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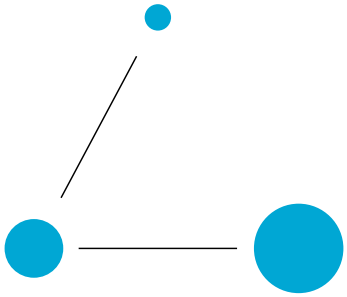

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# ON THE ALGEBRAIC DEPENDENCE OF HOLONOMIC FUNCTIONS

## SUR LA DÉPENDANCE ALGÉBRIQUE DES FONCTIONS HOLONOMES

Abstract. — We study the form of possible algebraic relations between functions satisfying linear differential equations. In particular, if $f$ and $g$ satisfy linear differential equations and are algebraically dependent, we give conditions on the differential Galois group associated to $f$ guaranteeing that $g$ is a polynomial in $f$. We apply this to hypergeometric functions and iterated integrals.

Résumé. — Nous nous intéressons aux relations algébriques vérifiées par des solutions d'équations différentielles linéaires. En particulier, si $f$ et $g$ satisfont des équations différentielles linéaires et sont algébriquement dépendantes, nous donnons des conditions sur le groupe de Galois différentiel de $f$ garantissant que $g$ est un polynôme en $f$. Nous appliquons cela aux fonctions hypergéométriques et aux intégrales itérées.

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## 1. Introduction

Let $K$ be a differential field of characteristic zero and let $k$ be an algebraically closed differential subfield of $K$ with the same field of constants $C$ :

$$C = \{f \in K \mid f' = 0\} = \{f \in k \mid f' = 0\}.$$

A typical example is the field of Puiseux series $K = \overline{C((x))}$ over the algebraically closed field $C$ endowed with the usual derivation $d/dx$ and $k = \overline{C(x)}$ the field of algebraic functions over $C(x)$.

The present paper is concerned with the following general question.

Question. — *We let $f$ and $g$ be elements of $K^\times$ satisfying some nontrivial homogeneous linear differential equations over $k$, say*

$$L(f) = 0 \quad \textit{and} \quad M(g) = 0,$$

*and we assume that $f$ and $g$ are algebraically dependent over $k$. What can be said about the form of the algebraic dependence?*

Differential Galois theory has been an effective tool in understanding the possible algebraic relations among solutions of linear differential equations (see, for example, [BBH88, HS85, HS86, Kol68, Roq14, Sin86, Spe86]). A celebrated illustration of this is the following result which is a special case of the Kolchin–Ostrowski Theorem characterizing the algebraic relations among exponentials and integrals. This answers Question 1 when $L$ and $M$ are first order equations:

Theorem 1.1 (Kolchin [Kol68]). — *If $L$ and $M$ have order 1, then $f$ and $g$ are algebraically dependent over $k$ if and only if there exists $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that*

$$f^m g^n \in k.$$

The present work started with the following question: what can be said if we assume that $L$ has order 1, but do not make any assumption on the order of $M$? Our answer reads as follows:

Theorem 1.2 (Theorem 3.2 in Section 3). — *If $L$ has order 1 and if $f \notin k$, then $f$ and $g$ are algebraically dependent over $k$ if and only if there exists $\theta$ in an algebraic extension of $K$ and a positive integer $n$ such that*

$$f = \theta^n \text{ and } g \in k\left[\theta, \theta^{-1}\right].$$

The rest of the paper is devoted to the case when the operator $L$ has order $\geqslant 2$. For any $f \in K^\times$ as above, we let $\mathcal{A}(f)$ be the $k$-algebra made up of the elements $g$ of $K$ holonomic and algebraically dependent on $f$ over $k$. In general, the obvious inclusion

$$k[f] \subset \mathcal{A}(f)$$

is not an equality. For example, let $K = \mathbb{C}(x)$ endowed with its usual derivation $d/dx$, $k = \mathbb{C}$, $f = x^2$ and $g = x$. Both $f$ and $g$ are holonomic over $\mathbb{C}$ and clearly algebraically dependent. Nonetheless $g \notin k[f]$.

The principal aim of Sections 4, 5 and 6 is to give criteria on the differential Galois group of $L$ over $k$ ensuring that $\mathcal{A}(f) = k[f]$. In this introduction, we only state the

following concrete consequence of these criteria and refer to Sections 4, 5 and 6 for more general statements.

THEOREM 1.3 (Consequence of Proposition 4.3 in Section 4 and Proposition 5.2 in Section 5). — *Assume that $L$ has order $n \geqslant 2$ and that its differential Galois group $G_L$ over $k$ is either* $\mathrm{SL}_n(C)$ *or* $\mathrm{GL}_n(C)$. *Then, we have*

$$\mathcal{A}(f) = k[f].$$

*Remark 1.4.* —

(1) We note that the hypothesis that $k$ is algebraically closed implies that the differential Galois group $G_L$ of any linear differential equation $L(y) = 0$ over $k$ is connected. Indeed, according to [S03, Proposition 1.34(2)], the quotient $H = G_L/G_L^\circ$ of $G_L$ by its identity component $G_L^\circ$ is isomorphic to the differential Galois group of some linear differential equation $P(y) = 0$ over $k$. But, since $H$ is finite, [S03, Proposition 1.34(3)] ensures that any Picard–Vessiot extension of $P(y) = 0$ over $k$ is a finite extension of $k$ and, hence, is equal to $k$. So, $H$ is reduced to the identity and $G_L = G_L^\circ$ as expected.
(2) The hypothesis that $k$ is algebraically closed also implies that the differential Galois group $G_L$ cannot be an intermediate subgroup between $\mathrm{SL}_n(C)$ and $\mathrm{GL}_n(C)$. Indeed, if $\mathrm{SL}_n(C) \subset G_L \subsetneq \mathrm{GL}_n(C)$, then $G_L/\mathrm{SL}_n(C)$ is finite and, hence, $\mathrm{SL}_n(C)$ contains the identity component $G_L^\circ$ of $G_L$. We thus have $G_L^\circ \subset \mathrm{SL}_n(C) \subset G_L$. But, we have seen that $G_L$ is connected, so $G_L^\circ = \mathrm{SL}_n(C) = G_L$ as expected.
(3) When $n = 2$, the fact that the differential Galois group over $k$ of $L$ is either $\mathrm{SL}_2(C)$ or $\mathrm{GL}_2(C)$ is equivalent to the irreducibility of $L$ over $k$ [NvdPT08]. We thus have the following statement : If $L$ has order 2 and is irreducible over $k$ then $\mathcal{A}(f) = k[f]$.
(4) Another simple consequence of Theorem 1.3 is: Assume that $L$ has order $n \geqslant 2$ and that its differential Galois group $G_L$ over $k$ is either $\mathrm{SL}_n(C)$ or $\mathrm{GL}_n(C)$. If $f \notin k$, then, for $m \geqslant 2$, $f^{1/m}$ does not satisfy any nontrivial linear differential equation over $k$. This follows easily form the fact that we cannot have $f^{1/m} \in k[f]$ unless $f$ is algebraic over $k$ and, hence, belongs to $k$. For related results, see [HS86, Sin86, Spe86].
(5) In [Kol68], Kolchin proves a general result characterizing when entries of fundamental matrices of linear differential equations having Galois groups $\mathrm{SL}_n(C)$ are algebraically dependent. His conclusion is that the entries of one such matrix can be expressed as $k$-linear combinations of the entries of the other matrix. In terms of scalar equations this means that the solutions and their derivatives of one equation can be expressed as $k$-linear combinations of solutions and their derivatives of the other equation. Our results in this paper start with the stronger hypothesis that one solution is algebraically dependent on a solution of the other equation and determines the form of the dependence without the intervention of derivatives.

Theorem 1.3 applies, for example, to many generalized hypergeometric series

$$ {}_pF_q\left[\boldsymbol{\alpha};\boldsymbol{\beta};x\right]=\sum_{k=0}^{+\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{x^k}{k!}\in\mathbb{C}((x)) $$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_p)\in\mathbb{C}^p$ and $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_q)\in(\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0})^q$ for some $p,q\in\mathbb{Z}_{\geqslant 0}$ and where the Pochhammer symbols $(t)_k$ are defined by $(t)_0=1$ and, for $k\in\mathbb{Z}_{\geqslant 1}$, $(t)_k=t(t+1)\cdots(t+k-1)$. Indeed, this series satisfies the generalized hypergeometric differential equation

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]\left({}_pF_q\left[\boldsymbol{\alpha};\boldsymbol{\beta};x\right]\right)=0 $$

where

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]=\delta\prod_{k=1}^{q}(\delta+\beta_k-1)-x\prod_{k=1}^{p}(\delta+\alpha_k) $$

with $\delta=x\frac{d}{dx}$ and the work of Beukers, Brownawell and Heckman [BBH88], Beukers and Heckman [BH89], Duval and Mitschi [DM89], Katz [Kat87, Kat90] and Mitschi [Mit96] give explicit conditions (holding for generic parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$) ensuring that the differential Galois group over $\overline{\mathbb{C}(x)}$ of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]$ is either $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$ (in these references, the differential Galois groups are computed over $\mathbb{C}(x)$, not $\overline{\mathbb{C}(x)}$; this is not a problem since if the differential Galois group over $\mathbb{C}(x)$ is connected then it does not change when one replaces the base field with $\overline{\mathbb{C}(x)}$ [Mag94, Proposition 5.28]. For instance, it follows from [Kat90, Theorem 3.6] that, for any $\beta\in\mathbb{Q}\setminus\mathbb{Z}_{\leqslant 0}$, the differential Galois group over $\overline{\mathbb{C}(x)}$ of ${}_0\mathscr{H}_1\left[-;\beta\right]$ is $\mathrm{SL}_2(\mathbb{C})$; so, for $K=\overline{\mathbb{C}((x))}$ endowed with the derivation $d/dx$ and $k=\overline{\mathbb{C}(x)}$, Theorem 1.3 ensures that:

Corollary 1.5. — *For any $\beta\in\mathbb{Q}\setminus\mathbb{Z}_{\leqslant 0}$, we have*

$$ \mathcal{A}\left({}_0F_1\left[-;\beta;x\right]\right)=\overline{\mathbb{C}(x)}\left[{}_0F_1\left[-;\beta;x\right]\right]. $$

Moreover, we show in Section 7.1 that our methods (combined with the local formal theory of differential equations) can be used in order to examine in detail the algebraic dependence relations between hypergeometric series. For instance, Corollary 1.5 ensures that if some generalized hypergeometric series ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ is algebraically dependent on ${}_0F_1\left[-;\beta;x\right]$, then it is a polynomial in ${}_0F_1\left[-;\beta;x\right]$; in Section 7.1, we establish a much more precise statement:

Theorem 1.6. — *Consider $\beta\in\mathbb{Q}\setminus\mathbb{Z}_{\leqslant 0}$. Consider $\boldsymbol{\gamma}\in(\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0})^p$ and $\boldsymbol{\delta}\in(\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0})^q$. If ${}_0F_1\left[-;\beta;x\right]$ and ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ are algebraically dependent over $\overline{\mathbb{C}(x)}$, then*

- *either*

$$ q=p+1\quad\text{and}\quad {}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]\in\mathbb{C}(x){}_0F_1\left[-;\beta;x\right]+\mathbb{C}(x); $$

- *or*

$$ q=p-1\quad\text{and}\quad {}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]\in\overline{\mathbb{C}(x)}. $$

*Remark 1.7.* — Whether or not the second condition holds can be decided by using Beukers and Heckman's [BH89] provided that $\gamma_i - \delta_j \notin \mathbb{Z}$ for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$.

As already mentioned, Theorem 1.3 is a consequence of results contained in Sections 4, 5 and 6, where the hypothesis that the differential Galois group $G_L$ of $L$ over $k$ is either $\mathrm{SL}_n(C)$ or $\mathrm{GL}_n(C)$ is replaced by the hypothesis that $G_L$ is either simply connected or reductive. These generalizations (more precisely, the generalization to the simply connected case) allow us to describe $\mathcal{A}(f)$ when $f$ is an iterated integral (i.e., when $f^{(n)} \in k$ for some positive integer $n$, see [Sri10]) and to study the algebraic relations among iterated integrals. Indeed, in Section 7.2 we prove the following results.

THEOREM 1.8 (Theorem 7.2 in Section 7.2). — *Let $f \in K^\times \setminus k$ be an iterated integral over $k$. Assume that $k$ contains an element $x$ with $x' = 1$. Then, we have*

$$\mathcal{A}(f) = k[f].$$

PROPOSITION 1.9 (Proposition 7.8 in Section 7.2). — *Assume that there is an element $x \in k$ such that $x' = 1$. Let $f_1, \ldots, f_n \in K$ be iterated integrals over $k$. If $f_1, \ldots, f_n$ are algebraically dependent over $k$ then there exist $u_1, \ldots, u_n \in k$, not all zero, such that*

$$u_1 f_1 + \cdots + u_n f_n \in k.$$

*Remark 1.10.* — Our proof of Proposition 1.9 does not use the fact that $k$ is algebraically closed; it only requires that $C$ be algebraically closed.

This paper is organized as follows. Section 3 is devoted to the case where $L$ has order 1 and contains a proof of Theorem 1.2 above. In Section 4 we give a result when the differential Galois group $G_L$ of $L$ over $k$ is simply connected, apply this to the case when $G_L = \mathrm{SL}_n(C)$ (proving the $\mathrm{SL}_n(C)$-case of Theorem 1.3 above). In Section 5 we consider the case when $G_L$ is $\mathrm{GL}_n(C)$ (proving the $\mathrm{GL}_n(C)$-case of Theorem 1.3 above) and in Section 6 the case when $G_L$ is a general reductive group. We have included a full treatment of the $\mathrm{GL}_n(C)$-case before considering the general reductive case because the proof is less technical in the $\mathrm{GL}_n(C)$-case but already contains most of the ideas used in the general reductive case. In Section 7.1 we apply the results of the previous sections to generalized hypergeometric series and prove Theorem 1.1 above. In Section 7.2 we study the algebraic relations among iterated integrals (proving Proposition 1.8 and Proposition 1.9 above). The Appendix A gives an informal introduction to the tannakian approach to the differential Galois theory - an approach that appears in several of our proofs.

## 2. Notations and conventions

In the whole paper, except where noted to the contrary (e.g., in Section 7.2):

- $K$ is a differential field of characteristic zero;
- $k$ is an algebraically closed differential subfield of $K$;

- we assume that $K$ and $k$ have the same field of constants $C$:

$$C = \{f \in K \mid f' = 0\} = \{f \in k \mid f' = 0\}.$$

Note that $C$ has characteristic zero and is algebraically closed because it is the field of constants of $k$ which itself has characteristic zero and is algebraically closed (see [S03, Exercise 1.5.2]). Also we recall from Remark 1.4 that the assumption that $k$ is algebraically closed implies that the differential Galois group of any Picard–Vessiot extension of $k$ is connected.

We will assume that the reader is familiar with some basic notions concerning linear algebraic groups (a general reference is [Hum75]) and the basic differential Galois theory (a general reference is [Mag94] or [S03]). In our proofs we will also sometimes use the tannakian correspondence. Briefly this states that for a differential equation $Y' = AY$, there is a correspondence between representations of the associated differential Galois group and equations gotten from the original equations via applying the constructions of linear algebra (direct sums, tensor products, duals, submodules and quotients) to the differential module associated with $Y' = AY$. In the Appendix A we give a fuller but still informal description of this. Formally this is given as an equivalence of categories; for precise details we refer to [DM81, Kat72, Kat87, S03]).

Except with explicit mention to the contrary, by "algebraic group", we will mean "linear algebraic group over $C$".

## 3. The first order case

We start this Section with a preliminary result concerning algebraic groups. This result will be used several times in this paper. Recall that an *isogeny* $\varphi : G_1 \to G_2$ is an epimorphism whose kernel is finite [Hum75, Section 32.1].

LEMMA 3.1. — *Let $\varphi : G_1 \to G_2$ be an isogeny between two algebraic groups $G_1$ and $G_2$. If $G_1$ is connected and if $G_2$ is a torus, then $G_1$ is a torus as well and $G_1$ and $G_2$ have the same dimension.*

*Proof.* — To show that the connected group $G_1$ is a torus it is enough to show that it consists of semisimple elements (c.f. [Hum75, Exercise 2 in Section 21.4]). It is equivalent to showing that $e_{G_1}$ is the unique unipotent element of $G_1$ (because any linear algebraic group over $C$ contains the semisimple and the unipotent parts of the Jordan decompositions of its elements). Since the image by $\varphi$ of a unipotent element of $G_1$ is unipotent and since the only unipotent element of $G_2$ is $e_{G_2}$, the unipotent elements of $G_1$ are all in $\ker \varphi$. Since $\ker \varphi$ is finite and unipotent elements have infinite order or are the identity, $G_1$ has a unique unipotent element, namely $e_{G_1}$.

It remains to prove that $G_1$ and $G_2$ have the same dimension. This follows from the fact that $\dim G_1 = \dim \ker \varphi + \dim \operatorname{im} \varphi$, [Hum75, Chapter 7.3, Proposition B] □

THEOREM 3.2. — *Let $f \in K^\times \setminus k$ be such that*

$$f' = af$$

*for some $a \in k$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. We then have that $f$ and $g$ are algebraically dependent over $k$ if and only if there exists $\theta$ in an algebraic extension of $K$ and an integer $n$ such that*

$$f = \theta^n \quad \textit{and} \quad g \in k\left[\theta, \theta^{-1}\right]. \tag{3.1}$$

*Proof.* — If $f$ and $g$ satisfy (3.1), they are clearly algebraically dependent over $k$.

Let us prove the converse implication. Let $L$ be a nonzero linear differential operator with coefficients in $k$ annihilating $g$ and of minimal order $r$ for this property. Let $Y' = A_L Y$ be the corresponding linear differential system.

We let $E$ be a Picard–Vessiot extension over $k$ for the differential system

$$Y' = \begin{pmatrix} a & 0 \\ 0 & A_L \end{pmatrix} Y \tag{3.2}$$

containing $f$ and $g$. Let $G_E$ be the corresponding differential Galois group. The differential system (3.2) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & g_1 & \cdots & g_r \\ 0 & g_1' & \cdots & g_r' \\ 0 & \vdots & \ddots & \vdots \\ 0 & g_1^{(r-1)} & \cdots & g_r^{(r-1)} \end{pmatrix} \in GL_{r+1}(E)$$

with $g_1 = g$ (thus, $E$ is the field extension of $k$ generated by the entries of $\mathfrak{Y}$). The field extension $E_1$ (resp. $E_2$) of $k$ generated by $f$ (resp. by the $g_i^{(j)}$) is a Picard–Vessiot extension over $k$ for $y' = ay$ (resp. $Y' = A_L Y$); we let $G_1$ (resp. $G_2$) be the corresponding differential Galois group over $k$.

By minimality of $L$, we have

$$\operatorname{span}_C Gg \;=\; \operatorname{span}_C G_2 g \;=\; \operatorname{span}_C \{g_1, \ldots, g_r\}$$

(note that $Gg = G_2 g$ because the restriction morphisms $G \to G_2$ is surjective). Moreover, the $C$-line generated by $f$ is left invariant by $G$. These facts, together with the fact that $g$ is algebraic over $k(f)$, imply that the $g_i^{(j)}$ are algebraic over $k(f)$ as well, i.e., $E$ is a finite extension of $E_1$.

The differential Galois theory implies that the restriction morphism $G \to G_1$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \to G_1 \to 0.$$

The kernel $H$ (which is the group of differential field automorphisms of $E$ over $E_1$) is finite because $E$ is a finite extension of $E_1$. Moreover, $G_1$ is a one dimensional torus (it is $C^\times$ in the representation given by $f$). Lemma 3.1 implies that $G$ itself is a one dimensional torus.

Therefore, there exists a basis of solutions $\widetilde{g}_1, \ldots, \widetilde{g}_r$ of $L$ in $E$ such that the image of the representation of the Galois group $G$ with respect to

$$\widetilde{\mathfrak{Y}} = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & \widetilde{g}_1 & \cdots & \widetilde{g}_r \\ 0 & \widetilde{g}_1' & \cdots & \widetilde{g}_r' \\ 0 & \vdots & \ddots & \vdots \\ 0 & \widetilde{g}_1^{(r-1)} & \cdots & \widetilde{g}_r^{(r-1)} \end{pmatrix} \in \mathrm{GL}_n(E)$$

is given by

$$\left\{ \mathrm{diag}\left(t^{n_0}, \ldots, t^{n_m}\right) \,\middle|\, t \in C^\times \right\}$$

for some $n_0, \ldots, n_m \in \mathbb{Z}$.

Therefore, if $\theta$ is such that $f = \theta^{n_0}$, then $\widetilde{g}_i = a_i \theta^{n_i}$ for some $a_i \in k^\times$ (because $(\widetilde{g}_i/\theta^{n_i})^{n_0}$ is fixed by $G$). This concludes the proof since $g$ is in the $C$-span of the $\widetilde{g}_i$. □

Corollary 3.3. — *Let $f \in K^\times$ be such that*

$$f' = af$$

*for some $a \in k$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that*

$$f = P(g)$$

*for some nonconstant polynomial $P$ with coefficients in $k$. Then, there exists $\theta$ in $K$ such that*

$$f = \theta^\ell \quad \text{and} \quad g = c + d\theta$$

*for some $\ell \in \mathbb{Z}_{\geqslant 1}$ and some $c, d \in k$.*

*Proof.* — If $f$ is algebraic over $k$, $g$ is algebraic over $k$ as well and, hence, belongs to $k$.

So, we can and will assume that $f$ is transcendental over $k$. Theorem 3.2 ensures that there exists $\theta$ and an integer $n \in \mathbb{Z}$ such that

$$f = \theta^n \quad \text{and} \quad g \in k\left[\theta, \theta^{-1}\right].$$

Since $k$ is algebraically closed, we may factor $P(Y)$ as a product of linear factors and, hence, the equation $P(g) = f$ can be rewritten as

$$e_1 \prod_{i=1}^{\ell} (g - d_i) = \theta^n$$

for some $\ell \in \mathbb{Z}_{\geqslant 1}$ and some $e_1$ and $d_i$ in $k$. Since $\theta$ is transcendental over $k$, the unique factorization property of $k[\theta, \theta^{-1}]$ implies that each factor in the above equality belongs to $k\theta^{\mathbb{Z}}$; in particular, we have for each $i$

$$g - d_i = e_i \theta^{r_i}$$

for some $e_i \in k^\times$ and some $r_i \in \mathbb{Z}$. For any pair $r_i, r_j, i \neq j$, we have $d_i - d_j = e_i\theta^{r_i} - e_j\theta^{r_j}$. Since $\theta$ is transcendental, we must have $d_i = d_j =: d, e_i = e_j =: e$ and $r_i = r_j =: r$. So $g = e\theta^r + d$ and $e, d \in k$.

In particular we have that $n = r\ell$. Whence the desired result where the $\theta$ of the conclusion is $\theta^r$ of this proof. □

COROLLARY 3.4. — *Let $f, g \in K$ satisfy some nonzero homogeneous linear differential equations over $k$. Assume that*

$$f \in k(g).$$

*Then,*

- *either $f \in k[g]$,*
- *or there exists $\theta$ in $K$ such that $\theta'/\theta \in k$ and*

$$g = c + d\theta \quad \text{and} \quad f \in k\left[\theta, \theta^{-1}\right]$$

*for some $c, d \in k$.*

*Proof.* — Consider $R \in k(X)$ such that $f = R(g)$. Let $A, B \in k[X]$ be such that $\gcd(A, B) = 1$ and $R = A/B$. Bézout's Identity states that there exist $U, V \in k[X]$ such that $AU + BV = 1$, so $RU + V = 1/B$. Since the set of elements satisfying homogeneous linear differential equations over $k$ forms a $k$-algebra, it follows that $fU(g) + V(g) = R(g)U(g) + V(g) = 1/B(g)$ satisfies a nonzero homogeneous linear differential equations over $k$. By [HS85] (see also [Spe86, Sin86]), we have $B(g)'/B(g) \in k$ and hence we can apply Corollary 3.3: if $B$ is nonconstant then there exists $\theta$ such that

$$B(g) = \theta^{\ell} \quad \text{and} \quad g = c + d\theta$$

for some $\ell \in \mathbb{Z}_{\geqslant 0}$ and some $c, d \in k$. Since $f = R(g) = A(g)/B(g) = A(c + d\theta)\theta^{-\ell}$, we have $f \in k[\theta, \theta^{-1}]$.

Of course, if $B$ is constant, then $f \in k[g]$. □

## 4. The simply connected case

We begin with some facts concerning simply connected[(1)] algebraic groups.

LEMMA 4.1. — *A simply connected algebraic group $G$ has no torus quotient, that is, there is no nontrivial homomorphism $\phi : G \to T = (C^*)^n$. In particular, any multiplicative character of $G$ is trivial.*

*Proof.* — From [Hoc81, Chapter XVIII, Proposition 3.2], we know that the radical $R(G)$ is unipotent. Let $\phi : G \to T = (C^*)^n$ be a homomorphism. Since homomorphisms preserve unipotent elements, we must have that $R(G) \subset \mathrm{Ker}(\phi)$. Therefore $\phi$ factors through $H := G/R(G)$. Since $H$ is semisimple[(2)], it coincides with its derived subgroup $H'$. Therefore, $\phi$ takes its values in the derived subgroup $T'$ of $T$. Since $T'$ is trivial, we have that $\phi$ is trivial. □

(1) Here, simply connected means: a connected algebraic group $G$ not admitting any non-trivial isogeny $\phi : H \to G$ where $H$ is also a connected algebraic group. For semisimple algebraic groups over the field of complex numbers this definition is equivalent to the topological one.

(2) $R(G)$ is the smallest closed normal subgroup $K$ of $G$ such that $G/K$ is semisimple, cf. [Hum75, Section 19.5 and Theorem 27.5].

In the proof of the following proposition, we give two arguments - one based on the Picard–Vessiot approach to the Galois theory and the other based on the tannakian approach. The tannakian approach plays the dominant role in what follows and this allows the reader to compare the two approaches in this simpler situation.

PROPOSITION 4.2. — *Let $f \in K^\times \setminus k$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over $k$ of order $n \geqslant 1$ and assume that the differential Galois group $G_L$ over $k$ of the latter equation is simply connected. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that $f$ and $g$ are algebraically dependent over $k$.*

(1) *If $F$ is a differential field extension of $k(f, g)$ with field of constants $C$ containing a basis of solutions $f_1 = f, f_2, \ldots, f_n$ of $L$, then*

$$g \in k\left[W\left(f_1, \ldots, f_n\right)\right]$$

*where $W(f_1, \ldots, f_n)$ denotes the Wronskian matrix associated to $f_1, \ldots, f_n$.*

(2) *Assume*

$$(*) \qquad k(f) \text{is relatively algebraically closed in } k\left(W(f_1, \ldots, f_n)\right).$$

*Then,*

$$g \in k[f].$$

*Proof.*

(1) Let $Y' = A_L Y$ be the linear differential system corresponding to $L$.

Let $M$ be a nonzero linear differential operator with coefficients in $k$ annihilating $g$ and of minimal order $m$ for this property. Let $Y' = A_M Y$ be the corresponding linear differential system.

We let $E$ be a Picard–Vessiot extension over $k$ of the differential system

$$Y' = \begin{pmatrix} A_L & 0 \\ 0 & A_M \end{pmatrix} Y \tag{4.1}$$

containing $f$ and $g$. Let $G$ be the corresponding differential Galois group over $k$. The differential system (4.1) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_L & 0 \\ 0 & \mathfrak{Y}_M \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where

$$\mathfrak{Y}_L = \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \in \mathrm{GL}_n(E)$$

and

$$\mathfrak{Y}_M = \begin{pmatrix} g_1 & \cdots & g_m \\ g_1' & \cdots & g_m' \\ \vdots & \ddots & \vdots \\ g_1^{(m-1)} & \cdots & g_m^{(m-1)} \end{pmatrix} \in \mathrm{GL}_m(E)$$

with $f_1 = f$ and $g_1 = g$ (thus, $E$ is the field extension of $k$ generated by the entries of $\mathfrak{Y}_L$ and $\mathfrak{Y}_M$). The field extension $E_L$ (resp. $E_M$) of $k$ generated by the $f_i^{(j)}$ (resp. by the $g_i^{(j)}$) is a Picard–Vessiot extension over $k$ for $Y' = A_L Y$ (resp. $Y' = A_M Y$); we let $G_L$ (resp. $G_M$) be the corresponding differential Galois group over $k$.

By minimality of $M$, we have

$$\operatorname{span}_C Gg \;=\; \operatorname{span}_C G_M g \;=\; \operatorname{span}_C \{g_1, \ldots, g_m\}$$

(note that $Gg = G_M g$ because the restriction morphism $\pi_M : G \to G_M$ is surjective). Moreover, $E_L$ is left invariant by $G$. These facts, together with the fact that $g$ is algebraic over $E_L$ (because $f$ and $g$ are algebraically dependent over $k$ and $f$ is transcendental over $k$), imply that the $g_i^{(j)}$ are algebraic over $E_L$ as well, i.e., $E$ is a finite extension of $E_L$.

Again, the differential Galois theory implies that the restriction morphism $\pi_L : G \to G_L$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \xrightarrow{\pi_L} G_L \to 0.$$

The kernel $H$ (which is the group of differential fields automorphisms of $E$ over $E_L$) is finite because $E$ is a finite extension of $E_L$.

Since $G_L$ is a connected and simply connected algebraic group, the kernel $H$ is trivial and $\pi_L : G \to G_L$ is an isomorphism. In particular, since $H$ is trivial, the Galois correspondence implies that $E = E_L$. The $C$-span of the entries of $\mathfrak{Y}_M$ form a $G = G_L$-invariant finite dimensional $C$-vector space. It follows from [S03, Corollary 1.38] that all of these entries, and in particular $g = g_1$, lie in the Picard–Vessiot ring $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$. Note that $G_L$ leaves the $C$-line spanned by $\det(\mathfrak{Y}_L)$ invariant and, since all characters of $G_L$ are trivial (see Lemma 4.1), it must leave $\det(\mathfrak{Y}_L)$ fixed. Therefore $\det(\mathfrak{Y}_L) \in k$ and so $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}] = k[\mathfrak{Y}_L]$. This proves the first claim (1) of the Proposition 4.2.

We now turn to a tannakian proof of the first claim. As already noted the fact that $G_L$ is a connected and simply connected algebraic group implies that the kernel $H$ is trivial and $\pi_L : G \to G_L$ is an isomorphism. It follows that there exists an algebraic group morphism $u : G_L \to G_M$ such that

$$G = \operatorname{im}(\pi_L \oplus \pi_M) = \operatorname{im}(\mathrm{id}_{G_L} \oplus u).$$

We now adopt the tannakian point of view (see Appendix A): the category of rational linear representations of $G$ is equivalent to the tannakian category generated by the differential system (4.1). In this equivalence, the differential system $Y' = A_L Y$ (resp. $Y' = A_M Y$) corresponds to the restriction morphism $\pi_L : G \to G_L$ (resp. $\pi_M : G \to G_M$). The factorization

$$\pi_M = u \circ \pi_L$$

ensures that $Y' = A_M Y$ belongs to the tannakian category generated by $Y' = A_L Y$. It follows that the entries of the solutions of $Y' = A_M Y$ belong to the Picard–Vessiot ring $R_L = k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$ of $Y' = A_L Y$. As before $G_L$ leaves the line spanned by $\det(\mathfrak{Y}_L)$ invariant and, since all characters of $G_L$ are trivial (see Lemma 4.1), it must leave $\det(\mathfrak{Y}_L)$ fixed. Therefore $\det(\mathfrak{Y}_L) \in k$ and so $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}] = k[\mathfrak{Y}_L]$. This yields the second proof of the first claim (1) of the Proposition 4.2.

(2) If $(*)$ is satisfied, then $g$ belongs to $k(f)$ (because $g$ is algebraic over $k(f)$). Corollary 3.4 implies that either $g \in k[f]$, or there exists $\theta$ in $K$ such that $\theta'/\theta \in k$ and $f = c + d\theta$ and $g \in k[\theta, \theta^{-1}]$ for some $c, d \in k$. We claim that the latter case is impossible. Indeed, otherwise the line $C\theta$ spanned by $\theta$ would be left invariant by $G_L$. But Lemma 4.1 states that any character of $G_L$ is trivial. So $G_L$ would act as the identity on $\theta$. By the Galois correspondence, this would imply that $\theta \in k$ and, hence, that $f \in k$. This is excluded by the assumption. □

In order to draw the conclusion of Proposition 4.2 (2), one needs to verify condition $(*)$. We will give two examples of this below but first mention a general approach.

Since we are assuming $k$ is algebraically closed, the Galois group $G$ of any Picard–Vessiot extension of $k$ is connected and the Picard–Vessiot ring of any linear differential equation over $k$ is $k$-isomorphic to the coordinate ring $k[G \otimes k]$ of the group $G$ over $k$ ([Mag94, Corollary 5.16], [S03, Proposition 1.31, Corollary 1.32]). In particular, this implies that the Picard–Vessiot field $K$ is isomorphic to the function field of $G \otimes k$. This latter field is known to be a purely transcendental extension of $k$ ([Bor91, Remark 14.14]). Therefore, if one can find algebraically independent elements $z_1 = f, z_2, \ldots, z_m$ such that $K = k(z_1, \ldots, z_m)$ then $k(f)$ will be relatively algebraically closed in $K$. Note that in Proposition 4.2 the field $k(W(f_1, \ldots, f_n))$ is the Picard–Vessiot field of the equation $L(y) = 0$.

### 4.1. When $G_L = \mathrm{SL}_n(C)$

In this section, we will prove

Proposition 4.3. — *Let $f \in K^\times \setminus k$ satisfy an $n^{th}$ order nonzero homogeneous linear differential equation $L(f) = 0$ over $k$ and assume that the differential Galois group $G_L$ over $k$ of this equation is $\mathrm{SL}_n(C)$ seen as a subgroup of $\mathrm{GL}_n(C)$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that $f$ and $g$ are algebraically dependent over $k$. Then,*

$$g \in k[f].$$

The proof will depend on the following Lemma.

Lemma 4.4. — *Let $K, f, L, G_L, g$ be as in Proposition 4.2. Assume that $G_L$, seen as a subgroup of $\mathrm{GL}_n(C)$ via the basis of solutions $f_1, \ldots, f_n$, also satisfies the following property : if we let*

$$k[G_L \otimes k] = k\left[(X_{i,j})_{1 \leqslant i,j \leqslant n}, \det(X_{i,j})^{-1}_{1 \leqslant i \leqslant n}\right]/I$$

*be the coordinate ring of $G_L \otimes k$, then, for any $a \in \mathrm{span}_k\{X_{i,1} \mid 1 \leqslant i \leqslant n\} \mod I$, $k(a)$ is relatively algebraically closed in $k(G_L \otimes k)$. Then,*

$$g \in k[f].$$

*Moreover, $g \in k[f]$ if $G_L$, seen as a subgroup of $\mathrm{GL}_n(C)$ via some basis of solutions of $L$, satisfies the following property : for any $a \in \mathrm{span}_k\{X_{i,j} \mid 1 \leqslant i,j \leqslant n\} \mod I$, $k(a)$ is relatively algebraically closed in $k(G_L \otimes k)$.*

*Proof.* — We know that the Picard–Vessiot ring $R_L$ of $L$ is isomorphic, as a $k$-algebra, to $k[G_L \otimes k]$, more precisely (c.f. [Mag94, Corollary 5.16], [S03, Proposition 1.31, Corollary 1.32]) there exists $v \in \mathrm{GL}_n(k)$ such that the $k$-algebra morphism

$$k\left[(X_{i,j})_{1\leqslant i,j\leqslant n}, \det\left((X_{i,j})_{1\leqslant i,j\leqslant n}\right)^{-1}\right] \to R_L$$

defined by $X \mapsto v\mathfrak{Y}_L$ induces an isomorphism

$$k\left[G_L \otimes k\right] = k\left[(X_{i,j})_{1\leqslant i,j\leqslant n}, \det\left(X_{i,j}\right)^{-1}_{1\leqslant i,j\leqslant n}\right]/I \xrightarrow{\sim} R_L.$$

In this isomorphism, $f$ corresponds to an element of $\mathrm{span}_k\{X_{i,1} \mid 1 \leqslant i \leqslant n\} \mod I$, so, the hypothesis on $k(G_L \otimes k)$ implies $k(f)$ is relatively algebraically closed in $k(W(f_1, \ldots, f_n))$. Proposition 4.2 implies the conclusion.

The final assertion follows from this. Indeed, the differential Galois group of $L$ seen as a group of matrices via some basis of solutions of $L$ is conjugate via some element of $\mathrm{GL}_n(C)$ to the same differential Galois group seen as a group of matrices via the basis of solutions $f_1, \ldots, f_n$. It follows that if the final hypothesis is satisfied then the previous hypothesis is satisfied as well. □

*Remark 4.5.* — The difference between the two parts of Lemma 4.4 is that, in the first part, $G_L$ is seen as a group of matrices via the basis of solutions $f_1, \ldots, f_n$ which has $f_1 = f$ as first element, whereas, in the second, $f$ is not required to be an element of the basis used to see $G_L$ as a group of matrices.

*Remark 4.6.* — The hypotheses of Lemma 4.4 concerning the coordinate ring of $G_L \otimes k$ are not hypotheses on the differential Galois group of $L$ as an "abstract" algebraic group but hypotheses on the incarnation $G_L$ of this differential Galois group as a subgroup of $\mathrm{GL}_n(C)$ *via* the linear representation given by the choice of a basis of solutions of $L$. In other words, these hypotheses are not necessarily invariant under isomorphisms of algebraic groups. For instance, this hypothesis is satisfied by $G_L = \mathrm{SL}_2(C) \subset \mathrm{GL}_2(C)$, but not by $\mathrm{Sym}^3(\mathrm{SL}_2(C)) \subset \mathrm{GL}_4(C)$, despite the fact that these two algebraic groups are isomorphic. The fact that the hypothesis is satisfied when $G_L = \mathrm{SL}_2(C)$ is shown in the proof of Proposition 4.3 below. Let us prove that it is not satisfied by $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. We have denoted by $\mathrm{Sym}^3(\mathrm{SL}_2(C))$ the image of $\mathrm{SL}_2(C)$ by its (faithful) 3rd symmetric power representation

$$\mathrm{Sym}^3 : \mathrm{SL}_2(C) \to \mathrm{GL}_4(C),$$

which has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & * & * & b^2d \\ ac^2 & * & * & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}.$$

Let

$$\begin{aligned} k\left[\mathrm{Sym}^3\left(\mathrm{SL}_2(C)\right) \otimes k\right] &= k\left[(X_{i,j})_{1\leqslant i,j\leqslant 4}, \det\left(X_{i,j}\right)^{-1}_{1\leqslant i,j\leqslant 4}\right]/I \\ &= k\left[(x_{i,j})_{1\leqslant i,j\leqslant 4}, \det\left(x_{i,j}\right)^{-1}_{1\leqslant i,j\leqslant 4}\right] \end{aligned}$$

be the coordinate ring of $\mathrm{Sym}^3(\mathrm{SL}_2(C))$. Then,

$$f = 3^{-1}x_{1,2}\left(x_{2,4}/x_{1,4} - x_{2,1}/x_{1,1}\right) \in k\left(\mathrm{Sym}^3\left(\mathrm{SL}_2(C)\right) \otimes k\right)$$

satisfies $f^3 = x_{1,1}$. Therefore, $f$ is algebraic over $k(x_{1,1})$ but does not belong to $k(x_{1,1})$ and, hence, $k(x_{1,1})$ is not relatively algebraically closed in $k(\mathrm{Sym}^3(\mathrm{SL}_2(C)) \otimes k)$.

This is not a surprise: the conclusion of Proposition 4.3 is not satisfied when $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. Indeed, let $M$ be a linear differential equation over $k$ of order 2 with differential Galois group $\mathrm{SL}_2(C)$. Consider the third symmetric power $L = M^{\circledS 3}$; this is a linear differential equation over $k$ of order 4 with differential Galois group $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. If $(u, v)$ is a basis of solutions of $M$, then $(u^3, u^2v, uv^2, v^3)$ is a basis of solutions of $L$. Therefore, $f = u^3$ is a solution of $L$, $g = u$ is a solution of $M$, $f \notin k$ and $f$ and $g$ are algebraically dependent over $k$ but $g \notin k(f)$.

*Proof of Proposition 4.3.* — We shall prove that the hypotheses of Lemma 4.4 are satisfied when $G_L = \mathrm{SL}_n(C)$. The fact that $\mathrm{SL}_n(C)$ is connected and simply connected is well-known (cf. [Hum75, Section 31.1]). It remains to prove that, if we denote by

$$\begin{aligned} k\left[\mathrm{SL}_n(C) \otimes k\right] &= k\left[(X_{i,j})_{1\leqslant i,j\leqslant n}, \det\left(X_{i,j}\right)^{-1}_{1\leqslant i,j\leqslant n}\right] / \left(\det\left(X_{i,j}\right)_{1\leqslant i,j\leqslant n} - 1\right) \\ &= k\left[(x_{i,j})_{1\leqslant i,j\leqslant n}\right] \end{aligned}$$

the coordinate ring of $\mathrm{SL}_n(C) \otimes k$, we have: for any

$$a = \sum_{1\leqslant i,j\leqslant n} \lambda_{i,j}\, x_{i,j} \;\in\; \mathrm{span}_k\left\{x_{i,j} \,|\, 1 \leqslant i, j \leqslant n\right\},$$

$k(a)$ is relatively algebraically closed in $k(\mathrm{SL}_n(C) \otimes k)$.

Let us first assume that at least one of the $\lambda_{i,j}$ is zero, say $\lambda_{i_0,j_0} = 0$. Then, we have

$$\begin{aligned} k\left(\mathrm{SL}_n(C) \otimes k\right) &= k\left((x_{i,j})_{1\leqslant i,j\leqslant n}\right) \\ &= k\left((x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0)}\right) \\ &= k\left(a, (x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0),(i_1,j_1)}\right) \end{aligned}$$

where $(i_1, j_1)$ is chosen such that $\lambda_{i_1,j_1} \neq 0$. The second equality above follows from the fact $x_{i_0,j_0} \in k((x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0)})$ which itself follows from the equality $\det(x_{i,j})_{1\leqslant i,j\leqslant n} = 1$. The third equality follows from the fact that $x_{i_1,j_1}$ belongs to the $k$-span of $a$ and $(x_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(i_0,j_0),(i_1,j_1)}$. Since

$$\mathrm{tr.deg}(k(\mathrm{SL}_n(C) \otimes k)/k) = n^2 - 1$$

and

$$\sharp\left\{a, (x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0),(i_1,j_1)}\right\} \leqslant n^2 - 1,$$

we get that the family $(a, (x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0),(i_1,j_1)})$ is algebraically independent over $k$. In particular, $k(a)$ is algebraically closed in $k(\mathrm{SL}_n(C) \otimes k)$.

Let us now assume that the $\lambda_{i,j}$ are nonzero. We have

$$k(\mathrm{SL}_n(C)\otimes k) = k\left((x_{i,j})_{1\leqslant i,j\leqslant n}\right) = k\left(a,\,(x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(1,1)}\right).$$

The latter equality follows from the fact that $x_{1,1}$ belongs to the $k$-span of $a$ and $(x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(1,1)}$. But, we have

$$x_{2,1} = \lambda_{2,1}^{-1}a - \sum_{\substack{1\leqslant i,j\leqslant n\\(i,j)\neq(1,1),(2,1)}} \lambda_{2,1}^{-1}\lambda_{i,j}x_{i,j} - \lambda_{2,1}^{-1}\lambda_{1,1}x_{1,1}.$$

The equality $\det(x_{i,j})_{1\leqslant i,j\leqslant n} = 1$ ensures that

$$x_{1,1} = -\frac{\delta_{2,1}}{\delta_{1,1}}x_{2,1} - \sum_{i=3}^{n}\frac{\delta_{i,1}}{\delta_{1,1}}x_{i,1} + \frac{1}{\delta_{1,1}}$$

where $\delta_{i,j}$ is the $(i,j)$ cofactor of the matrix $(x_{i,j})_{1\leqslant i,j\leqslant n}$. It follows that

$$\left(1-\lambda_{2,1}^{-1}\lambda_{1,1}\frac{\delta_{2,1}}{\delta_{1,1}}\right)x_{2,1} = \lambda_{2,1}^{-1}a - \sum_{\substack{1\leqslant i,j\leqslant n\\(i,j)\neq(1,1),(2,1)}} \lambda_{2,1}^{-1}\lambda_{i,j}x_{i,j} + \lambda_{2,1}^{-1}\lambda_{1,1}\left(\sum_{i=3}^{n}\frac{\delta_{i,1}}{\delta_{1,1}}x_{i,1} - \frac{1}{\delta_{1,1}}\right).$$

This shows that $x_{2,1}$ belongs to $k(a,(x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(1,1),(2,1)})$ and, hence, that

$$k\left(\mathrm{SL}_n(C)\otimes k\right) = k\left(a,(x_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(1,1),(1,2)}\right).$$

We can now conclude that $k(a)$ is algebraically closed in $k(\mathrm{SL}_n(C)\otimes k)$ by using a transcendence degree argument as we did above. □

## 5. A special case of reductive $G_L$: $\mathrm{GL}_n(C)$

We denote the center of a group $G$ by $Z_G$.

Lemma 5.1. — *Let $\varphi : G_1 \to G_2$ be an isogeny with $G_1$ connected. Then:*

- *$\varphi^{-1}(Z_{G_2}) = Z_{G_1}$, $\varphi(Z_{G_1}) = Z_{G_2}$ and $\varphi(Z_{G_1}^0) = Z_{G_2}^0$;*
- *if $G_2 = Z_{G_2}^0 G_2'$, then $G_1 = Z_{G_1}^0 G_1'$ [3];*
- *if $Z_{G_2}^0$ is a torus of dimension $d$, then $Z_{G_1}^0$ is a torus of dimension $d$ as well;*
- *if $G_2'$ is semi-simple, then $G_1'$ is semi-simple as well;*
- *if $G_2$ is reductive, then $G_1$ is reductive as well.*

*Proof.* — We first prove that $\varphi^{-1}(Z_{G_2}) \subset Z_{G_1}$. Consider $g \in \varphi^{-1}(Z_{G_2})$. For any $h \in G_1$, $\varphi([g,h]) = [\varphi(g),\varphi(h)] = e_{G_2}$, so $[g,h] \in \ker\varphi$. Since $G_1$ is connected, the image the morphism of algebraic varieties

$$\begin{aligned} G_1 &\to \ker\varphi\\ h &\mapsto [g,h]\end{aligned}$$

is contained in $(\ker\varphi)^0 = \{e_{G_1}\}$, so $g \in Z_{G_1}$ as expected.

[3] Recall that $G'$ is the derived subgroup of the algebraic group $G$.

The inclusion $Z_{G_1} \subset \varphi^{-1}(Z_{G_2})$ follows from the fact that $\varphi$ is onto and therefore $\varphi^{-1}(Z_{G_2}) = Z_{G_1}$.

The equality $\varphi(Z_{G_1}) = Z_{G_2}$ follows form this and from the fact that $\varphi$ is onto.

It follows that $\varphi(Z^0_{G_1}) = \varphi(Z_{G_1})^0 = Z^0_{G_2}$.

Since $\varphi$ is onto, we have $\varphi(G'_1) = G'_2$. We have seen that $\varphi(Z^0_{G_1}) = Z^0_{G_2}$. Therefore, $\varphi(Z^0_{G_1}G'_1) = Z^0_{G_2}G'_2 = G_2$. Since $\ker\varphi$ is central, we get $G_1 \subset (\ker\varphi)Z^0_{G_1}G'_1 \subset Z_{G_1}G'_1$, so $G_1 = Z_{G_1}G'_1$. But $Z^0_{G_1}G'_1$ is a closed subgroup of $G_1 = Z_{G_1}G'_1$ of finite index, so $G^0_1 = G_1 \subset Z^0_{G_1}G'_1$, whence the equality $G_1 = Z^0_{G_1}G'_1$.

Assume that $Z^0_{G_2}$ is a torus of dimension $d$. We have seen that $\varphi(Z^0_{G_1}) = Z^0_{G_2}$, so $\varphi$ induces an isogeny between $Z^0_{G_1}$ and $Z^0_{G_2}$. Lemma 3.1 implies that $Z^0_{G_1}$ is a torus of dimension $d$.

Since $\varphi(G'_1) = G'_2$, $\varphi$ induces an isogeny between $G'_1$ and $G'_2$. Therefore, $G'_1$ is semi-simple if and only if $G'_2$ is semi-simple.

The last assertion follows from the previous one. □

In the proof of the next result we will use the tannakian correspondence. Appendix A contains a review of this tool.

PROPOSITION 5.2. — *Let $f \in K^\times$ satisfy a nonzero homogeneous linear differential equation over $k$ of order $n$ and assume that the differential Galois group over $k$ of the latter equation is $\mathrm{GL}_n(C)$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that $f$ and $g$ are algebraically dependent over $k$. We have:*

(1) *if $n = 1$, then there exist $\theta$ and an integer $\ell$ such that*

$$f = \theta^\ell \quad \text{and} \quad g \in k\left[\theta, \theta^{-1}\right];$$

(2) *if $n > 1$, then*

$$g \in k[f].$$

*Proof.* — Let $L$ be a nonzero linear differential operator of order $n$ with coefficients in $k$ and with differential Galois group over $k$ equal to $\mathrm{GL}_n(C)$ annihilating $f$. Let $Y' = AY$ be the corresponding linear differential system.

Let $M$ be a nonzero linear differential operator with coefficients in $k$ annihilating $g$ and of minimal order $m$ for this property. Let $Y' = BY$ be the corresponding linear differential system.

We let $E$ be a Picard–Vessiot extension over $k$ of the differential system

$$Y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Y \tag{5.1}$$

containing $f$ and $g$. Let $G$ be the corresponding differential Galois group over $k$. The differential system (5.1) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_A & 0 \\ 0 & \mathfrak{Y}_B \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where

$$\mathfrak{Y}_A = \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \in \mathrm{GL}_n(E)$$

and

$$\mathfrak{Y}_B = \begin{pmatrix} g_1 & \cdots & g_m \\ g_1' & \cdots & g_m' \\ \vdots & \ddots & \vdots \\ g_1^{(m-1)} & \cdots & g_m^{(m-1)} \end{pmatrix} \in \mathrm{GL}_m(E)$$

with $f_1 = f$ and $g_1 = g$ (thus, $E$ is the field extension of $k$ generated by the entries of $\mathfrak{Y}_A$ and $\mathfrak{Y}_B$). The field extension $E_A$ (resp. $E_B$) of $k$ generated by the $f_i^{(j)}$ (resp. by the $g_i^{(j)}$) is a Picard–Vessiot extension over $k$ for $Y' = AY$ (resp. $Y' = BY$); we let $G_A$ (resp. $G_B$) be the corresponding differential Galois group over $k$.

By minimality of $M$, we have

$$\mathrm{span}_C\, Gg \;=\; \mathrm{span}_C\, G_B g \;=\; \mathrm{span}_C\, \{g_1, \ldots, g_m\}$$

(note that $Gg = G_B g$ because the restriction morphism $G \to G_B$ is surjective). Moreover, $E_A$ is left invariant by $G$. These facts, together with the fact that $g$ is algebraic over $E_A$ (because $f$ and $g$ are algebraically dependent over $k$ and $f$ is transcendental over $k$ because $G_A$ is $\mathrm{GL}_n(C)$) imply that the $g_i^{(j)}$ are algebraic over $E_A$ as well, i.e., $E$ is a finite extension of $E_A$.

It is a general fact that the restriction morphism $\pi_A : G \to G_A$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \xrightarrow{\pi_A} G_A \to 0.$$

The kernel $H$ (which is the group of differential field automorphisms of $E$ over $E_A$) is finite because $E$ is a finite extension of $E_A$.

Since $k$ is algebraically closed, $G$ is connected. Applying Lemma 5.1 to $\pi_A : G \to G_A$ (with $G_A = \mathrm{GL}_n(C) = Z^0_{G_A} G'_A$ with $Z^0_{G_A} = C^\times I_n$ and $G'_A = \mathrm{SL}_n(C)$), we get

$$G = Z_G G' = Z^0_G G'$$

and also that $Z^0_G$ is a one dimensional torus. There exists $P \in \mathrm{diag}(I_n, \mathrm{GL}_m(C)) \subset \mathrm{GL}_{n+m}(C)$ such that

$$P Z^0_G P^{-1} = \left\{ \mathrm{diag}\left(t^{\alpha_0} I_n, t^{\alpha_1} I_{m_1}, \ldots, t^{\alpha_l} I_{m_l}\right) \,\middle|\, t \in C^\times \right\}$$

for some $m_1, \ldots, m_l \in \mathbb{Z}_{\geqslant 1}$ and some $\alpha_0, \ldots, \alpha_l \in \mathbb{Z}$ with $\alpha_1, \ldots, \alpha_l$ pairwise distinct. Therefore, $PGP^{-1}$ is block diagonal:

$$PGP^{-1} \subset \mathrm{diag}\left(\mathrm{GL}_n(C), \mathrm{GL}_{m_1}(C), \ldots, \mathrm{GL}_{m_l}(C)\right).$$

It follows that there exist $B_1 \in M_{m_1}(k), \ldots, B_l \in M_{m_l}(k)$ such that $Y' = BY$ is equivalent over $k$ to

$$Y' = \mathrm{diag}\left(B_1, \ldots, B_l\right) Y.$$

So, up to changing $\mathfrak{Y}_B$ by $R\mathfrak{Y}_B$ for some $R \in \mathrm{GL}_m(k)$, we can and will assume that $B = \mathrm{diag}(B_1, \ldots, B_l)$ and that $P = I_n$.

Let $\widetilde{A} = A - \mathrm{tr}(A)/nI_n$ and consider the differential system

$$Y' = \mathrm{diag}\left(\mathrm{tr}(A)/nI_n, \widetilde{A}, B_1, \ldots, B_l\right) Y. \tag{5.2}$$

Since $\mathrm{tr}(\widetilde{A}) = 0$, the differential Galois group of $\widetilde{A}$ is unimodular ([S03, Exercise 1.35.5]). We let $H$ be the differential Galois group of (5.2). In the tannakian correspondence (cf., Example A.1), the equation

$$Y' = \mathrm{diag}\left(A, B_1, \ldots, B_l\right) Y$$

corresponds to the representation $\rho : H \to G$ given by

$$\rho : \mathrm{diag}\left(t, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \mathrm{diag}\left(t\widetilde{h}, h_1, \ldots, h_l\right).$$

Applying Lemma 5.1 to $\rho : H \to G$ (again the differential Galois theory implies that $\rho$ is onto, and that it has finite kernel), we have

$$H = Z_H^0 H'$$

and $Z_H^0 = \rho^{-1}(Z_G^0)^0$ is a one dimensional torus. The restriction

$$\rho_{|Z_H^0} : Z_H^0 = \rho^{-1}\left(Z_G^0\right)^0 \to Z_G^0$$

is given by

$$h \mapsto \left(\chi_0(h) I_n, \chi_1(h) I_{m_1}, \ldots, \chi_l(h) I_{m_l}\right)$$

for some characters $\chi_i : Z_H^0 \to C^\times$. Another morphism $Z_H^0 \to Z_G^0$ is given by

$$h = \mathrm{diag}\left(t, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \mathrm{diag}\left(t^{\alpha_0} I_n, t^{\alpha_1} I_{m_1}, \ldots, t^{\alpha_l} I_{m_l}\right).$$

Therefore (since $Z_K^0$ and $Z_G^0$ are one dimensional tori: see Lemma 6.1), there exist nonzero relatively prime integers $N, M$ such that, for all $i$,

$$\chi_i(h)^N = t^{\alpha_i M}.$$

Consider the differential system

$$Y' = \mathrm{diag}\left(\mathrm{tr}(A)/nN, \widetilde{A}, B_1, \ldots, B_l\right) Y.$$

Let $H_N$ be the differential Galois group of this system. In the tannakian correspondence, the equation

$$Y' = \mathrm{diag}\left(A, B_1, \ldots, B_l\right) Y$$

corresponds to

$$\varrho := \rho \circ p : H_N \to G$$

where $p : H_N \to H$ (which corresponds to $Y' = \mathrm{diag}(\mathrm{tr}(A)/n, \widetilde{A}, B_1, \ldots, B_l)Y$ in the tannakian correspondence) is given by

$$p : \mathrm{diag}\left(z, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \mathrm{diag}\left(z^N, \widetilde{h}, h_1, \ldots, h_l\right).$$

Applying Lemma 5.1 to $p : H_N \to H$, we get

$$H_N = Z_{H_N}^0 H_N'$$

and $Z^0_{H_N} = p^{-1}(Z^0_H)^0$ is a one dimensional torus.

If $h = \operatorname{diag}\left(z, \widetilde{h}, h_1, \ldots, h_l\right) \in Z^0_{H_N} = p^{-1}(Z^0_H)^0$, then

$$\chi_i(p(h))^N = z^{\alpha_i NM}$$

so

$$\chi_i(p(h)) = z^{\alpha_i M}.$$

So, we have

$$\varrho_{|Z^0_{H_N}} : \left(z, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \left(z^N I_n = z^{\alpha_0 M} I_n, z^{\alpha_1 M} I_{m_1}, \ldots, z^{\alpha_l M} I_{m_l}\right).$$

On the other hand, the representation corresponding to $Y' = AY$ in the tannakian correspondence is

$$\varpi_A = \pi_A \circ \varrho : H_N \to G_A.$$

It is surjective (differential Galois theory) and has finite kernel (because $\varrho$ and $\pi_A$ have finite kernels), so it is an isogeny. Therefore, it induces an isogeny

$$\varpi_{A|H'_N} = \pi_A \circ \varrho : H'_N \to G'_A.$$

Since $G'_A = \mathrm{SL}_n(C)$ is a connected and simply connected algebraic group,

$$\varpi_{A|H'_N} = \pi_A \circ \varrho : H'_N \to G'_A$$

is actually an isomorphism. Since $G = \operatorname{im}(\pi_A \circ \varrho \oplus \pi_B \circ \varrho)$, it follows that there exists an algebraic group morphism $u = (u_1, \ldots, u_l) : G'_A \to G_B$ such that

$$\pi_B \circ \varrho = u \circ (\pi_A \circ \varrho).$$

So,

$$\varrho_{|H'_N} : \left(1, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \left(\widetilde{h}, u_1(\widetilde{h}), \ldots, u_l(\widetilde{h})\right).$$

Finally, we see that the representation $\varpi_B = \pi_B \circ \varrho$ corresponding to $Y' = BY$ in the tannakian correspondence is given by

$$\varpi_B = \pi_B \circ \varrho : \left(z, \widetilde{h}, h_1, \ldots, h_l\right) \mapsto \left(z^{\alpha_1 M} u_1(\widetilde{h}), \ldots, z^{\alpha_l M} u_l(\widetilde{h})\right).$$

This ensures, by the tannakian correspondence, that $Y' = B_i Y$ is equivalent to a system of the form $Y' = (\mathrm{Constr}_i(\widetilde{A}) + \alpha_i M(\mathrm{tr}(A)/nN) I_{m_i})Y$ where $\mathrm{Constr}_i(\widetilde{A})$ is obtained from $\widetilde{A}$ by some construction of linear algebra. It follows that the entries of the solutions of $Y' = BY$ belong to

$$k\left[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}\right]$$

for some positive integer $q$; in particular, $g$ belongs to

$$k\left[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}\right].$$

For $n = 1$, this gives the desired result (which, of course, already follows from Corollary 3.3).

Let us now assume $n > 1$. We have that $R_L = k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$ is isomorphic, as a $k$-algebra, to $k[\mathrm{GL}_n(k)]$. More precisely there exists $v \in \mathrm{GL}_n(k)$ such that the $k$-algebra morphism

$$k\left[(X_{i,j})_{1\leqslant i,j\leqslant n}, \det\left((X_{i,j})_{1\leqslant i,j\leqslant n}\right)^{-1}\right] \to R_L$$

defined by $X \mapsto v\mathfrak{Y}_L$ is an isomorphism; it induces an isomorphism

$$k\left[(X_{i,j})_{1\leqslant i,j\leqslant n}, \sqrt[q]{\det\left((X_{i,j})_{1\leqslant i,j\leqslant n}\right)}^{-1}\right] \to k\left[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}\right]$$

In this isomorphism, $f$ corresponds to an element $a$ of $\mathrm{span}_k\{X_{i,1} \mid 1 \leqslant i \leqslant n\}$ mod $I$. The fact that $k(a)$ is relatively algebraically closed in

$$k\left((X_{i,j})_{1\leqslant i,j\leqslant n}, \sqrt[q]{\det\left((X_{i,j})_{1\leqslant i,j\leqslant n}\right)}^{-1}\right)$$

(see Lemma 5.3 below) implies that

$$g \in k(f).$$

Corollary 3.4 ensures that:

- either $g \in k[f]$,
- or there exists $\theta$ such that $\theta'/\theta \in k$ and

$$f = c + d\theta \quad \text{and} \quad g \in k\left[\theta, \theta^{-1}\right]$$

for some $c, d \in k$.

The latter possibility is incompatible with the fact that $G_L = \mathrm{GL}_n(C)$ acts irreducibly on the solution space of $L$. □

Lemma 5.3. — *For any $a \in \mathrm{span}_k\{X_{i,j} \mid 1 \leqslant i,j \leqslant n\}$, $k(a)$ is relatively algebraically closed in*

$$k\left((X_{i,j})_{1\leqslant i,j\leqslant n}, \Delta^{-1}\right) \quad \textit{where} \quad \Delta = \sqrt[q]{\det\left((X_{i,j})_{1\leqslant i,j\leqslant n}\right)}.$$

*Proof.* — Let

$$L = k\left((X_{i,j})_{1\leqslant i,j\leqslant n}, \Delta\right).$$

and let $a = \sum \lambda_{i,j} X_{i,j}$ be in the $k$-span of the $X_{i,j}$.

Let us first assume at least one of the $\lambda_{i,j}$ is zero, say $\lambda_{i_0,j_0} = 0$. Then, we have

$$\begin{aligned} L = k\left((X_{i,j})_{1\leqslant i,j\leqslant n}, \Delta\right) &= k\left((X_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0)}, \Delta\right) \\ &= k\left(a, (X_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0),(i_1,j_1)}, \Delta\right) \end{aligned}$$

where $(i_1, j_1)$ is such that $\lambda_{i_1,j_1} \neq 0$. The second equality follows from the fact that $X_{i_0,j_0} \in k((X_{i,j})_{1\leqslant i,j\leqslant n,\,(i,j)\neq(i_0,j_0)}, \Delta)$ which itself follows from the equality

$$\det(X_{i,j})_{1\leqslant i,j\leqslant n} = \sum_{j=1}^{n} X_{i_0,j}\delta_{i_0,j} = \Delta^q$$

where $\delta_{i,j}$ is the $(i,j)$ cofactor of $(X_{i,j})_{1\leqslant i,j\leqslant n}$. The third equality follows from the fact that $X_{i_1,j_1}$ belongs to the $k$-span of $a$ and $(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(i_0,j_0),(i_1,j_1)}$. Since

$$\mathrm{tr.deg}(L/k)=n^2 \quad \text{and} \quad \sharp\left\{a,(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(i_0,j_0),(i_1,j_1)},\Delta\right\}\leqslant n^2,$$

we get that the family $(a,(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(i_0,j_0),(i_1,j_1)},\Delta)$ is algebraically independent over $k$. In particular, $k(a)$ is algebraically closed in $L$.

Let us now assume that the $\lambda_{i,j}$ are nonzero. We have

$$L=k\left((X_{i,j})_{1\leqslant i,j\leqslant n},\Delta\right)=k\left(a,(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(1,1)},\Delta\right)$$

The latter equality follows from the fact that $X_{1,1}$ belongs to the $k$-span of $a$ and $(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(1,1)}$. We have

$$X_{2,1}=\lambda_{2,1}^{-1}a-\sum_{\substack{1\leqslant i,j\leqslant n\\(i,j)\neq(1,1),(2,1)}}\lambda_{2,1}^{-1}\lambda_{i,j}X_{i,j}-\lambda_{2,1}^{-1}\lambda_{1,1}X_{1,1}.$$

Furthermore the equality $\det(X_{i,j})_{1\leqslant i,j\leqslant n}=\Delta^q$ ensures that

$$X_{1,1}=-\frac{\delta_{2,1}}{\delta_{1,1}}X_{2,1}-\sum_{i=3}^{n}\frac{\delta_{i,1}}{\delta_{1,1}}X_{i,1}+\frac{\delta^q}{\delta_{1,1}}$$

where $\delta_{i,j}$ is the $(i,j)$ cofactor of $(X_{i,j})_{1\leqslant i,j\leqslant n}$. It follows that

$$\left(1+\lambda_{2,1}^{-1}\lambda_{1,1}\frac{\delta_{2,1}}{\delta_{1,1}}\right)X_{2,1}=$$
$$\lambda_{2,1}^{-1}a-\sum_{\substack{1\leqslant i,j\leqslant n\\(i,j)\neq(1,1),(2,1)}}\lambda_{2,1}^{-1}\lambda_{i,j}X_{i,j}+\lambda_{2,1}^{-1}\lambda_{1,1}\left(\sum_{i=3}^{n}\frac{\delta_{i,1}}{\delta_{1,1}}X_{i,1}-\frac{\delta^q}{\delta_{1,1}}\right).$$

This shows that $X_{2,1}$ belongs to $k((X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(1,1),(2,1)},\Delta)$ and, hence, that

$$L=k\left(a,(X_{i,j})_{1\leqslant i,j\leqslant n,(i,j)\neq(1,1),(1,2)},\Delta\right).$$

We can now conclude as above by a transcendence degree argument. □

## 6. The general reductive case

LEMMA 6.1. — *Let $T_1$ and $T_2$ be tori of dimension $d$. Let $\phi,\varphi:T_1\to T_2$ be two isogenies. Then, there exist $N\in\mathbb{Z}_{\geqslant 1}$ and an isogeny $\psi:T_2\to T_2$ such that $\varphi^N=\psi\circ\phi$.*

*Proof.* — Using the fact that $T_1$ and $T_2$ are isomorphic to $(C^\times)^d$, we can assume that $T=T_1=T_2=(C^\times)^d$. We let $X(T)$ be the character group of $T$; it is a free abelian group of rank $d$. We set $\phi=(\phi_1,\ldots,\phi_d)$ and $\varphi=(\varphi_1,\ldots,\varphi_d)$; the $\phi_i$ and $\varphi_i$ are thus elements of $X(T)$. Since $\phi:T\to T$ is an isogeny, $\phi_1,\ldots,\phi_d$ are multiplicatively independent elements of $X(T)$ and, hence, they generate a free subgroup $\langle\phi_1,\ldots,\phi_d\rangle$ of rank $d$ of $X(T)$. Thus, every element of $X(T)$ has a positive power belonging to $\langle\phi_1,\ldots,\phi_d\rangle$. In particular, there exist a positive integer $N$ such that the $\varphi_i^N$ belong to $\langle\phi_1,\ldots,\phi_d\rangle$. Therefore, there exists $\psi_i\in X(T)$ such that

$\varphi_i^N = \psi_i \circ \phi$. Thus, the morphism $\psi = (\psi_1, \ldots, \psi_d) : T \to T$ is such that $\varphi^N = \psi \circ \phi$. Since $\varphi^N$ is an isogeny, $\psi$ is an isogeny as well. □

PROPOSITION 6.2. — *Let $f \in K^\times \setminus k$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over $k$ of order $n$ and assume that the differential Galois group over $k$ of the latter equation is reductive and has a simply connected derived subgroup. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that $f$ and $g$ are algebraically dependent over $k$.*

(1) *If $F$ is a differential field extension of $k(f, g)$ with field of constants $C$ containing a basis of solutions $f_1 = f, f_2, \ldots, f_n$ of $L$, then*
$$g \in k\left[W(f_1, \ldots, f_n), y_1^{-1/q}, \ldots, y_s^{-1/q}\right]$$
*where $W(f_1, \ldots, f_n)$ denotes the Wronskian matrix associated to $f_1, \ldots, f_n$, the $y_i \in k[W(f_1, \ldots, f_n)]$ are such that $y_i'/y_i \in k$ and $q$ is a positive integer.*

(2) *Assume*

(∗∗) *$k(f)$is relatively algebraically closed in $k\left(W(f_1, \ldots, f_n), y_1^{-1/q}, \ldots, y_s^{-1/q}\right)$.*

*Then,*
$$g \in k[f].$$

*Remark 6.3.* — Actually, the proof of Proposition 6.2 gives the following more precise information. We let $L = L_1 \cdots L_r$ be a factorization of $L$ over $k$ into irreducible factors. The proof of Proposition 6.2 shows that $s \leqslant r$ and that the $y_i$ can be chosen such that $y_i$ are solutions of $y' = tr(A_i)y$ where $A_i$ is a certain matrix gauge equivalent to the companion matrix of $L_i(Y) = 0$. In particular, with these refinements in mind, we recover the main ingredients of the proof of Proposition 5.2 (the only additional ingredient of the proof of Proposition 5.2 is the fact that the hypothesis (∗∗) is satisfied under the assumptions of Proposition 5.2).

*Proof.* — Let $Y' = AY$ be the linear differential system corresponding to $L$. Its differential Galois group over $k$ (the same as $L$) being reductive, we have that $Y' = AY$ is equivalent over $k$ to
$$Y' = \overline{A}Y, \quad \overline{A} = \operatorname{diag}(A_1, \ldots, A_s)$$
for some $A_i \in M_{n_i}(k)$ such that $Y' = A_iY$ is irreducible over $k$.

Let $M$ be a nonzero linear differential operator with coefficients in $k$ annihilating $g$ and of minimal order $m$ for this property. Let $Y' = BY$ be the corresponding linear differential system.

We let $E$ be a Picard–Vessiot extension over $k$ of the differential system
$$Y' = \begin{pmatrix} \overline{A} & 0 \\ 0 & B \end{pmatrix} Y \tag{6.1}$$
containing $f$ and $g$. Let $G$ be the corresponding differential Galois group over $k$. The differential system (6.1) has a fundamental solution of the form
$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_{\overline{A}} & 0 \\ 0 & \mathfrak{Y}_B \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where $\mathfrak{Y}_{\overline{A}}$ is a fundamental solution of $Y' = \overline{A}Y$ and $\mathfrak{Y}_B$ is a fundamental solution of $Y' = BY$ (thus, $E$ is the field extension of $k$ generated by the entries of $\mathfrak{Y}_{\overline{A}}$ and $\mathfrak{Y}_B$). The field extension $E_{\overline{A}}$ (resp. $E_B$) of $k$ generated by the entries of $\mathfrak{Y}_{\overline{A}}$ (resp. by the $\mathfrak{Y}_B$) is a Picard–Vessiot extension over $k$ for $Y' = \overline{A}Y$ (resp. $Y' = BY$); we have $f \in E_{\overline{A}}$ and $g \in E_B$. We let $G_{\overline{A}}$ (resp. $G_B$) be the corresponding differential Galois group over $k$. We recall that $G_{\overline{A}}$ is reductive with simply connected derived subgroup $G'_{\overline{A}}$; we will denote by $d$ the dimension of the torus $Z^0_{G_{\overline{A}}}$.

By minimality of $M$, we have that $E_B$ is generated, as a differential field extension of $k$, by $Gg = G_B g$ (we recall that $Gg = G_B g$ because the restriction morphism $G \to G_B$ is surjective). Moreover, $E_{\overline{A}}$ is left invariant by $G$. These facts, together with the fact that $g$ is algebraic over $E_A$, imply that $E$ is a finite extension of $E_{\overline{A}}$.

We again have that the restriction morphism $\pi_{\overline{A}} : G \to G_{\overline{A}}$ is surjective and hence induces a short exact sequence

$$0 \to J \to G \xrightarrow{\pi_{\overline{A}}} G_{\overline{A}} \to 0.$$

The kernel $J$ (which is the group of differential field automorphisms of $E$ over $E_{\overline{A}}$) is finite because $E$ is a finite extension of $E_{\overline{A}}$.

Applying Lemma 5.1 to $\pi_{\overline{A}} : G \to G_{\overline{A}}$, we get that $G$ is reductive and that $Z^0_G$ is a torus of dimension $d$.

Let

$$\widetilde{A} = \overline{A} - \operatorname{diag}\Big(\operatorname{tr}(A_1)/n_1 I_{n_1}, \ldots, \operatorname{tr}(A_s)/n_s I_{n_s}\Big)$$

and consider the differential system

$$Y' = \operatorname{diag}\Big(\operatorname{tr}(A_1)/n_1, \ldots \operatorname{tr}(A_s)/n_s, \widetilde{A}, B\Big)Y.$$

We let $H$ be the differential Galois group over $k$ of this equation. In the tannakian correspondence, the equation (6.1) corresponds to the representation $\rho : H \to G$ given by

$$\rho : \operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(c_1\widetilde{g}_1, \ldots, c_s\widetilde{g}_s, g).$$

Applying Lemma 5.1 to the isogeny $\rho : H \to G$ (as before, $\rho$ is onto, and it is clear that it has finite kernel because the $\widetilde{g}_i$ have determinant 1), we have that $H$ is reductive and that $Z^0_H = \rho^{-1}(Z^0_G)^0$ is a torus of dimension $d$. We claim that there exists an isogeny $\Xi : Z^0_H \to Z^0_G$ of the form

$$\operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \Xi(c_1, \ldots, c_s),$$

an isogeny $\psi : Z^0_G \to Z^0_G$ and a positive integer $N$ such that

$$\rho^N_{|Z^0_H} = \psi \circ \Xi.$$

Indeed, the image of $Z^0_H$ by

$$\operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(c_1, \ldots, c_s)$$

is a torus of dimension $d$ (this is because $\pi_{\overline{A}} \circ \rho : H \to G_{\overline{A}}$, which is given by

$$\operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(c_1\widetilde{g}_1, \ldots, c_s\widetilde{g}_s)$$

is an isogeny and, hence, induces an isogeny $Z_H \to Z_{G_{\overline{A}}}$, but $Z_{G_{\overline{A}}}$ is made of diagonal matrices of the form

$$\operatorname{diag}(*I_{n_1}, \ldots, *I_{n_s}),$$

and the $\widetilde{g}_i$ have determinant 1, so $(\pi_{\overline{A}} \circ \rho_{|Z_H})^{n_1 \cdots n_s}$ can be described as follows:

$$\left(\pi_{\overline{A}} \circ \rho_{|Z_H}\right)^{n_1 \cdots n_s} : \operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(c_1 I_{n_1}, \ldots, c_s I_{n_s})^{n_1 \cdots n_s}$$

and hence the image of $Z_H$ by

$$\operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(c_1 I_{n_1}, \ldots, c_s I_{n_s})$$

has the same dimension as the image of $\pi_{\overline{A}} \circ \rho_{|Z_H}$, which is equal to the dimension of $Z_{G_{\overline{A}}}$), so there exists an isogeny $\Xi : Z_H^0 \to Z_G^0$ of the form

$$\operatorname{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \Xi(c_1, \ldots, c_s).$$

But $\rho_{|Z_H^0} : Z_H^0 \to Z_G^0$ is also an isogeny, so Lemma 6.1 ensures that there exist a positive integer $N$ and an isogeny $\psi : Z_G^0 \to Z_G^0$ such that

$$\psi \circ \Xi = \rho_{|Z_H^0}^N.$$

This proves our claim.

Consider the differential system

$$Y' = \operatorname{diag}\left(\operatorname{tr}(A_1)/(Nn_1), \ldots, \operatorname{tr}(A_s)/(Nn_s), \widetilde{A}, B\right) Y.$$

Let $H_N$ be the differential Galois group over $k$ of this system. In the tannakian correspondence, the equation

$$Y' = \operatorname{diag}(A, B) Y$$

corresponds to

$$\varrho := \rho \circ p : H_N \to G$$

where $p : H_N \to H$ (which corresponds to

$$Y' = \operatorname{diag}\left(\operatorname{tr}(A_1)/n_1, \ldots \operatorname{tr}(A_s)/n_s, \widetilde{A}, B\right) Y$$

in the tannakian correspondence) is given by

$$p : \operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}\left(e_1^N, \ldots, e_s^N, \widetilde{g}_1, \ldots, \widetilde{g}_s, g\right).$$

Applying Lemma 5.1 to $p : H_N \to H$, we get that $H_N$ is reductive and that $Z_{H_N}^0 = p^{-1}(Z_H^0)^0$ is torus of dimension $d$.

If $\operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \in Z_{H_N}^0$, then

$$\begin{aligned} \varrho(\operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g))^N &= \rho_{|Z_H^0}\left(\operatorname{diag}\left(e_1^N, \ldots, e_s^N, \widetilde{g}_1, \ldots, \widetilde{g}_s, g\right)\right)^N \\ &= \psi \circ \Xi\left(e_1^N, \ldots, e_s^N\right) = \psi \circ \Xi(e_1, \ldots, e_s)^N \end{aligned}$$

so

$$\varrho_{|Z_{H_N}^0} = \psi \circ \Xi.$$

Moreover, the representation of $H_N$ corresponding to $Y' = \overline{A}Y$ in the tannakian correspondence is

$$\varpi_{\overline{A}} = \pi_{\overline{A}} \circ \varrho : H_N \to G_{\overline{A}}.$$

It is surjective (general fact) and has finite kernel (because $\varrho$ and $\pi_{\overline{A}}$ have finite kernels), so it is an isogeny. Therefore, it induces an isogeny

$$\varpi_{\overline{A}|H'_N} = \pi_{\overline{A}} \circ \varrho : H'_N \to G'_{\overline{A}}.$$

Since $G'_{\overline{A}}$ is a connected and simply connected algebraic group,

$$\varpi_{\overline{A}|H'_N} = \pi_{\overline{A}} \circ \varrho : H'_N \to G'_{\overline{A}}.$$

is actually an isomorphism. Since $G = \mathrm{im}(\pi_{\overline{A}} \circ \varrho \oplus \pi_B \circ \varrho)$, it follows that there exists an algebraic group morphism $u : G'_{\overline{A}} \to G'_B$ such that

$$\pi_B \circ \varrho = u \circ (\pi_{\overline{A}} \circ \varrho)\,.$$

So, the restriction of $\varrho$ to the derived subgroup $H'_N$ of $H_N$ is given by

$$\varrho_{|H'_N} : (1, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \mathrm{diag}\left(\widetilde{g}_1, \ldots, \widetilde{g}_s, u\,(\widetilde{g}_1, \ldots, \widetilde{g}_s)\right).$$

Finally, we see that the representation $\varpi_B = \pi_B \circ \varrho$ of $H_N$ corresponding to $Y' = BY$ in the tannakian correspondence is given by

$$\varpi_B = \pi_B \circ \varrho : \mathrm{diag}\,(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \psi \circ \Xi\,(e_1, \ldots, e_s)\,u\,(\widetilde{g}_1, \ldots, \widetilde{g}_s)\,.$$

It follows that the differential system $Y' = BY$ is equivalent over $k$ to

$$Y' = \mathrm{diag}\,(B_1, \ldots, B_\ell)\,Y$$

where each $B_i \in M_{m_i}(k)$ has the form

$$\mathrm{Constr}_i\left(\widetilde{A}\right) + \sum_{j=1}^{s} \alpha_{i,\,j}\,\mathrm{tr}(A_j)/(n_j N) I_{m_j}$$

for some $\alpha_{i,\,j} \in \mathbb{Z}$ and some $\mathrm{Constr}_i(\widetilde{A})$ obtained from $\widetilde{A}$ by some construction of linear algebra. It follows that the entries of the solutions of $Y' = BY$ belong to $k[\mathfrak{Y}_{\overline{A}}, \sqrt[q]{y_1}^{-1}, \ldots, \sqrt[q]{y_s}^{-1}]$ for some positive integer $q$, where each $y_j$ satisfies $y'_j = \mathrm{tr}(A_j) y_j$. □

## 7. Applications

In this section we will apply the previous results to generalized hypergeometric series and to iterated integrals.

### 7.1. Generalized hypergeometric series

The aim of this Section is to prove Theorem 1.6 stated in Section 1.

### 7.1.1. The generalized hypergeometric series and equations

We recall that, for any $p, q \in \mathbb{Z}_{\geqslant 0}$, the generalized hypergeometric series with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0})^q$ is given by

$$ {}_pF_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}; x\right] = \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{x^k}{k!} \in \mathbb{C}((x)) $$

where the Pochhammer symbols $(t)_k$ are defined by $(t)_0 = 1$ and, for $k \in \mathbb{Z}_{\geqslant 1}$, $(t)_k = t(t+1)\cdots(t+k-1)$.

This series satisfies the generalized hypergeometric differential equation

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]\left({}_pF_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}; x\right]\right) = 0 $$

where

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] = \delta \prod_{k=1}^{q} (\delta + \beta_k - 1) - x \prod_{k=1}^{p} (\delta + \alpha_k) $$

with $\delta = x\frac{d}{dx}$.

Note that :

- if $q + 1 = p$, then ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ has order $q + 1 = p$ and has at most three singularities, namely 0, 1 and $\infty$, all regular;
- if $q+1 > p$, then ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ has order $q+1$ and has at most two singularities, namely 0 and $\infty$; 0 is regular whereas $\infty$ is irregular.
- if $q + 1 < p$, then ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ has order $p$ and has at most two singularities, namely 0 and $\infty$; 0 is irregular whereas $\infty$ is regular.

### 7.1.2. Formal structure at $\infty$ of the generalized hypergeometric equations when $q + 1 > p$

We shall now recall some basic facts concerning the formal structure at $\infty$ of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ when $q + 1 > p$. Consider the uniformiser at $\infty$ given by $t = x^{-1}$. We have

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] = t^{-1}\left((-1)^{q+1} t\delta_t \prod_{k=1}^{q} (\delta_t - \beta_k + 1) - (-1)^p \prod_{k=1}^{p} (\delta_t - \alpha_k)\right) $$

with $\delta_t = t\frac{d}{dt}$. By definition (see [S03, Definition 3.44]), the Newton polygon

$$ \mathcal{N}_\infty({}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]) \quad \text{of} \quad {}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] $$

at $\infty$ is the convex hull of

$$ \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \text{there is a monomial } t^m\delta_t^n \text{ in } {}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] \text{ with } (x, y) \geqslant (n, m)\right\} $$

where $(x_1, y_1) \geqslant (x_2, y_2)$ if and only if $x_1 \leqslant x_2$ and $y_1 \geqslant y_2$. A straightforward calculation shows that this polygon has three extremal points, namely $(0, 0)$, $(p, 0)$ and $(q + 1, 1)$. Thus, the slopes of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ at $\infty$ are

$$ \lambda_1 = 0 \text{ with multiplicity } p \quad \text{and} \quad \lambda_2 = 1/\sigma \quad \text{with multiplicity } \sigma, $$

where

$$\sigma = q - p + 1.$$

It follows from [S03, Theorem 3.48] that the differential module $\mathcal{M}$ over $\mathbb{C}(x)$ associated to ${}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ satisfies

$$\mathbb{C}((t)) \otimes_{\mathbb{C}(x)} \mathcal{M} = \widehat{\mathcal{M}}_{\lambda_1} \oplus \widehat{\mathcal{M}}_{\lambda_2} \tag{7.1}$$

where $\widehat{\mathcal{M}}_{\lambda_1}$ is a regular singular differential module over $\mathbb{C}((t))$ of rank $p$ and $\widehat{\mathcal{M}}_{\lambda_2}$ is an irregular differential module over $\mathbb{C}((t))$ of rank $\sigma$, with only one slope, namely $\lambda_2$. According to [S03, Remark 3.34], we have

$$\mathbb{C}\left(\left(t^{1/\sigma}\right)\right) \otimes_{\mathbb{C}((t))} \widehat{\mathcal{M}}_{\lambda_2} = \oplus_{j=1}^{\sigma} \widehat{\mathcal{R}} \otimes_{\mathbb{C}\left(\left(t^{1/\sigma}\right)\right)} \widehat{\mathcal{E}}\left(q_j\left(t^{-1/\sigma}\right)\right) \tag{7.2}$$

where:

- $\widehat{\mathcal{R}}$ is a regular singular differential module over $\mathbb{C}((t^{1/\sigma}))$ of rank one,
- for all $j \in \{1, \ldots, \sigma\}$, $q_j(t^{-1/\sigma}) = q(\zeta_j t^{-1/\sigma})$ with $\zeta_j = e^{\frac{2\pi i j}{\sigma}}$ for some $q(X) \in X\mathbb{C}[X]$ of $X$-adic valuation 1,
- $\widehat{\mathcal{E}}(q_j(t^{-1/\sigma}))$ is the rank one differential module over $\mathbb{C}((t^{1/\sigma}))$ defined by $\widehat{\mathcal{E}}(q_j(t^{-1/\sigma})) = \mathbb{C}((t^{1/\sigma}))e_j$ with $\partial(e_j) = q_j(t^{-1/\sigma})e_j$.

Recall from [S03, Remark 3.55] that the slopes of $\widehat{\mathcal{M}}_{\lambda_2}$ can be computed using the $q_j(t^{-1/\sigma})$: they are the negative of the $t$-adic valuations of the $q_j(t^{-1/\sigma})$, which are all equal to $\frac{\deg_X q(X)}{\sigma}$. Since $\widehat{\mathcal{M}}_{\lambda_2}$ has a unique slope $\lambda_2 = 1/\sigma$, we get $\deg_X q(X) = 1$ so $q(X) = aX$ is a nonconstant monomial and

$$q_j\left(t^{-1/\sigma}\right) = a\zeta_j t^{-1/\sigma}.$$

We shall now determine $a$. Set $w = t^{1/\sigma}$ and $\delta_w = w\frac{d}{dw} = \sigma\delta_t$. We claim that the equivalence class of the nonzero complex number $a$ modulo $\mu_\sigma = \{\zeta_1, \ldots, \zeta_\sigma\}$ is characterized by the fact that the Newton polygon (with respect to $w$ and at $w = 0$) of $e^{-aw^{-1}}\,{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{aw^{-1}}$ has height[(4)] $< q + 1$. Indeed, (7.1) and (7.2) above ensure that the differential module over $\mathbb{C}((w))$ associated to ${}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ has a decomposition of the form

$$\widehat{\mathcal{N}} \oplus \oplus_{j=1}^{\sigma} \widehat{\mathcal{S}} \otimes_{\mathbb{C}((w))} \widehat{\mathcal{E}}\left(a\zeta_j w^{-1}\right) \tag{7.3}$$

where $\widehat{\mathcal{N}}$ is a regular singular differential module over $\mathbb{C}((w))$ of rank $p$ and $\widehat{\mathcal{S}}$ is a regular singular differential module over $\mathbb{C}((w))$ of rank 1. Therefore, for any $b \in \mathbb{C}^\times$, the differential module over $\mathbb{C}((w))$ associated to $e^{-bw^{-1}}\,{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{bw^{-1}}$ has a decomposition of the form

$$\widehat{\mathcal{N}} \otimes_{\mathbb{C}((w))} \widehat{\mathcal{E}}\left(-bw^{-1}\right) \oplus \oplus_{j=1}^{\sigma} \widehat{\mathcal{S}} \otimes_{\mathbb{C}((w))} \widehat{\mathcal{E}}\left((a\zeta_j - b)\,w^{-1}\right).$$

But, according to [S03, Remark 3.55], the slopes of (7.3) are the negative of the $w$-adic valuations of the polynomials $-bw^{-1}$ and $(a\zeta_j - b)w^{-1}$ for $j \in \{1, \ldots, \sigma\}$, so:

(4) The height of the Newton polygon of a differential operator $L = \sum_{i=0}^{n} a_i(w)\delta_w$ of order $n$ with coefficients in $\mathbb{C}((w))$ is equal to the sum of the slopes of $L$ counted with multiplicities or, equivalently, to $\max_{i \in \{0,\ldots,n\}} v_w(a_n(w)) - v_w(a_i(w))$ where $v_w : \mathbb{C}((w)) \to \mathbb{Z} \cup \{-\infty\}$ is the $w$-adic valuation.

- if $b \notin a\mu_\sigma$, then 1 is the unique slope of (7.3) and, hence, the Newton polygon of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}}$ (with respect to $w$ and at $w=0$) has height $q+1$;
- if $b \in a\mu_\sigma$, then (7.3) has two slopes, namely 0 and 1, and, hence, the Newton polygon of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}}$ (with respect to $w$ and at $w=0$) has height $< q+1$.

Let us now determine the nonzero complex numbers $b$ such that the Newton polygon (with respect to $w$ and at $w=0$) of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}}$ has height $< q+1$. We have

$$ {}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right] = w^{-\sigma}\left((-1)^{q+1}w^\sigma\frac{\delta_w}{\sigma}\prod_{k=1}^{q}\left(\frac{\delta_w}{\sigma}-\beta_k+1\right)-(-1)^p\prod_{k=1}^{p}\left(\frac{\delta_w}{\sigma}-\alpha_k\right)\right) $$

and

$$ \begin{aligned} {}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}} = \\ w^{-\sigma}\left((-1)^{q+1}w^\sigma\left(\frac{\delta_w}{\sigma}-\frac{b}{\sigma}w^{-1}\right)\prod_{k=1}^{q}\left(\frac{\delta_w}{\sigma}-\frac{b}{\sigma}w^{-1}-\beta_k+1\right)\right. \\ \left.-(-1)^p\prod_{k=1}^{p}\left(\frac{\delta_w}{\sigma}-\frac{b}{\sigma}w^{-1}-\alpha_k\right)\right) \end{aligned} $$

The coefficient of $\delta_w^j$ in ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}}$ has $w$-adic valuation $\geqslant -q+j-1$ and the $w$-adic valuation of $\delta_w^{q+1}$ is equal to 0; therefore, we have to find the $b$ such that the coefficient of $\delta_w^0$ in ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]e^{bw^{-1}}$ has $w$-adic valuation $> -(q+1)$. But the latter coefficient is of the form

$$ \left(\frac{b}{\sigma}w^{-1}\right)^{q+1}-w^{-\sigma}\left(\frac{b}{\sigma}w^{-1}\right)^{p}+\text{ terms of higher degree in } w; $$

it has $w$-adic valuation $> -(q+1)$ if and only if $(\frac{b}{\sigma})^{q+1} = (\frac{b}{\sigma})^p$ if and only if $b \in \sigma\mu_\sigma$.

Therefore,

$$ a \in \sigma\mu_\sigma. $$

In conclusion, the list of determining polynomials[(5)] at $\infty$ of ${}_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]$ is

$$ \left(0 \text{ repeated } p \text{ times}, \zeta_1\sigma t^{-1/\sigma}, \ldots, \zeta_\sigma\sigma t^{-1/\sigma}\right) $$

with $\zeta_j = e^{\frac{2\pi ij}{\sigma}}$.

### 7.1.3. A preliminary remark concerning Proposition 4.2 in the $\mathrm{SL}_2(C)$ case

The hypothesis of Proposition 4.2 is:

> Let $f \in K^\times$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over $k$ of order $n \geqslant 1$ and assume that the differential Galois group $G_L$ over $k$ of the latter equation is simply connected. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation

(5) These "determining polynomials" are the "eigenvalues" of [S03, Definition 3.26].

over $k$. Assume that $f \notin k$ and that $f$ and $g$ are algebraically dependent over $k$.

With the notations of the proof of Proposition 4.2, we have

$$\pi_M = u \circ \pi_L.$$

We shall now focus on the case $G_L = \mathrm{SL}_2(\mathbb{C})$. Then, the classification of the representations of $\mathrm{SL}_2(\mathbb{C})$ shows that the representation $u : G_L \to G_M$ is conjugate to the direct sum of symmetric power representations

$$\mathrm{Sym}^{m_i} : G_L = \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{m_i+1}(\mathbb{C}).$$

Letting $\mathcal{M}$ and $\mathcal{L}$ denote the differential modules associated with $Y' = A_M Y$ and $Y' = A_L Y$, the tannakian correspondence implies that $\mathcal{M}$ is isomorphic to the direct sum of the $\mathrm{Sym}^{m_i}(\mathcal{L})$. We will now use this to study the algebraic relations between generalized hypergeometric series.

### 7.1.4. Proof of Theorem 1.6

The differential Galois group over $\overline{\mathbb{C}(x)}$ of $L = {}_0\mathscr{H}_1\left[-;\beta\right]$ is $\mathrm{SL}_2(\mathbb{C})$; see [Kat90, Theorem 3.6]. As explained in Section 7.1.3, the differential module $\mathcal{M}$ associated to the minimal nonzero differential equation $M$ with coefficients in $\overline{\mathbb{C}(x)}$ annihilating ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ is isomorphic to a direct sum of symmetric powers of $\mathcal{L}$, say

$$\mathcal{M} \cong \oplus_{i=1}^{r} \mathrm{Sym}^{m_i}(\mathcal{L}).$$

Let us first assume that $q+1 > p$. We have seen in Section 7.1.2 that $L$ is irregular at $\infty$: it has exactly one slope at $\infty$, namely $1/2$, and its list of determining polynomials at $\infty$ is $\pm 2x^{1/2}$. Therefore, the list of the determining polynomials of $\mathrm{Sym}^{m_i}(\mathcal{L})$ at $\infty$ is

$$-2m_i x^{1/2}, 2\left(-m_i+2\right)x^{1/2}, 2\left(-m_i+4\right)x^{1/2}, \ldots, 2\left(m_i-2\right)x^{1/2}, 2m_i x^{1/2}. \tag{7.4}$$

So, the list of determining polynomials of $\mathcal{M}$ is the concatenation of the lists (7.4) for $i$ varying in $\{1, \ldots, r\}$.

On the other hand, $M$ is a factor of ${}_p\mathscr{H}_q\left[\boldsymbol{\gamma};\boldsymbol{\delta}\right]$ so (see Section 7.1.3) the list of determining polynomials of $\mathcal{M}$ is a sublist of

$$0 \text{ with multiplicity } p, \zeta_1\sigma x^{1/\sigma}, \ldots, \zeta_\sigma \sigma x^{1/\sigma}$$

with $\sigma = q-p+1$ and $\zeta_j = e^{\frac{2\pi i j}{\sigma}}$.

Comparing the two preceding descriptions of the determining factors of $\mathcal{M}$, we find that $\sigma = 2$ and that we have

- either all the $m_i$ are equal to 0;
- or one of the $m_i$, say $m_r$, is equal to 1 and the other $m_i$ are equal to 0.

We claim that the first case cannot happen. Indeed, otherwise the entire function ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ would be algebraic and, hence, polynomial, which is false.

Therefore, we have

$$\mathcal{M} \cong \mathrm{Sym}^0(\mathcal{L}) \oplus \cdots \oplus \mathrm{Sym}^0(\mathcal{L}) \oplus \mathrm{Sym}^1(\mathcal{L}).$$

It follows that ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ is of the form $a+bg+cg'$ for some $a,b,c\in\overline{\mathbb{C}(x)}$, $b$ or $c\neq 0$, and some nonzero solution $g$ of $L$. Since the differential Galois group of $L$ over $\overline{\mathbb{C}(x)}$ is $\mathrm{SL}_2(\mathbb{C})$, the only possibility for $a+bg+cg'$ to be algebraically depend with ${}_0F_1\left[\alpha;\beta;x\right]$ is that $g=d\cdot{}_0F_1\left[-;\beta;x\right]$ for some $d\in\mathbb{C}$ and $c=0$. Therefore,

$$ {}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]\in\overline{\mathbb{C}(x)}{}_0F_1\left[-;\beta;x\right]+\overline{\mathbb{C}(x)}. $$

The fact that we can descend this linear relation to $\mathbb{C}(x)$ follows from the fact that ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ and ${}_0F_1\left[-;\beta;x\right]$ are entire functions and that ${}_0F_1\left[-;\beta;x\right]$ and 1 are linearly independent over $\overline{\mathbb{C}(x)}$.

Let us now consider the case $q+1\leqslant p$. In that case, we have seen that ${}_p\mathscr{H}_q\left[\boldsymbol{\gamma};\boldsymbol{\delta}\right]$ is regular at $\infty$. Since $M$ is a factor of ${}_p\mathscr{H}_q\left[\boldsymbol{\gamma};\boldsymbol{\delta}\right]$, $M$ and hence $\mathcal{M}$ are regular at $\infty$ as well. Since $\mathrm{Sym}^{m_i}(\mathcal{L})$ is irregular at $\infty$ if $m_i$ is nonzero, we infer that all the $m_i$ are equal to 0. Therefore, ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ belongs to $\overline{\mathbb{C}(x)}$. This excludes the case $q+1<p$, because the radius of convergence of ${}_pF_q\left[\boldsymbol{\gamma};\boldsymbol{\delta};x\right]$ is 0 in that case.

## 7.2. Iterated integrals

From [Hoc81, Chapter XVIII, Theorem 3.3] we know that a unipotent group is simply connected. This allows us to make an application of Proposition 4.3 when the Galois group is unipotent.

Definition 7.1 (Cf. [Sri10]). — *We say that $f\in K$ is an* iterated integral *over $k$ if, for some $n\in\mathbb{N}$, $f^{(n)}\in k$.*

Theorem 7.2. — *Let $f\in K^\times$ be an iterated integral over $k$ and let $g\in K$ satisfy a nonzero homogeneous linear differential equation over $k$. Assume that $k$ contains an element $x$ with $x'=1$, $f$ is transcendental over $k$ and that $f$ and $g$ are algebraically dependent over $k$. Then $g\in k[f]$.*

*Remark 7.3.* — The condition that $k$ contains an element $x$ such that $x'=1$ is needed. For example, let $k=\mathbb{C}$ and $K=\mathbb{C}(x),x'=1$. The element $x^2$ is an iterated integral over $k$, $x$ satisfies a linear differential equation over $k$ and these elements are algebraically dependent over $k$. Nonetheless, $x\notin\mathbb{C}[x^2]$.

We will use, again and again, the fact that if $E\subset F$ are differential fields with the same algebraically closed field of constants and $y\in F$ with $y'\in E$, then either $y\in E$ or $y$ is transcendental over $E$. This follows easily from the Galois theory and the fact that the only algebraic subgroups of $\mathbb{G}_a(C)$ are $\{0\}$ and $C$.

Lemma 7.4. — *Let $E\subset F$ be differential fields with the same algebraically closed subfield $C$ of constants. If $x\in E$ such that $x'=1$ and $y,z\in F$ such that*

(1) *$y'\in E$ and $y$ transcendental over $E$ and*
(2) *$z\in E(y)$ and $z'=y$,*

*then $z=Ay+B$ for some $A,B\in E$.*

*Proof.* — Let $z=p(y)/q(y)$ where $p$ and $q$ are relatively prime polynomials. As in the proof of Corollary 3.4, [HS85] implies that $q(y)'/q(y)$ is algebraic over $E$ and

therefore in $E$ (since $E$ is algebraically closed in $E(y)$). We then have that $E(q(y))$ is a Picard–Vessiot extension of $E$ whose differential Galois group $G$ is a subgroup of $\mathrm{GL}_1(C)$ and is a quotient of $\mathbb{G}_a(C)$. This implies that $G$ must be trivial so $q(y) \in E$.

Therefore we may write $z = a_m y^m + \ldots + a_0$ with the $a_i \in E$. Differentiating, we have

$$y = z' = a'_m y^m + \left(m a_m y' + a'_{m-1}\right) y^{m-1} + \ldots.$$

If $a'_m \neq 0$ we have $m = 1$ and our conclusion follows. If $a'_m = 0$ we must have $m \geqslant 2$ and we will derive a contradiction. We have that

$$m a_m y' + a'_{m-1} = c \in E$$

where $c = 0$ if $m > 2$ or $c = 1$ if $m = 2$. In either case, $y = \frac{1}{m a_m}(cx - a_{m-1} + d) \in E$ for some constant $d$, contradicting the fact that $y$ is transcendental over $E$. □

Lemma 7.5. — *Let $k \subset K$ be differential fields with the same algebraically closed subfield of constants $C$. Assume that there exists $x \in k$ such that $x' = 1$.*

1. *If $f$ is an iterated integral over $k$, then $k\langle f\rangle$ is a Picard–Vessiot extension of $k$ with unipotent Galois group.*
2. *If $f \notin k$, then there exist algebraically independent $y_1, \ldots y_{r-1}, y_r = f$ such that $k\langle f\rangle = k(y_1, \ldots, y_r)$. In particular, $k(f)$ is algebraically closed in $k\langle f\rangle$.*

*Proof.* — If $f^{(n)} = h \in k$ then $f$ satisfies

$$y^{(n+1)} - \frac{h'}{h} y^{(n)} = 0. \tag{7.5}$$

(1) If $f \in k$, then clearly $k\langle f\rangle$ is a Picard–Vessiot extension of $k$. If $f \notin k$ then $f, 1, x, \ldots x^{n-1}$ are linearly independent over $C$ and so form a basis of the solution space of (7.5) and $k\langle f\rangle$ is a Picard–Vessiot extension of $k$. In addition, if we define $k_i = k_{i-1}(f^{(n-i)})$, we have a tower of differential fields $k = k_0 \subset k_1 \subset \ldots \subset k_n = k\langle f\rangle$ where each $k_i$ is a Picard–Vessiot extension whose differential Galois group is either $\{0\}$ or $\mathbb{G}_a$. Therefore the differential Galois group of $k\langle f\rangle$ over $k$ is unipotent.

(2) Assume that the transcendence degree of $k\langle f\rangle$ over $k$ is $r$. By assumption $r \geqslant 1$. Using the construction of the $k_i$ we see that from the set $\{f^{(n-1)}, f^{(n-2)}, \ldots, f', f\}$ we may select a subset of elements

$$T = \left\{ y_i = f^{(n-n_i)} \,\middle|\, n_1 < n_2 < \ldots < n_r \right\}$$

that forms a transcendence basis. We wish to show that we can select these so that $n_r = n$. Suppose among all choices of the set $T$ we have selected one with $n_r$ maximal. If $n_r < n$, let $E = k(y_1, \ldots, y_{r-1}), y = y_r = f^{(n-n_r)}, z = f^{(n-(n_r+1))}$. Applying Lemma 7.4, we have that $y_r = az + b$ for some $a, b \in E$. Therefore

$$\begin{aligned} k\langle f\rangle &= k(f^{(n-n_1)}, \ldots, f^{(n-n_{r-1})}, f^{(n-n_r)}) \\ &= E(y_r) \\ &= E(z) \\ &= k(f^{(n-n_1)}, \ldots, f^{(n-n_{r-1})}, f^{(n-(n_r+1))}), \end{aligned}$$

contradicting the maximality of $n_r$. □

*Proof of Theorem 7.2.* — This now follows from Proposition 4.2. □

*For the rest of Section 7.2 we remove the assumption that $k$ is algebraically closed and only assume it is a differential field of characteristic zero with an algebraically closed subfield of constants.*

A consequence of the Kolchin–Ostrowski Theorem ([Kol68, Section 2]) is that if $k \subset K$ are differential fields with the same constants and $f_1, \ldots, f_n \in K$ with $f_i' \in k$ for $i = 1, \ldots n$ then the $f_i$ are algebraically dependent over $k$ if and only if there exist constants $c_i$, not all zero, such that $c_1 f_1 + \ldots + c_n f_n \in k$. Proposition 7.8 below gives a related result for iterated integrals.

COROLLARY 7.6. — *Let $k \subset K$ have the same field of constants and assume that there is an element $x \in k$ such that $x' = 1$. Let $f, g \in K$ be iterated integrals over $k$. If $f$ and $g$ are algebraically dependent over $k$ then there exist $u, v \in k$, not both zero, such that $uf + vg \in k$.*

*Proof.* — Let $E = k(f, f', \ldots, f^{(n-1)}, g, g', \ldots, g^{(m-1)})$. Proceeding in a manner similar to the proof of Lemma 7.5 (2), there is an algebraically independent set of elements $\{z_1, \ldots, z_t\} \subset \{f, f', \ldots, f^{(n-1)}, g, g', \ldots, g^{(m-1)}\}$ such that $E = k(z_1, \ldots, z_t)$, that is, $E$ is a purely transcendental extension of $k$. Therefore if either $f$ or $g$ are algebraic, then both of them will lie in $k$ and the conclusion follows. Therefore we may assume neither $f$ nor $g$ are algebraic over $k$.

Theorem 7.2 implies that $f = P(g)$ and $g = Q(f)$ for some polynomials $P, Q \in \bar{k}[Y]$ where $\bar{k}$ is the algebraic closure of $k$. This implies that $P$ and $Q$ are linear polynomials so $f$, $g$ and 1 are linearly dependent over $\bar{k}$. Since $E$ is a purely transcendental extension of $k$, it is linearly disjoint from $\bar{k}$ over $k$. Therefore $f$, $g$ and 1 are linearly dependent over $k$. □

*Example 7.7.* — Let $k = \mathbb{C}(x), K = \mathbb{C}(x, \log x)$ and let $f = \log x + x, g = x \log x$. We have $f', g'' \in k$ and $xf - g = x^2 \in k$.

PROPOSITION 7.8. — *Let $k \subset K$ have the same field of constants and assume that there is an element $x \in k$ such that $x' = 1$. Let $f_1, \ldots, f_n \in K$ be iterated integrals over $k$. If $f_1, \ldots, f_n$ are algebraically dependent over $k$ then there exist $u_1, \ldots, u_n \in k$, not all zero, such that $u_1 f_1 + \ldots + u_n f_n \in k$.*

*Proof.* — We can assume that for each pair $i, j,\ i \neq j$

- the set $\{f_1, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_n\}$ is algebraically independent over $k$,
- $f_i, f_j$ are transcendental over $F_{i,j} = k(f_1, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_n)$, and
- $f_i$ and $f_j$ are algebraically dependent over $F_{i,j}$.

Corollary 7.6 implies that

$$f_i = A_{i,j} f_j + B_{i,j} \text{ for some } A_{i,j}, B_{i,j} \in F_{i,j}.$$

In particular, $f_1 = A_{1,2} f_2 + B_{1,2}$ for some $A_{1,2}, B_{1,2} \in F_{1,2}$. We shall show that $A_{1,2} \in k$ and that $B_{1,2}$ is a $k$ linear combination of $f_3, \ldots, f_n, 1$. Note that $k(f_2, \ldots, f_n)$ is a purely transcendental extension of $k$. We will let $\partial_i$ denote the derivation $\frac{\partial}{\partial f_i}$ on this field.

$A_{1,2} \in k$**.** Since we also have $f_1 = A_{1,3}f_3 + B_{1,3}$ for some $A_{1,2}, B_{1,2} \in F_{1,3}$, $A_{1,3}f_3 + B_{1,3} = A_{1,2}f_2 + B_{1,2}$. Applying $\partial_3$, we have $A_{1,3} = \partial_3(A_{1,2})f_2 + \partial_3(B_{1,2})$. Since $A_{1,3}$ is independent of $f_3$ and $A_{1,2}$ is independent of $f_1, f_2$ we have $A_{1,2} \in k(y_4, \ldots, y_n)$ so $A_{1,2} = \alpha f_3 + \beta$, where $\alpha, \beta \in k(f_4, \ldots, f_n)$.

We also have $f_2 = A_{2,1}f_1 + B_{2,1}$ for some $A_{2,1}, B_{2,1} \in F_{1,2} = F_{2,1}$ and $f_2 = A_{2,3}f_3 + B_{2,3}$ for some $A_{2,3}, B_{2,3} \in F_{2,3}$. Comparing these two expressions in a manner similar to the above we have that $A_{2,1} = \bar{\alpha}f_3 + \bar{\beta}, \alpha, \beta \in k(f_4, \ldots, f_n)$. Since $A_{2,1} = A_{1,2}^{-1}$, we must have $\alpha = \bar{\alpha} = 0$, that is, $A_{1,2}$ is independent of $f_3$. Replacing $f_3$ with each $f_i, i = 4, \ldots, n$ in the above argument yields that $A_{1,2} \in k$.

$B_{1,2}$ **is a** $k$**-linear combination of** $f_3, \ldots, f_n, 1$**.** Note that the above argument can be modified to work equally well for $f_1, f_j,\ j > 1$ to conclude that $f_1 = A_{1,j}f_j + B_{1,j}$ where $A_{1,j} \in k$ and $B_{1,j} \in F_{1,j}$. Therefore $\partial_j(f_1) = A_{1,j} \in k$, that is, $f_1 \in k(f_2, \ldots, f_n)$ is linear in each of the $f_i, 2 \leqslant i \leqslant n$ and so the conclusion is obtained. □

## Appendix A. The tannakian correspondence

We give an informal description of this correspondence, enough to understand its use in our proofs. For a formal description see [DM81, Kat72, Kat87], [S03, Chapter 2 and Appendix B].

Let $k$ be a differential field of characteristic zero (not assumed to be algebraically closed) with algebraically closed field of constants $C$. Let

$$Y' = AY \tag{A.1}$$

with $A \in \mathrm{gl}_n(k)$ and let $K$ be its Picard–Vessiot extension. If $\mathfrak{Y} \in \mathrm{GL}_n(K)$ is a fundamental solution matrix of (A.1), we refer to the ring $k[\mathfrak{Y}, \det(\mathfrak{Y})^{-1}]$ generated by the entries of $\mathfrak{Y}$ and the inverse of its determinant as the Picard–Vessiot ring associated with (A.1). It is independent of our choice of fundamental solution matrix.

One can associate to (A.1) a differential module $M$, that is, a finite dimensional $k$-vector space $M$ endowed with a map $\partial : M \to M$ such that for all $f \in k$ and $m, n \in M$, $\partial(m + n) = \partial m + \partial n$ and $\partial(fm) = f'm + f\partial m$. This is done in the following way. Let $M = k^n$ and denote the standard basis by $\{e_i\}$. We define $\partial$ by setting $\partial(e_i) := -\sum_j a_{j,i}e_j$ where $A = (a_{i,j})$. Given any differential module together with a basis $\mathbf{e} = \{e_i\}$, one can reverse this process and produce a differential equation $Y' = A_{\mathbf{e}}Y$. If $\mathbf{f} = B\mathbf{e}$ is another basis of $M$ with $B \in \mathrm{GL}_n(k)$ then $A_{\mathbf{e}}$ and $A_{\mathbf{f}}$ are related by $A_{\mathbf{e}} = B^{-1}A_{\mathbf{f}}B - B^{-1}B'$. In this case we say that the equations are *gauge equivalent* or just *equivalent*.

Given differential modules $M_1, M_2$ one can define a differential module homomorphism as well as sub and quotient differential modules in the obvious way. One can also form direct sums, tensor products, and duals as in the following table:

| Construction | $\partial$ | Equation |
|---|---|---|
| $M_1 \oplus M_2$ | $\partial(m_1 \oplus m_2) = \partial_1 m_1 \oplus \partial_2 m_2$ | $Y' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} Y$ |
| $M_1 \otimes M_2$ | $\partial(m_1 \otimes m_2) = \partial_1 m_1 \otimes m_2 + m_1 \otimes \partial_2 m_2$ | $Y' = (A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2)\, Y$ |
| $M_1^*$ | $\partial(f)(m) = f(\partial_1(m))$ | $Y' = -A^T Y$ |

The collection of all differential modules forms a category closed under differential module homomorphism and the constructions of submodules, quotients, direct sums, tensor products, and duals, which we refer to as *the constructions of linear algebra*. We denote by $\{\{M\}\}$ the smallest subcategory containing $M$ and closed under these five constructions. Given any $N \in \{\{M\}\}$ and any basis $\mathbf{e}$ of $N$, we can form the associated differential equation $Y' = A_{N,\mathbf{e}} Y$, $A_{N,\mathbf{e}} \in \mathrm{gl}_m(k)$. It is known ([S03, Chapter 2.4]) that this equation has a fundamental matrix $\mathfrak{Y}_{N,\mathbf{e}} \in \mathrm{GL}_m(K)$ and, a fortiori, in the Picard–Vessiot ring $k[\mathfrak{Y}, \det(\mathfrak{Y})^{-1}]$ ([S03, Corollary 1.38]). The differential Galois group $G$ of $K$ over $k$ acts on the $C$-space $V$ spanned by the columns of $\mathfrak{Y}$ and so induces a representation of $G$. The space $V$ depends on our selection of bases of $N$ but it can be shown that the representation of $G$ is independent, up to $G$-isomorphism, of this choice of bases (see [S03, Chapter 2.4] for a basis free way of defining this representation). We will denote this representation of $G$ by $S(N)$.

We denote by $\mathrm{Repr}_G$ the category of representations of $G$, that is the category of finite dimensional $C$-vector spaces on which $G$ acts. In this category one also has a notion of the five constructions above. The tannakian correspondence [S03, Theorem 2.22] says that

> The map $N \mapsto S(N)$ is a bijective map between elements of $\{\{M\}\}$ (modulo isomorphism of differential modules) and representations of $G$ (modulo conjugacy of representations). This map is compatible with homomorphisms and the five constructions above.

One consequence of this is: if $k$ is algebraically closed, then there is an isomorphism $\varphi$ of the Picard–Vessiot ring $R = k[\mathfrak{Y}, \det(\mathfrak{Y})^{-1}]$ and the coordinate ring $k[G \otimes k]$ such that for $\sigma \in G$ and $z \in R$ we have

$$\varphi(\sigma(z)) = \rho_\sigma^*(\varphi(z))$$

where $\rho_\sigma^* : k[G \otimes k] \to k[G \otimes k]$ is the isomorphism which, when applied to a regular function $F \in k[G \otimes k]$, yields the regular function $\rho_\sigma^*(F) : G \otimes k \to G \otimes k$ given by $\rho_\sigma^*(F)(g) = F(g \cdot \sigma)$.

*Example A.1.* — Consider $Y' = AY$ with $A \in \mathrm{gl}_n(k)$ and let $\widetilde{A} = A - (\mathrm{tr}(A)/n) I_n$. Since $\mathrm{tr}(\widetilde{A}) = 0$, the differential Galois group of the equation $Y' = \widetilde{A} Y$ is unimodular ([S03, Exercise 1.35.5]) and this construction is often used to reduce questions about general linear differential equations to ones that have unimodular Galois groups. We now form the $2n \times 2n$ differential system

$$Y' = \mathrm{diag}\left((\mathrm{tr}(A)/n) I_n, \widetilde{A}\right) Y \tag{A.2}$$

and let $\widetilde{K}$ be its Picard-Vessiot extension with differential Galois group $H$. Let $K$ be the Picard–Vessiot extension for $Y' = AY$ and let $G$ be its differential Galois group.

If $\mathfrak{Y}$ is the fundamental solution matrix of $Y' = AY$, then $\mathfrak{Y} = \sqrt[n]{\det \mathfrak{Y}}$ is a solution of $y' = (\operatorname{tr} A/n)y$. Note that $\mathfrak{Y}$ need not lie in $K$ but $\mathfrak{Y}^n \in K$. A computation shows that if $\widetilde{\mathfrak{Y}}$ is a fundamental solution matrix of $y' = \widetilde{A}y$, then $\mathfrak{Y} I_n \cdot \widetilde{\mathfrak{Y}}$ is a fundamental solution matrix of $Y' = AY$. Therefore $K \subset \widetilde{K}$. The differential Galois theory implies that restricting $H$ to $K$ gives a short exact sequence

$$0 \to H_0 \to H \xrightarrow{\rho} G \to 0$$

where $H_0$ is a finite subgroup of $\mathrm{GL}_n(C)$ (since $\widetilde{K}$ is a simple radical extension of $K$) and the map $\rho$ is given by

$$\rho : \operatorname{diag}(t, h) \to th.$$

This defines a homomorphism of $H \subset \mathrm{GL}_{2n}(C)$ to $\mathrm{GL}_n(C)$ and so defines a representation of $H$.

This example appears in the proof of Proposition 5.2.

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