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ASYMPTOTIC SHAPE OF THE CONCAVE MAJORANT OF A LÉVY PROCESS

FORME ASYMPTOTIQUE DU MAJORANT CONCAVE D'UN PROCESSUS DE LÉVY

Abstract. — We establish distributional limit theorems for the shape statistics of a concave majorant (i.e. the fluctuations of its length, its supremum, the time it is attained and its value at T) of a Lévy process on $[0,T]$ as $T \to \infty$. The scale of the fluctuations of the length and other statistics, as well as their asymptotic dependence, vary significantly with the tail behaviour of the Lévy measure. The key tool in the proofs is the recent representation of the concave majorant for all Lévy processes using a stick-breaking representation.

Résumé. — Nous établissons des théorèmes distributionnels limites pour les statistiques de la forme d'un majorant concave (i.e. les fluctuations de sa longueur, son supremum, son temps d'atteinte et sa valeur en T) d'un processus de Lévy sur $[0,T]$ lorsque $T \to \infty$. L'ampleur des

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fluctuations de la longueur et d'autres statistiques, ainsi que leur comportement asymptotique, varient considérablement en fonction de la queue de la mesure de Lévy. L'outil clé dans les preuves est la représentation récente du majorant concave des processus de Lévy à l'aide d'un processus de bâton brisé.

1. Introduction and main results

Convex hulls of random walks and related processes have been of interest for many decades (see e.g. [AKMV20, KLM12, MW16, MW18, PUB12, RFZ20] and references therein). The main objective of the present paper is to understand the asymptotic shape of the concave majorant of a Lévy process as the time horizon tends to infinity (see Figure 1.1).

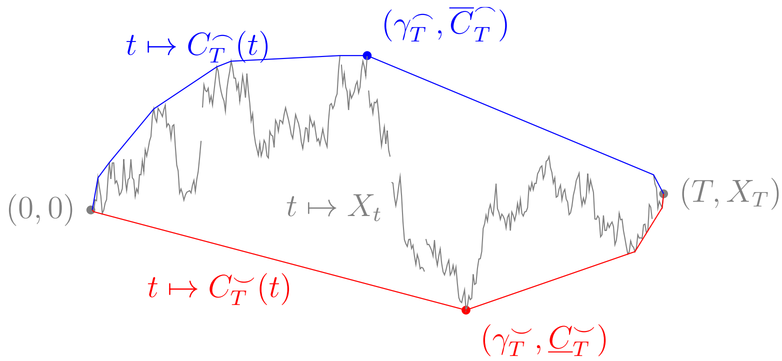

Figure 1.1. A sample path of a Lévy process X on the interval $[0,T]$, the graphs of the concave majorant $C_T^{\frown}$ and the convex minorant $C_T^{\smile}$ and the time and space position of their respective supremum $(\gamma_T^{\frown}, \overline{C}_T^{\frown})$ and infimum $(\gamma_T^{\smile}, \underline{C}_T^{\smile})$.

Let $X = (X_t)_{t \geqslant 0}$ be a one-dimensional Lévy process (see [Sat13, Definition 1.6, Chapter 1]) and fix a time interval $[0,T]$ for some positive time horizon $T > 0$. The concave majorant (resp. convex minorant) of a path of a Lévy process $(X_t)_{t \geqslant 0}$ is the smallest (resp. largest) function that is point-wise larger (resp. smaller) than the path of X, i.e. $C_T^{\frown}(t) \geqslant X_t$ (resp. $C_T^{\smile}(t) \leqslant X_t$) for all $t \in [0,T]$. Let $\Upsilon_T^{\frown}$ (resp. $\Upsilon_T^{\smile}$) denote the length of the graph of the concave (resp. convex) function $t \mapsto C_T^{\frown}(t)$ (resp. $t \mapsto C_T^{\smile}(t)$) over the interval $[0,T]$. The following inequalities are immediate from Figure 1.2 on page 785 below:

$$(1.1) \qquad 1 \leqslant \Upsilon_T^{\frown}/T \leqslant \Big(T + 2\overline{C}_T^{\frown} - C_T^{\frown}(T)\Big)/T, \qquad \text{where } \overline{C}_T^{\frown} := \sup_{t \in [0,T]} C_T^{\frown}(t).$$

If $\mathbb{E}|X_1|^{1+\epsilon} < \infty$ for some $\epsilon > 0$ and $\mathbb{E}X_1 = 0$, the bounds in (1.1) and [Sat13, Proposition 48.10] imply that $\Upsilon_T^{\frown}/T \to 1$ a.s. as $T \to \infty$ (note $\overline{C}_T^{\frown} = \sup_{t \in [0,T]} X_t$ and $C_T^{\frown}(T) = X_T$). Our main aim is to identify the precise asymptotic behaviour and the dependence of the shape parameters $\Upsilon_T^{\frown}$, supremum $\overline{C}_T^{\frown}$, time of supremum $\gamma_T^{\frown}$ and final position $C_T^{\frown}(T)$ of the concave majorant $C_T^{\frown}$. More precisely, we seek to identify the correct asymptotic mean, analyse the fluctuations of the length $\Upsilon_T^{\frown}$ around its asymptotic mean and study their dependence on other shape parameters.

If the second moment is infinite, we study analogous questions for X in the domain of attraction of a stable process.

Our main result describes the asymptotic dependence between the fluctuations of the length of the concave majorant, its supremum, final position and the time the supremum is attained, for Lévy processes that have zero mean and finite variance (see Theorem 1.1 below). We also describe this dependence in the case the process is in the domain of attraction of a stable law with stability parameter $\alpha \in (0,2] \setminus \{1\}$ (see Theorems 1.4, 1.6 and 1.7 for $\alpha \in (1,2)$ with zero mean, $\alpha \in (1,2]$ with nonzero mean and $\alpha \in (0,1)$, respectively). As we shall see, the dependence has very different structure in each of these cases, with Theorem 1.1 being the most subtle. In particular, for example, the dependence between the fluctuations of the length of the concave majorant and the other statistics weakens with increasing α. For a short overview of the results in this paper see the YouTube presentation [BGCM21b].

Before stating our results, recall that the concave majorant of a path of a Lévy process X is a piecewise linear function with countably many faces (see [GCM22, Theorem 11]). Each face is given by a horizontal length $l > 0$ and a vertical height $h \in \mathbb{R}$, thus having the slope h/l. Note that all the faces with slope equal to a given real value $s \in \mathbb{R}$ must lie next to each other in the graph of the concave majorant and can be concatenated into a *maximal* face with slope s. Let H_T equal the number of maximal faces with horizontal length l at least 1. Denote $(x)^+ := \max(0,x)$ for $x \in \mathbb{R}$ throughout.

Theorem 1.1. — *Let $X = (X_t)_{t\geqslant 0}$ be a Lévy process with Lévy measure ν. Assume that the Lévy process has zero mean $\mathbb{E}[X_1] = 0$ and finite positive variance $\sigma := \sqrt{\mathbb{E}[X_1^2]} \in (0,\infty)$. For $T > 0$ define $\Theta(T) := \frac{1}{2}\int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x)$. Then the following weak limit holds as $T \to \infty$:*

$$\left(\frac{\Upsilon_T^{\frown} - T - (\sigma^2/2)\,H_T + \Theta(T)}{\sqrt{\log T}}, \frac{H_T - \log T}{\sqrt{\log T}}, \frac{\overline{C}_T^{\frown}}{\sqrt{T}}, \frac{C_T^{\frown}(T)}{\sqrt{T}}, \frac{\gamma_T^{\frown}}{T}\right) \xrightarrow{d} \left(\tfrac{\sigma^2}{\sqrt{2}} Z_1, Z_2, \sigma \overline{B}_1, \sigma B_1, \rho\right), \tag{1.2}$$

where the standard Brownian motion $B = (B_t)_{t\,\geqslant\,0}$ is independent of the normal random vector (Z_1, Z_2) with zero mean, satisfying $\mathbb{E}Z_1^2 = \mathbb{E}Z_2^2 = 1$ and $\mathbb{E}[Z_1 Z_2] = 0$, $\overline{B}_1 := \sup_{t\,\in\,[0,1]} B_t$ and $\rho \in [0,1]$ is the a.s. unique time such that $B_\rho = \overline{B}_1$.

The weak limit in (1.2) shows that the asymptotic centering of the length $\Upsilon_T^{\frown}$ of the concave majorant $C_T^{\frown}$ is stochastic. Moreover, the fluctuations around the centering are asymptotically independent of the centering itself and the randomness in the centering is a function of the horizontal lengths of the faces of $C_T^{\frown}$ only. A linear transformation of the vector in (1.2) yields a deterministic centering of $\Upsilon_T^{\frown}$ at the cost of increasing the asymptotic variance. Put differently, the variance of the centering contributes $\sigma^4/4$ (recall $\sigma^2 = \mathbb{E}[X_1^2]$) to the total asymptotic variance of the length $\Upsilon_T^{\frown}$. For two functions f and g, write $f(T) = \mathrm{o}(g(T))$ as $T \to \infty$ if $\lim_{T\to\infty} f(T)/g(T) = 0$.

Corollary 1.2. — *Under the assumptions of Theorem 1.1, we have*

$$\frac{1}{\sqrt{\log T}}\left(\Upsilon_{\widehat{T}} - T - \frac{\sigma^2}{2}\log T + \Theta(T)\right) \xrightarrow{d} \frac{\sqrt{3}}{2}\sigma^2 Z, \qquad \text{as } T\to\infty, \tag{1.3}$$

where $\Theta(T) = \frac{1}{2}\int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x) = \mathrm{o}(\log T)$ *and* Z *is a standard normal variable. Moreover, if* $\int_{\mathbb{R}} x^2\log^+(|x|)^{1/2}\nu(\mathrm{d}x) < \infty$, *then* $\Theta(T) = \mathrm{o}(\sqrt{\log T})$, *and thus*

$$\frac{1}{\sqrt{\log T}}\left(\Upsilon_{\widehat{T}} - T - \frac{\sigma^2}{2}\log T\right) \xrightarrow{d} \frac{\sqrt{3}}{2}\sigma^2 Z, \qquad \text{as } T\to\infty. \tag{1.4}$$

Further remarks about Theorem 1.1 and Corollary 1.2 are in order.

Remark 1.3. —

(i) The limit in (1.2) reveals that the fluctuations of the asymptotic length of the concave majorant $C_{\widehat{T}}$ are independent of its asymptotic supremum, time of supremum and final position. In the case only the first moment of X_1 is finite, the dependence of these shape statistics persists in the limit (see Theorem 1.4 below), while if even the first moment of X_1 is infinite, the length $\Upsilon_{\widehat{T}}$ becomes a deterministic function of the asymptotic supremum and final position (see Theorem 1.7 below).

(ii) Corollary 1.2 is stated for the deterministic centering of the length only. However, the same linear transform yields a quintuple limit analogous to (1.2). Put differently, as $T\to\infty$, we have

$$\left(\frac{\Upsilon_{\widehat{T}} - T - (\sigma^2/2)\log T + \Theta(T)}{\sqrt{\log T}}, \frac{H_T - \log T}{\sqrt{\log T}}, \frac{\overline{C}_{\widehat{T}}}{\sqrt{T}}, \frac{C_{\widehat{T}}(T)}{\sqrt{T}}, \frac{\gamma_{\widehat{T}}}{T}\right) \xrightarrow{d} \left(\frac{\sigma^2}{\sqrt{2}}Z_1 + \frac{\sigma^2}{2}Z_2, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho\right).$$

The dependence structure of the length $\Upsilon_{\widehat{T}}$ and H_T is intractable for any finite $T>0$, but, as shown by this limit, is asymptotically rather simple.

(iii) There exist Lévy processes for which (1.3) holds and (1.4) does not. Indeed, by Fubini's theorem, the integral in the asymptotic mean satisfies

$$2\Theta(T) = \int_{\mathbb{R}} x^2\log^+\left(\min\left\{T, x^2\right\}\right)\nu(\mathrm{d}x) = \int_1^T \frac{1}{t}\int_{\mathbb{R}\setminus(-\sqrt{t},\sqrt{t})} x^2\nu(\mathrm{d}x)\mathrm{d}t,$$

(iv) Note that in the weak limit of Theorem 1.1 neither X nor its concave majorant $C_{\widehat{T}}$ are scaled before the length $\Upsilon_{\widehat{T}}$ is calculated. Since X is in the domain of attraction of Brownian motion, one could scale space by $1/\sqrt{T}$ and time by $1/T$ and *then* compute the length of the graph of the resulting concave majorant. This length would, by continuity, converge to the length of the concave majorant of a Brownian motion on $[0,1]$. For the original length $\Upsilon_{\widehat{T}}$, this approach only yields $\Upsilon_{\widehat{T}}/T \xrightarrow{d} 1$.

(v) To the best of our knowledge, Theorem 1.1 had been established neither for Brownian motion nor compound Poisson processes. Moreover, the marginal convergence in Corollary 1.2 does not follow easily from the random walk case, recently analysed in [AKMV20], since, for instance, the law of the length of the convex minorant is not invariant under stochastic time-changes, see Figure 1.3 below.

(vi) Consider the counting measure $h_T(A)$, where A is a Borel subset in $\mathbb{R}$, recording the number of maximal faces with horizontal lengths in A. In (1.2) we considered the variable $H_T = h_T([1,\infty))$, but the same weak limit holds for any $h_T([a,\infty))$ with $a \in (0,\infty)$. Moreover, for any bounded set A, the mean measure $\mathbb{E}[h_T(A)]$ equals $\int_A t^{-1}\mathrm{d}t$ for all $T>0$ satisfying $A \subset [0,T]$ by Lemma 2.1 below.

A Lévy process X is in the domain of attraction of an α-stable law for some $\alpha \in (0,2]$ if

$$X_T/a_T \xrightarrow{d} S_\alpha(1), \quad \text{as } T\to\infty, \text{for some positive function } a_T = T^{1/\alpha} l(T), \tag{1.5}$$

where l is slowly varying (i.e. $l(cx)/l(x)\to 1$ as $x\to\infty$ for all $c>0$) and $(S_\alpha(t))_{t\geqslant 0}$ is an α-stable process (see [Sat13, Chapter 3] for definition). We note that if X is as in Theorem 1.1 ($\mathbb{E}[X_1]=0$ and $\sigma=\sqrt{\mathbb{E}[X_1^2]}<\infty$), the standard CLT implies that X satisfies (1.5) with $\alpha=2$ and $a_T=\sqrt{T}$. Results analogous to Theorem 1.1 for Lévy process in the domain of attraction of an α-stable law will now be presented: the case $\alpha\in(1,2)$ with $\mathbb{E}[X_1]=0$ (resp. $\alpha\in(1,2]$ with $\mathbb{E}[X_1]\neq 0$; $\alpha\in(0,1)$) is considered in Theorem 1.4 (resp. Theorem 1.6; Theorem 1.7). The case $\alpha=2$ with $\mathbb{E}[X_1^2]=\infty$ and $\mathbb{E}[X_1]=0$ as well as the case $\alpha=1$ are not considered in this paper. To state these theorems, we recall that the uniform stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$ on $[0,1]$ is defined recursively by an iid-U$(0,1)$ sequence $(V_n)_{n\in\mathbb{N}}$ as follows: $L_0 \coloneqq 1$, $\ell_n \coloneqq V_n L_{n-1}$ and $L_n \coloneqq L_{n-1}-\ell_n$ for $n\in\mathbb{N}$. The process $(L_n)_{n\in\mathbb{N}\cup\{0\}}$ will be referred to as the stick-remainders.

Theorem 1.4. — *Assume X is in the domain of attraction of an $\alpha-$stable law with $\alpha\in(1,2)$ and $\mathbb{E}[X_1]=0$. Then, as $T\to\infty$, we have*

$$\begin{aligned}&\left(\frac{T}{a_T^2}\left(\Upsilon_T^{\frown}-T\right), \frac{\overline{C}_T^{\frown}}{a_T}, \frac{C_T^{\frown}(T)}{a_T}, \frac{\gamma_T^{\frown}}{T}\right)\\ &\quad\xrightarrow{d}\left(\frac{1}{2}\sum_{n=1}^\infty \ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2, \sum_{n=1}^\infty \ell_n^{1/\alpha}\left(S_\alpha^{(n)}\right)^+, \sum_{n=1}^\infty \ell_n^{1/\alpha}S_\alpha^{(n)}, \sum_{n=1}^\infty \ell_n \mathbb{1}_{\left\{S_\alpha^{(n)}>0\right\}}\right),\end{aligned} \tag{1.6}$$

where $(\ell_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process independent of the sequence $(S_\alpha^{(n)})_{n\in\mathbb{N}}$ of independent copies of $S_\alpha(1)$.

Under the assumptions of Theorem 1.4, the Lévy process X has infinite variance. By (1.6), the fluctuations of $\Upsilon_T^{\frown}$ about its centering function are typically of order $T^{2/\alpha-1}$, compared with the fluctuations of order $\sqrt{\log T}$ in the finite variance case (see Theorem 1.1 above). The last three coordinates of the limit law in (1.6) have the same law as $(\sup_{t\in[0,1]} S_\alpha(t), S_\alpha(1), \gamma^{\alpha\frown})$, where $\gamma^{\alpha\frown}$ is the time at which the supremum of $S_\alpha(t)$ over $t\in[0,1]$ is attained. We do not know of an interpretation of the law of the first coordinate as a simple functional of the path of the stable process S_α. In particular, it is *not* equal to the law of the length of the concave majorant of S_α on $[0,1]$. However, the tail decay of this coordinate can be characterised using the fact that the law of the series $\sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2$ satisfies a stochastic perpetuity equation.

Proposition 1.5. — *The following asymptotic equivalence holds*

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\frac{1}{2}\sum\limits_{n=1}^{\infty} \ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2 > x\right)}{\mathbb{P}\left(\left(S_\alpha^{(1)}\right)^2 > x\right)} = \lim_{x\to\infty} \frac{\mathbb{P}\left(\frac{1}{2}\sum\limits_{n=1}^{\infty} \ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2 > x\right)}{(c_+ + c_-)\, x^{-\alpha/2}} = \frac{2^{1-\alpha/2}}{2-\alpha},$$

for the constants $c_+, c_- \geqslant 0$ defined by $c_\pm \coloneqq \lim_{x\to\infty} \mathbb{P}(\pm S_\alpha^{(1)} > \sqrt{x})/x^{-\alpha/2}$, which satisfy $c_+ + c_- > 0$.

Note that in Theorems 1.1 and 1.4, we have assumed that X has a finite first moment and $\mathbb{E}[X_1] = 0$. If the mean is not zero, the behaviour in these cases is described by the following result. In this description, it is important to distinguish between the cases of positive and negative mean.

Theorem 1.6. — *Assume $\mu \coloneqq \mathbb{E}X_1 \neq 0$ and that X is in the domain of attraction of an α-stable law with $\alpha \in (1,2]$.*

(a) *Suppose $\mu > 0$, then, as $T \to \infty$, we have*

$$\left(\frac{\Upsilon_T^{\frown} - \sqrt{1+\mu^2}T}{a_T}, \frac{\overline{C}_T^{\frown} - \mu T}{a_T}, \frac{C_T^{\frown}(T) - \mu T}{a_T}\right) \xrightarrow{d} S_\alpha(1)\left(\frac{\mu}{\sqrt{1+\mu^2}}, 1, 1\right).$$

(b) *Suppose $\mu < 0$ and let $(\overline{X}_\infty, \gamma_\infty^{\frown})$ be the a.s. finite limit of the supremum and its time $(\overline{C}_T^{\frown}, \gamma_T^{\frown})$ as $T \to \infty$. Then, as $T \to \infty$, we have*

$$\left(\frac{\Upsilon_T^{\frown} - \sqrt{1+\mu^2}T}{a_T}, \overline{C}_T^{\frown}, \frac{C_T^{\frown}(T) - \mu T}{a_T}, \gamma_T^{\frown}\right) \xrightarrow{d} \left(\frac{\mu}{\sqrt{1+\mu^2}} S_\alpha(1), \overline{X}_\infty, S_\alpha(1), \gamma_\infty^{\frown}\right),$$

where $S_\alpha(1)$ and $(\overline{X}_\infty, \gamma_\infty^{\frown})$ are independent.

Note that the centering function of $\Upsilon_T^{\frown}$ in Theorem 1.6 equals the length of the graph of the linear function $t \mapsto \mu t$ on $[0,T]$. Moreover, the order of the fluctuations of $\Upsilon_T^{\frown}$ in this case is different than that in Theorems 1.1 and 1.4. Asymptotically, $\Upsilon_T^{\frown}$ and $C_T^{\frown}(T)$ are positively correlated when $\mu > 0$ and negatively correlated when $\mu < 0$.

When X is in the domain of attraction of an α-stable law with $\alpha \in (0,1)$, the tails of X are very heavy. The large jumps of X make its concave majorant thin and tall, implying that the length $\Upsilon_T^{\frown}$ will be well approximated by the extremes of X. Define $\underline{C}_T^{\smile} \coloneqq \inf_{t\in[0,T]} C_T^{\smile}(t)$ and let $\gamma_T^{\smile}$ be the time at which the infimum is attained (see Figure 1.1). Denote

$$\overline{S}_\alpha(1) \coloneqq \sup_{t\in[0,1]} S_\alpha(t), \quad \underline{S}_\alpha(1) \coloneqq \inf_{t\in[0,1]} S_\alpha(t)$$

and let $\gamma^{\alpha\frown}$ (resp. $\gamma^{\alpha\smile}$) be the time at which $(S_\alpha(t))_{t\in[0,1]}$ attains its supremum (resp. infimum).

Theorem 1.7. — *Let X be in the domain of attraction of the α-stable law $S_\alpha(1)$ for $\alpha \in (0,1)$. Define*

$$\Lambda_T^1 := \left(\frac{\Upsilon_T^{\frown}}{a_T}, \frac{\overline{C}_T^{\frown}}{a_T}, \frac{C_T^{\frown}(T)}{a_T}, \frac{\gamma_T^{\frown}}{T}\right), \quad \Lambda^1 := \left(2\overline{S}_\alpha(1) - S_\alpha(1), \overline{S}_\alpha(1), S_\alpha(1), \gamma^{\alpha\frown}\right),$$
$$\Lambda_T^2 := \left(\frac{\Upsilon_T^{\smile}}{a_T}, \frac{\underline{C}_T^{\smile}}{a_T}, \frac{C_T^{\smile}(T)}{a_T}, \frac{\gamma_T^{\smile}}{T}\right), \quad \Lambda^2 := \left(S_\alpha(1) - 2\underline{S}_\alpha(1), \underline{S}_\alpha(1), S_\alpha(1), \gamma^{\alpha\smile}\right).$$

Then the following joint convergence holds: $(\Lambda_T^1, \Lambda_T^2) \xrightarrow{d} (\Lambda^1, \Lambda^2)$ as $T \to \infty$.

The Lévy process X in Theorem 1.7 has a thin and tall concave majorant, so the asymptotic centering by T, present in Theorems 1.1 and 1.4, is no longer required. Moreover, note that in Theorems 1.1 and 1.4 the fluctuations of $\Upsilon_T^{\frown}$ about this centering were significantly smaller than T, which is no longer the case here. The proof of Theorem 1.7 in Section 3.2 below is based on and approximation of $C_T^{\frown}$ by simpler geometric figures such as the ones in Figure 1.2.

The concave majorant lies between two natural geometric figures. Under the concave majorant lies the 'hut' $C_T^{\wedge}$, defined as the linear path connecting the vertices: $(0,0)$, $(\gamma_T^{\frown}, \overline{X}_T)$ and (T, X_T), where $\gamma_T^{\frown} = \arg\inf\{t > 0 : X_t \vee X_{t-} = \overline{X}_T\}$ is the time X attains its supremum on $[0,T]$. Over the concave majorant lies the "box-top" $C_T^{\sqcap}$, defined as the linear path connecting the vertices: $(0,0)$, $(0, \overline{X}_T)$, $(T, \overline{X}_T)$ and (T, X_T).

Figure 1.2. The figure shows a sample of the path of X, the concave majorant $C_T^{\frown}$, the hut $C_T^{\wedge}$ and the box-top $C_T^{\sqcap}$.

Suppose that the lengths of the hut $C_T^{\wedge}$ and the box-top $C_T^{\sqcap}$ are $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$, respectively. It is clear from the triangle inequality that $\Upsilon_T^{\wedge} \leqslant \Upsilon_T^{\frown} \leqslant \Upsilon_T^{\sqcap}$. These lengths do not generally all have the same asymptotic behaviour. The next result provides a short comparison in the cases $\alpha \in (1,2]$ with $\mathbb{E}[X_1] = 0$ and $\alpha \in (0,1)$.

Proposition 1.8. — *Define $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ as before then the following statements hold as $T \to \infty$.*

(a) *Suppose $\mathbb{E}[X_1] = 0$ and $\sigma^2 = \mathbb{E}[X_1^2] < \infty$, then*

$$\left(\Upsilon_T^{\wedge} - T, \frac{1}{\sqrt{\log T}}\left(\Upsilon_T^{\frown} - T - \frac{\sigma^2}{2}\log T + \Theta(T)\right), \frac{1}{\sqrt{T}}\left(\Upsilon_T^{\sqcap} - T\right)\right)$$
$$\xrightarrow{d} \left(\frac{\sigma^2}{2}\left(\frac{\overline{B}_1^2}{\rho} + \frac{\left(\overline{B}_1 - B_1\right)^2}{1-\rho}\right), \frac{\sqrt{3}}{2}\sigma^2 Z, \sigma\left(2\overline{B}_1 - B_1\right)\right),$$

where Z is a standard normal variable independent of the standard Brownian motion $B = (B_t)_{t \geqslant 0}$, $\overline{B}_1 = \sup_{t \in [0,1]} B_t$ and $\rho \in [0,1]$ is the a.s. unique time such that $B_\rho = \overline{B}_1$.

(b) *Suppose the limit in* (1.5) *holds for some $\alpha \in (1,2)$ and $\mathbb{E}[X_1] = 0$, then*

$$\left(\frac{T}{a_T^2}\left(\Upsilon_T^{\wedge} - T\right), \frac{T}{a_T^2}\left(\Upsilon_T^{\frown} - T\right), \frac{1}{a_T}\left(\Upsilon_T^{\sqcap} - T\right)\right) \xrightarrow{d}$$
$$\frac{1}{2}\left(\left(\sum_{n=1}^{\infty} \ell_n^{1/\alpha}\left(S_\alpha^{(n)}\right)^+\right)^2 + \left(\sum_{n=1}^{\infty} \ell_n^{1/\alpha}\left(S_\alpha^{(n)}\right)^-\right)^2,\right.$$
$$\left.\sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2, 2\sum_{n=1}^{\infty} \ell_n^{1/\alpha}\left|S_\alpha^{(n)}\right|\right),$$

where $(\ell_n)_{n \in \mathbb{N}}$ is a uniform stick-breaking process independent of the sequence $(S_\alpha^{(n)})_{n \in \mathbb{N}}$ of independent copies of $S_\alpha(1)$.

(c) *Suppose the limit in* (1.5) *holds for some $\alpha \in (0,1)$, then*

$$\left(\frac{\Upsilon_T^{\wedge}}{a_T}, \frac{\Upsilon_T^{\frown}}{a_T}, \frac{\Upsilon_T^{\sqcap}}{a_T}\right) \xrightarrow{d} \left(2\overline{S}_\alpha(1) - S_\alpha(1)\right)(1,1,1).$$

Under the assumptions of either Theorem 1.1 or Theorem 1.4, the centering functions of $\Upsilon_T^{\wedge}$, $\Upsilon_T^{\frown}$ and $\Upsilon_T^{\sqcap}$ in Proposition 1.8 are of the form $T + o(T)$ as $T \to \infty$. However, even though the three statistics are closely related, the order of their fluctuations, measured via the scaling functions, exhibits a wide variety of behaviours, see Table 1.1 below. In the case $\alpha \in (0,1)$, the centering functions are all zero and the corresponding scaling functions coincide with the scale of the process.

Setting	Scaling of X_T	$\Upsilon_T^{\wedge}$	$\Upsilon_T^{\frown}$	$\Upsilon_T^{\sqcap}$
Theorem 1.1 ($\mathbb{E}[X_1^2] < \infty$)	$a_T = \sqrt{T}$	1	$\sqrt{\log T}$	$\sqrt{T}$
Theorem 1.4 ($1 < \alpha < 2$)	$a_T = T^{1/\alpha}l(T)$	$T^{2/\alpha-1}l(T)^2$	$T^{2/\alpha-1}l(T)^2$	$T^{1/\alpha}l(T)$
Theorem 1.7 ($0 < \alpha < 1$)	$a_T = T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$	$T^{1/\alpha}l(T)$

Table 1.1. The table shows the scaling functions (after centering) in the weak limits of the lengths $\Upsilon_T^{\wedge}$, $\Upsilon_T^{\frown}$ and $\Upsilon_T^{\sqcap}$ under the assumptions of the corresponding theorems with a_T as in (1.5).

Recall that $\Upsilon_T^{\wedge} \leqslant \Upsilon_T^{\frown} \leqslant \Upsilon_T^{\sqcap}$. Interestingly, for X with finite variance, by Proposition 1.8(a) the fluctuations of $\Upsilon_T^{\frown}$ are asymptotically independent of those of $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$, while the fluctuations of the sandwiching lengths $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ exhibit a strong asymptotic dependence, both being deterministic functions of the vector $(B_1, \overline{B}_1, \rho)$. Proposition 1.8(b)&(c) states that the dependence of the fluctuations of all three statistics persists in the limit when $\alpha < 2$.

1.1. Overview of the proofs

Our starting point is [GCM22, Theorem 11], which implies the following crucial identity for any Lévy process and time horizon $T > 0$:

$$\begin{aligned}(1.7)\qquad &\left(\Upsilon_T^{\frown}, H_T', C_T^{\frown}(T), \overline{C}_T^{\frown}, \gamma_T^{\frown}\right)\\ &\overset{d}{=} \sum_{n=1}^{\infty} \left(\sqrt{(T\ell_n)^2 + \xi_n^2}, \mathbb{1}_{\{T\ell_n \geqslant 1\}}, \xi_n, \xi_n^+, T\ell_n \mathbb{1}_{\{\xi_n > 0\}}\right),\end{aligned}$$

where $\xi_n := X_{TL_{n-1}} - X_{TL_n}$, H_T' is a random variable such that $|H_T - H_T'|$ is bounded in L^1 as $T \to \infty$ (see Lemma 2.6 below for details) and ℓ is a uniform stick-breaking process independent of X with stick-remainders $(L_n)_{n \in \mathbb{N} \cup \{0\}}$. This identity is essential in all that follows as it reduces the claims in Theorems 1.1, 1.4 and 1.6 to limit statements for the sum in (1.7), which is given in terms of the increments of the Lévy process over independent stick-breaking lengths. Establishing those limits as time horizon $T \to \infty$ turns out to be a delicate task.

In the case of finite variance and zero mean, the proof of Theorem 1.1 requires splitting the weak limits into three asymptotically independent weak limits. The faces of $C_T^{\frown}$ of length smaller than 1 do not contribute to the fluctuations of $(\Upsilon_T^{\frown}, H_T')$. However, all faces of $C_T^{\frown}$ of moderate size contribute in aggregate to its fluctuations, with any finite set of faces the of moderate size not surviving in the limiting fluctuations. In contrast, *only the largest few* faces of $C_T^{\frown}$ influence the scaling limit of the vector $(C_T^{\frown}(T), \overline{C}_T^{\frown}, \gamma_T^{\frown})$, making its limit independent of the limiting fluctuations of $(\Upsilon_T^{\frown}, H_T')$. Moreover, the CLT for $(\Upsilon_T^{\frown}, H_T')$ consists of two asymptotically independent weak limits. The first captures the fluctuations *due to* the stick-breaking process while the second describes the fluctuations *conditional on* a manifestation of the stick-breaking process. The remaining work in the proof of Theorem 1.1 is mostly concerned with establishing weak limits, conditional on the stick-breaking process, and crucially depends on [BGCM21a, Theorem 1.1].

In the case of finite first moment and infinite variance, the proofs of Theorems 1.4 and 1.6 split the sum in (1.7) into two sums according to whether the faces are shorter or longer than one. However, unlike in the finite-variance zero-mean case, here this is just a technical step: in the proof of Theorems 1.4 all the faces of the concave majorant survive in the limit, contributing both to the fluctuations of its length as well as the remaining statistics of $C_T^{\frown}$. It follows from the proof of Theorem 1.6 that only the vertical heights of the faces of $C_T^{\frown}$ in aggregate contribute to the fluctuations of its length, which are determined by the asymptotic behaviour of its final point $C_T^{\frown}(T) = X_T$ as $T \to \infty$.

In the infinite first moment case, Theorem 1.7 follows by a sandwiching argument involving the weak limits for the lengths $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ as in Proposition 1.8 above. As in the proof of Theorem 1.6, only the heights of the faces of $C_T^{\frown}$ in aggregate contribute to this limit.

1.2. Connections with the literature

Convex hulls of stochastic processes are of longstanding interest, see e.g. [APRUB11] and the references therein. Of particular interest are the geometric properties of convex hulls such as the length, area and diameter, see [AKMV20, MW18, RFZ20, WX15a, WX15b] for random walks and [KLM12] for isotropic stable process. Concave majorants of one-dimensional Lévy processes are also of interest in physics. In the monograph [Nag00, Chapter XI], for example, the problem of whether a quantum particle stays within the light cone is analysed using concave majorants of one-dimensional Lévy processes.

If the Lévy process is in a domain of attraction of a stable law, one can pose two types of question about the limiting behaviour of its convex hulls. A limit of a geometric quantity (e.g. perimeter) of the convex hull of the original process may be considered emphor the limit of the convex hull of the scaled process may be analysed. Since taking the convex hull of the graph of a function is a continuous mapping(1), in the latter case it is natural to expect that the limit will be given in terms of the corresponding geometric quantity of the convex hull of the stable limit, which is what happens in [MW16, Section 5]. The present paper considers the former type of question for the length of the concave majorant. It is clear from Theorems 1.1 and 1.4 above that in this case the asymptotic mean and the scale of the fluctuations around them are of different order than those of the process. Moreover, the limit is not given in terms of the corresponding quantity for the stable process. Differently put, we analyse the statistics describing the geometry of the convex hull of the original process as the time horizon tends to infinity without scaling the process first and then considering the limiting behaviour of such statistics.

The object of study in [MW16] is the convex hull of the scaled multi-dimensional Lévy process attracted to an isotropic α-stable process. This confers upon the convex hull a spatial homogeneity not enjoyed by the concave majorant, which is a one dimensional object in space-time that behaves very differently in space and time coordinates. A further difference with problem considered in [MW16] is that our aim is to understand the fluctuations around the asymptotic centering rather than obtaining the limit, which in our case is straightforward, see (1.1) above.

A related question about the fluctuations of the length of the convex minorant of a random walk, as the time horizon tends to infinity, was studied in the recent paper [AKMV20]. CLT-type results for the length of the convex minorant of a random walk ware established in [AKMV20] under hypothesis analogous to ours (i.e. the increments either have finite variance and zero mean or are in the domain of attraction of an α-stable law for $\alpha \in (0,2) \setminus \{1\}$). The joint limits for the shape statistics in the random walk case are not discussed in [AKMV20]. Moreover, we stress that the fluctuations of the length of the concave majorant in our Theorems 1.1, 1.4 and 1.6 cannot be deduced easily from the results of [AKMV20] even in the case of a compound Poisson process since the random time-change connecting it with a random walk distorts the concave majorant, see Figure 1.3 below.

(1) This mapping takes càdlgàg functions equipped with the Skorokhod topology to compact sets in $\mathbb{R}^2$ equipped with the Hausdorff distance.

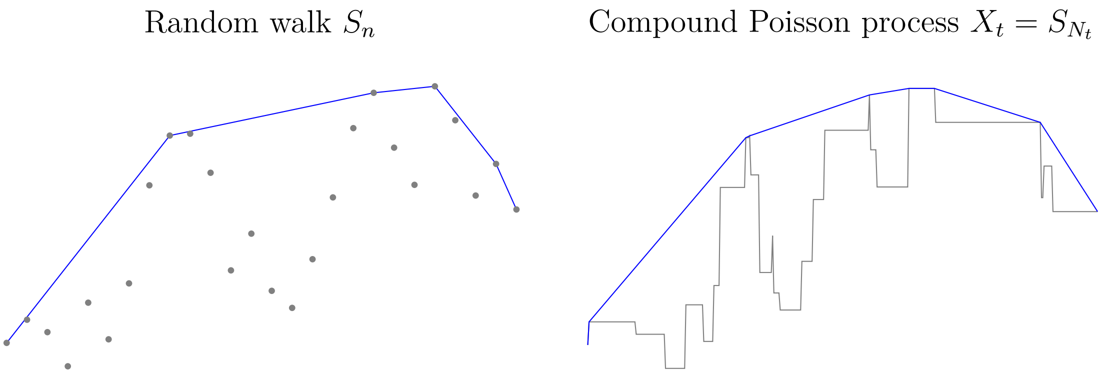

Figure 1.3. The figure shows a sample of the path of a random walk S_n (left) and that of the compound Poisson process $X_t = S_{N_t}$ (right), where N_t is a Poisson process independent of S_n. Note that both processes visit the same states and in the same order, but the random time-change induced by N_t distorts the shape of the concave majorant, since the two concave majorants have a different number of faces.

As mentioned in Section 1.1 above, a crucial structure used to establish our main results is the characterisation of the law of the concave majorant for all Lévy processes given in the recent article [GCM22, Theorem 11]. Note that the main result in [GCM22] generalises to emphall Lévy processes the characterisation of the law of the concave majorant established in [PUB12] for diffuse Lévy processes. This extension is important for the results in the present paper because it allows us to understand the asymptotic shape of the concave majorant of all Lévy processes, including Poisson processes with drift.

Finally we note that in [Sat13, Section 28], Sato explores the long time behaviour of a Lévy process and its supremum of the process. Since the concave majorant on $[0,T]$ always coincides with the process at times T and $\gamma_{\widehat{T}}$, our results may be viewed as an extension of those in [Sat13, Section 28].

The remainder of the paper is organised as follows. Section 2 proves Theorem 1.1. Section 3 proves Theorems 1.4, 1.6 and 1.7 as well as the two propositions in the introduction.

2. Proof of Theorem 1.1

Recall that $\xi_n = X_{TL_{n-1}} - X_{TL_n}$ and denote $t_n := T\ell_n$ for $n \in \mathbb{N}$, where $\ell = (\ell_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process on $[0,1]$, independent of the Lévy process X, and $L = (L_n)_{n\in\mathbb{N}\cup\{0\}}$ is its stick-remainder process. Note that the sequence $(t_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process on $[0,T]$. Define the following set of indices $\mathfrak{I}_T := \{n \in \mathbb{N} : t_n \geqslant 1\}$.

The strategy for the proof of Theorem 1.1 is the following. We will show that the cardinality of $\mathfrak{I}_T$ is by [GCM22, Theorem 11] closely related to the random variable H_T appearing in Theorem 1.1 (see Lemma 2.6 below for more details). Setting

$$\varpi_T := \left(0, \frac{|\mathfrak{I}_T| - \log T}{\sqrt{\log T}}, \frac{\sum_{n=1}^{\infty} \xi_n^+}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty} \xi_n}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty} t_n \mathbb{1}_{\{\xi_n > 0\}}}{T}\right),$$

and using the aforementioned close relationship and [GCM22, Theorem 11], we will find that Theorem 1.1 is equivalent to the vector

$$\left(\frac{\sum_{n=1}^{\infty}\left(\sqrt{\xi_n^2+t_n^2}-t_n\right) - \frac{\sigma^2}{2}|\mathfrak{I}_T| + \Theta(T)}{\sqrt{\log T}}, 0, 0, 0, 0\right) + \varpi_T, \tag{2.1}$$

converging weakly to $\zeta := (\sigma^2 Z_1/\sqrt{2}, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho)$ as $T \to \infty$. We next apply certain moment estimates for X and limit results for the stick-breaking process ℓ to show that the quintuple in (2.1) converges weakly to ζ if and only if the following weak limit holds as $T \to \infty$:

$$\left(\frac{\sum_{n \in \mathfrak{I}_T}\left(\xi_n^2/t_n - \sigma_{t_n}^2\right)}{2\sqrt{\log T}}, 0, 0, 0, 0\right) + \varpi_T \xrightarrow{d} \zeta, \tag{2.2}$$

where $\sigma_t^2 := \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$ for any $t \geqslant 1$ and some $\kappa \geqslant 1$ such that $\sigma_1 > 0$ (see Proposition 2.8 below for details). Note that the second coordinate in the quintuple in (2.2) is a deterministic function of the stick-breaking process ℓ and denote the remaining quadruple by ζ_T'. In order to establish (2.2), we condition ζ_T' on ℓ and prove that its weak limit under the conditional law is $(\sigma^2 Z_1/\sqrt{2}, \sigma\overline{B}_1, \sigma B_1, \rho)$. Since this limit law does not depend on ℓ, applying Proposition 2.3 below will complete the proof of Theorem 1.1.

The steps described in this strategy require a variety of technical results. The details of the proof of Theorem 1.1 are given after the technical results have been established (see page 800 below).

2.1. Limit properties of the stick-breaking process

The proof of Theorem 1.1 requires a detailed analysis of certain asymptotic properties of the stick-breaking process. We start with a compensation formula for the point process based on a stick-breaking process, analogous to Campbell's formula for Poisson point processes.

Lemma 2.1. — *Define the point process $\Xi_T := \sum_{n\in\mathbb{N}} \delta_{t_n}$, where δ_x is the Dirac measure at x. Then for any measurable function $f : [0,T] \to \mathbb{R}_+$ the following identities hold (with all quantities possibly equal to $+\infty$):*

$$\mathbb{E}\left[\int_{\mathbb{R}_+} f(x)\Xi_T(\mathrm{d}x)\right] = \mathbb{E}\left[\sum_{n\in\mathbb{N}} f(t_n)\right] = \int_0^T \frac{f(t)}{t}\mathrm{d}t. \tag{2.3}$$

The point process Ξ_T converges weakly as $T \to \infty$ to a Poisson point process Ξ_∞ on $(0,\infty)$ with intensity $t \mapsto t^{-1}$. Moreover, there exists a coupling of point processes

$\overline{\Xi}_\infty$ and $\overline{\Xi}_T$ for all $T > 0$ such that: $\overline{\Xi}_T \overset{d}{=} \Xi_T$ and $\overline{\Xi}_\infty \overset{d}{=} \Xi_\infty$, $\overline{\Xi}_T \to \overline{\Xi}_\infty$ a.s. in the vague topology and for every compact set $A \subset (0,\infty)$, we have $\overline{\Xi}_T|_A = \overline{\Xi}_\infty|_A$ for all sufficiently large T.

The distributional convergence in Lemma 2.1 holds in the vague topology of locally finite measures on $(0,\infty)$, see [Kal02, Chapter 16, p. 316] for definition. More specifically, the a.s. convergence $\overline{\Xi}_T \to \overline{\Xi}_\infty$ as $T \to \infty$ in the vague topology is equivalent to $\int f(x)\overline{\Xi}_T(\mathrm{d}x) \to \int f(x)\overline{\Xi}_\infty(\mathrm{d}x)$ for any continuous function f on $(0,\infty)$ that vanishes at 0 and ∞.

Proof. — Note that $-\log \ell_n$ is gamma distributed with density $t \mapsto t^{n-1}e^{-t}/(n-1)!$. Thus, Fubini's theorem implies (2.3):

$$\mathbb{E}\left[\sum_{n\in\mathbb{N}} f(t_n)\right] = \sum_{n\in\mathbb{N}} \int_0^\infty \frac{f\left(Te^{-t}\right)t^{n-1}}{(n-1)!}e^{-t}\mathrm{d}t = \int_0^\infty f\left(Te^{-t}\right)\mathrm{d}t = \int_0^T \frac{f(t)}{t}\mathrm{d}t.$$

To prove $\Xi_T \overset{d}{\to} \Xi_\infty$ as $T \to \infty$, it suffices to provide a coupling $(\overline{\Xi}_T, \overline{\Xi}_\infty)$ with $\overline{\Xi}_T \overset{d}{=} \Xi_T$ and $\overline{\Xi}_\infty \overset{d}{=} \Xi_\infty$ such that $\overline{\Xi}_T \to \overline{\Xi}_\infty$ a.s. as $T \to \infty$. To that end, let Y be a subordinator with infinite mean $\mathbb{E}[Y_1] = \infty$ and the convex minorant C_∞ on $\mathbb{R}_+$. By [PUB12, Cor. 3], for any enumeration of the horizontal lengths $(l_n)_{n\in\mathbb{N}}$ and vertical heights $(h_n)_{n\in\mathbb{N}}$ of the faces of C_∞, the point process $\widetilde{\Xi}_\infty := \sum_{n\in\mathbb{N}} \delta_{(l_n,h_n)}$ on $(0,\infty)^2$ is Poisson with mean measure $t^{-1}\mathbb{P}(Y_t \in \mathrm{d}x)\mathrm{d}t$, $(t,x) \in (0,\infty)^2$. Similarly, let $\widetilde{\Xi}_T$ be the point process of lengths and heights of the convex minorant C_T of Y on $[0,T]$.

For any $s > 0$ define the set $A_s := \{(t,x) \in (0,\infty)^2 : x/t < s\}$ and let T_s be the last time the right derivative of C_∞ was smaller than s, which is a.s. finite by [PUB12, Corollary 3]. It follows that $C_T = C_\infty$ on $[0,T_s]$ for any $T > T_s$, implying that $\widetilde{\Xi}_T$ and $\widetilde{\Xi}_\infty$ agree on A_s for any $T > T_s$. Since $\bigcup_{s>0} A_s = (0,\infty)^2$ and any compact set in $(0,\infty)^2$ is contained in some A_s, we have

$$\int_{(0,\infty)^2} f(y)\widetilde{\Xi}_T(\mathrm{d}y) = \int_{(0,\infty)^2} f(y)\widetilde{\Xi}_\infty(\mathrm{d}y), \qquad T > T_s,$$

for any compactly supported continuous function $f : (0,\infty)^2 \to \mathbb{R}_+$. Since $T_s < \infty$ for all $s > 0$, we therefore have $\widetilde{\Xi}_T \to \widetilde{\Xi}_\infty$ a.s. in the vague topology. Moreover, this implies that the projections

$$\overline{\Xi}_T := \widetilde{\Xi}_T(\cdot \times \mathbb{R}_+) \overset{d}{=} \Xi_T \qquad \text{converge to} \qquad \overline{\Xi}_\infty := \widetilde{\Xi}_\infty(\cdot \times \mathbb{R}_+) \overset{d}{=} \Xi_\infty$$

a.s. in the vague topology. □

Recall that $\mathfrak{I}_T = \{n \in \mathbb{N} : t_n \geqslant 1\}$ is the finite set of indices of sticks in $[0,T]$ of length greater than one and denote by $\mathfrak{I}_T^{\mathsf{c}} := \mathbb{N} \setminus \mathfrak{I}_T$ its infinite complement.

Corollary 2.2. — (a) *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $T \geqslant 1$. Then the following equalities hold:*

$$\mathbb{E}\sum_{n\in\mathfrak{I}_T} f(t_n) = \int_1^T \frac{f(t)}{t}\mathrm{d}t \qquad \textit{and} \qquad \mathbb{E}\sum_{n\in\mathfrak{I}_T^{\mathsf{c}}} f(t_n) = \int_0^1 \frac{f(t)}{t}\mathrm{d}t. \tag{2.4}$$

In particular, the first expectation in (2.4) *always has a (possibly infinite) limit as* $T \to \infty$ *and for any* $q > 0$ *we have* $\lim_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T} t_n^{-q} = 1/q$.

(b) *For any bounded and measurable function* $f : [1, \infty) \to \mathbb{R}$ *with* $\lim_{t\to\infty} f(t) = 0$ *we have* $\mathbb{E} \sum_{n\in\mathfrak{I}_T} f(t_n) = \mathrm{o}(\log T)$*, implying that*

$$(\log T)^{-1} \sum_{n\in\mathfrak{I}_T} f(t_n) \xrightarrow{L^1} 0.$$

Proof. —

(a) Note that $f(t_n)\mathbb{1}_{\{n\in\mathfrak{I}_T\}} = h(t_n)$ where $h(t) = f(t)\mathbb{1}_{\{T>1\}}$, so (2.4) follows from (2.3). The formulae for the power functions then follow easily.

(b) Let $T > 1$ and note that

$$\frac{1}{\log T} \mathbb{E} \sum_{n\in\mathfrak{I}_T} f(t_n) = \int_1^T \frac{f(t)}{t \log T} \mathrm{d}t = \mathbb{E}\left[f\left(Z_T\right)\right],$$

where Z_T has the density $t \mapsto (t \log T)^{-1}$, $t \in [1, T]$. Since $Z_T \xrightarrow{\mathbb{P}} \infty$, we have $f(Z_T) \xrightarrow{\mathbb{P}} 0$ and since the variables $|f(Z_T)|$ are bounded by $\sup_{t\in[1,\infty)} |f(t)|$, the dominated convergence theorem implies that $\mathbb{E}[f(Z_T)] \to 0$. □

We now prove the following CLT for the cardinality of the set $\mathfrak{I}_T$ defined above.

Proposition 2.3. — *The cardinality* $|\mathfrak{I}_T|$ *of the set* $\mathfrak{I}_T$ *satisfies the limits*

$$|\mathfrak{I}_T| / \log T \xrightarrow{L^1} 1 \qquad \textit{and} \qquad \left(|\mathfrak{I}_T| - \log T\right) / \sqrt{\log T} \xrightarrow{d} N(0, 1) \quad \textit{as} \quad T \to \infty.$$

Moreover, for any T *we have* $\mathfrak{I}_T \subset \{1, \dots, \tau(T)+1\}$ *and* $\mathbb{E}[\tau(T)] = \mathbb{E}[|\mathfrak{I}_T|] = \log^+(T)$*, where we define* $\tau(T) := |\{n \in \mathbb{N} : L_n \geqslant 1/T\}|$.

Proof. — Recall by definition of the stick-remainder that $L_n = \prod_{i=1}^n (1 - V_i)$ for an iid sequence $(V_i)_{i\in\mathbb{N}}$ of uniform random variables on the unit interval. Thus $S_n := -\log L_n$ is a random walk with exponential increments of unit mean or, equivalently, the jump times of a Poisson process with unit intensity. Thus, the definition of $\tau(T)$ implies that, for $T > 1$, $\tau(T)$ follows the marginal distribution of the Poisson process with unit intensity at time $\log T$. Put differently, $\tau(T)$ is Poisson distributed with mean $\log T$. In particular, we have $(\tau(T) - \log T)/\sqrt{\log T} \xrightarrow{d} N(0, 1)$ as $T \to \infty$.

Recall that $\ell_m = L_n \prod_{i=n+1}^m V_i < L_n$ for all $m > n$. Since $L_{\tau(T)+1} < 1/T$ we get $\ell_m < 1/T$ for all $m > \tau(T)+1$ and thus $\mathfrak{I}_T \subset \{1, \dots, \tau(T)+1\}$ and $\tau(T)+1-|\mathfrak{I}_T| \geqslant 0$. Corollary 2.2 (a) gives $\mathbb{E}[\tau(T) + 1 - |\mathfrak{I}_T|] = 1$ and thus $\mathbb{E}[|\tau(T) - |\mathfrak{I}_T||] \leqslant 2$ for all $T > 0$, implying $(\tau(T) - |\mathfrak{I}_T|)/\sqrt{\log T} \xrightarrow{L^1} 0$. Hence, the CLT for $\tau(T)$ yields the CLT for $|\mathfrak{I}_T|$. Since the random variables $\tau(T)/\log T$, $T \geqslant 2$, are uniformly integrable, we have $\tau(T)/\log T \xrightarrow{L^1} 1$ and thus

$$|\mathfrak{I}_T| / \log T = \Big(|\mathfrak{I}_T| - \tau(T)\Big) / \log T + \tau(T) / \log T \xrightarrow{L^1} 1. \qquad \square$$

Remark 2.4. — The law $|\mathfrak{I}_T|$ is much more complicated than that of $\tau(T)$, which follows a Poisson distribution with mean $\log T$ (for $T > 1$). The reason for this lies in the fact that $\tau(T)$ is a stopping time in a correct filtration, while $|\mathfrak{I}_T|$ is not, making its moments hard to control. In Proposition 2.3 we circumvent this problem by approximating $|\mathfrak{I}_T|$ with $\tau(T)$. We note that, even though the expectation $\mathbb{E}[|\tau(T)-|\mathfrak{I}_T||]\leqslant 2$ is bounded for all $T>0$, the difference $|\tau(T)-|\mathfrak{I}_T||$ takes arbitrarily large values with positive probability.

The following L^1 limit holds.

Proposition 2.5. — *Let $f:[1,\infty)\to\mathbb{R}_+$ be measurable and non-increasing with $\lim_{t\to\infty}f(t)=0$. Then, as $T\to\infty$,*

$$\frac{1}{\sqrt{\log T}}\left(\sum_{n\in\mathfrak{I}_T}f(t_n)-\mathbb{E}\sum_{n\in\mathfrak{I}_T}f(t_n)\right)=\frac{1}{\sqrt{\log T}}\left(\sum_{n\in\mathfrak{I}_T}f(t_n)-\int_1^T\frac{f(t)}{t}\mathrm{d}t\right)\xrightarrow{L^1}0.$$

Proof. — Define for every T the random variables

$$A_T\coloneqq\sum_{n\in\mathfrak{I}_T}f(t_n)-\sum_{n=1}^{\tau(T)}f(TL_n),\qquad\text{and}\qquad B_T\coloneqq\sum_{n=1}^{\tau(T)}f\left(TL_n\right)-\int_1^T\frac{f(t)}{t}\mathrm{d}t.$$

Note that it suffices to show that $\mathbb{E}|A_T|$ is bounded for $T>1$ and $B_T/\sqrt{\log T}\xrightarrow{L^1}0$.

By Lemma 2.1 and the equality in law $t_n\overset{d}{=}TL_n$, we have

$$\mathbb{E}[A_T]=\sum_{n\in\mathbb{N}}\mathbb{E}\left[f(t_n)\mathbb{1}_{\{t_n\geqslant 1\}}\right]-\sum_{n\in\mathbb{N}}\mathbb{E}\left[f\left(TL_n\right)\mathbb{1}_{\{TL_n\geqslant 1\}}\right]=0.\tag{2.5}$$

Since the function f is non-increasing and $t_n\leqslant TL_{n-1}$ for all $n\in\mathbb{N}$, we have

$$C_T\coloneqq\sum_{n\in\mathfrak{I}_T}\left(f(t_n)-f(TL_{n-1})\right)\leqslant 0.$$

Similarly, as f is non-increasing, by Proposition 2.3

$$\begin{aligned}|C_T-A_T|&=\left|\sum_{n=2}^{\tau(T)+1}f\left(TL_{n-1}\right)-\sum_{n\in\mathfrak{I}_T}f\left(TL_{n-1}\right)\right|\\&=\left|-f(T)+\sum_{n\in\{1,\dots,\tau(T)+1\}\setminus\mathfrak{I}_T}f\left(TL_{n-1}\right)\right|\leqslant f(1)\Big|\tau(T)+2-|\mathfrak{I}_T|\Big|.\end{aligned}$$

Thus (2.5) and Proposition 2.3 yield $\mathbb{E}[|C_T|]=-\mathbb{E}[C_T]=\mathbb{E}[A_T-C_T]\leqslant 2f(1)$, implying that $\mathbb{E}|A_T|$ is bounded by $4f(1)$ for all $T>1$.

It remains to show that $B_T/\sqrt{\log T}\xrightarrow{L^1}0$. Let $S_n\coloneqq-\log L_n$ and note that $\Xi_T\coloneqq\sum_{i=1}^{\tau(T)}\delta_{S_i}$ is a random measure with atoms at the jump times on the interval $[0,\log T]$ of a Poisson process with unit intensity. Thus Ξ_T is a Poisson point process on $[0,\log T]$ with the Lebesgue measure as its mean measure. By the reflection and translation invariance of the Lebesgue measure, the mapping theorem for Poisson point processes [Kin93, Section 2.3] gives

$$\Xi_T\overset{d}{=}\sum_{i=1}^{\tau(T)}\delta_{\log T-S_i},$$

implying

$$D_T := \sum_{n=1}^{\tau(T)} f\left(e^{S_n}\right) \overset{d}{=} \sum_{n=1}^{\tau(T)} f\left(e^{\log T - S_n}\right) = \sum_{n=1}^{\tau(T)} f(TL_n) = B_T + \int_1^T \frac{f(t)}{t}\mathrm{d}t.$$

Campbell's formula (see [Kin93, p. 28]) yields

$$\mathbb{E}[D_T] = \int_0^{\log T} f(e^x)\mathrm{d}x = \int_1^T \frac{f(t)}{t}\mathrm{d}t, \quad \mathrm{Var}[D_T] = \int_0^{\log T} f(e^x)^2\mathrm{d}x = \int_1^T \frac{f(t)^2}{t}\mathrm{d}t.$$

Thus, it suffices to show that $\mathbb{E}[B_T^2]/\log T = \mathrm{Var}[D_T]/\log T \to 0$ as $T \to \infty$. Consider the distribution functions $g_T(t) = \log t/\log T$ for $t \in [1,T]$ and define $Z_T := g_T^{-1}(U) = T^U$ for all $T > 1$ and some fixed uniform random variable U on $(0,1)$. Then $Z_T \to \infty$ a.s. and hence $f(Z_T)^2 \to 0$ a.s. as $T \to \infty$. By the dominated convergence theorem,

$$\mathrm{Var}[D_T]/\log T = \mathbb{E}\left[f\left(Z_T\right)^2\right] \to 0 \quad \text{as} \quad T \to \infty.$$

□

2.2. A conditional limit theorem and the proof of Theorem 1.1

Recall from the first paragraph of Section 2 that $(t_n)_{n\in\mathbb{N}}$ denotes a uniform stick-breaking process on $[0,T]$, independent of X, and that $\mathfrak{I}_T$ denotes the set $\{n \in \mathbb{N} : t_n \geqslant 1\}$. Each horizontal length t_n has an associated slope given by ξ_n/t_n, where $\xi_n = X_{TL_{n-1}} - X_{TL_n}$ is the corresponding vertical height. Aggregate all the horizontal lengths with a common slope in the sequence $(t_n)_{n\in\mathbb{N}}$ into a maximal horizontal length corresponding to that slope. Consider the set $\mathfrak{F}_T$ of the maximal horizontal lengths with size at least 1. Note that, by [GCM22, Theorem 11], $|\mathfrak{F}_T| \overset{d}{=} H_T$, where H_T is the number of all horizontal lengths greater or equal to 1 of the maximal faces of the concave majorant $t \mapsto C_T^{\frown}(t)$. The analysis of the set $\mathfrak{F}_T$ is based on the properties of the $\mathfrak{I}_T$ established in Subsection 2.1 above. This strategy is feasible because the difference of sets $\mathfrak{F}_T$ and $\{t_n : n \in \mathfrak{I}_T\}$ is bounded in L^1 in the following sense.

Lemma 2.6. — *For any bounded function $f : [1,\infty) \to \mathbb{R}$, the following holds*

$$\sup_{T>0} \mathbb{E}\left|\sum_{t\in\mathfrak{F}_T} f(t) - \sum_{n\in\mathfrak{I}_T} f(t_n)\right| < \infty.$$

Proof. — Suppose X is not compound Poisson with drift. Then, by Doeblin's diffuseness lemma [Kal02, Lemma 15.22] and [GCM22, Theorem 11], no two slopes in the sequence $(\xi_n/t_n)_{n\in\mathbb{N}}$ coincide, implying the identity $\mathfrak{F}_T = \{t_n : n \in \mathfrak{I}_T\}$ a.s. The claim then follows since both random sums are equal a.s.

Suppose X is compound Poisson with drift γ (see [Sat13, p. 39] for the definition of the drift of a Lévy processes of finite variation). Consider two horizontal lengths t_n and t_m such that the corresponding slopes ξ_n/t_n and ξ_m/t_m are equal with positive probability. Since the pair (t_n, t_m) has a density $f_{n,m} : (0,T)\times(0,T)\to(0,\infty)$, the result in [Sat13, Proposition 27.6] implies

$$\mathbb{P}\left(\frac{\xi_n}{t_n}=\frac{\xi_m}{t_m}\right)=\int_{(0,T)^2}\mathbb{P}\left(\frac{X_s}{s}=\frac{X'_u}{u}\right)f_{n,m}(s,u)\mathrm{d}s\mathrm{d}u=\mathbb{P}\left(\frac{\xi_n}{t_n}=\gamma=\frac{\xi_m}{t_m}\right),$$

where $X'\overset{d}{=}X$ is a Lévy process independent of X. Thus all slopes ξ_n/t_n different from γ are also different from each other with probability one and therefore the corresponding faces are already maximal. Hence the set equality $\{t_n : n\in\mathfrak{I}_T\}\setminus\mathfrak{F}_T = \{t_n : n\in\mathfrak{I}_T,\, \xi_n=\gamma t_n\}$ holds a.s.

To complete the proof, it is sufficient to show that the number of faces with length at least 1 and slope $\xi_n/t_n=\gamma$ is bounded in L^1. By Corollary 2.2 (a), we have

$$\begin{aligned}\mathbb{E}\left|\left\{n\in\mathfrak{I}_T : \xi_n/t_n=\gamma\right\}\right| &= \mathbb{E}\sum_{n\in\mathfrak{I}_T}\mathbb{P}\left(X_{t_n}=\gamma t_n|t_n\right)\\ &=\int_1^T\frac{\mathbb{P}\left(X_t=\gamma t\right)}{t}\mathrm{d}t\xrightarrow[T\to\infty]{}\int_1^\infty\frac{\mathbb{P}\left(X_t=\gamma t\right)}{t}\mathrm{d}t,\end{aligned}$$

where the limit is finite by [Sat13, Lemma 48.3]. □

Remark 2.7. — The proof of Lemma 2.6 implies that the only maximal face of the concave majorant $C_T^\frown$ of a compound Poisson process X with drift γ that corresponds to more than one face in the representation in [GCM22, Theorem 11] is the face whose slope equals γ. All the other faces in this representation are finite in number and have slopes different from each other.

The following result, a conditional CLT given ℓ, is the final ingredient for the proof of Theorem 1.1.

Proposition 2.8. — *Suppose* $\mathbb{E}[X_1]=0$ *and* $\sigma:=\sqrt{\mathbb{E}[X_1^2]}\in(0,\infty)$. *If* $\nu\neq0$, *choose* $\kappa\geqslant1$ *such that* $\nu((-\kappa,\kappa))\in(0,\infty]$ *and otherwise set* $\kappa:=1$ *and recall that*

$$\sigma_t^2=\sigma^2-\int_{\mathbb{R}\setminus\left(-\kappa\sqrt{t},\kappa\sqrt{t}\right)}x^2\nu(\mathrm{d}x)\quad\textit{for}\quad t>0.$$

Then we have the following limit in probability as $T\to\infty$*:*

$$\Sigma_T-\frac{\sum_{n=1}^{\infty}\left(\sqrt{\xi_n^2+t_n^2}-t_n\right)-\frac{1}{2}\left(\sigma^2|\mathfrak{I}_T|-\int_{\mathbb{R}}x^2\log^+\left(\min\{T,x^2\}\right)\nu(\mathrm{d}x)\right)}{\sqrt{\log T}}\xrightarrow{\mathbb{P}}0, \tag{2.6}$$

$$\textit{where}\qquad \Sigma_T:=\frac{1}{2\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\left(\frac{\xi_n^2}{t_n}-\sigma_{t_n}^2\right).$$

Proof. — Define for every $T > 1$, the random variables

$$\Sigma_T^{(1)} := \frac{1}{\sqrt{\log T}} \sum_{n \in \mathfrak{I}_T^{\mathsf{c}}} \left(\sqrt{t_n^2 + \xi_n^2} - t_n \right),$$

$$\Sigma_T^{(2)} := \frac{1}{\sqrt{\log T}} \sum_{n \in \mathfrak{I}_T} \left(\sqrt{t_n^2 + \xi_n^2} - t_n - \frac{\xi_n^2}{2t_n} \right),$$

and $$\Sigma_T^{(3)} := \frac{1}{2\sqrt{\log T}} \left(\sum_{n \in \mathfrak{I}_T} \left(\sigma^2 - \sigma_{t_n}^2 \right) - \int_{\mathbb{R}} x^2 \log^+ \left(\min \left\{ T, x^2 \right\} \right) \nu(\mathrm{d}x) \right),$$

and note that, since $\mathbb{N} = \mathfrak{I}_T^{\mathsf{c}} \cup \mathfrak{I}_T$, (2.6) states that $\Sigma_T^{(3)} - \Sigma_T^{(1)} - \Sigma_T^{(2)} \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. It is therefore sufficient to prove that the expectations $\mathbb{E}[|\Sigma_T^{(1)}|]$, $\mathbb{E}[|\Sigma_T^{(2)}|^{1/2}]$ and $\mathbb{E}[|\Sigma_T^{(3)}|]$ all tend to 0 as $T \to \infty$.

Since

$$\sqrt{t_n^2 + \xi_n^2} - t_n \leqslant |\xi_n| \qquad \text{and} \qquad \mathbb{E}\left[|\xi_n| | \ell \right] \leqslant \mathbb{E}\left[\xi_n^2 \middle| \ell \right]^{1/2} = \sigma \sqrt{t_n},$$

by Corollary 2.2 (a),

$$\mathbb{E}\left[\left| \Sigma_T^{(1)} \right| \right] \leqslant \frac{1}{\sqrt{\log T}} \mathbb{E} \sum_{n \in \mathfrak{I}_T^{\mathsf{c}}} \mathbb{E}\left[|\xi_n| | \ell \right] \leqslant \frac{1}{\sqrt{\log T}} \mathbb{E} \sum_{n \in \mathfrak{I}_T^{\mathsf{c}}} \sigma \sqrt{t_n} \xrightarrow[T \to \infty]{} 0.$$

Taylor's theorem for the function $x \mapsto \sqrt{1 + x^2}$ around $x = 0$ applied to $\sqrt{1 + \xi_n^2 / t_n^2}$ yields

$$\left| \Sigma_T^{(2)} \right| = \frac{1}{\sqrt{\log T}} \left| \sum_{n \in \mathfrak{I}_T} \frac{\xi_n^4}{8t_n^3} \cdot \theta \left(|\xi_n| / t_n \right) \right| \leqslant \frac{1}{\sqrt{\log T}} \sum_{n \in \mathfrak{I}_T} \frac{\xi_n^4}{8t_n^3},$$

where $\theta : [0, \infty) \to [0, 1]$ is a bounded function. Recall that $\mathbb{E}[X_t^2] = \mathrm{Var}(X_t) = \sigma^2 t$ for all $t \geqslant 0$. Since $x \mapsto \sqrt{x}$ is concave and starts at 0, we have

$$\begin{aligned} \mathbb{E}\left[\left| \frac{1}{\sqrt{\log T}} \sum_{n \in \mathfrak{I}_T} \frac{\xi_n^4}{t_n^3} \right|^{1/2} \right] &\leqslant \frac{\mathbb{E} \sum_{n \in \mathfrak{I}_T} t_n^{-3/2} \xi_n^2}{\log^{1/4} T} = \frac{\mathbb{E} \sum_{n \in \mathfrak{I}_T} \mathbb{E}\left[t_n^{-3/2} \xi_n^2 \middle| \ell \right]}{\log^{1/4} T} \\ &= \sigma^2 \frac{\mathbb{E} \sum_{n \in \mathfrak{I}_T} t_n^{-1/2}}{\log^{1/4} T} = 2\sigma^2 \frac{1 - T^{-1/2}}{\log^{1/4} T} \xrightarrow[T \to \infty]{} 0, \end{aligned}$$

where the last equality follows from Corollary 2.2 (a). This implies $\mathbb{E}[|\Sigma_T^{(2)}|^{1/2}] \to 0$ as $T \to \infty$.

It remains to prove that $\mathbb{E}[|\Sigma_T^{(3)}|] \to 0$ as $T \to \infty$. Applying Corollary 2.2 (a) and Fubini's theorem, for any $T > 1$ we obtain

$$\begin{aligned}\mathbb{E} \sum_{n \in \mathfrak{I}_T} \left(\sigma^2 - \sigma_{t_n}^2\right) \\ &= \int_1^T \frac{1}{t} \int_{\mathbb{R}\setminus\left(-\kappa\sqrt{t},\kappa\sqrt{t}\right)} x^2 \nu(\mathrm{d}x)\mathrm{d}t \\ &= \int_{\mathbb{R}\setminus(-\kappa,\kappa)} \int_1^{T\wedge\left(x^2/\kappa^2\right)} \frac{\mathrm{d}t}{t} x^2\nu(\mathrm{d}x) = \int_{\mathbb{R}} x^2 \log^+\left(\min\left\{T, x^2/\kappa^2\right\}\right)\nu(\mathrm{d}x).\end{aligned}$$

Moreover, since $\kappa \geqslant 1$, we have

$$\begin{aligned}0 &\leqslant \int_{\mathbb{R}} x^2 \log^+\left(\min\left\{T, x^2\right\}\right)\nu(\mathrm{d}x) - \mathbb{E}\sum_{n \in \mathfrak{I}_T}\left(\sigma^2 - \sigma_{t_n}^2\right) \\ &= \int_{\mathbb{R}} \left(\log\left(\kappa^2\right) \mathbb{1}_{\left\{|x| < \sqrt{T}\right\}} + \log\left(T\kappa^2/x^2\right) \mathbb{1}_{\left\{\sqrt{T}\leqslant|x|<\kappa\sqrt{T}\right\}}\right) x^2\nu(\mathrm{d}x) \\ &\leqslant \log\left(\kappa^2\right) \int_{\mathbb{R}} x^2\nu(\mathrm{d}x) < \infty.\end{aligned}$$

Thus, Proposition 2.5 implies that, as $T \to \infty$,

$$\begin{aligned}\Sigma_T^{(3)} &= \frac{1}{2\sqrt{\log T}}\left(\sum_{n \in \mathfrak{I}_T}\left(\sigma^2 - \sigma_{t_n}^2\right) - \mathbb{E}\sum_{n \in \mathfrak{I}_T}\left(\sigma^2 - \sigma_{t_n}^2\right)\right) \\ &\quad + \frac{1}{2\sqrt{\log T}}\left(\mathbb{E}\sum_{n \in \mathfrak{I}_T}\left(\sigma^2 - \sigma_{t_n}^2\right) - \int_{\mathbb{R}} x^2 \log^+\left(\min\left\{T, x^2\right\}\right)\nu(\mathrm{d}x)\right) \xrightarrow{L^1} 0. \quad \square\end{aligned}$$

The conditional limit result is a key ingredient for the proof of Theorem 1.1 is the following conditional limit result.

Proposition 2.9. — *Let Σ_T be as in* (2.6) *in Proposition 2.8. Then the following conditional limit holds: for any $x \in \mathbb{R}$,*

$$\mathbb{P}\left(\Sigma_T \leqslant x|\ell\right) \xrightarrow{L^1} \Phi\left(\sqrt{2}x/\sigma^2\right), \qquad \text{as } T \to \infty, \tag{2.7}$$

where Φ is the distribution function of a standard normal random variable.

The limit law in (2.7) is $N(0, \sigma^4/2)$ and the convergence in L^1 is equivalent to the convergence in probability since the random variables $\mathbb{P}(\Sigma_T \leqslant x|\ell)$ are bounded. In particular, (2.7) implies the weak convergence $\mathbb{P}(\Sigma_T \leqslant x) \to \Phi(\sqrt{2}x/\sigma^2)$ for all $x \in \mathbb{R}$. The proof of Proposition 2.8 requires certain limit results for stick-breaking processes from Subsection 2.1 and [BGCM21a, Theorem 1.1].

Proof. — The proof of Proposition 2.9 consists of three steps.

Step 1: Let $Z \sim N(0,1)$ be independent of the stick-breaking process ℓ. Fix $r > 0$ and $\gamma \geqslant 0$, let

$$\begin{aligned}g_T(t) &:= \\ &(\log T)^{-\gamma/2}\left|\mathbb{E}\left[\left|X_t^2/t\right|^\gamma \mathbb{1}\left\{X_t^2/t \leqslant r\sqrt{\log T}\right\} - \left|\sigma_t^2 Z^2\right|^\gamma \mathbb{1}\left\{\sigma_t^2 Z^2 \leqslant r\sqrt{\log T}\right\}\right]\right|,\end{aligned}$$

for $t > 0$, where we recall that $\sigma_t^2 = \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$. In this step we establish the following limit:

$$\sum_{n\in\mathfrak{I}_T} g_T(t_n) \xrightarrow{L^1} 0, \quad \text{as } T\to\infty. \tag{2.8}$$

The integration-by-parts formula implies that for any non-negative random variable ζ and constant $a\in(0,\infty)$ we have

$$a^{-\gamma}\mathbb{E}\left[\zeta^\gamma \mathbb{1}_{\{\zeta\leqslant a\}}\right] = \mathbb{P}\left(\zeta\leqslant a\right) - \gamma\int_0^1 x^{\gamma-1}\mathbb{P}\left(\zeta\leqslant ax\right)\mathrm{d}x.$$

Applying the identity in the previous display twice yields

$$0\leqslant g_T(t)\leqslant r^\gamma K_T(t)\leqslant 2r^\gamma K(t), \quad \text{where} \tag{2.9}$$
$$\begin{aligned} K_T(t) &\coloneqq \left|\mathbb{P}\left(X_t^2/t\leqslant r\sqrt{\log T}\right) - \mathbb{P}\left(\sigma_t^2 Z^2\leqslant r\sqrt{\log T}\right)\right| \\ &\quad + \gamma\int_0^1 x^{\gamma-1}\left|\mathbb{P}\left(X_t^2/t\leqslant xr\sqrt{\log T}\right) - \mathbb{P}\left(\sigma_t Z\leqslant xr\sqrt{\log T}\right)\right|\mathrm{d}x \end{aligned}$$

and

$$K(t) := \sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(X_t/\sqrt{t}\leqslant x\right) - \mathbb{P}\left(\sigma_t Z\leqslant x\right)\right|.$$

Since the normal distribution has a bounded density, the weak limits

$$X_t/\sqrt{t}\xrightarrow{d} N\left(0,\sigma^2\right) \qquad \text{and} \qquad \sigma_t Z\xrightarrow{d} N\left(0,\sigma^2\right)$$

as $t\to\infty$ hold in the Kolmogorov distance by [Pet95, 1.8.31 & 1.8.32, p. 43], implying $\lim_{t\to\infty} K(t) = 0$. Moreover, by the dominated convergence theorem, we have $\lim_{T\to\infty} K_T(t) = 0$ and thus $\lim_{T\to\infty} g_T(t) = 0$ for all $t > 0$.

Let $\overline{\Xi}_T$ and $\overline{\Xi}_\infty$ be the coupled point processes described in Lemma 2.1 and recall that $\overline{\Xi}_T\to\overline{\Xi}_\infty$ in the vague topology and, for any $N > 1$, we have $\overline{\Xi}_\infty([1,N]) < \infty$ and $\overline{\Xi}_T|_{[1,N]} = \overline{\Xi}_\infty|_{[1,N]}$ for all sufficiently large T. By the definition of vague topology, we have $\int_{[1,\infty)} K(x)\overline{\Xi}_T(\mathrm{d}x)\to\int_{[1,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x)$ a.s. Since $g_T(t)\to 0$ as $T\to\infty$ for every atom t of $\overline{\Xi}_\infty|_{[1,N]}$, we have

$$\begin{aligned} &\limsup_{T\to\infty}\int_{[1,\infty)} g_T(x)\overline{\Xi}_T(\mathrm{d}x) \\ &\leqslant \limsup_{T\to\infty}\int_{[1,N]} g_T(x)\overline{\Xi}_T(\mathrm{d}x) + \limsup_{T\to\infty}\int_{(N,\infty)} 2r^\gamma K(x)\overline{\Xi}_T(\mathrm{d}x) \\ &= \limsup_{T\to\infty}\int_{[1,N]} g_T(x)\overline{\Xi}_\infty(\mathrm{d}x) + \int_{(N,\infty)} 2r^\gamma K(x)\overline{\Xi}_\infty(\mathrm{d}x) = 2r^\gamma\int_{(N,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x). \end{aligned}$$

By [BGCM21a, Theorem 1.1] we have $\mathbb{E}\int_{[1,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x) = \int_1^\infty t^{-1}K(t)\mathrm{d}t < \infty$. Therefore, the display above and Fatou's lemma imply, as $N\to\infty$,

$$\begin{aligned} \limsup_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T} g_T(t_n) &\leqslant \mathbb{E}\limsup_{T\to\infty}\int_{[1,\infty)} g_T(x)\overline{\Xi}_T(\mathrm{d}x) \\ &\leqslant 2r^\gamma\mathbb{E}\int_{(N,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x) = 2r^\gamma\int_N^\infty \frac{K(x)}{x}\mathrm{d}x\to 0, \end{aligned}$$

thus proving (2.8).

Step 2: Denote $S_{n,T} \coloneqq \xi_n^2/(2t_n\sqrt{\log T})$ for all $n \in \mathbb{N}$ and $T > 1$. Assume that the following limits in probability hold as $T \to \infty$:

$$\sum_{n\in\mathfrak{I}_T} \mathbb{P}_\ell\left(S_{n,T} \geqslant \epsilon\right) \xrightarrow{\mathbb{P}} 0, \qquad \text{for every } \epsilon > 0, \tag{2.10}$$

$$\sum_{n\in\mathfrak{I}_T} \mathrm{Var}_\ell\left(S_{n,T}\mathbb{1}_{\{S_{n,T}\leqslant r\}}\right) \xrightarrow{\mathbb{P}} \frac{\sigma^4}{2}, \qquad \text{for some } r > 0, \tag{2.11}$$

$$\sum_{n\in\mathfrak{I}_T} \left(\mathbb{E}_\ell\left[S_{n,T}\mathbb{1}_{\{S_{n,T}\leqslant r'\}}\right] - \frac{\sigma_{t_n}^2}{2\sqrt{\log T}}\right) \xrightarrow{\mathbb{P}} 0, \qquad \text{for some } r' > 0, \tag{2.12}$$

where we denote $\mathbb{P}_\ell(\cdot) = \mathbb{P}(\cdot|\ell)$, $\mathbb{E}_\ell[\cdot] = \mathbb{E}[\cdot|\ell]$ and $\mathrm{Var}_\ell(\cdot) \coloneqq \mathrm{Var}(\cdot|\ell)$. We now prove that (2.10)–(2.12) imply the L^1 limit in (2.7).

Since the random variables in (2.7) are bounded, it suffices to prove the limit in probability. Fix a sequence $(T_k)_{k\in\mathbb{N}}$ such that $T_k \to \infty$. By a diagonal argument, there exists a subsequence, again denoted $(T_k)_{k\in\mathbb{N}}$ for ease of notation, such that the limit in (2.10) holds for all positive rational ϵ as $T_k \to \infty$ almost surely. Thus, the limit in (2.10) holds for *all* $\epsilon > 0$ as $T_k \to \infty$ a.s. Moreover, we may assume that the limits in (2.11)–(2.12) hold a.s. as $T_k \to \infty$. Recall that, given the stick-breaking process ℓ, the variables $\{S_{n,T_k} : n \in \mathfrak{I}_{T_k}\}$ are independent, making

$$\left(\{S_{n,T_k} : n \in \mathfrak{I}_{T_k}\}\right)_{k\in\mathbb{N}}$$

a triangular array of row-wise independent random variables. Applying the CLT for triangular arrays in [Pet75, Theorem 18, Chapter IV, § 4], we deduce that (2.7) holds a.s. as $T_k \to \infty$.

Step 3: In this step we prove (2.10)–(2.12). Recall that $Z \sim N(0,1)$ is independent of ℓ. By (2.8) with $\gamma = 0$ and $r = \epsilon$, Markov's inequality and Proposition 2.3, we have

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}\sum_{n\in\mathfrak{I}_T} \mathbb{P}_\ell\left(\frac{\xi_n^2/t_n}{\sqrt{\log T}} > \epsilon\right) &= \lim_{T\to\infty} \mathbb{E}\sum_{n\in\mathfrak{I}_T} \mathbb{P}_\ell\left(\frac{\sigma_{t_n}^2 Z^2}{\sqrt{\log T}} > \epsilon\right)\\ &\leqslant \lim_{T\to\infty} \mathbb{E}\sum_{n\in\mathfrak{I}_T} \frac{\sigma_{t_n}^6\,\mathbb{E}\left[Z^6\right]}{(\epsilon\sqrt{\log T})^3} \leqslant \lim_{T\to\infty}\frac{15\sigma^6\mathbb{E}\left|\mathfrak{I}_T\right|}{\epsilon^3(\log T)^{3/2}} = 0,\end{aligned}$$

for all $\epsilon > 0$, implying (2.10) (recall that $S_{n,T} = \xi_n^2/(2t_n\sqrt{\log T})$).

To prove the limit in (2.11), first note that $|a^2 - b^2| \leqslant (a+b)|a-b|$ for $a, b \geqslant 0$, implying

$$\begin{aligned}&\left|\mathbb{E}_\ell\left[\frac{1}{2}t_n^{-1}\xi_n^2\mathbb{1}_{\{\xi_n^2\leqslant 2t_n\epsilon\sqrt{\log T}\}}\right]^2 - \mathbb{E}_\ell\left[\frac{1}{2}\sigma_{t_n}^2 Z^2\mathbb{1}_{\{\sigma_{t_n}^2 Z^2\leqslant 2\epsilon\sqrt{\log T}\}}\right]^2\right|\\ &\qquad\leqslant 2\epsilon\sqrt{\log T}\left|\mathbb{E}_\ell\left[\frac{1}{2}t_n^{-1}\xi_n^2\mathbb{1}_{\{\xi_n^2\leqslant 2t_n\epsilon\sqrt{\log T}\}}\right] - \mathbb{E}_\ell\left[\frac{1}{2}\sigma_{t_n}^2 Z^2\mathbb{1}_{\{\sigma_{t_n}^2 Z^2\leqslant 2\epsilon\sqrt{\log T}\}}\right]\right|.\end{aligned}$$

Thus, by applying (2.8) with $\gamma = 1$ and $\gamma = 2$ and $r = 2\epsilon$, we find (all limits are taken in L^1):

$$\begin{aligned}
&\lim_{T\to\infty} \frac{\sum_{n\in\mathfrak{I}_T} \mathrm{Var}_\ell\left(\frac{1}{2}t_n^{-1}\xi_n^2 \mathbb{1}_{\{\xi_n^2 \leqslant 2t_n\epsilon\sqrt{\log T}\}}\right)}{\log T} \\
&\quad = \lim_{T\to\infty} \frac{\sum_{n\in\mathfrak{I}_T} \mathrm{Var}_\ell\left(\frac{1}{2}\sigma_{t_n}^2 Z^2 \mathbb{1}_{\{\sigma_{t_n}^2 Z^2 \leqslant 2\epsilon\sqrt{\log T}\}}\right)}{\log T} = \lim_{T\to\infty} \frac{\sum_{n\in\mathfrak{I}_T} \mathrm{Var}_\ell\left(\frac{1}{2}\sigma_{t_n}^2 Z^2\right)}{\log T} \\
&\quad = \lim_{T\to\infty} \frac{1}{2\log T}\left(\sigma^4\left|\mathfrak{I}_T\right| + \sum_{n\in\mathfrak{I}_T}\left(\sigma_{t_n}^4 - \sigma^4\right)\right) = \frac{\sigma^4}{2},
\end{aligned}$$

where the first equality in the second line follows from the fact that $\mathrm{Var}(Z^2) = 2$ and the last equality in the same line follows from Proposition 2.3 and Corollary 2.2 (b) applied to the bounded function $t \mapsto \sigma_t^4 - \sigma^4$ with zero limit as $t \to \infty$. This establishes (2.11) since $S_{n,T} = \xi_n^2/(2t_n\sqrt{\log T})$.

It remains to prove (2.12). Markov's inequality, the equality $\mathbb{E}[Z^2] = 1$ and Proposition 2.3 imply

$$\begin{aligned}
&\frac{1}{\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T} \left|\mathbb{E}_\ell\left[\frac{1}{2}\sigma_{t_n}^2 Z^2 \mathbb{1}_{\{\sigma_{t_n}^2 Z^2 \leqslant 2\epsilon\sqrt{\log T}\}}\right] - \frac{\sigma_{t_n}^2}{2}\right| \\
&\quad = \frac{1}{\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T} \mathbb{E}_\ell\left[\frac{1}{2}\sigma_{t_n}^2 Z^2 \mathbb{1}_{\{\sigma_{t_n}^2 Z^2 > 2\epsilon\sqrt{\log T}\}}\right] \leqslant \frac{1}{\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T} \frac{\mathbb{E}_\ell\left[\sigma_{t_n}^8 Z^8\right]}{\left(2\epsilon\sqrt{\log T}\right)^3} \\
&\quad = \frac{1}{8\epsilon^3\log^2 T} \sum_{n\in\mathfrak{I}_T} \sigma_{t_n}^8 \mathbb{E}\left[Z^8\right] \leqslant \frac{105\sigma^8}{8\epsilon^3\log^2 T}\left|\mathfrak{I}_T\right| \xrightarrow{L^1} 0.
\end{aligned}$$

The display above and (2.8) with $\gamma = 1$ and $r = 2\epsilon$ imply (2.12), completing the proof.

□

Proof of Theorem 1.1. — The proof of Theorem 1.1 consists of several steps.

Step 1: In this step we show that (1.2) follows from the limits in (2.14) below. By [GCM22, Theorem 11], Lemma 2.6 and Proposition 2.8, the weak limit in (1.2) of Theorem 1.1 is equivalent to the following limit as $T \to \infty$:

$$\zeta_T := \left(\frac{\sum_{n\in\mathfrak{I}_T}\left(\xi_n^2/t_n - \sigma_{t_n}^2\right)}{2\sqrt{\log T}}, \frac{\left|\mathfrak{I}_T\right| - \log T}{\sqrt{\log T}}, \frac{\sum_{n=1}^{\infty}\xi_n^+}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty}\xi_n}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty} t_n \mathbb{1}_{\{\xi_n>0\}}}{T}\right) \xrightarrow{d} \zeta, \tag{2.13}$$

where $\zeta = (\sigma^2 Z_1/\sqrt{2}, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho)$, the standard Brownian motion B, the stick-breaking process ℓ and the standard normal variables Z_1 and Z_2 are all independent.

Define $\eta_n \coloneqq \xi_n/\sqrt{t_n}$ for $n \in \mathbb{N}$ and note that

$$\zeta_T = \left(\frac{\sum_{n \in \mathfrak{I}_T} \left(\eta_n^2 - \sigma_{t_n}^2\right)}{2\sqrt{\log T}}, \frac{|\mathfrak{I}_T| - \log T}{\sqrt{\log T}}, \sum_{n=1}^{\infty} \ell_n^{1/2} \eta_n^+, \sum_{n=1}^{\infty} \ell_n^{1/2} \eta_n, \sum_{n=1}^{\infty} \ell_n \mathbb{1}_{\{\eta_n > 0\}} \right).$$

Let $W_1, W_2 \ldots$ be an iid sequence of standard normal random variables independent of ℓ, Z_1 and Z_2. For $k \in \mathbb{N}$ and $T > 1$ define the random variables $\chi_{k,T}$ and χ_k as

$$\left(\frac{\sum_{n=k}^{\infty} (\eta_n^2 - \sigma_{t_n}^2) \mathbb{1}_{\{t_n \geqslant 1\}}}{2\sqrt{\log T}}, \frac{\sum_{n=k}^{\infty} \mathbb{1}_{\{t_n \geqslant 1\}} - \log T}{\sqrt{\log T}}, \sum_{n=1}^{k-1} \ell_n^{1/2} \eta_n^+, \sum_{n=1}^{k-1} \ell_n^{1/2} \eta_n, \sum_{n=1}^{k-1} \ell_n \mathbb{1}_{\{\eta_n > 0\}} \right),$$

$$\text{and} \quad \left(\frac{\sigma^2}{\sqrt{2}} Z_1, Z_2, \sum_{n=1}^{k-1} \ell_n^{1/2} \sigma W_n^+, \sum_{n=1}^{k-1} \ell_n^{1/2} \sigma W_n, \sum_{n=1}^{k-1} \ell_n \mathbb{1}_{\{\sigma W_n > 0\}} \right),$$

respectively. By [Bil99, Theorem 3.2], (2.13) will follow if we prove that the following limits hold:

$$\begin{aligned} &\text{(a) } \chi_{k,T} \xrightarrow[T \to \infty]{d} \chi_k, \\ (2.14) \qquad &\text{(b) } \chi_k \xrightarrow[k \to \infty]{d} \zeta, \\ &\text{(c) } \lim_{k \to \infty} \limsup_{T \to \infty} \mathbb{P}\left(\|\chi_{k,T} - \zeta_T\| > \epsilon\right) = 0, \quad \forall\, \epsilon > 0, \end{aligned}$$

where $\|x\| = \sum_{i=1}^d |x_i|$ denotes the ℓ^1-norm in $\mathbb{R}^d$, $d \geqslant 1$.

Step 2: In this step we establish (2.14a). Define $\ell^{(k)} \coloneqq (\ell_1, \ldots, \ell_{k-1})$. To prove (2.14a), it suffices to show that $\mathbb{E}[\phi(\chi_{k,T})|\ell^{(k)}] \to \mathbb{E}[\phi(\chi_k)|\ell^{(k)}]$ a.s. as $T \to \infty$ for any continuous and bounded function $\phi : \mathbb{R}^5 \to \mathbb{R}$. With this in mind, denote by $\mathbb{P}^{(k)}$ the conditional probability measure $\mathbb{P}$ given $\ell^{(k)}$.

Under $\mathbb{P}^{(k)}$, the process $(\ell_k, \ell_{k+1}, \ldots)$ is a uniform stick-breaking process on $[0, L_{k-1}]$ independent of the variables $(\eta_n)_{n<k}$. Thus the first two coordinates of $\chi_{k,T}$ independent under $\mathbb{P}^{(k)}$ of the last three coordinates. Moreover, since $X_t/\sqrt{t} \xrightarrow{d} \sigma Z_1$ as $t \to \infty$, then, under $\mathbb{P}^{(k)}$, we have $(\eta_1, \ldots, \eta_{k-1}) = (\xi_1/\sqrt{t_1}, \ldots, \xi_{k-1}/\sqrt{t_{k-1}}) \xrightarrow{d} (\sigma W_1, \ldots, \sigma W_{k-1})$ as $T \to \infty$ (recall that $t_n = T\ell_n$). Thus, to prove (2.14a), it suffices to show that the first two coordinates of $\chi_{k,T}$ converge weakly to the first two coordinates of χ_k under $\mathbb{P}^{(k)}$.

Recall that, under $\mathbb{P}^{(k)}$, the process $(\ell_k, \ell_{k+1}, \ldots)$ is a uniform stick-breaking process on $[0, L_{k-1}]$ and $\sum_{n=k}^{\infty} t_n = TL_{k-1}$. Thus, Proposition 2.3 implies that

$$\frac{\sum_{n=k}^{\infty} \mathbb{1}_{\{t_n \geqslant 1\}} - \log(TL_{k-1})}{\sqrt{\log(TL_{k-1})}} \xrightarrow{d} Z_2, \quad \text{as } T \to \infty \quad \text{under} \mathbb{P}^{(k)}.$$

Since $\log(TL_{k-1}) = \log T + \log L_{k-1}$, where L_{k-1} is deterministic under $\mathbb{P}^{(k)}$, then

$$M_T \coloneqq \frac{\sum_{n=k}^{\infty} \mathbb{1}_{\{t_n \geqslant 1\}} - \log T}{\sqrt{\log T}} \xrightarrow{d} Z_2, \quad \text{as } T \to \infty \text{ under } \mathbb{P}^{(k)}.$$

Moreover, since $\mathbb{P}^{(k)}(\cdot|\ell) = \mathbb{P}(\cdot|\ell)$, Proposition 2.9 implies that $\mathbb{P}^{(k)}(\Sigma_T \leqslant x|\ell) \xrightarrow{L^1} \mathbb{P}(\sigma^2 Z_1/\sqrt{2} \leqslant x)$ for all $x \in \mathbb{R}$ as $T \to \infty$, where Σ_T is as in (2.6). Denote by $\mathbb{E}^{(k)}$ the expectation under $\mathbb{P}^{(k)}$. Thus, taking limits in the following identity

$$\begin{aligned}\mathbb{E}^{(k)}\left[\mathbb{1}_{\{M_T \leqslant y\}}\mathbb{P}^{(k)}\left(\Sigma_T \leqslant x|\ell\right)\right] &= \mathbb{P}^{(k)}\left(M_T \leqslant y\right)\mathbb{P}^{(k)}\left(\sigma^2 Z_1/2 \leqslant x\right)\\ &\quad + \mathbb{E}^{(k)}\left[\mathbb{1}_{\{M_T \leqslant y\}}\left(\mathbb{P}^{(k)}\left(\Sigma_T \leqslant x|\ell\right) - \mathbb{P}^{(k)}\left(\sigma^2 Z_1/\sqrt{2} \leqslant x\right)\right)\right],\end{aligned}$$

implies $\mathbb{P}^{(k)}(M_T \leqslant y, \Sigma_T \leqslant x) \to \mathbb{P}^{(k)}(Z_2 \leqslant y)\mathbb{P}^{(k)}(\sigma^2 Z_1/\sqrt{2} \leqslant x)$ as $T \to \infty$. To see that the first two coordinates of $\chi_{k,T}$ converge weakly to those of χ_k under $\mathbb{P}^{(k)}$, note that

$$\mathbb{E}^{(k)}\sum_{n=1}^{k-1}\left|\eta_n^2 - \sigma_{t_n}^2\right|/\sqrt{\log T} \leqslant 2(k-1)\sigma^2/\sqrt{\log T} \to 0 \quad \text{as} \quad T \to \infty.$$

Step 3: In this step we establish (2.14b)–(2.14c). To prove (2.14b), it suffices to show the convergence for the last three coordinates. Note that

$$\begin{aligned}&\sum_{n=1}^{k-1}\left(\sqrt{\ell_n}\sigma W_n^+, \sqrt{\ell_n}\sigma W_n, \ell_n \mathbb{1}_{\{\sigma W_n > 0\}}\right)\\ &\qquad\qquad \xrightarrow[k\to\infty]{a.s.} \sum_{n=1}^{\infty}\left(\sqrt{\ell_n}\sigma W_n^+, \sqrt{\ell_n}\sigma W_n, \ell_n \mathbb{1}_{\{\sigma W_n > 0\}}\right),\end{aligned}$$

where the limit has the same law as $(\sigma\overline{B}_1, \sigma B_1, \rho)$ by the scaling property of Brownian motion and (1.7) applied to σB, implying (2.14b).

If we prove $\lim_{m\to\infty}\limsup_{T\to\infty}\mathbb{E}\left\|\chi_{k,T} - \zeta_T\right\| = 0$, (2.14c) will follow by Markov's inequality. Moreover, the previous limit is a consequence of the following limits

$$\limsup_{T\to\infty}\frac{\mathbb{E}\sum_{n=1}^{k-1}\left|\eta_n^2 - \sigma_{t_n}^2\right|\mathbb{1}_{\{t_n \geqslant 1\}}}{2\sqrt{\log T}} = 0, \qquad \limsup_{T\to\infty}\frac{\mathbb{E}\sum_{n=1}^{k-1}\mathbb{1}_{\{t_n \geqslant 1\}}}{2\sqrt{\log T}} = 0,$$

$$\lim_{k\to\infty}\limsup_{T\to\infty}\mathbb{E}\sum_{n=k}^{\infty}\sqrt{\ell_n}\left|\eta_n\right| = 0, \qquad \lim_{k\to\infty}\mathbb{E}\sum_{n=k}^{\infty}\ell_n = 0.$$

The first two limits in the display are obvious. The fourth limit holds since $\sum_{n=k}^{\infty}\ell_k = L_{k-1}$ and $\mathbb{E}L_{k-1} = 2^{1-k}$. Finally, the third limit in the display above follows from the bounds

$$\mathbb{E}\sum_{n=k}^{\infty}\sqrt{\ell_n}|\eta_n| \leqslant \sum_{n=k}^{\infty}\mathbb{E}\left[\sqrt{\ell_n}\mathbb{E}_\ell\left[\eta_n^2\right]^{1/2}\right] = \sigma\sum_{n=k}^{\infty}\mathbb{E}\sqrt{\ell_n} = \sigma\sum_{n=k}^{\infty}(2/3)^n = 3\sigma(2/3)^k,$$

implying (2.14c) and completing the proof.

□

Proof of Corollary 1.2. — By Theorem 1.1, it suffices to prove the claims on $\int_{\mathbb{R}} x^2\log^+(\min\{T, x^2\})\nu(\mathrm{d}x)$. Since $x^2\log^+(\min\{T, x^2\})/\log T$ tends to 0 pointwise on x as $T \to \infty$ and is upper bounded by the ν-integrable function $x \mapsto x^2$, the dominated convergence theorem implies that the integral is $\mathrm{o}(\log T)$. Similarly, the integral is $\mathrm{o}(\sqrt{\log T})$ if $x \mapsto x^2(\log^+|x|)^{1/2}$ is ν-integrable. □

3. Stable domain of attraction

This section is dedicated to proving Theorems 1.4, 1.6 and 1.7, stated in Section 1. Assume that the limit in (1.5) holds for some $\alpha \in (0,2] \setminus \{1\}$. Recall that this is equivalent to

$$(X_{tT}/a_T)_{t\in[0,1]} \xrightarrow{d} (S_\alpha(t))_{t\in[0,1]}, \qquad \text{as } T \to \infty, \tag{3.1}$$

in the Skorokhod space $\mathcal{D}[0,1]$ equipped with the J_1-topology [Bil99, Chapter 3], where a_T is as in (1.5). Since $a_T \to \infty$ as $T \to \infty$, we assume without loss of generality that $a_T > 1$ is locally bounded for all $T \geqslant 1$. The following lemma provides a key step in the proofs of Theorems 1.4 and 1.6.

Lemma 3.1. — *Suppose a Lévy process X satisfies* (3.1) *for some $\alpha \in (0,2]$. Then, for every $p \in [0,\alpha)$, there exists a constant $C_p \in (0,\infty)$ such that $\mathbb{E}[|X_t/a_t|^p] \leqslant C_p$ for all $t \geqslant 1$.*

Proof. — By the the concavity of $x \mapsto x^p$ (when $p \in [0,1]$) and Jensen's inequality (when $p \in (1,\alpha)$), we have $(a+b)^p \leqslant 2^{(p-1)^+}(a^p + b^p)$ for any $a, b \geqslant 0$. Thus, $\mathbb{E}[|X_t|^p] \leqslant 2^{(p-1)^+}(\mathbb{E}[|X_{\lfloor t\rfloor}|^p] + \mathbb{E}[|X_{t-\lfloor t\rfloor}|^p])$ for all $t \geqslant 1$, where $\lfloor t\rfloor \coloneqq \sup\{m \in \mathbb{N} : m \leqslant t\}$. By [IL71, Lemma 5.2.2], $\mathbb{E}[|X_n/a_n|^p]$ is bounded for all $n \in \mathbb{N}$. By the regular variation of $a_t \geqslant 1$, we have

$$1 \leqslant \liminf_{t\to\infty} \frac{a_t}{a_{\lfloor t\rfloor}} \leqslant \limsup_{t\to\infty} \frac{a_t}{a_{\lfloor t\rfloor}} \leqslant \limsup_{t\to\infty} \frac{a_t}{a_{ct}} = c^{-1/\alpha},$$

for any $c \in (0,1)$, implying $a_t/a_{\lfloor t\rfloor} \to 1$ as $t \to \infty$. Thus, it suffices to show that $\mathbb{E}[|X_s|^p]$ is bounded for $s \in [0,1]$. This bound follows from [GCMUB22, Lemma 2] and the inequality $\mathbb{E}[|X_s|^p] \leqslant \mathbb{E}[\overline{X}_s^p] + \mathbb{E}[|\underline{X}_s|^p]$ implied by $|X_s|^p \leqslant \max\{\overline{X}_s^p, |\underline{X}_s|^p\}$. □

Remark 3.2. — An explicit upper bound in Lemma 3.1 can be obtained in terms of the characteristics of X and the regularly varying function a_t by using methods analogous to the ones in the proof of [GCMUB22, Lemma 2]. Since the explicit value of the upper bound C_p is not important in our context, we only provide the short proof above.

3.1. The case of finite mean

Proof of Theorem 1.4. — Recall $\mathbb{P}_\ell(\cdot) = \mathbb{P}(\cdot|\ell)$ and $\mathbb{E}_\ell[\cdot] = \mathbb{E}[\cdot|\ell]$, where ℓ is the stick-breaking process on $[0,1]$, and $t_n = T\ell_n$. Denote $\eta_n \coloneqq \xi_n/a_{t_n}$ and $\varrho_n \coloneqq a_{t_n}/a_T$ for $n \in \mathbb{N}$ and note that $\sqrt{t_n^2+\xi_n^2} - t_n = \xi_n^2/(t_n + \sqrt{t_n^2+\xi_n^2})$. Thus, by (1.7), we have

$$\left(\frac{\Upsilon_T^{\frown} - T}{a_T^2/T}, \frac{\overline{C}_T^{\frown}}{a_T}, \frac{C_T^{\frown}(T)}{a_T}, \frac{\gamma_T^{\frown}}{T}\right)$$
$$\stackrel{d}{=} \sum_{n=1}^{\infty}\left(\frac{\varrho_n^2\eta_n^2}{\ell_n + \sqrt{\ell_n^2 + \varrho_n^2\eta_n^2 a_T^2/T^2}}, \varrho_n\eta_n^+, \varrho_n\eta_n, \ell_n\mathbb{1}_{\{\varrho_n\eta_n>0\}}\right).$$

By [Bil99, Theorem 3.2], (1.6) will follow if we prove the following limits: for any $k \in \mathbb{N}$,

$$\begin{aligned}(3.2)\quad &\sum_{n=1}^{k-1}\left(\frac{\varrho_n^2\eta_n^2}{\ell_n+\sqrt{\ell_n^2+\varrho_n^2\eta_n^2a_T^2/T^2}}, \varrho_n\eta_n^+, \varrho_n\eta_n, \ell_n\mathbb{1}_{\{\varrho_n\eta_n>0\}}\right)\\ &\qquad\xrightarrow[T\to\infty]{d}\sum_{n=1}^{k-1}\left(\frac{1}{2}\ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2, \ell_n^{1/\alpha}\left(S_\alpha^{(n)}\right)^+, \ell_n^{1/\alpha}S_\alpha^{(n)}, \ell_n\mathbb{1}_{\left\{S_\alpha^{(n)}>0\right\}}\right),\end{aligned}$$

and, for all $\epsilon > 0$,

$$(3.3)\quad \lim_{k\to\infty}\limsup_{T\to\infty}\mathbb{P}\left(\sum_{n=k}^{\infty}\left\|\left(R_n, \varrho_n\eta_n^+, \varrho_n\eta_n, \ell_n\mathbb{1}_{\{\varrho_n\eta_n>0\}}\right)\right\|>\epsilon\right)=0,$$

$$\text{where}\quad R_n := \frac{\varrho_n^2\eta_n^2}{\ell_n+\sqrt{\ell_n^2+\varrho_n^2\eta_n^2a_T^2/T^2}},$$

and $\|x\| = \sum_{i=1}^d |x_i|$ denotes the ℓ^1-norm in $\mathbb{R}^d$, $d \geqslant 1$.

To prove (3.2), it suffices to show that the weak convergence holds conditional on ℓ. By assumption, we have $X_t/a_t \xrightarrow{d} S_\alpha^{(1)}$, $a_{ct}/a_t \to c^{1/\alpha}$ and $a_t/t \to 0$ as $t \to \infty$. Thus, given ℓ, the random variables $\eta_1, \ldots, \eta_k$ are independent and we have the following convergences as $T \to \infty$: $(\eta_1, \ldots, \eta_k) \xrightarrow{d} (S_\alpha^{(1)}, \ldots, S_\alpha^{(k)})$, $(\varrho_1, \ldots, \varrho_k) \to (\ell_1^{1/\alpha}, \ldots, \ell_k^{1/\alpha})$ and $a_T/T \to 0$. The continuous mapping theorem then yields the weak convergence in (3.2) conditional on ℓ.

Next we prove (3.3). Note that $\sum_{n=k}^{\infty}\ell_k = L_{k-1}$ and $\mathbb{P}(L_{k-1} > \epsilon) \leqslant \epsilon^{-1}\mathbb{E}L_{k-1} \to 0$ as $k \to \infty$, so it suffices to show that, for all $\epsilon > 0$, the following limits hold as $k \to \infty$:

$$(3.4)\qquad \limsup_{T\to\infty}\mathbb{P}\left(\sum_{n=k}^{\infty}R_n>\epsilon\right)\to 0,\quad \limsup_{T\to\infty}\mathbb{P}\left(\sum_{n=k}^{\infty}\varrho_n|\eta_n|>\epsilon\right)\to 0.$$

We will prove both limits via Markov's inequality $\mathbb{P}(|\zeta| > \epsilon) \leqslant \epsilon^{-p}\mathbb{E}[|\zeta|^p]$ for $p > 0$, and bounding the first moment by splitting the summation over the sets $\mathfrak{I}_T$ and $\mathfrak{I}_T^{\mathsf{c}}$ (recall that $\mathfrak{I}_T = \{n \in \mathbb{N} : t_n \geqslant 1\}$). First note that $R_n \leqslant |\xi_n|(T/a_T^2)$ and $\rho_n|\eta_n| = |\xi_n|/a_T$, where $a_T \to \infty$ and $a_T^2/T \to \infty$ as $T \to \infty$. There exists a constant K such $\mathbb{E}[|X_t|] \leqslant K\sqrt{t}$ for all $t \leqslant 1$ (see, e.g. [GCMUB22, Lemma 2]), so Corollary 2.2(a) yields

$$\limsup_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T^{\mathsf{c}},n\geqslant k}\varrho_n\mathbb{E}_\ell|\eta_n| \leqslant \limsup_{T\to\infty}\frac{K}{a_T}\mathbb{E}\sum_{n\in\mathfrak{I}_T^{\mathsf{c}}}\ell_n^{1/2} = \limsup_{T\to\infty}\frac{2K}{a_T}=0,$$

and

$$\limsup_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T^{\mathsf{c}},n\geqslant k}R_n \leqslant \limsup_{T\to\infty}\frac{KT}{a_T^2}\mathbb{E}\sum_{n\in\mathfrak{I}_T^{\mathsf{c}}}\ell_n^{1/2} = \limsup_{T\to\infty}\frac{2KT}{a_T^2}=0.$$

It remains to consider the summation sets $\mathfrak{I}_T \cap \{k, k+1, \ldots\}$. By Lemma 3.1, for any $p \in (0, \alpha)$, we have $\mathbb{E}_\ell[|\eta_n|^p] \leqslant C_p$ for some $C_p > 0$. Since $t \mapsto a_t$ is regularly varying at infinity with index $1/\alpha$, Potter's theorem [BGT87, Theorem 1.5.6] implies that for all $q \in (0,1/\alpha)$ there exists a constant $C'_q > 0$ such that $a_s/a_t \leqslant C'_q(s/t)^q$ for all $t > s \geqslant 1$. Thus, the second limit in (3.4) follows from the limit

$$\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T, n\geqslant k} \varrho_n \mathbb{E}_\ell|\eta_n| \leqslant C_1 C'_{1/2} \sum_{n=k}^{\infty} \mathbb{E}\left[\ell_n^{1/2}\right] = 3C_1C'_{1/2}(2/3)^{k-1} \xrightarrow[k\to\infty]{} 0.$$

Fix any $p \in (0,\alpha/2)$ and $q \in (1/2, 1/\alpha)$ and note that $R_n \leqslant \varrho_n^2\eta_n^2/\ell_n$. By Markov's inequality and the subadditivity of $x \mapsto x^p$, the first limit in (3.4) follows from

$$\begin{aligned}\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T, n\geqslant k} R_n^p &\leqslant C_{2p}\left(C'_q\right)^{2p} \sum_{n=k}^{\infty} \mathbb{E}\left[\ell_n^{p(2q-1)}\right] \\ &= \frac{C_{2p}\left(C'_q\right)^{2p}(1+p(2q-1))^{1-k}}{p(2q-1)} \xrightarrow[k\to\infty]{} 0. \quad \square\end{aligned}$$

Asymptotic equivalence $f(x) \sim g(x)$ as $x \to \infty$ is defined as $\lim_{x\to\infty} f(x)/g(x) = 1$.

Proof of Proposition 1.5. — Note that $Q := \frac{1}{2}\sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2$ satisfies

$$\begin{aligned}2Q &= \ell_1^{2/\alpha-1}\left(S_\alpha^{(1)}\right)^2 + \sum_{i=2}^{\infty} \ell_n^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2 \\ &= \ell_1^{2/\alpha-1}\left(S_\alpha^{(1)}\right)^2 + L_1^{2/\alpha-1}\sum_{i=2}^{\infty}\left(\frac{\ell_n}{L_1}\right)^{2/\alpha-1}\left(S_\alpha^{(n)}\right)^2.\end{aligned}$$

Let $A := L_1^{2/\alpha-1}$, $B := \frac{1}{2}\ell_1^{2/\alpha-1}(S_\alpha^{(1)})^2$ and $Q' := \frac{1}{2}\sum_{i=2}^{\infty}(\ell_n/L_1)^{2/\alpha-1}(S_\alpha^{(n)})^2$ and note that $Q = AQ' + B$. Since $(\ell_n/L_1)_{n\geqslant 2}$ is a stick-breaking process on $[0,1]$ independent of L_1 and $S_\alpha^{(1)}$, we conclude that $Q' \overset{d}{=} Q$ is independent of (A,B).

By [BDM16, Theorem 2.4.3] it follows that $\mathbb{P}(Q > x) \sim (1-\mathbb{E}[A^{\alpha/2}])^{-1}\mathbb{P}(B > x)$, as $x \to \infty$. Furthermore, by [BDM16, Lemma B.5.1], we have

$$\mathbb{P}(B > x) \sim \mathbb{E}\left[\left(\frac{1}{2}\ell_1^{2/\alpha-1}\right)^{\alpha/2}\right]\mathbb{P}\left(\left(S_\alpha^{(1)}\right)^2 > x\right), \quad \text{as } x \to \infty.$$

Recall that $L_1 = 1 - \ell_1 \sim U(0,1)$. Similarly, we have that $\ell_1 \sim U(0,1)$. Thus, it follows that

$$\begin{aligned}\left(1-\mathbb{E}\left[A^{\alpha/2}\right]\right)^{-1}\mathbb{E}\left[\left(\frac{1}{2}\ell_1^{2/\alpha-1}\right)^{\alpha/2}\right] &= 2^{-\alpha/2}\left(1-\mathbb{E}\left[V_1^{1-\alpha/2}\right]\right)^{-1}\mathbb{E}\left[V_1^{1-\alpha/2}\right] \\ &= 2^{-\alpha/2}\left(1-\frac{2}{4-\alpha}\right)^{-1}\frac{2}{4-\alpha} = \frac{2^{1-\alpha/2}}{2-\alpha}.\end{aligned}$$

Thus we have $\mathbb{P}(Q > x) \sim 2^{1-\alpha/2}\mathbb{P}((S_\alpha^{(1)})^2 > x)/(2-\alpha)$, as $x \to \infty$. The last asymptotic equivalence in Proposition 1.5 follows from the identity $\mathbb{P}((S_\alpha^{(1)})^2 > x) = \mathbb{P}(S_\alpha^{(1)} > \sqrt{x}) + \mathbb{P}(-S_\alpha^{(1)} > \sqrt{x})$. $\square$

Proof of Theorem 1.6. —
(a) Assume $\mu > 0$. We assume without loss of generality that $t \mapsto a_t$ is continuous and $a_t \geqslant 1$ for all $t > 0$. Define

$$\begin{aligned} Z_T &\coloneqq \left(\frac{\mu}{\sqrt{1+\mu^2}}, 1, 1\right) \frac{X_T - \mu T}{a_T}, \\ Z'_T &\coloneqq \frac{1}{a_T}\left(\Upsilon_T^{\frown} - \sqrt{1+\mu^2}T, X_T - \mu T, \overline{X}_T - \mu T\right). \end{aligned}$$

Since $Z_T \xrightarrow{d} (\mu/\sqrt{1+\mu^2}, 1, 1)S_\alpha(1)$ as $T \to \infty$, it suffices to show that $\|Z_T - Z'_T\| \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Define

$$\Delta_T \coloneqq \Upsilon_T^{\frown} - \sqrt{1+\mu^2}T - \frac{\mu}{\sqrt{1+\mu^2}}\left(X_T - \mu T\right), \qquad T > 0.$$

Note that $|\underline{X}_T/a_T| \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$ since the positive drift $\mu > 0$ implies that $-\underline{X}_T \to -\underline{X}_\infty < \infty$ a.s. as $T \to \infty$. Since $\|Z'_T - Z_T\| = a_T^{-1}\|(\Delta_T, 0, (\overline{X}_T - X_T)/a_T)\|$, and $\overline{X}_T - X_T \overset{d}{=} -\underline{X}_T$, part (a) will follow if we show that $\Delta_T/a_T \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$.

By (1.7), we have

$$\left(\Upsilon_T^{\frown} - T, X_T - \mu T\right) \overset{d}{=} \sum_{n=1}^{\infty}\left(\sqrt{t_n^2 + \xi_n^2} - t_n, \widetilde{\xi}_n\right),$$

where we define $\widetilde{\xi}_n \coloneqq \xi_n - \mu t_n$. Thus we have $\Delta_T \overset{d}{=} \sum_{n \in \mathbb{N}} \zeta_n$, where

$$\begin{aligned} \zeta_n &\coloneqq \sqrt{t_n^2 + \xi_n^2} - \sqrt{1+\mu^2}t_n - \frac{\mu}{\sqrt{1+\mu^2}}\widetilde{\xi}_{t_n} \\ &= \sqrt{1+\mu^2}t_n\left(\left(1 + \frac{\widetilde{\xi}_n^2 + 2\mu t_n \widetilde{\xi}_n}{t_n^2(1+\mu^2)}\right)^{1/2} - 1 - \frac{\mu}{1+\mu^2}\frac{\widetilde{\xi}_n}{t_n}\right). \end{aligned}$$

To prove that $\Delta_T/a_T \xrightarrow{\mathbb{P}} 0$, we again split the summation set with $\mathfrak{I}_T$ and $\mathfrak{I}_T^c$. Define: $\Delta_T^{(1)} \coloneqq \sum_{n \in \mathfrak{I}_T} \zeta_n$ and $\Delta_T^{(2)} \coloneqq \sum_{n \in \mathfrak{I}_T^c} \zeta_n$ and note that $\Delta_T \overset{d}{=} \Delta_T^{(1)} + \Delta_T^{(2)}$.

Fix some $p \in (0, \alpha/2)$ and use the inequality $\sqrt{1+z} \leqslant 1 + z/2$ for $z \geqslant -1$ and the subadditivity of $x \mapsto x^p$ to obtain

$$\mathbb{E}\left[\left|\Delta_T^{(1)}/a_T\right|^p\right] \leqslant \mathbb{E}\left[\left|\sum_{n \in \mathfrak{I}_T} \frac{\widetilde{\xi}_n^2}{2a_T\sqrt{1+\mu^2}t_n}\right|^p\right] \leqslant \mathbb{E}\sum_{n \in \mathfrak{I}_T} \frac{\left|\widetilde{\xi}_n\right|^{2p}}{a_T^p t_n^p}.$$

Recall that $(X_t - \mu t)/a_t \xrightarrow{d} S_\alpha(1)$ as $t \to \infty$. Thus, by Lemma 3.1, there exists a constant $C_{2p} > 0$ such that $\mathbb{E}[|X_t - \mu t|^{2p}] \leqslant C_{2p}a_t^{2p}$ for all $t \geqslant 1$. Therefore $\mathbb{E}_\ell[|\widetilde{\xi}_n|^{2p}] \leqslant C_{2p}a_{t_n}^{2p}$ for $n \in \mathfrak{I}_T$.

Suppose $\alpha \in (1, 2)$. Pick $q \in (1/2, 1/\alpha)$ and apply Potter's Theorem [BGT87, Theorem 1.5.6] to obtain $a_t/a_T \leqslant C'_q(t/T)^q$ for all $T > t \geqslant 1$ and some $C'_q > 0$. Thus, Corollary 2.2 (a) yields

$$\begin{aligned}\mathbb{E}\left[\left|\Delta_T^{(1)}/a_T\right|^p\right] &\leqslant C_{2p}\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T}\frac{a_{t_n}^{2p}}{a_T^p t_n^p} = C_{2p}\left(\frac{a_T}{T}\right)^p\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T}\ell_n^{-p}\left(\frac{a_{t_n}}{a_T}\right)^{2p}\\ &\leqslant C_{2p}\left(C_q'\right)^{2p}\left(\frac{a_T}{T}\right)^p\mathbb{E}\sum_{n=1}^{\infty}\ell_n^{p(2q-1)} = \frac{C_{2p}\left(C_q'\right)^{2p}}{p(2q-1)}\left(\frac{a_T}{T}\right)^p,\end{aligned}$$

which tends to 0 as $T\to\infty$, implying $\Delta_T^{(1)}/a_T \xrightarrow{\mathbb{P}} 0$.

Suppose $\alpha=2$. We may assume $a_t=\sqrt{t}l(t)$ for a locally bounded and slowly varying function l. Thus, by [BGT87, Proposition 1.5.9a], $\widetilde{l}(T):=\int_1^T t^{-1}l(t)^{2p}\mathrm{d}t$ is also slowly varying and Corollary 2.2 (a) yields

$$\mathbb{E}\left[\left|\Delta_T^{(1)}/a_T\right|^p\right] \leqslant C_{2p}\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T}\frac{a_{t_n}^{2p}}{a_T^p t_n^p} = C_{2p}\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T}\frac{l(t_n)^{2p}}{a_T^p} = C_{2p}\frac{\widetilde{l}(T)}{a_T^p}\xrightarrow[T\to\infty]{} 0.$$

It remains to show that $\Delta_T^{(2)}/a_T \xrightarrow{\mathbb{P}} 0$ as $T\to\infty$. The inequality $\sqrt{1+x+y}\geqslant 1+y/2$ for $x\geqslant y^2/4$ and $x+y\geqslant -1$ shows that $\Delta_T^{(2)}\geqslant 0$ a.s. By the subadditivity of $x\mapsto\sqrt{x}$, we obtain

$$\begin{aligned}\frac{1}{a_T}\mathbb{E}\left[\left|\Delta_T^{(2)}\right|\right] &\leqslant \frac{\sqrt{1+\mu^2}}{a_T}\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T^{\mathrm{c}}} t_n\left|\frac{\widetilde{\xi}_n}{t_n\sqrt{1+\mu^2}}+\frac{\left(2|\mu|\left|\widetilde{\xi}_n\right|\right)^{1/2}}{\sqrt{t_n\left(1+\mu^2\right)}}-\frac{\mu}{1+\mu^2}\frac{\widetilde{\xi}_n}{t_n}\right|\\ &\leqslant \frac{1}{a_T}\mathbb{E}\sum_{n\,\in\,\mathfrak{I}_T^{\mathrm{c}}}\left(\left(1-\frac{\mu}{\sqrt{1+\mu^2}}\right)\left|\widetilde{\xi}_n\right|+\sqrt{2|\mu|t_n\left|\widetilde{\xi}_n\right|}\right).\end{aligned}$$

By [GCMUB22, Eq. (24)] and Jensen's inequality, there exists a constant $C>0$ such that $\mathbb{E}[|X_t-\mu t|]\leqslant C\sqrt{t}$ for all $t\leqslant 1$. Thus, Corollary 2.2 (a) yields $\Delta_T^{(2)}/a_T \xrightarrow{L^1} 0$ as $T\to\infty$, completing the proof of part (a).

(b) Note that $\overline{X}_T\to\overline{X}_\infty<\infty$ a.s. and $\gamma_T^{\frown}\to\gamma_\infty^{\frown}<\infty$ a.s. as $T\to\infty$. We next split the length of the concave majorant in two at the time of the supremum, so the total length $\Upsilon_T^{\frown}$ up to time T is equal to the sum of the length $\Delta_T^{(1)}$ up to time $\gamma_T^{\frown}$ and the length $\Delta_T^{(2)}$ from $\gamma_T^{\frown}$ to T. It follows that $\Delta_T^{(1)}\to\Upsilon_{\gamma_\infty^{\frown}}^{\frown}$ a.s. as $T\to\infty$, implying $\Delta_T^{(1)}/a_T\to 0$ a.s. Thus, it suffices to consider $\Delta_T^{(2)}$ for the weak limit of $\Upsilon_T^{\frown}$. Since the post-supremum process is independent of the pre-supremum process by [Ber93, Theorem 2.3], as in part (a) we conclude that, as $T\to\infty$,

$$\begin{aligned}\left(\frac{\Delta_T^{(2)}-\left(T-\gamma_T^{\frown}\right)}{a_T},\frac{\left(C_T^{\frown}(T)-\overline{X}_T\right)-\mu\left(T-\gamma_\infty^{\frown}\right)}{a_T}\right)&\Bigg|\left(\overline{X}_\infty,\gamma_\infty^{\frown}\right)\\ &\xrightarrow{d}\left(\frac{\mu}{\sqrt{1+\mu^2}},1\right)S_\alpha(1).\end{aligned}$$

Note here that the limit law does not depend on $(\overline{X}_\infty,\gamma_\infty^{\frown})$, so the limit is independent of $(\overline{X}_\infty,\gamma_\infty^{\frown})$. Since we also have $|\overline{X}_\infty-\overline{X}_T|\to 0$ and $|\gamma_\infty^{\frown}-\gamma_T^{\frown}|\to 0$ a.s. as $T\to\infty$, the result follows. □

3.2. Sandwiching the concave majorant

When the tails of X are sufficiently heavy for it not to have the first moment, the asymptotic behaviour of the boundary of its convex hull is straightforward.

Proof of Theorem 1.7. — The supremum, infimum and the times at which they are attained are functionals that are continuous a.s. in J_1-topology with respect to the law of an α-stable process, since the times at which the extrema are attained are a.s. unique (see [Kal02, Lemma 14.12] and [PUB12, Theorem 2]). Thus, by the continuous mapping theorem, it suffices to prove $|\Upsilon_T^{\frown} - (2\overline{C}_T^{\frown} - C_T^{\frown}(T))|/a_T \to 0$ and $|\Upsilon_T^{\smile} - (C_T^{\smile}(T) - 2\underline{C}_T^{\smile})|/a_T \to 0$ a.s. as $T \to \infty$. Recall $X_T = C_T^{\frown}(T) \leqslant \overline{X}_T = \overline{C}_T^{\frown}$ and $\gamma_T^{\frown} \in [0,T]$. Hence, by Figure 1.2, the following inequalities hold:

$$\begin{aligned} 2\overline{X}_T - X_T &\leqslant \left((\gamma_T^{\frown})^2 + \left(\overline{X}_T\right)^2\right)^{1/2} + \left((T-\gamma_T^{\frown})^2 + \left(\overline{X}_T - X_T\right)^2\right)^{1/2} \\ &\leqslant \Upsilon_T^{\frown} \leqslant 2\overline{X}_T - X_T + T. \end{aligned}$$

Since $\alpha \in (0,1)$ we have $\lim_{T\to\infty} T/a_T = 0$, implying $|\Upsilon_T^{\frown} - (2\overline{C}_T^{\frown} - C_T^{\frown}(T))|/a_T \to 0$ a.s. as $T \to \infty$. The proof of the second limit is analogous. ☐

Proof of Proposition 1.8. —
(a) & (b) In part (a), define $a_T := \sqrt{T}$ for all $T > 0$. Note that $\Upsilon_T^{\sqcap} - T = 2\overline{X}_T - X_T$ and

$$\Upsilon_T^{\wedge} - T = \left(\sqrt{(\gamma_T^{\frown})^2 + \overline{X}_T^2} - \gamma_T^{\frown}\right) + \left(\sqrt{(T-\gamma_T^{\frown})^2 + \left(\overline{X}_T - X_T\right)^2} - (T-\gamma_T^{\frown})\right).$$

We will show that

$$\frac{T}{a_T^2}\left|\Upsilon_T^{\wedge} - T - \frac{\overline{X}_T^2}{2\gamma_T^{\frown}} - \frac{\left(\overline{X}_T - X_T\right)^2}{2\left(T-\gamma_T^{\frown}\right)}\right| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } T\to\infty. \tag{3.5}$$

The conclusions of parts (a) & (b) will then follow from (3.5), an application of the continuous mapping theorem and Theorems 1.1 & 1.4, respectively.

To prove (3.5), by symmetry, it suffices to establish, as $T\to\infty$, the limit

$$Ta_T^{-2}\left|\left((\gamma_T^{\frown})^2 + \overline{X}_T^2\right)^{1/2} - \gamma_T^{\frown} - \overline{X}_T^2/\left(2\gamma_T^{\frown}\right)\right| \xrightarrow{\mathbb{P}} 0.$$

Taylor's theorem yields $\sqrt{1+x^2} = 1 + x^2/2 + x^4\theta(|x|)/8$, where $\theta : [0,\infty) \to [0,1]$ is a bounded function. Thus, the limit in probability is implied by the limit $Ta_T^{-2}\overline{X}_T^4/(\gamma_T^{\frown})^3 \xrightarrow{\mathbb{P}} 0$ as $T\to\infty$, which is itself a direct consequence of the fact that $a_T/T \to 0$, the continuous mapping theorem and the weak limits

$$\gamma_T^{\frown}/T \xrightarrow{d} \gamma^{\alpha\frown} \quad \text{and} \quad \overline{X}_T/a_T \xrightarrow{d} \overline{S}_\alpha(1) \quad \text{as} \quad T\to\infty.$$

(c) The proof follows as in the proof of Theorem 1.7, using the triangle inequality to obtain

$$2\overline{X}_T - X_T \leqslant \Upsilon_T^{\wedge} \leqslant \Upsilon_T^{\sqcap} = T + 2\overline{X}_T - X_T,$$

and then using the fact that $T/a_T \to 0$ as $T\to\infty$. ☐

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