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Notes on Umemura polynomials

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To the memory of Professor Hiroshi Umemura

ABSTRACT. — We give a survey on special polynomials associated with algebraic solutions of the sixth Painlevé equation, and formulate a conjecture regarding a combinatorial formula for Umemura polynomials associated with a class of algebraic solutions of P_{VI} with two discrete parameters.

RÉSUMÉ. — Nous donnons un survol des polynômes spéciaux associés aux solutions algébriques de la sixième équation de Painlevé, et formulons une conjecture concernant une formule combinatoire pour les polynômes d'Umemura associés à une classe de solutions algébriques de P_{VI} avec deux paramètres discrets.

1. Introduction

The *Umemura polynomials* refer to the special polynomials associated with algebraic solutions of the sixth Painlevé equation P_{VI}. In this paper we give a survey on special polynomials associated with algebraic solutions of the sixth Painlevé equation as discussed in [10], and formulate a conjecture regarding a combinatorial formula for Umemura polynomials associated with a class of algebraic solutions obtained by a two-parameter family of Bäcklund transformations as investigated by Masuda [8].

Consider the following recurrence formula of Toda type for the sequence $U_n = U_n(x)$ $(n \in \mathbb{Z})$ of rational functions in x:

$$U_{n-1}U_{n+1} = (x^2 - 1)^2 \Big(x\, U_n'' U_n + U_n' U_n - x\, (U_n')^2 \Big)$$
$$- \Big(a^2(x+1)^2 - b^2(x-1)^2 - (2n-1)^2 x \Big) U_n^2, \quad (1.1)$$

with initial condition $U_0 = U_1 = 1$, where $' = \mathrm{d}/\mathrm{d}x$ stands for the derivation with respect to x, and a, b are parameters. This recurrence formula originates from the Toda equation for a sequence of τ functions associated with a certain algebraic solution of the sixth Painlevé equation. Leaving aside the question how this recurrence formula was derived, we make here some comments on remarkable features of this recurrence formula.

With the initial data $U_0 = U_1 = 1$, the second member is given by $U_2 = a^2(x+1)^2 - b^2(x-1)^2 - x$, and also U_3 is a polynomial of degree 6 in x. When we try to determine U_{n+1} inductively, we realize that the right-hand side of the recurrence formula is always divisible by U_{n-1}, and as a result U_{n+1} becomes a polynomial of degree $n(n+1)$. Although (1.1) is of rational nature, we see that this recurrence formula is *special* in the sense that it produces polynomials only. Since U_n is of degree $n(n-1)$, the degree grows rapidly and U_n are gigantic polynomials already for small n.

Note that, if $f(x)$ is a polynomial in x of degree $\leqslant d$, it can be written as $f(x) = F(x+1, x-1)$ by a unique homogenous polynomial $F(u, v)$ in (u, v) of degree d; in fact, $F(u, v)$ is determined by $F(u, v) = y^d f(x/y) = (\frac{u-v}{2})^d f(\frac{u+v}{u-v})$. If we rewrite U_n as a homogenous polynomial of $u = x+1$ and $v = x-1$ of degree $n(n-1)$, we see that U_n can be expressed in a concise form, and that something more interesting is happening.

$$
\begin{aligned}
U_2 &= -\frac{1}{4}(2a-1)(2a+1)(x+1)^2 + \frac{1}{4}(2b-1)(2b+1)(x-1)^2 \\
&= \frac{1}{2^2}\big(-(c-1)(c+1)u^2 + (d-1)(d+1)v^2\big),
\end{aligned}
\tag{1.2}
$$

where we have set $2a = c$ and $2b = d$. Similarly, we have

$$
\begin{aligned}
2^6 U_3 = {}&-(c-3)(c-1)^2(c+1)^2(c+3)u^6 \\
&+ 3(c-3)(c-1)(c+1)(c+3)(d-1)(d+1)u^4v^2 \\
&- 3(c-1)(c+1)(d-3)(d-1)(d+1)(d+3)u^2v^4 \\
&+ (d-3)(d-1)^2(d+1)^2(d+3)v^6. \quad (1.3)
\end{aligned}
$$

We now introduce *bilateral shifted factorials*

$$
c_k = \prod_{i=0}^{k-1}(c-k+1+2i), \quad d_k = \prod_{i=0}^{k-1}(d-k+1+2i) \quad (k = 0, 1, 2, \ldots), \tag{1.4}
$$

so that

$$
\begin{aligned}
&c_0 = 1, \quad c_1 = c, \quad c_2 = (c-1)(c+1), \quad c_3 = (c-2)c(c+2), \\
&c_4 = (c-3)(c-1)(c+1)(c+3), \quad \ldots.
\end{aligned}
\tag{1.5}
$$

(This definition of c_k is different from that of [10].) Then we have

$$
\begin{aligned}
2^2 U_2 &= -c_2 u^2 + d_2 v^2 \\
2^6 U_3 &= -c_4 c_2 u^6 + 3 c_4 d_2 u^4 v^2 - 3 c_2 d_4 u^2 v^4 + d_4 d_2 v^6.
\end{aligned}
\tag{1.6}
$$

The coefficients $1, 3, 3, 1$ are binomial coefficients. If we proceed to the next step,

$$
\begin{aligned}
2^{12} U_4 = {}& c_6 c_4 c_2 u^{12} - 6 c_6 c_4 d_2 u^{10} v^2 + 15 c_6 c_2 d_4 u^8 v^4 \\
& - 10 c_6 d_4 d_2 u^6 v^6 - 10 c_4 c_2 d_6 u^6 v^6 + 15 c_4 d_6 d_2 u^4 v^8 \\
& \qquad\qquad\qquad\qquad - 6 c_2 d_6 d_4 u^2 v^{10} + d_6 d_4 d_2 v^{12}, \quad (1.7)
\end{aligned}
$$

where the binomial coefficient $\binom{6}{3} = 20$ has split into two factorized terms with the same monomial $u^6 v^6$. Let's check one more:

$$
\begin{aligned}
2^{20} U_5 = {}& c_8 c_6 c_4 c_2 u^{20} - 10 c_8 c_6 c_4 d_2 u^{18} v^2 + 45 c_8 c_6 c_2 d_4 u^{16} v^4 \\
& - 50 c_8 c_6 d_4 d_2 u^{14} v^6 - 70 c_8 c_4 c_2 d_6 u^{14} v^6 + 175 c_8 c_4 d_6 d_2 u^{12} v^8 \\
& + 35 c_6 c_4 c_2 d_8 u^{12} v^8 - 126 c_8 c_2 d_6 d_4 u^{10} v^{10} - 126 c_6 c_4 d_8 d_2 u^{10} v^{10} \\
& + 35 c_8 d_6 d_4 d_2 u^8 v^{12} + 175 c_6 c_2 d_8 d_4 u^8 v^{12} - 70 c_6 d_8 d_4 d_2 u^6 v^{14} \\
& - 50 c_4 c_2 d_8 d_6 u^6 v^{14} + 45 c_4 d_8 d_6 d_2 u^4 v^{16} \\
& \qquad\qquad\qquad\qquad - 10 c_2 d_8 d_6 d_4 u^2 v^{18} + d_8 d_6 d_4 d_2 v^{20}. \quad (1.8)
\end{aligned}
$$

Seeing these formulas, one would expect that the general formula could be

$$
2^{(n+1)n} U_{n+1} = \sum_{I \sqcup J = \{2,4,\ldots,2n\}} (-1)^{\frac{1}{2} \sum_{i \in I} i} \, n_{I,J} \prod_{i \in I} c_i u^i \prod_{j \in J} d_j v^j
\tag{1.9}
$$

with some positive integers $n_{I,J}$, where the summation is taken over all divisions of the set $\{2, 4, \ldots, 2n\}$ into the disjoint union of two subsets.

A remarkable fact about these *Umemura polynomials* U_{n+1} is that each coefficient $n_{I,J}$ in (1.9) represents the dimension of an irreducible polynomial representation of $\mathrm{GL}(n+1)$. Recall that the irreducible polynomial representations of the general linear group $\mathrm{GL}(n+1)$ are parametrized by the partitions λ with $l(\lambda) \leqslant n+1$, i.e. non-increasing sequences of nonnegative integers

$$
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}), \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n+1} \geqslant 0.
\tag{1.10}
$$

For each partition λ, we denote by $L_{\mathrm{GL}(n+1)}(\lambda)$ the irreducible representation of highest weight λ. Then the dimension of $L_{\mathrm{GL}(n+1)}(\lambda)$ is given by the hook-length formula

$$
\dim_{\mathbb{C}} L_{\mathrm{GL}(n+1)}(\lambda) = \prod_{s \in \lambda} \frac{n + 1 + c_\lambda(s)}{h_\lambda(s)},
\tag{1.11}
$$

where for each box $s = (i, j)$ in the Young diagram λ, $c_\lambda(s) = j - i$ and $h_\lambda(s) = \lambda_i - i + \lambda'_j - j + 1$ denote the content and the hook-length of s respectively (for the combinatorial terminology for partitions, see Macdonald's book [7]). Returning to the formula (1.9), let (I, J) be a pair of subsets of $\{2, 4, \ldots, 2n\}$ such that $I \sqcup J = \{2, 4, \ldots, 2n\}$. We express the subsets I and J in the form $I = \{2i_1, 2i_2, \ldots, 2i_r\}$, $J = \{2j_1, 2j_2, \ldots, 2j_s\}$ by two strict partitions $i_1 > \cdots > i_r$ and $j_1 > \cdots > j_s$ such that $\{i_1, \ldots, i_r\} \sqcup \{j_1, \ldots, j_s\} = \{1, \ldots, n\}$. Then, to the strict partition $i_1 > i_2 > \cdots > i_r$ we attach a partition

$$\lambda_I = (i_1 - 1, \ldots, i_r - 1 | i_1, i_2, \ldots, i_r) \tag{1.12}$$

in terms of the Frobenius symbol. The Frobenius symbol $\lambda = (a_1, \ldots, a_r \mid b_1, \ldots, b_r)$ with $a_1 > \cdots > a_r \geqslant 0$, $b_1 > \cdots > b_r \geqslant 0$, represents the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i - i = a_i$ for $i = 1, \ldots, r$, and $\lambda'_j - j = b_j$ for $j = 1, \ldots, r$. The observation made by [10] was that

$$n_{I,J} = \dim_{\mathbb{C}} L_{\mathrm{GL}(n+1)}(\lambda_I) = \dim_{\mathbb{C}} L_{\mathrm{GL}(n+1)}(\lambda_J). \tag{1.13}$$

As an example, let's look at the coefficient of $c_8 c_4 d_6 d_2 u^{12} v^8$ in (1.8) for $2^{20} U_5$. In this case, $n = 4$, $I = \{8, 4\}$, $r = 2$. Since $(i_1, i_2) = (4, 2)$, the corresponding partition is $\lambda_I = (3, 1|4, 2) = (4, 3, 2, 2, 1)$.

$$\lambda_I = (3, 1|4, 2) = (4, 3, 2, 2, 1)$$

$$\tag{1.14}$$

Then the dimension of $L_{\mathrm{GL}(5)}(\lambda_I)$ is computed by the hook-length formula (1.11) as

$$= 5 \cdot 5 \cdot 7 = 175. \tag{1.15}$$

The conjecture (1.9), (1.13) of [10] proposed by Soichi Okada, Kazuo Okamoto, Hiroshi Umemura and the author has been proved by Taneda [15]. On the basis of this work, Kirillov–Taneda [4, 5] investigated a generalization of Umemura polynomials from the combinatorial point of view. However, it seems that they did not specify how their generalized Umemura polynomials

should be related to the sixth Painlevé equation. In the present paper, after reviewing the affine Weyl group symmetry of P_{VI}, we formulate a precise conjecture concerning combinatorial formula for Umemura polynomials associated with a larger class of algebraic solutions of P_{VI} with two discrete parameters in the spirit of Kirillov–Taneda.

2. Symmetry of the sixth Painlevé equation

In this section, we give a summary of basic facts concerning the discrete symmetry of the sixth Painlevé equation.

The sixth Painlevé equation

$$(P_{\mathrm{VI}}) \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t^2-1)}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \quad (2.1)$$

for $y = y(t)$ is expressed as a Hamiltonian system

$$(H_{\mathrm{VI}}) \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q} \quad (2.2)$$

with Hamiltonian $H = H(q, p; t)$ defined by

$$t(t-1)H = p^2 q(q-1)(q-t)$$
$$- p\{(\alpha_0 - 1)q(q-1) + \alpha_3 q(q-t) + \alpha_4(q-1)(q-t)\}$$
$$+ \alpha_2(\alpha_1 + \alpha_2)(q-t), \quad (2.3)$$

or explicitly,

$$t(t-1)\frac{\mathrm{d}q}{\mathrm{d}t} = 2pq(q-1)(q-t) - (\alpha_0 - 1)q(q-1)$$
$$- \alpha_3 q(q-t) - \alpha_4(q-1)(q-t),$$
$$t(t-1)\frac{\mathrm{d}p}{\mathrm{d}t} = -p^2\big((q-1)(q-t) + q(q-t) + q(q-1)\big) \qquad (2.4)$$
$$+ p\{(\alpha_0 - 1)(2q-1) + \alpha_3(2q-t) + \alpha_4(2q-1-t)\}$$
$$- \alpha_2(\alpha_1 + \alpha_2).$$

Here we use the complex parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ subject to the linear relation $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$. One can directly verify that the Hamiltonian system (H_{VI}) is equivalent to the sixth Painlevé equation (P_{VI}) for $y(t) = q(t)$ with parameters

$$\alpha = \frac{1}{2}\alpha_1^2, \quad \beta = -\frac{1}{2}\alpha_4^2, \quad \gamma = \frac{1}{2}\alpha_3^2, \quad \delta = \frac{1}{2}(1 - \alpha_0^2). \quad (2.5)$$

For two rational functions φ, ψ in (q, p, t), we define their Poisson bracket by

$$\{\varphi, \psi\} = \frac{\partial \varphi}{\partial p} \frac{\partial \psi}{\partial q} - \frac{\partial \varphi}{\partial q} \frac{\partial \psi}{\partial p}, \tag{2.6}$$

so that $\{p, q\} = 1$, $\{q, t\} = 0$, $\{p, t\} = 0$. Then, (2.2) implies that, for any $\varphi = \varphi(q, p, t)$, the differential equation to be satisfied by φ is expressed as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \{H, \varphi\} + \frac{\partial \varphi}{\partial t}. \tag{2.7}$$

We introduce parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ such that

$$\alpha_0 = 1 - \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3,$$
$$\alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = \varepsilon_3 + \varepsilon_4 \tag{2.8}$$

and regard $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ as the coordinates of the four dimensional affine space $V = \mathbb{C}^4$. We identify this parameter space V with the Cartan subalgebra of the simple Lie algebra $\mathfrak{so}(8)$ of type D_4, and regard the parameters α_j $(j = 0, 1, 2, 3, 4)$ as the *simple affine roots* of type $D_4^{(1)}$ identifying the *null root* $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ with the constant function 1 on V.

$$\tag{2.9}$$

It has been known since the pioneering work of Kazuo Okamoto [14] that the Hamiltonian system (H_{VI}) admits a group of *Bäcklund transformations* which is isomorphic to the extended affine Weyl group $\widetilde{W}(D_4^{(1)})$ of type D_4. This group $\widetilde{W}(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4; r_1, r_3, r_4 \rangle$ is generated by *simple reflections* s_i $(i = 0, 1, \ldots, 4)$ attached to the simple affine roots α_i and the Dynkin diagram automorphisms r_1, r_3 and $r_4 = r_1 r_3$ corresponding to the permutations $\sigma_1 = (01)(34)$, $\sigma_3 = (03)(14)$, $\sigma_4 = (04)(13)$ of $\{0, 1, 3, 4\}$: these generators are subject to the fundamental relations

$$s_i^2 = 1 \ (i = 0, 1, \ldots, 4); \quad s_i s_j = s_j s_i \ (i, j = 0, 1, 3, 4); \quad s_i s_2 s_i = s_2 s_i s_2$$
$$(i = 0, 1, 3, 4) \tag{2.10}$$

and

$$r_i^2 = 1 \ (i = 1, 3); \quad r_1 r_3 = r_3 r_1; \quad r_k s_i = s_{\sigma_k(i)} r_k$$
$$(k = 1, 3 : i = 0, 1, 3, 4). \tag{2.11}$$

Table 2.1. Fundamental Bäcklund transformations of (H_{VI})

	α_0	α_1	α_2	α_3	α_4	q	p
s_0	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	q	$p - \dfrac{\alpha_0}{q-t}$
s_1	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	q	p
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	$q + \dfrac{\alpha_2}{p}$	p
s_3	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	q	$p - \dfrac{\alpha_3}{q-1}$
s_4	α_0	$-\alpha_1$	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	q	$p - \dfrac{\alpha_4}{q}$
r_1	α_1	α_0	α_2	α_4	α_3	$\dfrac{t(q-1)}{q-t}$	$-\dfrac{(q-t)((q-t)p + \alpha_2)}{t(t-1)}$
r_3	α_3	α_4	α_2	α_0	α_1	$\dfrac{t}{q}$	$-\dfrac{q(qp + \alpha_2)}{t}$
r_4	α_4	α_3	α_2	α_1	α_0	$\dfrac{q-t}{q-1}$	$\dfrac{(q-1)((q-1)p + \alpha_2)}{t-1}$

Let $\mathcal{K} = \mathbb{C}(\alpha; q, p, t)$ the field of rational functions in $(\alpha_1, \alpha_2, \alpha_3, \alpha_4; q, p, t)$ and define the derivation $\partial_t = \mathrm{d}/\mathrm{d}t : \mathcal{K} \to \mathcal{K}$ by

$$\partial_t(\alpha_i) = 0 \quad (i = 0, 1, \ldots, 4);$$

$$\partial_t(q) = \frac{\partial H}{\partial p}, \quad \partial_t(p) = -\frac{\partial H}{\partial q}, \quad \partial_t(t) = 1. \quad (2.12)$$

By a *Bäcklund transformation* of (H_{VI}), we mean an automorphism w of the differential field $(\mathcal{K}, \partial_t)$, i.e. a $\mathbb{C}$-automorphism $w : \mathcal{K} \to \mathcal{K}$ of the field that commutes with ∂_t: $w(\partial_t(\varphi)) = \partial_t(w(\varphi))$ for all $\varphi \in \mathcal{K}$. If we regard an automorphism of $\mathcal{K}$ as representing a birational mapping of the seven dimensional affine space with coordinates $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; q, p, t)$, then a Bäcklund transformation is a birational transformation which leaves the differential equation (H_{VI}) invariant.

The Hamiltonian system (H_{VI}) has eight *fundamental* Bäcklund transformations s_0, s_1, s_2, s_3, s_4 and r_1, r_3, r_4 as specified in Table 2.1 in terms of the action on the generators of $\mathcal{K}$. Note also that these transformations preserve the Poisson bracket, namely $w(\{\varphi, \psi\}) = \{w(\varphi), w(\psi)\}$ for all $\varphi, \psi \in \mathcal{K}$.

We give below some comments on how to find these Bäcklund transformations. Leaving this question aside, one can directly verify that these birational transformations leave (H_{VI}) invariant, and also that they satisfy the fundamental relations (2.10), (2.11) of the generators of the extended affine Weyl group $\widetilde{W}(D_4^{(1)})$. We denote by $\widetilde{W} = \langle s_0, s_1, s_2, s_3, s_4, r_1, r_3, r_4 \rangle \subset \mathrm{Aut}(\mathcal{K}, \partial_t)$ the group of Bäcklund transformations generated by s_i and r_i. Note that the action of $\widetilde{W}$ stabilizes the subfield $\mathbb{C}(\alpha) \subset \mathcal{K}$, and is identical to the action of

$\widetilde{W}(D_4^{(1)})$ on the field of rational functions of V induced from its standard action on the affine space V. In what follows, we denote by $\langle\,\cdot\,,\cdot\,\rangle : V^* \times V^* \to \mathbb{C}$ the scalar product (symmetric $\mathbb{C}$-bilinear form) defined by $\langle\varepsilon_i,\varepsilon_j\rangle = \delta_{i,j}$ $(i,j=1,\ldots,4)$. Also, we use the notation

$$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4 \subset P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3 \oplus \mathbb{Z}\omega_4 \subset V^* \quad (2.13)$$

the *root lattice* and the *weight lattice* of type D_4; the *fundamental weights* are specified as

$$\omega_1 = \varepsilon_1, \ \ \omega_2 = \varepsilon_1+\varepsilon_2, \ \ \omega_3 = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4), \ \ \omega_4 = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4).$$
$$(2.14)$$

For each $\mu \in P$, we define the *translation* $T_\mu : \mathbb{C}(\alpha) \to \mathbb{C}(\alpha)$ by μ as the $\mathbb{C}$-automorphism such that $T_\mu(\varepsilon_i) = \alpha - \langle\mu,\varepsilon_i\rangle$ $(i=1,2,3,4)$. Then we obtain the two groups of translations

$$T(Q) = \{T_\alpha \mid \alpha \in Q\} \subset T(P) = \{T_\mu \mid \mu \in P\} \subset \mathrm{Aut}(\mathbb{C}(\alpha)) \qquad (2.15)$$

by the root lattice and by the weight lattice; both $T(Q)$ and $T(P)$ are free abelian groups of rank 4. It is known that $T(P)$ is contained in the extended affine Weyl group $\widetilde{W} = \langle s_0,s_1,s_2,s_3,s_4;r_1,r_3,r_4\rangle$, and $\widetilde{W}$ is expressed as the semidirect product $\widetilde{W} = T(P) \rtimes W_0$ of $T(P)$ and the finite Weyl group $W_0 = W(D_4) = \langle s_1,s_2,s_3,s_4\rangle$ acting on $T(P)$. We remark that the subgroup $W = \langle s_0,s_1,s_2,s_3,s_4\rangle \subset \widetilde{W}$ is the standard affine Weyl group $W(D_4^{(1)})$, and is expressed as the semidirect product $W = T(Q) \rtimes W_0$. In terms of the generators of $\widetilde{W}$, the translations by the simple roots and the fundamental weights are expressed as follows.

$$
\begin{array}{ll}
T_{\alpha_1} = s_2s_0s_3s_4s_2s_0s_3s_4s_2s_1 & T_{\omega_1} = s_0s_2s_3s_4s_2s_0r_1 \\[2pt]
T_{\alpha_2} = s_0s_1s_3s_4s_2s_0s_1s_3s_4s_2 & T_{\omega_2} = s_0s_2s_1s_3s_4s_2s_1s_3s_4s_2 \\[2pt]
T_{\alpha_3} = s_2s_0s_1s_4s_2s_0s_1s_4s_2s_3 & T_{\omega_3} = s_0s_2s_1s_4s_2s_0r_3 \\[2pt]
T_{\alpha_4} = s_2s_0s_1s_3s_2s_0s_1s_3s_2s_4 & T_{\omega_4} = s_0s_2s_1s_3s_2s_0r_4.
\end{array}
\qquad (2.16)
$$

We already know that the action of the extended Weyl group $\widetilde{W} = \langle s_0,s_1,s_2,s_3,s_4;r_1,r_3,r_4\rangle$ on $\mathbb{C}(\alpha)$ can be lifted to $\mathcal{K} = \mathbb{C}(\alpha;q,p,t)$ preserving the fundamental relations of the generators. From the representation $\widetilde{W} = T(P) \rtimes W_0$, we obtain a commutative family T_μ $(\mu \in P)$ of Bäcklund transformations parametrized by the weight lattice. These translations $T_\mu : \mathcal{K} \to \mathcal{K}$ are also called the *Schlesinger transformations*.

We remark that, besides this transformation group $\widetilde{W} \subset \mathrm{Aut}(\mathcal{K},\partial_t)$, the Hamiltonian system (H_{VI}) has birational canonical transformations (that leave the 2-form $\Omega = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t$ invariant) corresponding to the permutation group $\mathfrak{S}_{\{0,1,3,4\}}$ of the indexing set $\{0,1,3,4\}$. We describe in Table 2.2 the birational canonical transformations σ_{ij} corresponding to the

Table 2.2. Birational canonical transformations that realize $\mathfrak{S}_{\{0,1,3,4\}}$

	α_0	α_1	α_2	α_3	α_4	t	∂_t	q	p
σ_{01}	α_1	α_0	α_2	α_3	α_4	$1-t$	$-\partial_t$	$\dfrac{(1-t)q}{q-t}$	$\dfrac{(q-t)((q-t)p+\alpha_2)}{t((t-1)}$
σ_{03}	α_3	α_1	α_2	α_0	α_4	$\dfrac{1}{t}$	$-t^2\partial_t$	$\dfrac{q}{t}$	tp
σ_{04}	α_4	α_1	α_2	α_3	α_0	$\dfrac{t}{t-1}$	$-(t-1)^2\partial_t$	$\dfrac{q-t}{1-t}$	$(1-t)p$
σ_{13}	α_0	α_3	α_2	α_1	α_4	$\dfrac{t}{t-1}$	$-(t-1)^2\partial_t$	$\dfrac{q}{q-1}$	$-(q-1)((q-1)p+\alpha_2)$
σ_{14}	α_0	α_4	α_2	α_3	α_1	$\dfrac{1}{t}$	$-t^2\partial_t$	$\dfrac{1}{q}$	$-q(qp+\alpha_2)$
σ_{34}	α_0	α_1	α_2	α_4	α_3	$1-t$	$-\partial_t$	$1-q$	$-p$

transpositions $(ij) \in \mathfrak{S}_{\{0,1,3,4\}}$. Notice that the Bäcklund transformations r_1, r_3, r_4 are obtained from these birational canonical transformations as compositions $r_1 = \sigma_{01}\sigma_{34}$, $r_3 = \sigma_{03}\sigma_{14}$, $r_4 = \sigma_{04}\sigma_{13}$. If we include these birational canonical transformations that represent $\mathfrak{S}_{\{0,1,3,4\}}$ as well, the symmetry group of (H_{VI}) extends to an affine Weyl group of type F_4.

Recall that the Hamiltonian system (H_{VI}) arises from the monodromy preserving deformation

$$
\begin{aligned}
\left(\partial_z^2 + a_1(z;t)\partial_z + a_2(z;t)\right)u(z;t) &= 0, \\
\partial_t u(z;t) &= (b_1(z;t)\partial_z + b_2(z;t))\,u(z;t)
\end{aligned}
\tag{2.17}
$$

of a second order Fuchsian differential equation for $u = u(z;t)$ on $\mathbb{P}^1$ with four regular singular points and an apparent singularity, where $\partial_z = \partial/\partial z$ and $\partial_t = \partial/\partial t$. The first equation is assumed to be Fuchsian with Riemann scheme

$$
\left\{
\begin{array}{ccccc}
z=0 & z=1 & z=t & z=q & z=\infty \\
0 & 0 & 0 & 0 & \alpha_2 \\
\alpha_4 & \alpha_3 & \alpha_0 & 2 & \alpha_1+\alpha_2
\end{array}
\right\},
$$
$$
\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \quad (2.18)
$$

and the singularity $z = q$ with characteristic exponents $0, 2$, which may depend on t, is assumed to be non-logarithmic. Under these assumptions, the coefficients $a_1(z;t)$ and $a_2(z;t)$ are determined uniquely by $p =$

$\text{Res}_{z=q}(a_2(z;t)\mathrm{d}z)$ as

$$
\begin{aligned}
a_1(z;t) &= \frac{1-\alpha_4}{z} + \frac{1-\alpha_3}{z-1} + \frac{1-\alpha_0}{z-t} - \frac{1}{z-q}, \\
a_2(z;t) &= \frac{1}{z(z-1)}\left(-\frac{t(t-1)H}{z-t} + \frac{q(q-1)p}{z-q} + \alpha_2(\alpha_1+\alpha_2)\right),
\end{aligned}
\tag{2.19}
$$

In particular the Hamiltonian is obtained as $H = -\text{Res}_{z=t}(a_2(z;t)\mathrm{d}z)$. The compatibility condition of (2.17) is then equivalent to the Hamiltonian system (H_{VI}). In this context of monodromy preserving deformation, the Bäcklund transformations s_0, s_1, s_3, s_4 except s_2 arise from gauge transformations $u(z;t) \to g(z;t)u(z;t)$ by power functions in z branching only at $z = 0, 1, t, \infty$. Also, the birational canonical transformations of Table 2.2 are obtained from the coordinate changes in z by fractional linear transformations that permutes the singular points $\{0, 1, t, \infty\}$.

Where does the fundamental Bäcklund transformation s_2 come from? We give a comment on how Okamoto obtained nontrivial Bäcklund transformations, recalling some of the key ideas of [14]. Incidentally, s_2 is sometimes called the *Okamoto transformation*. Okamoto renormalizes the Hamiltonian by setting

$$
\begin{aligned}
h &= t(t-1)H(q,p;t) + c_1 t + c_0, \\
c_1 &= \varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_3, \\
c_0 &= -\frac{1}{2}(\varepsilon_1\varepsilon_2 - \varepsilon_1\varepsilon_3 - \varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3 - \varepsilon_2\varepsilon_4 + \varepsilon_3\varepsilon_4),
\end{aligned}
\tag{2.20}
$$

and shows that the Hamiltonian system (H_{VI}) is equivalent to a single differential equation

$$
\frac{\mathrm{d}h}{\mathrm{d}t}\left(t(t-1)\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)^2 + \left(2h\frac{\mathrm{d}h}{\mathrm{d}t} - (2t-1)\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\right)^2
$$
$$
= \left(\frac{\mathrm{d}h}{\mathrm{d}t}+\varepsilon_1^2\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}+\varepsilon_2^2\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}+\varepsilon_3^2\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}+\varepsilon_4^2\right) \quad (2.21)
$$

for the Hamiltonian function $h = h(t)$ under the assumption that $\mathrm{d}^2h/\mathrm{d}t^2 \neq 0$. Also, if $h = h(t)$ is a generic solution of (2.21), the variables q, p can be recovered from h, $h' = \mathrm{d}h/\mathrm{d}t$ and $h'' = \mathrm{d}^2h/\mathrm{d}t^2$ by the relations

$$
\begin{aligned}
2(h'+\varepsilon_1^2)(h'+\varepsilon_2^2)q &= -(\varepsilon_1+\varepsilon_2)t(t-1)h'' \\
&\quad + ((2t-1)h' - 2h - \varepsilon_1\varepsilon_2)(h'-\varepsilon_1\varepsilon_2) \\
&\quad + (h'+\varepsilon_1^2+\varepsilon_1\varepsilon_2+\varepsilon_2^2)(h'+\varepsilon_3\varepsilon_4), \\
(\varepsilon_1+\varepsilon_2)q(q-1)p &= h + (q-t)h' + c_1 q + c_0.
\end{aligned}
\tag{2.22}
$$

Note that the differential equation (2.21) is manifestly invariant under the action of the Weyl group $W(D_4)$ of type D_4 on the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ (the

permutations and the even number of sign changes of ε_i). (In fact Okamoto uses the coordinates $b = (b_1, b_2, b_3, b_4)$ of V such that $b_1 = \varepsilon_3$, $b_2 = \varepsilon_4$, $b_3 = -\varepsilon_2$, $b_4 = -\varepsilon_1$.) Analyzing the differential equation (2.21), Okamoto then constructs birational canonical transformations lifting the simple reflections s_i on the parameter space, as well as those lifting parallel translations on the parameters. In fact, the birational canonical transformation corresponding to s_2 (transposition of $\varepsilon_2, \varepsilon_3$) is determined as

$$s_2 : \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; q, p, t, H) \quad \longrightarrow$$
$$\left(\varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_4; q + \frac{\varepsilon_2 - \varepsilon_3}{p}, p, t, H + \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)}{t} + \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_1 + \varepsilon_4)}{t - 1} \right).$$
$$(2.23)$$

As to parallel translations, Okamoto deals with the shift with respect to ε_2 as an example; the corresponding canonical transformation is given by

$$T_{\varepsilon_2} : \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; q, p, t, H) \quad \longrightarrow$$
$$\left(\varepsilon_1, \varepsilon_2 - 1, \varepsilon_3, \varepsilon_4; \widetilde{q}, \widetilde{p}, t, H + \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)}{t} + \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_1 + \varepsilon_4)}{t - 1} \right),$$
$$(2.24)$$

where $\widetilde{q}$, $\widetilde{p}$ are rational functions in (q, p) which do not specify here. Also, Okamoto constructs birational canonical transformations corresponding to the permutations of the simple affine roots $\alpha_0, \alpha_1, \alpha_3, \alpha_4$. In this way, Okamoto clarified that the transformation group of (H_{VI}) of birational canonical transformations that leaves ∂_t invariant form an extension of the affine Weyl group of type D_4, and that, together with the birational canonical transformations lifting the Dynkin diagram automorphisms, this transformation group extends to an affine Weyl group of type F_4.

In order to describe the action of translations in a systematic way, Okamoto introduces the notion of τ function for the Hamiltonian system (H_{VI}) as a dependent variable such that

$$H = \frac{\mathrm{d}}{\mathrm{d}t} \log \tau = \frac{1}{\tau} \frac{\mathrm{d}\tau}{\mathrm{d}t}. \tag{2.25}$$

Note that this τ function is determined up to multiplication by a constant which may depend on the parameters. For each $m \in \mathbb{Z}$, we denote the m-th iteration of the birational canonical transformation (2.24) by

$$T_{\varepsilon_2}^m : \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; q, p, t, H) \to (\varepsilon_1, \varepsilon_2 - m, \varepsilon_3, \varepsilon_4; q_m, p_m, t, H_m) \tag{2.26}$$

and by τ_m the corresponding τ function. Investigating the differential equation (2.21) for the Hamiltonian function, Okamoto proves that this sequence

of τ functions $(\tau_m)_{m\in\mathbb{Z}}$ satisfies the Toda equation

$$\frac{\mathrm{d}}{\mathrm{d}t}t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\log\tau_m + (\varepsilon_1+\varepsilon_2-m)(\varepsilon_2-\varepsilon_3-m) = c_m\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2} \quad (m\in\mathbb{Z}),$$
(2.27)

where c_m are constants which depend on the renormalization of the τ functions. If we specify the initial τ functions τ_0, τ_1, in the generic setting all the other τ functions τ_m are determined by this Toda equation.

Remark 2.1. — In the context of the monodromy preserving deformation of second order Fuchsian differential equations, it is known that the Okamoto transformation s_2 can be interpreted in terms of an integral transformation of Euler type, called the *middle convolution* (see Filipuk [1], Novikov [13]; for related works by Boalch, Harnad, and Mazzocco, see Haraoka–Filipuk [2] and references therein). Also, in the framework of the 8×8 Lax pair of Noumi–Yamada [12], the Bäcklund transformations discussed above naturally arise from the action of the affine Weyl group associated with $\widehat{\mathfrak{so}}(8)$.

3. Umemura polynomials with two discrete parameters

After his works on the irreducibility and the classification of transcendental classical solutions of the Painlevé equations ([16, 17, 18]), Hiroshi Umemura proposed around 1995 to investigate the special polynomials arising from the sequences of τ functions of the sixth Painlevé equation ([10, 20]). His interest was directed in particular to those associated with algebraic solutions appearing as fixed points of Bäcklund transformations. Special polynomials in this case can be regarded as counterparts for P_{VI} of the Yablonski–Vorobiev polynomials for P_{II} and the Okamoto polynomials for P_{IV} (see [9] for the cases of P_{II} and P_{IV}). In this section we reproduce the construction of Umemura polynomials starting from a simpler seed solution than the one Umemura originally worked on. After that, we introduce *Umemura polynomials $U_{m,n}(x;c,d)$ with two discrete parameters* following the work of Tetsu Masuda [8]. We will not touch here upon the classical solutions of hypergeometric type that appear on the walls of the affine Weyl group.

Keeping the notations of the previous section, we consider the extended affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, s_3, s_4; r_1, r_3, r_4\rangle$ acting on the differential field $(\mathcal{K}, \partial_t)$, $\mathcal{K} = \mathbb{C}(\alpha; q, p, t)$, as a group of Bäcklund transformations for (H_{VI}). As we remarked already, this transformation group can also be expressed as the semidirect product $\widetilde{W} = T(P) \rtimes W_0$ of the abelian group of translations by the weight lattice P (Schlesinger transformations) and the

finite Weyl group $W_0 = W(D_4) = \{s_1, s_2, s_3, s_4\}$. Let $w \in \widetilde{W}$ and set

$$w(\alpha_i) = \alpha_{w,i} \quad (i = 0, 1, \dots, 4),$$
$$w(q) = \psi_w(\alpha; q, p, t), \quad w(p) = \varphi_w(\alpha; q, p, t), \quad (3.1)$$

where $\alpha_{w,i}$ are linear functions in α_j, and $\psi_w(\alpha; q, p, t)$, $\varphi_w(\alpha; q, p, t)$ are rational functions in (q, p, t) with coefficients in $\mathbb{C}(\alpha)$. Let $(q(t), p(t))$ be a solution of (H_{VI}) with parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and consider the functions

$$q_w(t) = \psi_w(\alpha; q(t), p(t), t), \quad p_w(t) = \varphi_w(\alpha; q(t), p(t), t), \quad (3.2)$$

supposing $(q(t), p(t))$ is generic in the sense that these substitutions make sense. The assumption that w commutes with ∂_t then implies that $(q_w(t), p_w(t))$ gives rise to a solution of (H_{VI}) with parameters $\alpha_w = (\alpha_{w,0}, \dots, \alpha_{w,4})$. In this way, we obtain a large class of solutions $(q_w(t), p_w(t))$ $(w \in \widetilde{W})$ starting from a generic *seed solution* $(q(t), p(t))$.

Suppose now that $w \in \widetilde{W}$ has a fixed point $(\alpha; q, p)$, $q = q(t)$, $p = p(t)$:

$$\alpha_{w,i} = \alpha_i \ (i = 0, 1, \dots, 4), \quad \psi_w(\alpha; q, p) = q, \quad \varphi_w(\alpha; q, p, t) = p. \quad (3.3)$$

Note that the assumption that w commute with ∂_t can be written as

$$\begin{aligned}
\frac{\partial H}{\partial p} \frac{\partial \psi_w}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial \psi_w}{\partial p} + \frac{\partial \psi_w}{\partial t} &= w\left(\frac{\partial H}{\partial p}\right), \\
\frac{\partial H}{\partial p} \frac{\partial \varphi_w}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial \varphi_w}{\partial p} + \frac{\partial \varphi_w}{\partial t} &= -w\left(\frac{\partial H}{\partial q}\right).
\end{aligned} \quad (3.4)$$

On the other hand, the assumption that $(q(t), p(t))$ is a fixed point of w implies

$$\frac{dq}{dt} \frac{\partial \psi_w}{\partial q} - \frac{dp}{dt} \frac{\partial \psi_w}{\partial p} + \frac{\partial \psi_w}{\partial t} = \frac{dq}{dt}, \quad \frac{dq}{dt} \frac{\partial \varphi_w}{\partial q} - \frac{dp}{dt} \frac{\partial \varphi_w}{\partial p} + \frac{\partial \varphi_w}{\partial t} = \frac{dp}{dt}. \quad (3.5)$$

Noting that $w(\partial H / \partial p) = \partial H / \partial p$, $w(\partial H / \partial q) = \partial H / \partial q$ at the fixed point, and using $\{\varphi_w, \psi_w\} = 1$, one can show that $(q(t), p(t))$ must be a solution of (H_{VI}) whenever $\partial \psi_w / \partial q + \partial \varphi_w / \partial p \neq 2$.

We now take the Dynkin diagram automorphism r_3 for w. From the data of Table 2.1, the condition of a fixed point of r_3 is given by

$$\alpha_0 = \alpha_3, \ \alpha_1 = \alpha_4, \ t/q = q, \ -q(qp + \alpha_2)/t = p. \quad (3.6)$$

Solving these equations, we see that, for the parameter values

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(a, b, \frac{1}{2} - a - b, a, b\right) \quad (a, b \in \mathbb{C}), \quad (3.7)$$

there exist simple algebraic solutions

$$(q(t), p(t)) = \left(t^{\frac{1}{2}}, \frac{1}{2}\left(a + b - \frac{1}{2} \right) t^{-\frac{1}{2}} \right).\tag{3.8}$$

We investigate below the class of algebraic solutions that are obtained by a two-parameter family of Bäcklund transformations from this seed solution.

For the moment, we take the fundamental weight $\omega_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$ and use the m-th iteration of the translation T_{ω_3}:

$$(T_{\omega_3}^m(\alpha_0), \ldots, T_{\omega_3}^m(\alpha_4)) = (\alpha_0 + m, \alpha_1, \alpha_2, \alpha_3 - m, \alpha_4) \qquad (m \in \mathbb{Z}).\tag{3.9}$$

Applying the Bäcklund transformation $w = T_{\omega_3}^m$ to the seed solution mentioned above, we obtain algebraic solutions $(q_m(t), p_m(t))$ $(m \in \mathbb{Z})$ with α parameters $(a + m, b, \frac{1}{2} - a - b, a - m, b)$ for each $m \in \mathbb{Z}$. For example, we first compute

$$T_{\omega_3}(q) = \frac{t(p(q-t) - \alpha_0)(pq(q-t) - (\alpha_0 + \alpha_4)q + t\alpha_4)}{(pq(q-t) + \alpha_2 q - (\alpha_0 + \alpha_2)t)(pq(q-t) + (\alpha_1 + \alpha_2)q - (\alpha_0 + \alpha_1 + \alpha_2)t)}$$

$$T_{\omega_3}(p) = -\frac{(pq(q-t) + \alpha_2 q - (\alpha_0 + \alpha_2)t)\,(pq(q-t) + (\alpha_1 + \alpha_2)q - (\alpha_0 + \alpha_1 + \alpha_2)t)}{t\,(p(q-t) - \alpha_0)}$$

$$\cdot \frac{(p^2 q(q-t) + p(\alpha_1 + 2\alpha_2)q - (\alpha_0 + \alpha_1 + 2\alpha_2)t + \alpha_2(\alpha_1 + \alpha_2))}{(p^2 q(q-t)^2 + p(q-t)((\alpha_1 + 2\alpha_2)q + \alpha_4 t) + \alpha_2(\alpha_1 + \alpha_2)q - ((\alpha_0 + \alpha_2)^2 + (\alpha_0\alpha_1 + \alpha_2\alpha_1 + \alpha_0\alpha_4))t)},\tag{3.10}$$

and then substitute the seed solution $(q, p) = (x, \frac{1}{2}(a + b - \frac{1}{2})x^{-1})$, $x = t^{\frac{1}{2}}$, with parameters $\alpha = (a, b, \frac{1}{2} - a - b, a, b)$ to obtain an algebraic solution

$$q_1 = \frac{x\left((a - b - \frac{1}{2})x + a + b + \frac{1}{2} \right)\left((a + b - \frac{1}{2})x + a - b + \frac{1}{2} \right)}{\left((a - b + \frac{1}{2})x + a + b - \frac{1}{2} \right)\left((a + 2b + \frac{1}{2})x + a - b - \frac{1}{2} \right)}$$

$$p_1 = \frac{(a + b - \frac{1}{2})((a - b + \frac{1}{2})x + a + b - \frac{1}{2})((a + b - \frac{3}{2})x + a - b - \frac{1}{2})((a + b + \frac{1}{2})x + a - b - \frac{1}{2})}{2x((a + b - \frac{1}{2})x + a - b + \frac{1}{2})((a - \frac{1}{2})^2 - b^2)x^2 + 2((a - \frac{1}{2})^2 + b^2 - \frac{1}{2})x + (a - \frac{1}{2})^2 - b^2)}\tag{3.11}$$

with parameters $(a + 1, b, \frac{1}{2} - a - b, a - 1, b)$. In this example, one can already observe a characteristic phenomenon of factorization of the action of $\widetilde{W}$. The Umemura polynomials arise as the special polynomials in $x = t^{\frac{1}{2}}$ associated with the sequence of τ functions for these algebraic solutions with parameters $(a + m, b, \frac{1}{2} - a - b, a - m, b)$ $(m \in \mathbb{Z})$.

We now consider to extend the action of $\widetilde{W} = \langle s_0, s_1, s_2, s_3, s_4; r_1, r_3, r_4 \rangle$ on $\mathcal{K} = \mathbb{C}(\alpha; q, p, t)$ to the level of τ functions. We introduce four τ functions $\tau_0, \tau_1, \ldots, \tau_4$ corresponding to the four nodes of the Dynkin diagram (more precisely, they are *multiplicative* variables attached to the affine fundamental weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_4$). We need to specify the differential equations to be satisfied by τ_i as well as how they should behave under the action of $\widetilde{W}$. From the viewpoint of Weyl group actions of Noumi–Yamada [11], it is natural to

assume that the simple reflections s_i $(i = 0, 1, \ldots, 4)$ act on the τ functions τ_j $(j = 0, 1, \ldots, 4)$ as

$$s_i(\tau_j) = \tau_j \quad (i, j \in \{0, 1, 2, 3, 4\}; \ j \neq i); \quad s_0(\tau_0) = c_0(t)(q - t)\frac{\tau_2}{\tau_0},$$

$$s_1(\tau_1) = c_1(t)\frac{\tau_2}{\tau_1}, \quad s_2(\tau_2) = c_2(t)p\frac{\tau_0\tau_1\tau_3\tau_4}{\tau_2}, \tag{3.12}$$

$$s_3(\tau_3) = c_3(t)(q - 1)\frac{\tau_0}{\tau_3}, \quad s_4(\tau_4) = c_4(t)\frac{\tau_2}{\tau_4},$$

with some nonzero functions $c_i(t)$ of t which do not depend on α parameters. The factors $f_0 = q - t$, $f_1 = 1$, $f_2 = p$, $f_3 = q - 1$, $f_4 = q$ are the "invariant divisors" chosen as the generators of the Poisson algebra $\mathbb{C}[q, p, t]$ that are subject to the Serre relations of type $D_4^{(1)}$ with respect to Poisson adjoint actions. (The τ factors on the right-hand sides come from the action of simple affine reflections s_i on the affine fundamental weights Λ_j.) Also, the differential equations for τ_i should be expressed as

$$t(t - 1)\partial_t(\tau_i) = h_i\tau_i, \quad \text{namely,} \quad h_i = \frac{t(t - 1)\partial_t(\tau_i)}{\tau_i} \quad (i = 0, 1, \ldots, 4) \tag{3.13}$$

with appropriate Hamiltonian functions $h_0, h_1, \ldots, h_4 \in \mathbb{C}(\alpha)[q, p, t]$. Although the choice of $c_i(t)$ and $h_0, h_1, \ldots, h_4$ is not unique, in this paper we simply follow the convention of Tetsu Masuda [8], where $c_i(t) = 1$ $(i = 0, 1, 3, 4)$, $c_2(t) = t^{-\frac{1}{2}}$; we do no specify $h_0, h_1, \ldots, h_4$ here, and refer to [8] for their definitions. Also, the action of r_1, r_3, r_4 can be extended in the form

$$r_k(\tau_i) = d_{k,i}(t)\tau_{\sigma_k(i)} \quad (i = 0, 1, 3, 4);$$
$$r_1(\tau_2) = e_1(t)(q - t)\tau_2, \quad r_3(\tau_2) = e_3(t)(q - 1)\tau_2, \quad r_4(\tau_2) = e_4(t)\, q\tau_2 \tag{3.14}$$

with some functions $c_{k,i}(t), d_k(t) \in \mathbb{C}(t^{\frac{1}{4}}, (t - 1)^{\frac{1}{4}})$; we do not specify these functions since they depend on the convention. (In [8], r_1, r_3, r_4 are denoted by s_5, s_6, s_7 respectively.) The derivation ∂_t on $\mathcal{K} = \mathbb{C}(\alpha; q, p, t)$ is extended in this way to the ring of Laurent polynomials $\mathcal{L} = \widetilde{\mathcal{K}}[\tau^{\pm 1}] = \widetilde{\mathcal{K}}[\tau_0^{\pm 1}, \tau_1^{\pm 1}, \ldots, \tau_4^{\pm 1}]$, where $\widetilde{\mathcal{K}} = \mathbb{C}(\alpha; q, p, t, t^{\frac{1}{4}}, (t - 1)^{\frac{1}{4}})$. Also, the automorphisms $s_0, s_1, \ldots, s_4$, r_1, r_3, r_4 on $\mathcal{K}$ are extended to $\mathcal{L} = \widetilde{\mathcal{K}}[\tau^{\pm 1}]$ so that they commute with ∂_t. The automorphisms $s_0, s_1, s_2, s_3, s_4 : \mathcal{L} \to \mathcal{L}$ satisfy the fundamental relations of the affine Weyl group $W = \langle s_0, s_1, s_2, s_3, s_4 \rangle$. However, it should be noted that $r_1, r_3, r_4 : \mathcal{L} \to \mathcal{L}$ do not satisfy all the fundamental relations of the Dynkin diagram automorphisms when they act on τ functions. We denote by $\widehat{W}$ the subgroup of $\mathrm{Aut}(\mathcal{L})$ generated by s_i and r_i. Then $\widehat{W}$ is a bigger group than $\widetilde{W}$, and $\widetilde{W}$ is a quotient of $\widehat{W}$. It is a subtle question whether or not one can preserve the whole set of fundamental relations of the extended affine Weyl group $\widetilde{W}$, in lifting its action on $\mathcal{K}$ to the level of τ functions.

We take the lift

$$\widehat{T}_{\omega_3} = r_3 s_3 s_2 s_1 s_4 s_2 s_3 : \ \widetilde{\mathcal{K}}[\tau^{\pm 1}] \to \widetilde{\mathcal{K}}[\tau^{\pm 1}] \tag{3.15}$$

of the translation $T_{\omega_3} : \mathcal{K} \to \mathcal{K}$, and consider the sequence of τ functions $\tau^{(m)} = \widehat{T}_{\omega_3}^m(\tau_0)$ $(m \in \mathbb{Z})$. By the action of $\widehat{W}$ on $\mathcal{L} = \widetilde{\mathcal{K}}[\tau^{\pm 1}]$, one can directly verify

$$t^{\frac{1}{2}} \frac{\widehat{T}_{\omega_3}(\tau) \widehat{T}_{\omega_3}^{-1}(\tau_0)}{\tau_0^2} = (t-1)^2 \partial_t \left(\frac{t \partial_t(\tau)}{\tau} \right) + \frac{1}{4}(\alpha_0 - \alpha_3 - 1)^2 + \frac{1}{2} \tag{3.16}$$

Applying $\widehat{T}_{\omega_3}^m$ to this equality, we obtain the Toda equation

$$t^{\frac{1}{2}} \frac{\tau^{(m+1)} \tau^{(m-1)}}{(\tau^{(m)})^2} = (t-1)^2 \partial_t \left(\frac{t \partial_t(\tau^{(m)})}{\tau^{(m)}} \right) + \frac{1}{4}(\alpha_0 - \alpha_3 + 2m - 1)^2 + \frac{1}{2} \quad (m \in \mathbb{Z}) \tag{3.17}$$

for the sequence of τ functions $\tau^{(m)} = \widehat{T}_{\omega_3}^m(\tau_0)$ with parameter values

$$(T_{\omega_3}^m(\alpha_0), \ldots, T_{\omega_3}^m(\alpha_4)) = (\alpha_0 + m, \alpha_1, \alpha_2, \alpha_3 - m, \alpha_4). \tag{3.18}$$

We specialize this Toda equation to the seed solution

$$(q, p) = \left(t^{\frac{1}{2}}, \frac{1}{2}\left(a + b - \frac{1}{2} \right) t^{-\frac{1}{2}} \right)$$

$$\text{with} \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(a, b, \frac{1}{2} - a - b, a, b \right) \tag{3.19}$$

at the fixed point of r_3. Note that, for this seed solution, τ_0, τ_3 can be taken as

$$\tau_0 = t^{-\frac{1}{2}((a-\frac{1}{2})^2 + b^2 + \frac{1}{8})} (t^{\frac{1}{2}} - 1)^{(a-\frac{1}{2})^2 + \frac{1}{2}} (t^{\frac{1}{2}} + 1)^{b^2 + \frac{1}{2}}, \quad \tau_3 = t^{-\frac{1}{4}} \tau_0. \tag{3.20}$$

By this specialization, we obtain the Toda equation

$$t^{\frac{1}{2}} \frac{\tau^{(m+1)} \tau^{(m-1)}}{(\tau^{(m)})^2} = (t-1)^2 \partial_t \left(\frac{t \partial_t(\tau^{(m)})}{\tau^{(m)}} \right) + \frac{1}{4}(2m-1)^2 + \frac{1}{2} \quad (m \in \mathbb{Z}) \tag{3.21}$$

for the τ functions $\tau^{(m)}$ of the algebraic solutions with parameters $(a + m, b, \frac{1}{2} - a - b, a - m, b)$. Furthermore, by setting $\tau^{(m)} = \phi_m \tau_0^{1-m} \tau_3^m t^{\frac{1}{4}(3m - 2m^2)}$, we obtain the recurrence formula for $\phi_m = \phi_m(x)$, $x = t^{\frac{1}{2}}$:

$$4\phi_{m+1}\phi_{m-1} = (x^2 - 1)^2 \left(x \phi_m'' \phi_m + \phi_m' \phi_m - x(\phi_m')^2 \right)$$

$$- \left(\left(a - \frac{1}{2} \right)^2 (x+1)^2 - b^2(x-1)^2 - (2m-1)^2 x \right) \phi_m^2, \tag{3.22}$$

where $' = \mathrm{d}/\mathrm{d}x$. Noting that $\phi_0, \phi_1 \in \mathbb{C}$, we set $\phi_m = c_m U_m$ with the constants c_m $(m \in \mathbb{Z})$ determined by

$$c_0 = \phi_0, \quad c_1 = \phi_1, \quad 4c_{m+1}c_{m-1} = c_m^2 \quad (m \in \mathbb{Z}). \tag{3.23}$$

Then the Toda equation for $U_m = U_m(x)$ is expressed as

$$U_{m+1}U_{m-1} = (x^2 - 1)^2\big(xU_m''U_m + U_m'U_m - x(U_m')^2\big)$$

$$- \left(\left(a - \frac{1}{2}\right)^2 (x+1)^2 - b^2(x-1)^2 - (2m-1)^2 x\right)U_m^2 \quad (m \in \mathbb{Z}),$$

$$(3.24)$$

with the initial condition $U_0 = U_1 = 1$. This is the recurrence formula for the Umemura polynomials U_m that we discussed in Section 1 with a replaced by $a - \frac{1}{2}$.

Following Masuda [8], we make use of two translations T_{ε_4}, $T_{-\varepsilon_3}$ in order to generate a two-parameter family of algebraic solutions from the seed solution

$$(q,p) = \left(x, \frac{1}{2}\left(a+b-\frac{1}{2}\right)x^{-1}\right), \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(a, b, \frac{1}{2} - a - b, a, b\right)$$

$$(3.25)$$

at the fixed point of r_3, where $x = t^{\frac{1}{2}}$. The translations T_{ε_4} and $T_{-\varepsilon_3}$ act on the parameter as

$$
\begin{array}{c||ccccc}
 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hline\hline
T_{\varepsilon_4} & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 + 1 & \alpha_4 - 1 \\
T_{-\varepsilon_3} & \alpha_0 & \alpha_1 & \alpha_2 - 1 & \alpha_3 + 1 & \alpha_4 + 1
\end{array}. \tag{3.26}
$$

Hence, for each $(m,n) \in \mathbb{Z}^2$, by the Bäcklund transformation

$$q_{m,n} = T_{\varepsilon_4}^m T_{-\varepsilon_3}^n(q), \quad p_{m,n} = T_{\varepsilon_4}^m T_{-\varepsilon_3}^n(p) \tag{3.27}$$

we obtain an algebraic solution $(q_{m,n}, p_{m,n})$ with parameter values

$$\left(a, b, \frac{1}{2} - a - b - n, a + m + n, b - m + n\right). \tag{3.28}$$

In order to define τ functions for this family, we use the lifts

$$\widehat{T}_{\varepsilon_4} = s_3 s_2 s_0 s_1 s_2 s_3 r_1, \quad \widehat{T}_{-\varepsilon_3} = s_3 s_4 s_2 s_0 s_1 s_2 r_1 : \quad \widetilde{\mathcal{K}}[\tau^{\pm 1}] \to \widetilde{\mathcal{K}}[\tau^{\pm 1}], \tag{3.29}$$

and set

$$\tau_{m,n} = \widehat{T}_{\varepsilon_4}^m \widehat{T}_{-\varepsilon_3}^n(\tau_0) \qquad (m,n \in \mathbb{Z}). \tag{3.30}$$

It is known by Masuda [8] that, when specialized to the seed solution (3.25), these τ functions are expressed in the form

$$\tau_{m,n} = c_{m,n} x^{\lambda_{m,n}} (x+1)^{\mu_{m,n}} (x-1)^{\nu_{m,n}} V_{m,n}(x; a, b), \tag{3.31}$$

where $c_{m,n}$, $\lambda_{m,n}$, $\mu_{m,n}$, $\nu_{m,n}$ are constants determined by a, b and m, n, and $V_{m,n}(x; a, b)$ is a polynomial in x with parameters (a, b). These polynomials $V_{m,n}(x; a, b)$ are by definition the *Umemura polynomials with two discrete parameters* m, n for this family of algebraic solutions. Furthermore,

the algebraic solution $(q_{m,n}, p_{m,n})$ with parameter values $(a, b, \frac{1}{2} - a - b - n,$ $a + m + n, b - m + n)$ is expressed as

$$q_{m,n} = x \, \frac{V_{m,n-1}(x; a+1, b)V_{m-1,n}(x; a+1, b)}{V_{m-1,n-}(x; a+1, b+1)V_{m,n-1}(x; a+1, b+1)}$$

$$p_{m,n} = \frac{1}{2}\left(a + b + n - \frac{1}{2}\right)$$

$$\cdot x^{-1} \frac{V_{m-1,n}(x; a+1, b-1)V_{m,n-1}(x; a+1, b+1)V_{m,n-1}(x; a, b)}{V_{m,n}(x; a, b)V_{m-1,n-1}(x; a+1, b)V_{m,n-1}(x; a+1, b)},$$

$$(3.32)$$

for each $m, n \in \mathbb{Z}$. Also, these polynomials $V_{m,n} = V_{m,n}(x; a, b)$ $(m, n \in \mathbb{Z})$ are characterized by the Toda equations

$$\frac{1}{2}\left(a - b + m - \frac{1}{2}\right)V_{m+1,n}V_{m-1,n}$$

$$= x(x^2 - 1)(\ddot{V}_{m,n}V_{m,n} - \dot{V}_{m,n}^2) + (3x^2 - 1)\dot{V}_{m,n}V_{m,n}$$

$$+ \left\{\left(a - b + m - \frac{1}{2}\right)\left[\left(a - b + 3m + \frac{1}{2}\right)x + \left(a + b + n - \frac{1}{2}\right)\right] - n(n+1)x\right\}V_{m,n}^2$$

$$\frac{1}{2}\left(a + b + n - \frac{1}{2}\right)V_{m,n+1}V_{m,n-1}$$

$$= x(x^2 - 1)(\ddot{V}_{m,n}V_{m,n} - \dot{V}_{m,n}^2) + (3x^2 - 1)\dot{V}_{m,n}V_{m,n}$$

$$+ \left\{\left(a + b + n - \frac{1}{2}\right)\left[\left(a + b + 3n + \frac{1}{2}\right)x + \left(a - b + m - \frac{1}{2}\right)\right] - m(m+1)x\right\}V_{m,n}^2,$$

$$(3.33)$$

with initial conditions

$$V_{-1,-1} = V_{-1,0} = V_{0,-1} = V_{0,0} = 1, \qquad (3.34)$$

where $\dot{V}_{m,n} = dV_{m,n}/dx$ and $\ddot{V}_{m,n} = d^2V_{m,n}/dx^2$ ([8, Proposition 4.4]). For other characterization of these polynomials see [8]. One of the main results of Masuda [8] is an explicit determinant formula for $V_{m,n}(x; a, b)$ (as a universal character) in terms of certain Jacobi polynomials, but we will not get into this topic here.

We remark that this family of Umemura polynomials $V_{m,n}(x; a, b)$ satisfies

$$V_{m,n}(x; a, b) = V_{n,m}(x; a, -b) \qquad (m, n \in \mathbb{Z}). \qquad (3.35)$$

From the combinatorial point of view, it seems appropriate to reparametrize these polynomials as

$$U_{m,n}(x; c, d) = V_{m,n}(x; a, b)$$

$$\text{with } \begin{cases} c = 2a + m + n - 1, & d = 2b - m + n, \\ a = \frac{1}{2}(c - m - n + 1), & b = \frac{1}{2}(d + m - n). \end{cases} \qquad (3.36)$$

Then one can show that

$$U_{m,n}(x; c, d) = U_{n,m}(x; c, -d),$$
$$U_{m,n}(x; c, d) = (-1)^{\binom{m+1}{2} + \binom{n+1}{2}} U_{m,n}(x; -c, -d), \tag{3.37}$$

and

$$U_{m,n}(x; c, d) = U_{-1-m,n}(x; c, d)$$
$$= U_{m,-1-n}(x; c, d) = U_{-1-m,-1-n}(x; c, d). \tag{3.38}$$

Through the parameterization (3.36), these polynomials $U_{m,n}(x; c, d)$ correspond to the parameter values

$$\frac{1}{2}(c + 1 - m - n, d + m - n, -c - d, c + 1 + m + n, d - m + n)$$
$$= \frac{1}{2}(c+1, d, -c-d, c+1, d) + \frac{m}{2}(-1, 1, 0, 1, -1) + \frac{n}{2}(-1, -1, 0, 1, 1). \tag{3.39}$$

In particular, the subfamily $U_{-m,-m}(x; c, d)$ correspond to algebraic solutions with parameters

$$\left(\frac{1}{2}(c + 1) + m, \frac{1}{2}d, -\frac{1}{2}(c + d), \frac{1}{2}(c + 1) - m, \frac{1}{2}d \right). \tag{3.40}$$

We remark that $U_{-m,-m}(x; c, d)$ coincides with a constant multiple of the Umemura polynomial $U_m(x; c, d)$ that we discussed before in (3.24) with $a - \frac{1}{2} = \frac{1}{2}c$, $b = \frac{1}{2}d$.

In the next section, we formulate a conjecture on the combinatorial formula for this family $U_{m,n}(x; c, d)$ of special polynomials.

Remark 3.1. — There are many related subjects that we could not cover here. The classification of algebraic solutions of P_{VI} has been completed by contribution of many researchers (see Lisovyy–Tykhyy [6] and references therein). Also, interesting developments are going on in understanding the τ functions for general solutions of Painlevé equations in connection with the conformal field theory. It would also be an intriguing problem to study special polynomials of Umemura type for the discrete Painlevé equations in Sakai's list.

4. Combinatorial Umemura polynomials: a conjecture

In this section we introduce a class of combinatorial polynomials $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$, which we call the *combinatorial Umemura polynomials*. We also formulate a conjecture of combinatorial formula for Umemura polynomials $U_{m,n}(x; c, d)$ with two discrete parameters, in terms of these combinatorial polynomials. This part is based on the collaboration of the author with

Hiroshi Kageyama and Takuya Yoshida who were students in the master's course of Kobe University.

Let $\boldsymbol{c} = (c_1, c_2, \dots)$ and $\boldsymbol{d} = (d_1, d_2, \dots)$ be two sequences of parameters. For each finite set $\Lambda = \{l_1, \dots, l_n\}$ of positive integers with $|\Lambda| = n$, we define a polynomial $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$ in two variables (u, v) by

$$
\begin{aligned}
& G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d}) \\
&= \sum_{I \sqcup J = \Lambda} (-1)^{\binom{|I|}{2} + \sum_{i \in I}(i-1)} \prod_{i \in I; j \in J} \frac{j + i}{j - i} \prod_{i \in I} c_i u^i \prod_{j \in J} d_j v^j \\
&= \sum_{I \sqcup J = \{1, \dots, n\}} (-1)^{\binom{|I|}{2} + \sum_{i \in I}(l_i - 1)} \prod_{i \in I; j \in J} \frac{l_j + l_i}{l_j - l_i} \prod_{i \in I} c_{l_i} u^{l_i} \prod_{j \in J} d_{l_j} v^{l_j}. \quad (4.1)
\end{aligned}
$$

We call $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$ the *combinatorial Umemura polynomial* attached to Λ. Note that Λ can be identified with a *strict partition* $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 > \dots > \lambda_n$, by rearranging the elements of Λ in the decreasing order. As a polynomial is (u, v), $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$ is homogenous of degree $|\lambda| = \lambda_1 + \dots + \lambda_n$.

Besides the combinatorial formula (4.1), two determinant formulas are known for $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$; one *of Vandermonde type*, due to Yoshida [21], and the other *of Cauchy type*, due to Kirillov–Taneda [5] and Kageyama [3].

THEOREM 4.1 (Yoshida [21]). — *For any finite set $\Lambda = \{l_1, l_2, \dots, l_n\}$ of positive integers with $|\Lambda| = n$, we have*

$$
G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d}) = \frac{1}{\Delta(l_1, \dots, l_n)} \det\left(l_j^{n-i}\left((-1)^{n-i+l_j-1} c_{l_j} u^{l_j} + d_{l_j} v^{l_j}\right)\right)_{i,j=1}^n, \tag{4.2}
$$

where $\Delta(l_1, \dots, l_n) = \prod_{1 \leqslant i < j \leqslant n} (l_i - l_j)$ *stands for the difference product.*

This theorem follows from the following determinant formula of rational functions in the variables $(x_1, \dots, x_n)$ with parameters $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n)$ and t.

LEMMA 4.2. —

$$
\sum_{I \sqcup J = \{1, \dots, n\}} t^{\binom{|I|}{2}} \prod_{i \in I; j \in J} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j
$$
$$
= \frac{\det\left(x_j^{n-i}(\beta_j + t^{n-i}\alpha_j)\right)_{i,j=1}^n}{\Delta(x_1, \dots, x_n)} \quad (4.3)
$$

In fact, formula (4.2) is a special case of (4.3) where $t = -1$ and

$$
x_j = l_j, \quad \alpha_j = (-1)^{l_j-1} c_{l_j} u^{l_j}, \quad \beta_j = d_{l_j} v^{l_j} \quad (j = 1, \dots, n). \tag{4.4}
$$

Proof of Lemma 4.2. — Note first that

$$\frac{\prod_{i \in I} T_{t,x_i} \Delta(x_1, \ldots, x_n)}{\Delta(x_1, \ldots, x_n)} = t^{\binom{|I|}{2}} \prod_{i \in I; \; j \notin I} \frac{tx_i - x_j}{x_i - x_j} \qquad (4.5)$$

for any subset $I \subseteq \{1, \ldots, n\}$. Here, for each $i = 1, \ldots, n$, T_{t,x_i} stands for the multiplicative shift operator in x_i by t: $T_{t,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, tx_i, \ldots, x_n)$. Hence we have

$$\sum_{I \sqcup J = \{1, \ldots, n\}} t^{\binom{|I|}{2}} \prod_{i \in I; \; j \in J} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j$$

$$= \sum_{I \sqcup J = \{1, \ldots, n\}} \prod_{j \in j} \beta_j \prod_{i \in I} \alpha_i \frac{\prod_{i \in I} T_{t,x}^I \Delta(x_1, \ldots, x_n)}{\Delta(x_1, \ldots, x_n)}$$

$$= \frac{1}{\Delta(x_1, \ldots, x_n)} \left(\sum_{I \sqcup J = \{1, \ldots, n\}} \prod_{j \in J} \beta_j \prod_{i \in I} \alpha_i \, T_{t,x_i} \right) \Delta(x_1, \ldots, x_n)$$

$$= \frac{1}{\Delta(x_1, \ldots, x_n)} \prod_{k=1}^{n} \big(\beta_k + \alpha_k T_{t,x_k} \big) \Delta(x_1, \ldots, x_n). \qquad (4.6)$$

By $\Delta(x_1, \ldots, x_n) = \det \left(x_j^{n-i} \right)_{i,j=1}^{n}$, we obtain

$$\sum_{I \sqcup J = \{1, \ldots, n\}} t^{\binom{|I|}{2}} \prod_{i \in I; \; j \in J} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j$$

$$= \frac{1}{\Delta(x_1, \ldots, x_n)} \prod_{k=1}^{n} \big(\beta_k + \alpha_k T_{t,x_k} \big) \det \left(x_j^{n-i} \right)_{i,j=1}^{n}$$

$$= \frac{1}{\Delta(x_1, \ldots, x_n)} \det \left(\big(\beta_j + \alpha_j T_{t,x_j} \big) x_j^{n-i} \right)_{i,j=1}^{n}$$

$$= \frac{1}{\Delta(x_1, \ldots, x_n)} \det \left(x_j^{n-i} \big(\beta_j + t^{n-i} \alpha_j \big) \right)_{i,j=1}^{n}. \qquad (4.7)$$

$\square$

THEOREM 4.3 (Kirillov–Taneda [5], Kageyama [3]). — *For any finite set $\Lambda = \{l_1, l_2, \ldots, l_n\}$ of positive integers with $|\Lambda| = n$, we have*

$$G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$$

$$= \det \left(d_i v^i \delta_{i,j} + (-1)^{l_i - 1} c_{l_i} u^{l_i} \frac{2l_i}{l_i + l_j} \prod_{1 \leqslant k \leqslant n; \; k \neq i} \frac{l_k + l_i}{l_k - l_i} \right)_{i,j=1}^{n}. \qquad (4.8)$$

This theorem follows from the following determinant formula of rational functions in the variables $(x_1, \ldots, x_n)$ and parameters $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n)$.

LEMMA 4.4. —

$$\sum_{I \sqcup J=\{1,\ldots,n\}} (-1)^{\binom{|I|}{2}} \prod_{\substack{i\in I \\ j\in J}} \frac{1-x_ix_j}{x_i-x_j} \prod_{i\in I}\alpha_i \prod_{j\in J}\beta_j$$

$$= \det\left(\beta_i\delta_{i,j} + \alpha_i\frac{1-x_i^2}{1-x_ix_j} \prod_{\substack{1\leqslant k\leqslant n \\ k\neq i}} \frac{1-x_ix_k}{x_i-x_k}\right)_{i,j=1}^n \quad (4.9)$$

In this formula, set $x_k = 1 - q^{l_k}$ $(k = 1,\ldots,n)$ and take the limit as $q \to 1$ to obtain

$$\sum_{I \sqcup J=\{1,\ldots,n\}} (-1)^{\binom{|I|}{2}} \prod_{i\in I}\alpha_i \prod_{j\in J}\beta_j \prod_{i\in I;j\in J} \frac{l_j+l_i}{l_j-l_i}$$

$$= \det\left(\beta_i\delta_{i,j} + \alpha_i\frac{2l_i}{l_i+l_j} \prod_{\substack{1\leqslant k\leqslant n \\ j\neq i}} \frac{l_k+l_i}{l_k-l_i}\right)_{i,j=1}^n . \quad (4.10)$$

Setting

$$\alpha_i = (-1)^{l_i-1}c_{l_i}u^{l_i}, \quad \beta_i = d_{l_i}v^{l_i} \quad (i=1,\ldots,n), \quad (4.11)$$

we obtain

$$\sum_{I \sqcup J=\{1,\ldots,n\}} (-1)^{\binom{|I|}{2}+\sum_{i\in I}(l_i-1)} \prod_{i\in I}c_{l_i}u^{l_i} \prod_{j\in J}d_{l_j}v^{l_j} \prod_{i\in I;j\in J} \frac{l_j+l_i}{l_j-l_i}$$

$$= \det\left(d_{l_i}v^{l_i}\delta_{i,j} + (-1)^{l_i-1}c_{l_i}u^{l_i}\frac{2l_i}{l_i+l_j} \prod_{1\leqslant k\leqslant n;k\neq i} \frac{l_k+l_i}{l_k-l_i}\right)_{i,j=1}^n . \quad (4.12)$$

Proof of Lemma 4.4. — In Cauchy's determinant formula

$$\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1}^n = \frac{\Delta(x_1,\ldots,x_n)\Delta(y_1,\ldots,y_n)}{\prod_{i,j=1}^n(1-x_iy_j)}, \quad (4.13)$$

multiply each row by $1-x_i^2$ for $i=1,\ldots,n$ to obtain

$$\det\left(\frac{1-x_i^2}{1-x_iy_j}\right)_{i,j=1}^n = \prod_{i=1}^n(1-x^2)\frac{\Delta(x_1,\ldots,x_n)\Delta(y_1,\ldots,y_n)}{\prod_{i,j=1}^n(1-x_iy_j)}. \quad (4.14)$$

Setting $y_j = x_j$ $(j=1,\ldots,n)$ we obtain

$$\det\left(\frac{1-x_i^2}{1-x_ix_j}\right)_{i,j=1}^n = (-1)^{\binom{n}{2}} \prod_{1\leqslant i,j\leqslant n;\ i\neq j} \frac{x_i-x_j}{1-x_ix_j}. \quad (4.15)$$

When $I \sqcup J = \{1, \ldots, n\}$,

$$\prod_{i \in I; j \in J} \frac{1 - x_i x_j}{x_i - x_j} = \prod_{i \in I; j \in \{1, \ldots, n\}; i \neq j} \frac{1 - x_i x_j}{x_i - x_j} \prod_{i, j \in I; i \neq j} \frac{1 - x_i x_j}{x_i - x_j}$$

$$= (-1)^{\binom{|I|}{2}} \det \left(\frac{1 - x_i^2}{1 - x_i x_j} \right)_{i, j \in I} \prod_{i \in I; k \in \{1, \ldots, n\}; k \neq i} \frac{1 - x_i x_k}{x_i - x_k} \quad (4.16)$$

$$= (-1)^{\binom{|I|}{2}} \det \left(\frac{1 - x_i^2}{1 - x_i x_j} \prod_{1 \leqslant k \leqslant n; k \neq i} \frac{1 - x_i x_k}{x_i - x_k} \right)_{i, j \in I}$$

Hence we have

$$\det \left(\beta_i \delta_{i,j} + \alpha_i \frac{1 - x_i^2}{1 - x_i x_j} \prod_{1 \leqslant k \leqslant n; k \neq i} \frac{1 - x_i x_k}{x_i - x_k} \right)_{i,j=1}^{n}$$

$$= \sum_{I \sqcup J = \{1, \ldots, n\}} \prod_{j \in J} \beta_j \, \det \left(\alpha_i \frac{1 - x_i^2}{x_i - x_j} \prod_{1 \leqslant k \leqslant n; k \neq i} \frac{1 - x_i x_k}{x_i - x_k} \right)_{i,j \in I} \quad (4.17)$$

$$= \sum_{I \sqcup J = \{1, \ldots, n\}} (-1)^{\binom{|I|}{2}} \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j \prod_{i \in I; j \in J} \frac{1 - x_i x_j}{x_i - x_j}.$$

$\square$

In the context of the sixth Painlevé equation, we specify the sequence of parameters c_k, d_k $(k = 0, 1, 2, \ldots)$ as the bilateral shifted factorials

$$c_k = \prod_{i=0}^{k-1} (c - k + 1 + 2i) = (c - k + 1)(c - k + 2) \cdots (c + k - 1),$$

$$d_k = \prod_{i=0}^{k-1} (d - k + 1 + 2i) = (d - k + 1)(d - k + 2) \cdots (d + k - 1). \quad (4.18)$$

for $k = 0, 1, 2, \ldots$. Note that

$$c_0 = 1, \quad c_1 = c, \quad c_2 = (c - 1)(c + 1),$$
$$c_3 = (c - 2)c(c + 2), \quad c_4 = (c - 3)(c - 1)(c + 1)(c + 3) \ldots. \quad (4.19)$$

In terms of the ordinary shifted factorials $(c)_k = c(c + 1) \cdots (c + k - 1)$, the bilateral shifted factorials defined above can be expressed as

$$c_{2k} = (-1)^k 2^{2k} \left(\frac{1}{2} + c \right)_k \left(\frac{1}{2} - c \right)_k, \quad c_{2k+1} = (-1)^k 2^{2k} c \, (1 + c)_k (1 - c)_k. \quad (4.20)$$

In this situation, we use the notation $G_\Lambda(u, v; c, d)$ in place of $G_\Lambda(u, v; \boldsymbol{c}, \boldsymbol{d})$.

In the previous section, we introduced the Umemura polynomials $U_{m,n}(x; c, d)$ $(m, n \in \mathbb{Z})$ with two discrete parameters as

$$U_{m,n}(x; c, d) = V_{m,n}(x; a, b)$$
$$\text{with} \quad c = 2a + m + n - 1, \quad d = 2b - m + n, \quad (4.21)$$

reparametrizing Masuda's polynomials $V_{m,n}(x; a, b)$ $(m, n \in \mathbb{Z})$. They have the following symmetries with respect to the discrete parameters $(m, n) \in \mathbb{Z}$:

$$U_{m,n}(x; c, d) = U_{n,m}(x; c, -d),$$
$$U_{m,n}(x; c, d) = (-1)^{\binom{m+1}{2} + \binom{n+1}{2}} U_{m,n}(x; -c, -d), \quad (4.22)$$

and

$$U_{m,n}(x; c, d) = U_{-1-m,n}(x; c, d)$$
$$= U_{m,-1-n}(x; c, d) = U_{-1-m,-1-n}(x; c, d). \quad (4.23)$$

Note also that

$$U_{0,0}(x; c, d) = U_{0,-1}(x; c, d) = U_{-1,0}(x; c, d) = U_{-1,-1}(x; c, d) = 1. \quad (4.24)$$

CONJECTURE 4.5. — *Supposing that $n \geqslant m \geqslant 0$, we take the indexing set*

$$\Lambda_{m,n} = \{n+m, n+m-2, \ldots, n-m+2, n-m, n-m-1, \ldots, 1\} \quad (4.25)$$

corresponding to the sum $(m, m-1, \ldots, 1) + (n, n-1, \ldots, 1)$ of two staircase partitions. Then we have

$$U_{m,n}(x; c, d) = (-1)^{\binom{m+1}{2}} G_{\Lambda_{m,n}}(u, v; c, d) \quad (4.26)$$

under the identification of variables $u = x + 1$, $v = x - 1$.

Note that the original Umemura polynomials that we discussed in Section 1 are the special cases $U_{n-1,n-1}(x; c, d) = \text{const.} \, U_n(x; c, d)$ $(n \in \mathbb{Z})$. It means that this conjecture has been proved for $U_{n,n}(x; c, d)$ $(n \in \mathbb{Z})$ by Taneda. We include in Table 4.1 a list of Umemura polynomials $U_{m,n}(x; c, d)$ in the variables (u, v) with small $m, n \geqslant 0$ with $m \leqslant n$. For small m, n, one can verify by computer that $U_{m,n}(x; c, d)$ coincides up to sign with $G_{\Lambda_{m,n}}(u, v; c, d)$ defined by the combinatorial formula (4.1).

To conclude this paper, I would like to tell a personal story about Umemura polynomials. As I mentioned already, it was around 1995 that Umemura started to investigate τ functions associated with the Schlesinger transformations of algebraic solutions appearing as fixed points of Dynkin diagram automorphisms. He was very much excited in recognizing that the recurrence formula (1.1) generates polynomials only. When he brought us U_4 and U_5 in the original coordinate in the spring of 1996, U_4 was already a lengthy polynomial of one page, and U_5 occupied more than five pages. It was a kind of spectacular to see, every time we run the computer, a gigantic

Table 4.1. Umemura polynomials $U_{m,n} = U_{m,n}(x; c, d)$ $(u = x + 1,$ $v = x - 1)$

$U_{0,0} = 1$

$U_{0,1} = c_1 u + d_1 v$

$U_{0,2} = c_2 c_1 u^3 + 3c_2 d_1 u^2 v + 3c_1 d_2 u v^2 + d_2 d_1 v^3$

$U_{0,3} = c_3 c_2 c_1 u^6 + 6c_3 c_2 d_1 u^5 v + 15 c_3 c_1 d_2 u^4 v^2 + 10 c_3 d_2 d_1 u^3 v^3$
$\qquad + 10 c_2 c_1 d_3 u^3 v^3 + 15 c_2 d_3 d_1 u^2 v^4 + 6 c_1 d_3 d_2 u v^5 + d_3 d_2 d_1 v^6$

$U_{0,4} = c_4 c_3 c_2 c_1 u^{10} + 10 c_4 c_3 c_2 d_1 u^9 v + 45 c_4 c_3 c_1 d_2 u^8 v^2 + 50 c_4 c_3 d_2 d_1 u^7 v^3$
$\qquad + 70 c_4 c_2 c_1 d_3 u^7 v^3 + 175 c_4 c_2 d_3 d_1 u^6 v^4 + 35 c_3 c_2 c_1 d_4 u^6 v^4 + 126 c_4 c_1 d_3 d_2 u^5 v^5$
$\qquad + 126 c_3 c_2 d_4 d_1 u^5 v^5 + 35 c_4 d_3 d_2 d_1 u^4 v^6 + 175 c_3 c_1 d_4 d_2 u^4 v^6 + 70 c_3 d_4 d_2 d_1 u^3 v^7$
$\qquad + 50 c_3 c_2 d_1 d_4 u^3 v^7 + 45 c_2 d_4 d_3 d_1 u^2 v^8 + 10 c_1 d_4 d_3 d_2 u v^9 + d_4 d_3 d_2 d_1 v^{10}$

$U_{1,1} = c_2 u^2 - d_2 v^2$

$U_{1,2} = c_3 c_1 u^4 + 2c_3 d_1 u^3 v - 2c_1 d_3 u v^3 - d_3 d_1 v^4$

$U_{1,3} = c_4 c_2 c_1 u^7 + 5 c_4 c_2 d_1 u^6 v + 9 c_4 c_1 d_2 u^5 v^2 + 5 c_4 d_2 d_1 u^4 v^3$
$\qquad - 5 c_2 c_1 d_4 u^3 v^4 - 9 c_2 d_4 d_1 u^2 v^5 - 5 c_1 d_4 d_2 u v^6 - d_4 d_2 d_1 v^7$

$U_{1,4} = c_5 c_3 c_2 c_1 u^{11} + 9 c_5 c_3 c_2 d_1 u^{10} v + 35 c_5 c_3 c_1 d_2 u^9 v^2 + 35 c_5 c_3 d_2 d_1 u^8 v^3$
$\qquad + 40 c_5 c_2 c_1 d_3 u^8 v^3 + 90 c_5 c_2 d_3 d_1 u^7 v^4 + 56 c_5 c_1 d_3 d_2 u^6 v^5 - 14 c_3 c_2 c_1 d_5 u^6 v^5$
$\qquad + 14 c_5 d_3 d_2 d_1 u^5 v^6 - 56 c_3 c_2 d_5 d_1 u^5 v^6 - 90 c_3 c_1 d_5 d_2 u^4 v^7 - 40 c_3 d_5 d_2 d_1 u^3 v^8$
$\qquad - 35 c_2 c_1 d_5 d_3 u^3 v^8 - 35 c_2 d_5 d_3 d_1 u^2 v^9 - 9 c_1 d_5 d_3 d_2 u v^{10} - d_5 d_3 d_2 d_1 v^{11}$

$U_{2,2} = c_4 c_2 u^6 - 3 c_4 d_2 u^4 v^2 + 3 c_2 d_4 u^2 v^4 - d_4 d_2 v^6$

$U_{2,3} = c_5 c_3 c_1 u^9 + 3 c_5 c_3 d_1 u^8 v - 8 c_5 c_1 d_3 u^6 v^3 - 6 c_5 d_3 d_1 u^5 v^4$
$\qquad + 6 c_3 c_1 d_5 u^4 v^5 + 8 c_3 d_5 d_1 u^3 v^6 - 3 c_1 d_5 d_3 u v^8 - d_5 d_3 d_1 v^9$

$U_{2,4} = c_6 c_4 c_2 c_1 u^{13} + 7 c_6 c_4 c_2 d_1 u^{12} v + 18 c_6 c_4 c_1 d_2 u^{11} v^2 + 14 c_6 c_4 d_2 d_1 u^{10} v^3$
$\qquad - 25 c_6 c_2 c_1 d_4 u^9 v^4 - 63 c_6 c_2 d_4 d_1 u^8 v^5 - 50 c_6 c_1 d_4 d_2 u^7 v^6 + 14 c_4 c_2 c_1 d_6 u^7 v^6$
$\qquad - 14 c_6 d_4 d_2 d_1 u^6 v^7 + 50 c_4 c_2 d_6 d_1 u^6 v^7 + 63 c_4 c_1 d_6 d_2 u^5 v^8 + 25 c_4 d_6 d_2 d_1 u^4 v^9$
$\qquad - 14 c_2 c_1 d_6 d_4 u^3 v^{10} - 18 c_2 d_6 d_4 d_1 u^2 v^{11} - 7 c_1 d_6 d_4 d_2 u v^{12} - d_6 d_4 d_2 d_1 v^{13}$

$U_{3,3} = c_6 c_4 c_2 u^{12} - 6 c_6 c_4 d_2 u^{10} v^2 + 15 c_6 c_2 d_4 u^8 v^4 - 10 c_6 d_4 d_2 u^6 v^6$
$\qquad - 10 c_4 c_2 d_6 u^6 v^6 + 15 c_4 d_6 d_2 u^4 v^8 - 6 c_2 d_6 d_4 u^2 v^{10} + d_6 d_4 d_2 v^{12}$

$U_{3,4} = c_7 c_5 c_3 c_1 u^{16} + 4 c_7 c_5 c_3 d_1 u^{15} v - 20 c_7 c_5 c_1 d_3 u^{13} v^3 - 20 c_7 c_5 d_3 d_1 u^{12} v^4$
$\qquad + 36 c_7 c_3 c_1 d_5 u^{11} v^5 + 64 c_7 c_3 d_5 d_1 u^{10} v^6 - 20 c_5 c_3 c_1 d_7 u^9 v^7 - 45 c_7 c_1 d_5 d_3 u^8 v^8$
$\qquad - 45 c_5 c_3 d_7 d_1 u^8 v^8 - 20 c_7 d_5 d_3 d_1 u^7 v^9 + 64 c_5 c_1 d_7 d_3 u^6 v^{10} + 36 c_5 d_7 d_3 d_1 u^5 v^{11}$
$\qquad - 20 c_3 c_1 d_7 d_5 u^4 v^{12} - 20 c_3 d_7 d_5 d_3 u^3 v^{13} + 4 c_1 d_7 d_5 d_3 u v^{15} + d_7 d_5 d_3 d_1 v^{16}$

$U_{4,4} = c_8 c_6 c_4 c_2 u^{20} - 10 c_8 c_6 c_4 d_2 u^{18} v^2 + 45 c_8 c_6 c_2 d_4 u^{16} v^4 - 50 c_8 c_6 d_4 d_2 u^{14} v^6$
$\qquad - 70 c_8 c_4 c_2 d_6 u^{14} v^6 + 175 c_8 c_4 d_6 d_2 u^{12} v^8 + 35 c_6 c_4 c_2 d_8 u^{12} v^8 - 126 c_8 c_2 d_6 d_4 u^{10} v^{10}$
$\qquad - 126 c_6 c_4 d_8 d_2 u^{10} v^{10} + 35 c_8 d_6 d_4 d_2 u^8 v^{12} + 175 c_6 c_2 d_8 d_4 u^8 v^{12} - 70 c_6 d_8 d_4 d_2 u^6 v^{14}$
$\qquad - 50 c_4 c_2 d_8 d_6 u^6 v^{14} + 45 c_4 d_8 d_6 d_2 u^4 v^{16} - 10 c_2 d_8 d_6 d_4 u^2 v^{18} + d_8 d_6 d_4 d_2 v^{20}$

polynomial of the right-hand side divides by another gigantic polynomial. I was completely stunned by these polynomials.

There were two workshops at CRM, Montreal, in two consecutive weeks of May, 1996; one on Painlevé equations and the other on q-special functions. It was lucky that Okamoto, Umemura and I had a chance to stay at CRM together in the same period. We discussed many things about Painlevé equations walking around the university. When we started to talk on Umemura's computation, Okamoto suggested: If the sixth Painlevé equation is the counterpart of the hypergeometric equation in the world of nonlinear differential equations, its particular solutions should share the best properties of hypergeometric functions; if it is the case, we know $x = \pm 1$ are the singular points in the case of Umemura's polynomials, and so, the values of U_n at these points should factorize! It was a rainy weekend in Montreal, I was troubled by a heavy toothache, but I had to go to work at an office of CRM as I had promised with Okamoto. It was a surprise to both Okamoto and me to see the values of U_n at $x = \pm 1$ factorize, but in an unexpected way. We realized that these values are products of quadratic (bilateral) shifted factorials; we were actually seeing the coefficients at the both ends $u^{n(n-1)}$ and $v^{n(n-1)}$ of U_n. I am not sure whether we already knew that Jacobi polynomials on the interval $[-1, 1]$ have nicely factorized coefficients when expressed as a homogenous polynomial of $x + 1$ and $x - 1$. In any event, Okamoto and I decided to rewrite Umemura's polynomials in the symmetric, homogenous form in terms of $u = x + 1$ and $v = x - 1$. As I have already shown in Introduction, the results were astonishing. The coefficients of U_{n+1} in the homogenous expansion also seemed to be products of quadratic shifted factorials; they were built up according how to divide the indexing set $\{1, \ldots, n\}$ into two parts. Once we decided to express U_{n+1} as a sum over the pairs (I, J) of subsets with $I \sqcup J = \{1, \ldots, n\}$, it was a fun for us to find the way to the final form of formula (1.9).

Umemura was amazed with these computations of Okamoto and me. The gigantic dragon of U_5 is now a lovely polynomial of four lines! But, what are the mysterious positive integers $n_{I,J}$ appearing in the coefficients? Umemura would bring these new formulas for U_n to anyone who might be interested in combinatorics to ask this question. It was a half year later that we met a third surprise. Towards the end of 1996, we held a workshop on hypergeometric systems at Kobe University. On that occasion, I had a chance to show those new formulas for U_n to Okada who was a participant of the workshop. The next morning, Okada came to me and said with all smiles: These are dimensions of irreducible representations of GL. Wow! Again, I was completely knocked out by this.

This was how Umemura, Okamoto, Okada and I came to write the joint paper "Special polynomials, II". I think that this story of Umemura polynomials was one of the most exciting events in my life. The NOOU conjecture was proven in a couple of years by Taneda, who was a student of Umemura. Referring to it, Umemura wrote in [19]:

> This theorem should be counted as a landmark in the history of studies of the Painlevé equations. It clarified the relationship between combinatorics of Young diagrams and the Painlevé equations that cannot be imagined at all in view of the origin of Painlevé equations. At this point in time, I thought we could definitely say that the Painlevé equations define special functions.

To understand the mystery of Umemura polynomials – it was also the main motivation of my later works on Painlevé equations with Yamada and other colleagues. Although more than twenty years have passed since I met Umemura polynomials, I feel there are so many things, or I should say more things than before, that we do not really understand yet about Painlevé equations. The conjecture that I formulated in the present paper is one of the things being left unsolved around myself. I would respectfully pray for Professor Umemura, recollecting irreplaceable experiences with him.

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