C^d_m)} = \frac{1+\frac{\sigma_n(C^d_m\setminus\Omega_m)}{\sigma_n(\Omega_m)}}{1+\frac{\sigma'_n(C^d_m\setminus S^{-1}\Omega_m)}{\sigma'_n(S^{-1}\Omega_m)}} \xrightarrow[n\to\infty]{} \frac{1+\frac{\sigma(C^d_m\setminus\Omega_m)}{\sigma(\Omega_m)}}{1+\frac{\sigma(C^d_m\setminus S^{-1}\Omega_m)}{\sigma(S^{-1}\Omega_m)}}.$$

This yields

$$\sigma(S^{-1}B)=\sigma(B)\frac{1+\frac{\sigma(C^d_m\setminus\Omega_m)}{\sigma(\Omega_m)}}{1+\frac{\sigma(C^d_m\setminus S^{-1}\Omega_m)}{\sigma(S^{-1}\Omega_m)}}.$$

But the above argument is also valid if, at the beginning, we start with m' instead of m (and keep the same m'-box B.) This gives

$$\sigma(S^{-1}B)=\sigma(B)\frac{1+\frac{\sigma(C^d_{m'}\setminus\Omega_{m'})}{\sigma(\Omega_{m'})}}{1+\frac{\sigma(C^d_{m'}\setminus S^{-1}\Omega_{m'})}{\sigma(S^{-1}\Omega_{m'})}}.$$

Moreover, as $\sigma(C^d_m)>0$, we can choose the m'-box B in such a way that $\sigma(B)>0$, and comparing the last two equalities, we get

$$\frac{1+\frac{\sigma(C^d_m\setminus\Omega_m)}{\sigma(\Omega_m)}}{1+\frac{\sigma(C^d_m\setminus S^{-1}\Omega_m)}{\sigma(S^{-1}\Omega_m)}} = \frac{1+\frac{\sigma(C^d_{m'}\setminus\Omega_{m'})}{\sigma(\Omega_{m'})}}{1+\frac{\sigma(C^d_{m'}\setminus S^{-1}\Omega_{m'})}{\sigma(S^{-1}\Omega_{m'})}}.$$

But the ratio on the right-hand side can be made arbitrarily close to 1 by choosing m' large enough, hence it is equal to 1. This proves that, for any $m'\geqslant m$ and any m'-box $B\subset\Omega_m$, $\sigma(S^{-1}B)=\sigma(B)$. We thus get as in the proof of Lemma 3.2 that σ and $S_*(\sigma)$ coincide on $\bigcup_m\Omega_m = X\setminus X^d_{\mathbf{0}}$. And since both measures are equal to 0 on $X^d_{\mathbf{0}}$, this concludes the proof. □

The first two assumptions in Proposition 6.3 will be given by applications of Proposition 5.7 and Proposition 5.9. The following simple example presents the main ideas of how to construct sequences of measures (σ_n) and (σ'_n) satisfying the third requirement of Proposition 6.3.

Example 6.4. — Let $n\notin\{n_\ell:\ell\geqslant 1\}$. Let J be an interval contained in an n-crossing, set $J':=J+h_n$, assume that J' is also contained in an n-crossing. Finally, assume that, for each $j\in J$, $\Big\{t_n(T^jx_i): i=1,\dots,d\Big\}=\{1,2\}$.

Define $\sigma_n:=\gamma_J$, and $\sigma'_n:=\gamma_{J'}$. For an arbitrary $j\in J$, consider the partition $\{1,\dots,d\}=G_0\sqcup G_1$ into two nonempty subsets, where $G_0:=\Big\{i\in\{1,\dots,d\}: t_n(T^jx_i)=1\Big\}$, and $G_1:=\Big\{i\in\{1,\dots,d\}: t_n(T^jx_i)=2\Big\}$. Note that, since J is contained in an n-crossing, this partition does not depend on the choice of $j\in J$. Let S_n be the twisting transformation associated with (G_0,G_1). Then for each $m\leqslant n$ and each m-box $B\subset\Omega_m$, we have $\sigma'_n(S_n^{-1}B)=\sigma_n(B)$.

Indeed, consider first $i\in G_0$. Then, for $j\in J$, $T^j(x_i)$ is in the first subcolumn of tower n, hence when the orbit of x_i reaches the top of tower n, it will see no spacer before coming back to C_n. We thus have $j_n(T^{j+h_n}x_i) = j_n(T^j x_i)$, and in fact we have the equality $j_m(T^{j+h_n}x_i) = j_m(T^j x_i)$ for each $m \leqslant n$ (remember that j_n determines j_m for $m \leqslant n$).

On the other hand, if $i \in G_1$, the orbit of x_i will pass through the spacer above the middle subcolumn before coming back to C_n, and we have, for each $m \leqslant n$, $j_m(T^{j+h_n}x_i) = j_m(T^j x_i) - 1$ (provided $j_m(T^j x_i) \neq 0$).

Now, if $B \subset \Omega_m$ is an m-box for some $m \leqslant n$, the above argument shows that, for each $j \in J$, $(T^{\times d})^j x \in B \iff (T^{\times d})^{j+h_n} x \in S^{-1}B$.

7. End of the proof of the main result

Now we come back to the last part of the proof of Theorem 3.10. We interpret Proposition 3.17 as follows:

- either σ is a graph measure arising from powers of T,
- or there exist infinitely many integers n such that there exists at least one substantial n-crossing which is not synchronized.

It only remains to show how this latter property implies that σ can be decomposed as a product measure, as explained in Section 3.3 and with the tools of Section 6.

From now on, we thus assume that for infinitely many integers n, there exists at least one substantial n-crossing which is not synchronized. We have to study different cases, according to the relative positions of these integers n with respect to the sequence (n_ℓ).

7.1. The case $n_{(\ell-1)} \leqslant n \leqslant n_\ell - \ell$

Here we first assume that there exist infinitely many integers n for which

- there exists at least one substantial n-crossing which is not synchronized.
- $\exists\, \ell : n_{(\ell-1)} \leqslant n \leqslant n_\ell - \ell$.

Let us consider such an n. To unify the treatments of the cases $n = n_{(\ell-1)}$ and $n_{(\ell-1)} < n \leqslant n_\ell - \ell$, we set

$$\tilde{h}_n := \begin{cases} h_n + h_{n(\ell-1)-k(\ell-1)} & \text{if } n = n_{(\ell-1)}, \\ h_n & \text{if } n_{(\ell-1)} < n \leqslant n_\ell - \ell. \end{cases}$$

In this way, as long as we stay inside the interval $[-100h_n, 100h_n]$ (which is contained in a single n_ℓ-crossing as $n \leqslant n_\ell - \ell$), if j is in some n-crossing and $j_n(T^j x_i) > 0$, we have

$$j_n\left(T^{j+\tilde{h}_n}x_i\right) = \begin{cases} j_n\left(T^j x_i\right) & \text{if } t_n(T^j x_i) = 1, \\ j_n\left(T^j x_i\right) - 1 & \text{if } t_n(T^j x_i) = 2, \\ \text{one or other of the above values} & \text{if } t_n(T^j x_i) = 3. \end{cases}$$

Let J be a substantial n-crossing which is not synchronized. Then for $j \in J$, $\{t_n(T^j x_i) : i = 1, \dots, d\}$ contains at least two different values (which do not depend on the choice of $j \in J$ since $j \mapsto t_n(T^j x_i)$ is constant on an n-crossing).

We first assume that $\{1,2\} \subset \{t_n(T^j x_i) : i = 1,\dots,d\}$. Then, by the above formula, for $j \in J \setminus \min J$, the difference $j_n(T^j x_i) - j_n\left(T^{j+\tilde{h}_n} x_i\right)$ takes both values 0 and 1 as i runs over $\{1,\dots,d\}$. Set, for $a = 0,1$

$$G_a := \left\{ i : j_n\left(T^j x_i\right) - j_n\left(T^{j+\tilde{h}_n} x_i\right) = a \right\}.$$

Then we can define a twisting transformation S_n with this partition. We also define the interval $J' := J + \tilde{h}_n$, and the two measures $\sigma_n := \gamma_J$, $\sigma'_n := \gamma_{J'}$.

As explained in Example 6.4, if for some B is an m-box for some $m \leqslant n$ with $B \subset \Omega_m$, we then have

$$\sigma'_n(B) = \sigma_n(S_n^{-1}(B)). \tag{7.1}$$

Let us explain how we construct S_n, σ_n and σ'_n when $\{2,3\} = \{t_n(T^j x_i) : i = 1,\dots,d\}$ for $j \in J$. Then, for $j \in J \setminus \{\max J\}$, the difference $j_n\left(T^{j-\tilde{h}_n} x_i\right) - j_n(T^j x_i)$ takes both values 0 and 1 as i runs over $\{1,\dots,d\}$. In this case we define the partition by

$$G_a := \left\{ i : j_n\left(T^{j-\tilde{h}_n} x_i\right) - j_n\left(T^j x_i\right) = a \right\}, \quad a = 0,1,$$

and the corresponding twisting transformation S_n. We consider $J' := J - \tilde{h}_n$, $\sigma_n := \gamma_{J'}$ and $\sigma'_n := \gamma_J$, and we get (7.1) for any m-box $B \subset \Omega_m$, $m \leqslant n$.

Finally we consider the case when $\{1,3\} = \{t_n(T^j x_i) : i = 1,\dots,d\}$ for $j \in J$. Then, for $j \in J \setminus \{\min J\}$, there are two options:

- either there exists $i \in \{1,\dots,d\}$ with $t_n(T^j x_i) = 3$, and $j_n(T^j x_i) - j_n\left(T^{j+\tilde{h}_n} x_i\right) = 1$ (we see one spacer above the third column for at least one coordinate),
- or for each $i \in \{1,\dots,d\}$ such that $t_n(T^j x_i) = 3$, we have $j_n(T^j x_i) - j_n\left(T^{j+\tilde{h}_n} x_i\right) = 0$ (we see no spacer above the third column).

In the first option, we do the same construction as in the case $\{1,2\} \subset \{t_n(T^j x_i) : i = 1,\dots,d\}$. In the second option, we observe that

$$j_n\left(T^j x_i\right) - j_n\left(T^{j+2\tilde{h}_n} x_i\right) = \begin{cases} 1 & \text{if } t_n(T^j x_i) = 1, \\ 0 & \text{if } t_n(T^j x_i) = 3. \end{cases}$$

We then set $J' := J + 2\tilde{h}_n$, and construct S_n, σ_n and σ'_n as before.

Since we assume that there are infinitely many integers n with these properties, we can apply Proposition 5.7 to prove that $\sigma_n \to \sigma$ and $\sigma'_n \to \sigma$. Indeed, J and J' are both contained in $\{-5h_n,\dots,5h_n\}$ which is contained in an n_ℓ-crossing. Since J is a substantial n-crossing, we have $\eta h_n \leqslant |J| = |J'| \leqslant h_n$, and we have $\gamma_{\{-5h_n,\dots,5h_n\}} \xrightarrow[n\to\infty]{} \sigma$. Then Proposition 6.3 shows that σ can be decomposed as a product of two Radon measures to which we can apply the induction hypothesis.

We are now reduced to study the case where, for each ℓ large enough and each $n_{\ell-1} \leqslant n \leqslant n_\ell - \ell$, all substantial n-crossings are synchronized, but still there exist infinitely many integers n for which at least one substantial n-crossing is not synchronized.

7.2. The case $n_\ell - \ell < n < n_\ell - k(\ell)$

This case cannot be treated as the preceding one since, for such an n, we are not sure any more that an interval around 0 and of size of order h_n is completely contained in an n_ℓ-crossing. Hence on such an interval, when the orbit of some x_i leaves C_n, it may stay out of C_n for a long time (up to $h_{n_\ell - k(\ell)} + 1$, which may be much larger than h_n).

The following lemma is introduced to remedy this problem.

Lemma 7.1. — *For each large enough ℓ, there exists an integer*

$$n_{good}(\ell) \in \{n_\ell - k(\ell) + p_1 + 2p_2, \dots, n_\ell - k(\ell) + p_1 + 2p_2 + d\}$$

such that $\{h_{n_{\text{good}}(\ell)}, \dots, 2h_{n_{\text{good}}(\ell)}\}$ is contained in an n_ℓ-crossing.

Proof. — Assume that $\ell > \max_i \ell(x_i)$, and that $n_\ell - k(\ell) + p_1 + 2p_2 + d < n_\ell$. We say that the coordinate $i \in \{1, \dots, d\}$ is *bad for n* if there exists some $j \in \{h_n, \dots, 2h_n\}$ such that $T^j x_i \notin C_{n_\ell}$. We observe that if $\{h_n, \dots, 2h_n\}$ is not contained in an n_ℓ-crossing, then at least one coordinate is bad for n. To prove the lemma, it is sufficient to show that for each $i = 1, \dots, d$, there is at most one $n \in \{n_\ell - k(\ell) + p_1 + 2p_2, \dots, n_\ell - k(\ell) + p_1 + 2p_2 + d\}$ for which i is bad. So assume that i is bad for some n in this interval, and let $j \in \{h_n, \dots, 2h_n\}$ such that $T^j x_i \notin C_{n_\ell}$. The orbit of x_i comes back to C_{n_ℓ} before $j + h_{n_\ell - k(\ell)} + 1$, then stays in C_{n_ℓ} on an interval of length h_{n_ℓ}. But we have $j + h_{n_\ell - k(\ell)} + 1 < h_{n+1}$ and $j + h_{n_\ell} > 2h_{n_\ell - k(\ell) + p_1 + 2p_2 + d}$, hence i cannot be bad for any $n' > n$ in the interval $\{n_\ell - k(\ell) + p_1 + 2p_2, \dots, n_\ell - k(\ell) + p_1 + 2p_2 + d\}$. □

Remark 7.2. — It follows from Proposition 5.7 that we have the following convergence:

$$\gamma_{\{h_{n_{\text{good}}(\ell)}, \dots, 2h_{n_{\text{good}}(\ell)}\}} \xrightarrow[\ell \to \infty]{} \sigma.$$

Indeed, this proposition applies where ℓ plays the role of $\ell_m - 1$, $n_{\text{good}}(\ell)$ is $n(m)$, $\{h_{n_{\text{good}}(\ell)}, \dots, 2h_{n_{\text{good}}(\ell)}\}$ is J_m, and $\{0, \dots, 2h_{n_{\text{good}}(\ell)}\}$ is I_m.

We will also need the following result, which will also be useful in the next section. We consider here an integer n such that $n_\ell - \ell < n \leqslant n_\ell - k(\ell)$ for some ℓ, and we set $n' := n_\ell - k(\ell)$. As in the proof of Lemma 5.1 we introduce the fake n'-tower, and the fake n-tower that mimicks the structure of tower n inside tower n'. (Note that this is possible as long as $n \leqslant n'$.) $\tilde{C}_n$ is the union of the levels of the fake n-tower, and $\overline{C}_n := C_n \sqcup \tilde{C}_n$. Recall that j_n indicates the level of tower n to which a point in C_n belongs. We extend this definition to points in $\overline{C}_n$: $\overline{j}_n$ indicates the level of tower n (possibly fake) to which a point in $\overline{C}_n$ belongs.

Lemma 7.3. — *For each large enough ℓ, for each n such that $n_\ell - \ell < n \leqslant n_\ell - k(\ell)$, for each integer r such that $|rh_n| < 10h_{n_\ell}$, for each $i = 1, \dots, d$, we have*

- $x_i \in C_n$ *and* $4^\ell < j_n(x_i) < h_n - 1 - 4^\ell$,
- $T^{rh_n} x_i \in \overline{C}_n$,
- $j_n(x_i) - 4^\ell \leqslant \overline{j}_n\left(T^{rh_n} x_i\right) \leqslant j_n(x_i) + 4^\ell$.

Proof. — If $\ell - 1 \geqslant \max_i \ell(x_i)$ (cf. Lemma 2.4), we have $x_i \in C_{n_{(\ell-2)}} \subset C_n$ for $i = 1, \ldots, d$. Moreover, x_i is not in the first hundred occurrences of tower $n_{(\ell-1)} - (\ell-1)$ inside tower $n_{(\ell-1)}$. Hence, as $n_\ell/\ell \to \infty$ as $\ell \to \infty$, and remembering (2.5), we have for ℓ large enough

$$j_n(x_i) \geqslant j_{n_{(\ell-1)}}(x_i) \geqslant 100 h_{n_{(\ell-1)}-(\ell-1)} > 100 \times 3^{n_{(\ell-1)}-(\ell-1)} > 100 \times 3^{n_{(\ell-2)}} > 4^\ell.$$

By a symmetric argument, we also get for ℓ large enough $j_n(x_i) < h_n - 1 - 4^\ell$.

We observe that, since $|rh_n| < 10h_{n_\ell}$, $\{-|rh_n|, \ldots, |rh_n|\}$ is contained in an $n_{(\ell+1)}$-crossing. Hence when the orbit of some coordinate leaves C_n on this interval, it comes back after 0, 1, $h_{n'}$ or $h_{n'} + 1$ iterations of the transformation. If we consider the enlarged tower $\overline{C}_n$ instead of C_n, then $T^j x_i$ comes back to $\overline{C}_n$ after 0 or 1 iteration of the transformation. Hence $\overline{j}_n\left(T^{h_n} x_i\right) \in \{j_n(x_i) - 1, j_n(x_i)\}$, and by a simple induction we get $\overline{j}_n\left(T^{rh_n} x_i\right) \in \{j_n(x_i) - |r|, \ldots, j_n(x_i) + |r|\}$. The result then follows from the fact that $|r| < 4^\ell$ (indeed, by hypothesis we have $n > n_\ell - \ell$, hence $h_n > 10h_{n_\ell}/4^\ell$ for ℓ large enough). $\square$

Remark 7.4. — If, as in the case we are currently studying, we have the strict inequality $n < n_\ell - k(\ell)$, then the number of occurrences of the fake n-tower inside the fake n'-tower is a multiple of 3. So we can extend the function t_n to a function $\overline{t}_n$ defined on $\overline{C}_n$ in such a way that, for each r such that $|rh_n| < h_{n_\ell}$ and each $i = 1, \ldots, d$

$$\overline{t}_n\left(T^{rh_n} x_i\right) = \overline{t}_n(x_i) + r \bmod 3.$$

We consider now an integer n with $n_\ell - \ell < n < n_\ell - k(\ell)$ for some ℓ, where ℓ is large enough to apply the preceding lemmas, and we assume that there is at least one substantial n-crossing which is not synchronized. With the assumption stated at the end of Section 7.1, we can also assume that for each $n_{(\ell-1)} \leqslant m \leqslant n-1$, all substantial m-crossings are synchronized. Then, as in the second part of the proof of Proposition 3.17, we can construct inductively a family $(J_m)_{n_{(\ell-1)} \leqslant m \leqslant n}$ where

- $J_{n_{(\ell-1)}}$ is a substantial n-crossing of length $\geqslant (1 - (d+2)\eta) h_{n_{(\ell-1)}}$;
- for each $m > n_{(\ell-1)}$, J_m is a substantial m-crossing extending J_{m-1} and of size $|J_m| \geqslant h_m - (d+2)\eta h_{n_{(\ell-1)}}$ (see Remark 3.18).

In particular, the size of the n-crossing J_n satisfies $|J_n| \geqslant h_n - (d+2)\eta h_{n_{(\ell-1)}}$, and we can assume that ℓ is large enough so that this implies $|J_n| \geqslant (1 - \eta/100) h_n$. Since we assume that there is at least one substantial n-crossing which is not synchronized, this ensures that J_n itself is not synchronized. Indeed, assume that there is another substantial n-crossing J'_n which is not synchronized. Because the length of J_n is so close to h_n, the orbit of each coordinate has to pass through the top of tower n between J_n and the other substantial n-crossing J'_n. But J'_n intersects $\boldsymbol{I}_n$, hence the distance between J_n and J'_n is less than h_n. This shows that, for each $i = 1, \ldots, d$, $t_n(x_i)$ increases by 1 mod 3 between the two substantial n-crossings. Then, as J'_n is not synchronized, J_n itself is not synchronized.

Moreover, by Remark 2.6, the $(n_\ell - \ell)$-crossing containing 0 covers the interval $\{-100h_{n_{(\ell-1)}}, \ldots, 100h_{n_{(\ell-1)}}\}$. In particular, it contains $J_{(\ell-1)}$ hence it is $J_{(n_\ell-\ell)}$. As J_n extends $J_{(n_\ell-\ell)}$, this proves that J_n contains 0.

Consider the set

$$R := \left\{ r \geqslant 1 : J_n + rh_n \subset \{h_{n_{\text{good}}(\ell)}, \dots, 2h_{n_{\text{good}}(\ell)}\} \right\}$$

Since $2h_{n_{\text{good}}(\ell)} < h_{n_\ell}$, Lemma 7.3 applies to each $r \in R$. In particular, for each $r \in R$ and each $i = 1, \dots, d$ we have $T^{rh_n} x_i \in \overline{C}_n$. But by choice of $n_{\text{good}}(\ell)$, we also know that $T^{rh_n} x_i \in C_{n_\ell}$. Since C_{n_ℓ} is disjoint from the fake n'-tower, $T^{rh_n} x_i \notin \tilde{C}_n$, and finally $T^{rh_n} x_i \in C_n$. Let $\tilde{J}_n$ be the interval obtained by removing the first 4^ℓ elements of the n-crossing J_n. Then, by Lemma 7.3, we have $0 \in \tilde{J}_n$, and for each $r \in R$, $\tilde{J}_n + rh_n$ is contained in an n-crossing. Note that the size of $\tilde{J}_n$ is $\geqslant (1 - \eta/100)h_n - 4^\ell > (1-\eta)h_n$.

By Remark 7.4, for each $r \in R$ and each $i = 1, \dots, d$, we have $t_n(T^{rh_n} x_i) = t_n(x_i) + r \bmod 3$. In particular, as J_n is not synchronized, for each $r \in R$, $t_n(T^{rh_n} x_i)$ takes at least 2 values as i varies. We define

$$r_0 := \min\{r \in R : t_n(T^{rh_n} x_i) \text{ takes both values 1 and 2 as } i = 1, \dots, d\}.$$

We have $\min R \leqslant r_0 \leqslant \min R + 2$.

Now let us consider r such that both r and $r+1$ are in R. For $j \in \tilde{J}_n$, we want to compare the position in tower n of $T^{j+rh_n} x_i$ and $T^{j+(r+1)h_n} x_i$ for each coordinate.

- If i is such that $t_n(T^{rh_n} x_i) = 1$, the orbit of x_i will not pass through a spacer between $\tilde{J}_n + rh_n$ and $\tilde{J}_n + (r+1)h_n$. Hence in this case we have $j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i) = 0$.
- If i is such that $t_n(T^{rh_n} x_i) = 2$, the orbit of x_i will pass through one spacer between $\tilde{J}_n + rh_n$ and $\tilde{J}_n + (r+1)h_n$. Hence in this case we have $j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i) = 1$.
- If i is such that $t_n(T^{rh_n} x_i) = 3$, we have $j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i) \in \{0, 1\}$, depending on the position of $T^{rh_n} x_i$ in the subsequent towers.

More precisely, in every case the value of $j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i)$ is determined as follows: let m be the smallest integer, $m \geqslant 0$, such that $t_{n+m}(T^{rh_n} x_i) \neq 3$. Note that $n + m < n_\ell$ since $\tilde{J}_n + rh_n$ and $\tilde{J}_n + (r+1)h_n$ are contained in the same n_ℓ-crossing. Then we have

$$j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i) = \begin{cases} 0 & \text{if } t_{n+m}(T^{rh_n} x_i) = 0, \\ 1 & \text{if } t_{n+m}(T^{rh_n} x_i) = 1. \end{cases} \tag{7.2}$$

The difficulty which arises here is that, when $t_n(x_i) = 3$, the value of this difference may vary with r. This is why we need the following lemma.

Lemma 7.5. — *There exists an integer s, $0 \leqslant s < 3^{d-1}$, such that*

- $s = 0 \bmod 3$,
- *for each $i = 1, \dots, d$, there exists a smaller integer m_i, $0 \leqslant m_i \leqslant d-2$, satisfying $t_{n+m_i}(T^{(r_0+s)h_n} x_i) \neq 3$.*

Proof. — We first remark that for each $i = 1, \dots, d$ and each $m \geqslant 0$, the map $r \in R \mapsto t_{n+m}(T^{rh_n} x_i)$ has a very regular behaviour. Indeed, it is constant on intervals of length 3^m, and if both r and $r + 3^m$ are in R, we have

$$t_{n+m}(T^{(r+3^m)h_n} x_i) = t_{n+m}(T^{rh_n} x_i) + 1 \bmod 3. \tag{7.3}$$

If $\Big\{i \in \{1,\dots,d\} : t_n(T^{r_0h_n}x_i)=3\Big\}=\emptyset$, we just have to set $s:=0$ and we get the result with $m_i=0$ for each i. Otherwise, we consider

$$i_1 := \min\Big\{i \in \{1,\dots,d\} : t_n(T^{r_0h_n}x_i)=3\Big\}.$$

Then we choose $s_1 \in \{0,1,2\}$ such that $t_{n+1}(T^{(r_0+3s_1)h_n}x_{i_1})=1$, which is possible by (7.3). We note that replacing r_0 by (r_0+3s_1) does not affect the values of the $t_n(T^{rh_n}x_i)$. Now, if $\Big\{i \in \{1,\dots,d\} : t_{n+1}(T^{(r_0+3s_1)h_n}x_i)=3\Big\}=\emptyset$, we have the result with $s=3s_1$. Otherwise, we set

$$i_2 := \min\Big\{i \in \{1,\dots,d\} : t_{n+1}(T^{(r_0+3s_1)h_n}x_i)=3\Big\}.$$

(Note that $i_2>i_1$.) Then we choose $s_2 \in \{0,1,2\}$ such that

$$t_{n+2}(T^{(r_0+3s_1+9s_2)h_n}x_{i_2})=1.$$

Again, replacing (r_0+3s_1) by $(r_0+3s_1+9s_2)$ does not affect the values of the $t_{n+m}(T^{rh_n}x_i)$, $m=0,1$.

We continue in this way until we have found $s_1,\dots,s_k \in \{0,1,2\}$ such that, for each $i=1,\dots,d$, there exists m, $0\leqslant m\leqslant k$ such that

$$t_{n+m}(T^{(r_0+3s_1+\cdots+3^ks_k)h_n}x_i)\neq 3.$$

Since the algorithm also produces an strictly increasing sequence $i_1<i_2<\cdots$ in $\{1,\dots,d\}$, we are guaranteed that it will stop in $k\leqslant d$ steps. Moreover, since the sequence $i_1<\cdots<i_k$ contains no i such that $t_n(T^{r_0h_n}x_i)\in\{1,2\}$, we have in fact $k\leqslant d-2$. We then get the announced result by setting $s:=3s_1+\cdots+3^ks_k\leqslant 3^{d-1}$. $\square$

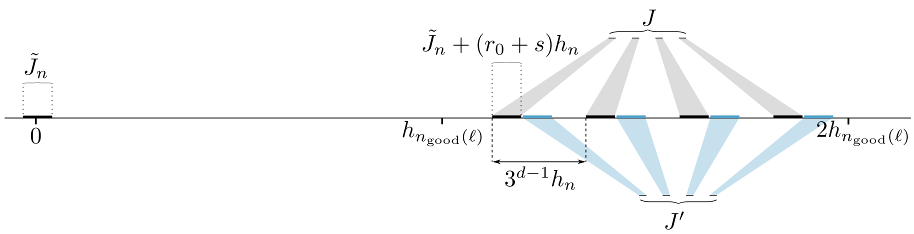

Figure 7.1. The choice of J and J' when $n_\ell-\ell<n<n_\ell-k(\ell)$

Now, with s defined in Lemma 7.5, we set

$$R_1 := \{r\in R : (r+1)\in R \text{ and } r=r_0+s \bmod 3^{d-1}\}.$$

Observe that $R_1\neq\emptyset$, as R is an interval of size

$$|R|\geqslant \lfloor h_{n_{\mathrm{good}}(\ell}/h_n\rfloor \geqslant 3^{n_{\mathrm{good}}(\ell)-n}\geqslant 3^{p_1}\geqslant 3^d.$$

Recall that for each $0\leqslant m\leqslant d-2$, and each $i=1,\dots,d$, the map $r\in R\mapsto t_{n+m}(T^{rh_n}x_i)$ is 3^{d-1}-periodic. Hence, by choice of s, for each $i=1,\dots,d$ the difference

$$j_n(T^{j+rh_n}x_i)-j_n(T^{j+(r+1)h_n}x_i)$$

depends neither on $j \in \tilde{J}_n$ nor on $r \in R_1$. Moreover, by choice of r_0, this difference takes both values 0 and 1 as i varies. Therefore we can construct the following partition $\{1, \dots, d\} = G_0 \sqcup G_1$, where for $a = 0, 1$,

$$G_a := \left\{ i : \forall\, j \in \tilde{J}_n, \forall\, r \in R_1, j_n(T^{j+rh_n} x_i) - j_n(T^{j+(r+1)h_n} x_i) = a \right\},$$

Then we denote by S_n the corresponding twisting transformation. We also consider the two disjoint subsets J and J' of $\{h_{n_{\mathrm{good}}(\ell)}, \dots, 2h_{n_{\mathrm{good}}(\ell)}\}$ defined by

$$J := \bigsqcup_{r \in R_1} \tilde{J}_n + rh_n, \qquad \text{and } J' := J + h_n.$$

(See Figure 7.1) Then, as in Example 6.4, the measures $\sigma_n := \gamma_J$ and $\sigma'_n := \gamma_{J'}$ satisfy (7.1) for each m-box $B \subset \Omega_m$, $m \leqslant n$.

Assuming the existence of infinitely many integers n with these properties, we can apply Proposition 5.9 to prove that $\sigma_n \to \sigma$ and $\sigma'_n \to \sigma$. Indeed, J and J' are both contained in $\{h_{n_{\mathrm{good}}(\ell)}, \dots, 2h_{n_{\mathrm{good}}(\ell)}\}$ which is contained in an n_ℓ-crossing. They both have the structure required in the assumptions of this proposition, with $M = 3^{d-1}$. Moreover, we also know by Remark 7.2 that $\gamma_{\{h_{n_{\mathrm{good}}(\ell)}, \dots, 2h_{n_{\mathrm{good}}(\ell)}\}} \xrightarrow[n\to\infty]{} \sigma$.

Then Proposition 6.3 shows that σ can be decomposed as a product of two Radon measures to which we can apply the induction hypothesis.

We are now reduced to study the case where, for each ℓ large enough and each $n_{\ell-1} \leqslant n < n_\ell - k(\ell)$, all substantial n-crossings are synchronized, but still there exist infinitely many integers n for which at least one substantial n-crossing is not synchronized.

7.3. The case $n = n_\ell - k(\ell)$

We consider now an integer n of the form $n = n_\ell - k(\ell)$ for some ℓ, where ℓ is large enough. We assume that there is at least one substantial n-crossing which is not synchronized, and also that for each $n_{(\ell-1)} \leqslant m \leqslant n-1$, all substantial m-crossings are synchronized. Then, as in Section 7.2, we prove that the n-crossing J_n containing 0 is of size $|J_n| \geqslant (1 - \eta/100)h_n$, and is not synchronized. We also define $\tilde{J}_n \subset J_n$ as in the previous section: $\tilde{J}_n$ contains 0 and $|\tilde{J}_n| \geqslant (1-\eta)h_n$.

We still work with the fake tower n, as introduced before Lemma 7.3 which is still valid in this case. The new difficulty here is that we cannot anymore extend t_n to $\overline{C}_n$.

We consider integers r with $0 \leqslant r \leqslant 10d$, and we assume that ℓ is large enough so that $k(\ell) > 10d$, thus $10d < h_{n_\ell}/h_n$ and the results of Lemma 7.3 are valid for these integers r. In particular, for each such r, either $\tilde{J}_n + rh_n$ is contained in an n-crossing (we then say that r corresponds to a *true n-crossing*), or there is one coordinate x_i such that $T^{rh_n}x_i$ is in the fake tower n $\tilde{C}_n$ (in this case we say that r corresponds to a *fake n-crossing*). Observe that for each $i = 1, \dots, d$, there is at most one integer r, $0 \leqslant r < \lfloor h_{n_\ell}/h_n \rfloor$, such that $T^{rh_n}x_i$ is in the fake tower n. Indeed, as everything takes place in a single $C_{n_{(\ell+1)}}$-crossing, when the orbit of x_i leaves C_{n_ℓ}, it comes back to C_{n_ℓ} after at most $h_n + 1$ units of time, and then stays in C_{n_ℓ} for h_{n_ℓ} units of time.

If both $T^{rh_n}x_i$ and $T^{(r+1)h_n}x_i$ are in C_n, then $t_n(T^{(r+1)h_n}x_i) = t_n(T^{rh_n}x_i)+1 \bmod 3$. If $T^{rh_n}x_i$ is in the fake tower n, then $T^{(r-1)h_n}x_i$ and $T^{(r+1)h_n}x_i$ are in C_n, and we have $t_n(T^{(r-1)h_n}x_i) = 3$, and $t_n(T^{(r+1)h_n}x_i) = 1$.

With these facts in mind, we can prove the following lemma.

Lemma 7.6. — *There exist two consecutive integers, $-2 \leqslant r < r+1 \leqslant 10d$, such that*

- *$\tilde{J}_n + rh_n$ is contained in an n-crossing,*
- *$\tilde{J}_n + (r+1)h_n$ is contained in an n-crossing,*
- *$\Big\{i \in \{1,\dots,d\} : t_n(T^{rh_n}x_i) = 1\Big\} \neq \emptyset$,*
- *$\Big\{i \in \{1,\dots,d\} : t_n(T^{rh_n}x_i) = 2\Big\} \neq \emptyset$.*

Proof. — There is at most d integers r, $0 \leqslant r \leqslant 10d$, such that $\tilde{J}_n+rh_n$ corresponds to a fake n-crossing (indeed, each coordinate can be responsible for only one r for which this property fails). Hence there is a smaller integer r_0, $0 \leqslant r_0 \leqslant 10d-2$, such that r_0, (r_0+1), (r_0+2) and (r_0+3) correspond to true n-crossings.

If $r_0 = 0$, since J_n is not synchronized, there are two coordinates x_{i_1} and x_{i_2} such that $t_n(x_{i_1}) \neq t_n(x_{i_2})$. If $\{t_n(x_{i_1}), t_n(x_{i_2})\} = \{1,2\}$, we just have to take $r = 0$. If $\{t_n(x_{i_1}), t_n(x_{i_2})\} = \{1,3\}$, we set $r = 1$, and if $\{t_n(x_{i_1}), t_n(x_{i_2})\} = \{2,3\}$, we set $r = 2$. In all these cases we get

$$\{t_n(T^{rh_n}x_{i_1}), t_n(T^{rh_n}x_{i_2})\} = \{1,2\}.$$

If $r_0 > 0$ and the n-crossing containing r_0h_n is not synchronized, then we can proceed as in the previous case, replacing 0 by r_0.

If $r_0 > 0$ and the n-crossing containing r_0h_n is synchronized, then by definition of r_0, (r_0-1) corresponds to a fake n-crossing, hence there is at least one coordinate x_{i_1} such that $t_n(T^{r_0h_n}x_{i_1}) = 1$. Since the corresponding n-crossing is assumed to be synchronized, we have $t_n(T^{r_0h_n}x_i) = 1$ for each $i = 1,\dots,d$. We also observe that there exists at least one coordinate x_{i_2} such that $T^{(r_0-1)h_n}x_{i_2} \in C_n$. Indeed, otherwise all coordinates would be in the fake n-tower at the same time, and this would imply that the n-crossing J_n containing 0 is synchronized. Now we take $r := r_0 - 3$. Then for each coordinate x_i such that $T^{(r_0-1)h_n}x_i \in C_n$, we have $t_n(T^{rh_n}x_i) = 1$, and for each coordinate x_i such that $T^{(r_0-1)h_n}x_i \notin C_n$, we have $t_n(T^{rh_n}x_i) = 2$. □

Now, with r provided by Lemma 7.6, we consider the two measures $\sigma_n := \gamma_{\tilde{J}_n+rh_n}$ and $\sigma'_n := \gamma_{\tilde{J}_n+(r+1)h_n}$ (see Figure 7.2). We can show by the same argument as in Section 7.1 that there exists a twisting transformation S_n such that (7.1) holds whenever B is an m-box in Ω_m for some $m \leqslant n$.

Finally, if we have infinitely many integers n to which the above arguments apply, Proposition 5.7 shows that $\sigma_n \xrightarrow[n\to\infty]{} \sigma$ and $\sigma'_n \xrightarrow[n\to\infty]{} \sigma$. Then Proposition 6.3 shows that σ can be decomposed as a product of two Radon measures to which we can apply the induction hypothesis.

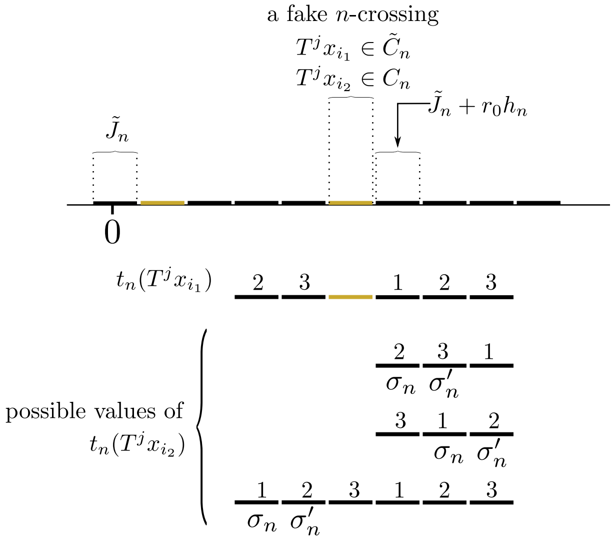

Figure 7.2. The three possible cases for the choice of the measures σ_n and σ'_n when $n = n_\ell - k(\ell)$. Here the orbit of the coordinate x_{i_1} is in the fake Rokhlin tower $\tilde{C}_n$ on the interval $\tilde{J}_n + (r_0 - 1)h_n$, whereas on the same interval the orbit of the coordinate x_{i_2} is in C_n.

7.4. The case $n_\ell - k(\ell) < n < n_\ell$

It only remains now to study the case where, for each ℓ large enough and each $n_{\ell-1} \leqslant n \leqslant n_\ell - k(\ell)$, all substantial n-crossings are synchronized, but still there exist infinitely many integers n for which at least one substantial n-crossing is not synchronized.

We consider now an integer n with $n_\ell - k(\ell) < n < n_\ell$ for some large ℓ, and we assume that there is at least one substantial n-crossing which is not synchronized. We can also assume that for each $n_{(\ell-1)} \leqslant m \leqslant n-1$, all substantial m-crossings are synchronized. Then, as in Section 7.2, we construct a family (J_m) of intervals, $n_{\ell-1} \leqslant m \leqslant n$, where J_m is the m-crossing containing 0, and is of size $|J_m| \geqslant h_m - (d+2)\eta h_{n_{(\ell-1)}}$. We set $n' := n_\ell - k(\ell)$. We have $|J_{n'}| \geqslant (1-\eta/100)h_{n'}$, provided ℓ is large enough.

As in Section 7.3, we apply Lemma 7.3 for n'. We consider all integers $r \geqslant 0$ such that $rh_{n'} \leqslant 4h_n$: each such integer r corresponds either to a true n'-crossing (if for each $i = 1, \dots, d$, $T^{rh_{n'}} x_i \in C_{n'}$), or to a fake n'-crossing (if there exists i such that $T^{rh_{n'}} x_i \in \tilde{C}_{n'}$). If $T^{rh_{n'}} x_i \in C_{n'}$, then we can consider $t_{n'}(T^{rh_{n'}} x_i)$ which evolves according to the rules stated in Section 7.3. We can even precise a little bit more these rules by considering also the position of $T^j x_i$ relatively to tower n: If $T^{rh_{n'}} x_i$ is in the fake tower n', then $T^{(r-1)h_{n'}} x_i$ and $T^{(r+1)h_{n'}} x_i$ are in $C_{n'} \subset C_n$, and we have $t_{n'}(T^{(r-1)h_{n'}} x_i) = t_n(T^{(r-1)h_{n'}} x_i) = 3$, and $t_{n'}(T^{(r+1)h_{n'}} x_i) = t_n(T^{(r+1)h_{n'}} x_i) = 1$.

Let us first consider the case where each $0 \leqslant r \leqslant \lfloor 4h_n/h_{n'} \rfloor$ corresponds to a true n'-crossing. Then the interval $\{0, \dots, 4h_n\}$ is contained in a single n_ℓ-crossing. We denote by $\tilde{J}_n$ the interval obtained by removing from J_n its first 3 points. Then, as in the proof of Lemma 7.3, we prove that $0 \in \tilde{J}_n$, and that the intervals $\tilde{J}_n$, $\tilde{J}_n + h_n$, $\tilde{J}_n + 2h_n$, $\tilde{J}_n + 3h_n$ are each contained in some n-crossing. Since J_n is not synchronized, we show by similar arguments as in the proof of Lemma 7.6 that for some $s \in \{0, 1, 2\}$,

$$\{1, 2\} \subset \Big\{t_n(T^{sh_n x_i}) : i = 1, \dots, d\Big\}.$$

Then we construct the measures $\sigma_n := \gamma_{\tilde{J}_n + sh_n}$ and $\sigma'_n := \gamma_{\tilde{J}_n+(s+1)h_n}$: by similar arguments as before, we construct a twisting transformation S_n such that (7.1) holds whenever B is an m-box in Ω_m for some $m \leqslant n$. If this can be done for infinitely many integers n, then Proposition 5.7 shows that $\sigma_n \xrightarrow[n\to\infty]{} \sigma$ and $\sigma'_n \xrightarrow[n\to\infty]{} \sigma$. Then Proposition 6.3 shows that σ can be decomposed as a product of two Radon measures to which we can apply the induction hypothesis.

Now we consider the case where there exists some r, $1 \leqslant r \leqslant \lfloor 4h_n/h_{n'} \rfloor$, which corresponds to a fake n'-crossing. This case is illustrated on Figure 7.3. We define r_0 as the smallest integer with this property. Then we know that there exists $i \in \{1, \dots, d\}$ such that $T^{r_0 h_{n'}} x_i \in \tilde{C}_{n'}$. Each such i is called an *outgoing* coordinate. Note that for each outgoing coordinate i, we have for each $n' \leqslant m \leqslant n$ $t_m(T^{(r_0-1)h_{n'}} x_i) = 3$ (indeed, the orbit of the outgoing coordinate has to reach the top of tower m before leaving C_{n_ℓ}).

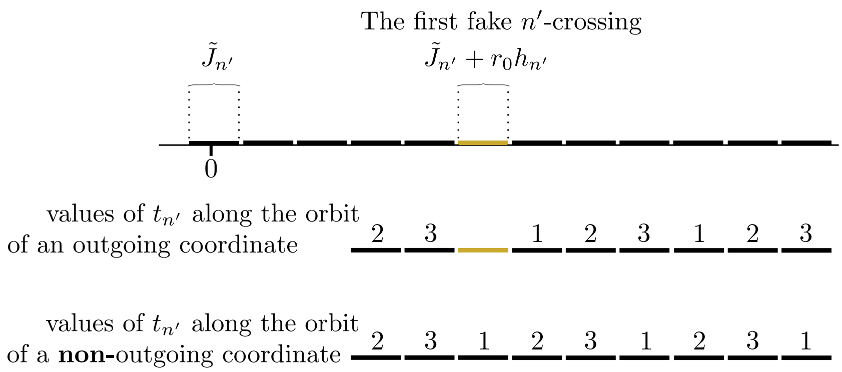

Figure 7.3. The behaviour of $t_{n'}$ along the orbits of outgoing and non-outgoing coordinates

We also observe that, since the interval $\{0, \dots, (r_0 - 1)h_{n'}\}$ is contained in an n_ℓ-crossing, we have for each $i_1, i_2 \in \{1, \dots, d\}$, each $0 \leqslant r \leqslant r_0 - 1$ and each $n' \leqslant m \leqslant n$

$$t_m(T^{rh_{n'}} x_{i_1}) - t_m(T^{rh_{n'}} x_{i_2}) = t_m(x_{i_1}) - t_m(x_{i_2}). \tag{7.4}$$

For $n' \leqslant m < n$, the above difference vanishes. Hence, we have $t_m(T^{(r_0-1)h_{n'}} x_{i_1}) = t_m(T^{(r_0-1)h_{n'}} x_{i_2})$ for each i_1, i_2. Taking into account the outgoing coordinates, we see that for each $i = 1, \dots, d$, $t_m(T^{(r_0-1)h_{n'}} x_i) = 3$. This proves that at time $(r_0 - 1)h_{n'}$, each coordinate is in the last occurrence of tower n' inside tower n.

Now, since the n-crossing J_n containing 0 is not synchronized, there exist i_1, i_2 such that the difference in (7.4) does not vanish for $m = n$, and this implies that there exist some $i \in \{1, \dots, d\}$ such that $t_n(T^{(r_0-1)h_{n'}} x_i) \neq 3$. In particular such an i is *not* an outgoing coordinate. At time $r_0 h_{n'}$, the orbit of a non-outgoing coordinate is in the first occurrence of tower n' inside tower n. When r runs over the set $R := \{r_0, \dots, r_0 + 3^{n-n'} - 1\}$, we get that for each non-outgoing coordinate i, $T^{rh_{n'}} x_i$

successively belongs to successive occurrences of tower n' inside tower n, and we have $t_{n'}(T^{rh_{n'}}x_i) = r - r_0 + 1 \bmod 3$.

On the other hand, if i is an outgoing coordinate, the orbit of x_i falls into the first occurrence of tower n' inside tower n only at time $(r_0+1)h_n$. And we have, for $r \in R \setminus \{r_0\}$, $t_{n'}(T^{rh_{n'}}x_i) = r - r_0 \bmod 3$.

Set $R_1 := \{r \in R : r - r_0 = 1 \bmod 3\}$. Then $R_1 \neq \emptyset$ because $n > n'$, and for $r \in R_1$, we have

$$t_{n'}(T^{rh_{n'}}x_i) = \begin{cases} 1 & \text{if } i \text{ is an outgoing coordinate,} \\ 2 & \text{otherwise.} \end{cases}$$

Let $\tilde{J}_{n'}$ be the interval obtained after removing the first 4^ℓ points from $J_{n'}$, and set $J := \bigsqcup_{r\in R_1} \tilde{J}_{n'} + rh_{n'}$, $J' := J + h_{n'}$, and let I be the smallest interval containing J and J'. Then J and J' have inside I the structure required in Proposition 5.9, with $M = 3$. We consider the two measures $\sigma_n := \gamma_J$ and $\sigma'_n := \gamma_{J'}$. Then there exists a twisting transformation S_n, defined from the partition of $\{1, \ldots, d\}$ into outgoing and non-outgoing coordinates, such that (7.1) holds whenever B is an m-box in Ω_m for some $m \leqslant n'$.

If we have infinitely many integers n for which the above construction is possible, then Proposition 5.7 ensures that $\gamma_I \xrightarrow[n\to\infty]{} \sigma$, then Proposition 5.9 yields $\sigma_n \xrightarrow[n\to\infty]{} \sigma$ and $\sigma'_n \xrightarrow[n\to\infty]{} \sigma$. Finally, Proposition 6.3 shows that σ can be decomposed as a product of two Radon measures to which we can apply the induction hypothesis.

This concludes the proof of Theorem 3.10.

8. Further properties and a question

We now provide some direct consequences of our main result on the commutant and factors of the nearly finite Chacon transformation T. The proof of the following proposition is directly derived from Section 5 of [JRdlR18] (see in particular Remark 5.7 therein).

Proposition 8.1. —

- *If S is an invertible μ-preserving transformation of X commuting with T, then $S = T^k$ for some $k \in \mathbb{Z}$.*
- *If $\pi : (X, \mu, T) \to (Y, \nu, R)$ is a factor map to another invertible, σ-finite measure preserving dynamical system (Y, ν, R), then π is in fact an isomorphism between the two systems.*

For some applications in the study of Poisson suspensions developed in [JRdlR17], we also need an extra property which is the existence of a measurable law of large numbers.

Definition 8.2. — A measurable law of large numbers *for a conservative, ergodic, measure preserving dynamical system $(X, \mathscr{A}, \mu, T)$ is a measurable function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that for all $B \in \mathscr{A}$, for μ-almost every $x \in X$,*

$$L\left(\mathbb{1}_B(x), \mathbb{1}_B(Tx), \ldots\right) = \mu(B).$$

Definition 8.3. — *A conservative, ergodic, measure preserving dynamical system $(X, \mathscr{A}, \mu, T)$ is* rationally ergodic *if there exists a set $B \in \mathscr{A}$, $0 < \mu(B) < \infty$, and a constant $M > 0$ such that, for any $r \geqslant 1$,*

$$\int_B \left(\sum_{0 \leqslant j \leqslant r-1} \mathbb{1}_B(T^j x) \right)^2 \mathrm{d}\mu(x) \leqslant M \left(\int_B \sum_{0 \leqslant j \leqslant r-1} \mathbb{1}_B(T^j x) \, \mathrm{d}\mu(x) \right)^2 .$$

According to Theorem 3.3.1 in [Aar97], a measurable law of large numbers exists for T as soon as T is rationally ergodic.

Proposition 8.4. — *The nearly finite Chacon transformation is rationally ergodic, hence admits a measurable law of large numbers.*

Proof. — In the cutting-and-stacking construction of the Nearly Finite Chacon transformation, we always cut the tower into 3 subcolumns, hence T is rank one with *bounded cuts*. But it is proved in [BSS$^+$15, Theorem 2.3] that this property implies that T is rationally ergodic. □

Let us finally mention the following observation. The original definition by Rudolph [Rud79] of *minimal self joinings* for finite measure preserving transformations considered any ergodic $T^{\ell_1} \times \cdots \times T^{\ell_d}$-invariant measure on the d-fold Cartesian product for $\ell_1, \ldots, \ell_d \in \mathbb{Z} \setminus \{0\}$. Although the terminology which was finally adopted refers only to $T^{\otimes d}$-invariant measures, it is nevertheless interesting to consider the action of $T^{\ell_1} \times \cdots \times T^{\ell_d}$. In the context of the nearly finite Chacon transformation, we can then ask whether there exists, for some d and some $\ell_1, \ldots, \ell_d \in \mathbb{Z} \setminus \{0\}$, an ergodic $T^{\ell_1} \times \cdots \times T^{\ell_d}$-invariant measure (say, supported on X_∞^d) different than those described in the statement of Theorem 3.10.

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