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Endomorphisms and bijections of the character variety $\chi(\mathbf{F}_2, \mathsf{SL}_2(\mathbf{C}))$ (*)

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Abstract. — We answer a question of Gelander and Souto in the special case of the free group of rank 2. The result may be stated as follows. If $\mathbf{F}$ is a free group of rank 2, and $\mathbf{G}$ is a proper subgroup of $\mathbf{F}$, the restriction of homomorphisms $\mathbf{F} \to \mathsf{SL}_2(\mathbf{C})$ to the subgroup $\mathbf{G}$ defines a map from the character variety $\chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$ to the character variety $\chi(\mathbf{G}, \mathsf{SL}_2(\mathbf{C}))$; this algebraic map never induces a bijection between these two character varieties.

Résumé. — Le résultat suivant, qui répond à une question de Gelander et Souto dans un cas particulier, est démontré : si $\mathbf{F}$ est le groupe libre de rang 2 et $\mathbf{G}$ est un sous-groupe de $\mathbf{F}$, la restriction des homomorphismes $\mathbf{F} \to \mathsf{SL}_2(\mathbf{C})$ au sous-groupe $\mathbf{G}$ fournit une application de la variété des caractères $\chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$ vers la variété des caractères $\chi(\mathbf{G}, \mathsf{SL}_2(\mathbf{C}))$; cette application algébrique n'est bijective que si $\mathbf{G}$ coïncide avec $\mathbf{F}$.

1. Representations and character varieties

Consider the free group of rank 2,

$$\mathbf{F} = \langle a, b \,|\, \emptyset \rangle, \tag{1.1}$$

and an algebraic group H. Every representation $\rho \colon \mathbf{F} \to H$ is defined by prescribing the images of the generators $A = \rho(a)$ and $B = \rho(b)$ in H. Thus, the variety of representations $\mathit{Rep}(\mathbf{F}, H)$ is just the product $H \times H$. The group H acts on this variety by conjugation, and the quotient, in the sense of geometric invariant theory, is called the character variety of $(\mathbf{F}, H)$; we shall denote it $\chi(\mathbf{F}, H)$.

Assume now that H is the special linear group SL_2 (the field of definition will be specified later). Since traces of matrices are polynomial functions in

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the coefficients of the matrices and are invariant under conjugacy, the three functions

$$x = \mathsf{tr}(A),\ y = \mathsf{tr}(B),\ z = \mathsf{tr}(AB) \tag{1.2}$$

provide regular functions on the character variety $\chi(\mathbf{F}, \mathsf{SL}_2)$. The following result, due to Fricke, is proven in details in [5].

FRICKE'S THEOREM. — *The character variety* $\chi(\mathbf{F}, \mathsf{SL}_2)$ *is the affine space of dimension* 3*; its ring of regular functions are the polynomial functions in the coordinates* $(x, y, z) = (\mathsf{tr}(A), \mathsf{tr}(B), \mathsf{tr}(AB))$.

We did not specify the field because this theorem works for any algebraically closed field. Examples of invariant functions are given by traces of words in the matrices A and B, for instance by the function $\mathsf{tr}(A^3B^{-2}AB)$; in fact, the theorem of Fricke is based on the fact that these traces can be expressed as polynomial functions of x, y, and z with integer coefficients. This follows easily from Cayley-Hamilton theorem. For instance $A - \mathsf{tr}(A)\, Id + A^{-1} = 0$, which shows that $\mathsf{tr}(A^{-1}B) = xy - z$. A classical example is given by the trace of the commutator of A and B:

$$\mathsf{tr}(ABA^{-1}B^{-1}) = x^2 + y^2 + z^2 - xyz - 2. \tag{1.3}$$

The level sets of this polynomial function are the cubic surfaces

$$S_\kappa = \{(x, y, z)\,;\, x^2 + y^2 + z^2 = xyz + \kappa\}. \tag{1.4}$$

The surface S_0 is known as the Markoff surface, and S_4 as the Cayley cubic (see [1, §2.8] and [2, §1.5]).

2. Restrictions

Now, consider a subgroup $\mathbf{G}$ of $\mathbf{F}$. It is a free group, and we assume that $\mathbf{G}$ has rank two, as $\mathbf{F}$. Fixing a basis (u, v) of $\mathbf{G}$, we have:

(1) u and v are elements of $\mathbf{F}$, hence they are words $u = u(a, b)$ and $v = v(a, b)$ in the generators a and b and their inverses;
(2) $\mathbf{G} = \langle u, v\rangle$, with no relations between u and v.

Since u is a word in a and b, we know from the theorem of Fricke that there is a polynomial function $P \in \mathbf{Z}[X, Y, Z]$ with the following property. For every pair (A, B) of elements of SL_2,

$$P(x, y, z) = \mathsf{tr}(u(A, B)) \quad \text{where} \quad (x, y, z) = (\mathsf{tr}(A), \mathsf{tr}(B), \mathsf{tr}(AB)). \tag{2.1}$$

Similarly, there are polynomial functions Q and R such that

$$Q(x, y, z) = \mathsf{tr}(v(A, B)) \quad \text{and} \quad R(x, y, z) = \mathsf{tr}(u(A, B)v(A, B)). \tag{2.2}$$

Every representation ρ of $\mathbf{F}$ into SL_2 gives a representation of $\mathbf{G}$: the restriction of ρ to $\mathbf{G}$. Thus, we get a map $res\colon \chi(\mathbf{F}, \mathsf{SL}_2) \to \chi(\mathbf{G}, \mathsf{SL}_2)$. Once these character varieties have been identified to affine spaces of dimension 3 using the coordinates $(x, y, z) = (\mathsf{tr}(A), \mathsf{tr}(B), \mathsf{tr}(AB))$ and $(r, s, t) = (\mathsf{tr}(U), \mathsf{tr}(V), \mathsf{tr}(UV))$, this map res corresponds to the algebraic endomorphism $\mathbb{A}^3 \to \mathbb{A}^3$ defined by

$$(x, y, z) \mapsto (P(x, y, z), Q(x, y, z), R(x, y, z)). \tag{2.3}$$

Our goal is to understand whether this map can be a bijection (resp. an isomorphism of algebraic varieties) when $\mathbf{G}$ is a strict subgroup of $\mathbf{F}$. This was the question raised by Gelander and Souto, in its simpler form.

To restate this question more precisely, we adopt another equivalent viewpoint. Consider the endomorphism $\varphi\colon \mathbf{F} \to \mathbf{F}$ that maps a to $u(a, b)$ and b to $v(a, b)$. Its image is $\mathbf{G}$. Given any representation ρ of $\mathbf{F}$, $\varphi_*\rho = \rho \circ \varphi$ is a new representation of $\mathbf{F}$; this determines an algebraic endomorphism

$$\Phi\colon \chi(\mathbf{F}, \mathsf{SL}_2) \to \chi(\mathbf{F}, \mathsf{SL}_2). \tag{2.4}$$

Then, res is a bijection if and only if Φ is a bijection (these two maps are actually the same maps in affine coordinates). Thus, the question may be stated as follows.

Questions. — *Given an endomorphism $\varphi\colon \mathbf{F} \to \mathbf{F}$ of the free group $\mathbf{F}$, under what condition does it induce an automorphism $\Phi\colon \chi(\mathbf{F}, \mathsf{SL}_2) \to \chi(\mathbf{F}, \mathsf{SL}_2)$ of the algebraic variety $\chi(\mathbf{F}, \mathsf{SL}_2)$? Given an endomorphism $\varphi\colon \mathbf{F} \to \mathbf{F}$, and a field $\mathbf{k}$, under what condition does φ induce a bijection $\Phi\colon \mathbb{A}^3(\mathbf{k}) \to \mathbb{A}^3(\mathbf{k})$ of the set of $\mathbf{k}$ points of $\chi(\mathbf{F}, \mathsf{SL}_2) = \mathbb{A}^3$?*

For the second version of the question, it is crucial to indicate over which field one works. If the field is too small, for instance if it is a finite field, there are many endomorphisms φ that induce bijections on the set of representations into $\mathsf{SL}_2(\mathbf{k})$. Indeed, consider a finite group H, for example $H = \mathsf{SL}_2(\mathbf{k})$ for some finite field $\mathbf{k}$, and denote by n the number of elements of H. Then, every element $h \in H$ satisfies $h^n = e_H$. Now, pick positive integers ℓ and ℓ' and consider the endomorphism φ of $\mathbf{F}$ that maps a to $a^{\ell n+1}$ and b to $b^{\ell' n+1}$. Then, $\rho(\varphi(a)) = \rho(a)$ and $\rho(\varphi(b)) = \rho(b)$ for every representation $\rho\colon \mathbf{F} \to H$; thus, φ induces an injection (hence a bijection) of the finite set of representations of $\mathbf{F}$ into H.

If we assume that $\mathbf{k}$ is algebraically closed and of characteristic 0, the two questions are actually equivalent, as the following classical statement shows.

Bijectivity Theorem. — *Let $\Phi\colon \mathbb{A}^d \to \mathbb{A}^d$ be a regular endomorphism of an affine space, defined over an algebraically closed field $\mathbf{k}$ of characteristic 0. If Φ is an injective transformation of $\mathbb{A}^d(\mathbf{k})$ then Φ is an automorphism of $\mathbb{A}^d$: it is bijective and its inverse is also defined by polynomial formulas.*

This theorem fails over the field of real numbers, as $x \mapsto x + x^3$ shows. It also fails in positive characteristic, as the Frobenius morphism shows. For a proof of the Bijectivity Theorem see the book [4]. Note also that this result holds in much greater generality, and can therefore be applied to character varieties of higher rank free groups.

3. The main theorem

Theorem A. — *Let* $\mathbf{F}$ *be the free group of rank* 2*, and* $\varphi\colon \mathbf{F} \to \mathbf{F}$ *be an endomorphism of* $\mathbf{F}$*. If the algebraic endomorphism*

$$\Phi\colon \chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C})) \to \chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$$

induced by φ *is injective, then* $\varphi\colon \mathbf{F} \to \mathbf{F}$ *is an automorphism of the free group* $\mathbf{F}$*.*

Corollary 3.1. — *Let* $\mathbf{F}$ *be the free group of rank* 2*. If* $\mathbf{G}$ *is a proper subgroup of* $\mathbf{F}$*, the restriction res:* $\rho \mapsto \rho_{|\mathbf{G}}$ *does not induce a bijection from the character variety* $\chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$ *to the character variety* $\chi(\mathbf{G}, \mathsf{SL}_2(\mathbf{C}))$*.*

Proof of the corollary. — For *res* to be a bijection, $\mathbf{G}$ should have rank 2 (the dimension of $\chi(\mathbf{G}, \mathsf{SL}_2(\mathbf{C}))$ is $3\mathsf{rk}(\mathbf{G}) - 3$). The previous section shows that *res* is a bijection if and only if the endomorphism $\varphi\colon \mathbf{F} \to \mathbf{F}$ determined by any isomorphism between $\mathbf{F}$ and $\mathbf{G}$ induces a bijection on the character variety of $\mathbf{F}$. Theorem A shows that φ must be an isomorphism, hence $\mathbf{G} = \mathbf{F}$. □

4. The proof

To prove Theorem A, one first makes use of the Bijectivity Theorem, and deduce that the polynomial endomorphism Φ which is determined by φ is a polynomial automorphism of the character variety $\chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$. In what follows, S_κ is the complex affine surface defined by Equation (1.4) (it may be better to denote it $S_\kappa(\mathbf{C})$).

4.1. Automorphisms of the surfaces S_κ

In what follows, we denote by $\mathsf{Aut}(W)$ the group of automorphisms of the algebraic variety W. (Note that we play with two distint notions of automorphisms and endomorphisms, one for groups, one for algebraic varieties.)

One can identify the group $\mathsf{Out}(\mathbf{F})$ with $\mathsf{GL}_2(\mathbf{Z})$ (see [7, Prop. I.4.5]). This group acts on the character variety $\chi(\mathbf{F}, \mathsf{SL}_2)$. The function $\mathsf{tr}([A, B])$ is invariant under this action because every automorphism of the group $\mathbf{F}$ maps $aba^{-1}b^{-1}$ to a conjugate of itself or its inverse (see [7, Prop. I.5.1] for instance). This gives an embedding

$$\mathsf{GL}_2(\mathbf{Z}) \to \mathsf{Aut}(\chi(\mathbf{F}, \mathsf{SL}_2)), \tag{4.1}$$

i.e. in $\mathsf{Aut}(\mathbb{A}^3)$, that preserves the polynomial function $x^2+y^2+z^2-xyz-2$ and its level sets S_κ.

El'Huti's Theorem. — *Let κ be a complex number. The group $\mathsf{GL}_2(\mathbf{Z}) = \mathsf{Out}(\mathbf{F})$ provides a subgroup of index 4 in the group of all automorphisms of the complex affine surface S_κ: every automorphism of S_κ is the composition of an element of $\mathsf{Out}(\mathbf{F})$ and a linear map $(x, y, z) \mapsto (\epsilon_1 x, \epsilon_2 y, \epsilon_3 z)$ where each $\epsilon_i = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3 = 1$.*

Let us explain how this result follows from the main theorems of [3]. First, note that the image of the homomorphism $\mathsf{GL}_2(\mathbf{Z}) \to \mathsf{Aut}(S_\kappa)$ contains the finite group of permutations of the coordinates. For instance, the permutation $(x, y, z) \mapsto (z, y, x)$ is induced by the automorphism of $\mathbf{F}$ mapping a and b to $(ab)^{-1}$ and b.

To describe more precisely El'Huti's work, we compactify S_κ by taking its closure $\overline{S_\kappa}$ in the projective space $\mathbb{P}^3_{\mathbf{C}}$. In homogeneous variables $[x : y : z : w]$, this surface is defined by the cubic equation

$$(x^2 + y^2 + z^2)w = xyz + \kappa w^3. \tag{4.2}$$

It intersects the plane at infinity $\{w = 0\}$ into a triangle $\{xyz = 0\}$. If f is an automorphism of S_κ, it extends as a birational map $\overline{f}$ of $\overline{S_\kappa}$, typically with indeterminacy points on the triangle at infinity.

There are three obvious involutions on S_κ. Indeed, if one projects S_κ onto the (x, y)-plane one gets a 2-to-1 cover because the equation of S_κ has degree 2 with respect to the z-variable; the deck transformation of this cover is the involution

$$\sigma_z(x, y, z) = (x, y, xy - z). \tag{4.3}$$

Geometrically, σ_z is the following birational transformation of $\overline{S_\kappa}$: if $[x : y : z : w]$ is a point of $\overline{S_\kappa}$, draw the line joining this point to the point "at infinity" $[0 : 0 : 1 : 0] \in \overline{S_\kappa}$; this line intersects $\overline{S_\kappa}$ in exactly three points, and the third point of intersection is precisely $\sigma_z[x : y : z : w]$. Permuting the variables, we obtain three involutions σ_x, σ_y, σ_z and Theorem 1 of [3] says that the group generated by those three involutions is a free product

$\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$. Now, note that the element

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathsf{GL}_2(\mathbf{Z}) \tag{4.4}$$

is represented by the automorphism of $\mathbf{F}$ mapping the generators a and b to a and b^{-1}, and its action on traces corresponds to σ_z because $\mathsf{tr}(B^{-1}) = \mathsf{tr}(B)$ and $\mathsf{tr}(AB^{-1}) = -\mathsf{tr}(AB) + \mathsf{tr}(A)\mathsf{tr}(B)$ for elements of SL_2 (see Section 1). Using permutations of coordinates, we see that the image of $\mathsf{GL}_2(\mathbf{Z})$ in $\mathsf{Aut}(S_\kappa)$ contains the three involutions σ_x, σ_y, and σ_z, hence the group $\langle \sigma_x, \sigma_y, \sigma_z \rangle$ that they generate.

Theorem 2 of [3] states that the automorphism group $\mathsf{Aut}(S_\kappa)$ is generated by two groups: the group $\langle \sigma_x, \sigma_y, \sigma_z \rangle \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$, and the group $W(S_\kappa)$ of projective transformations of $\mathbb{P}^3_{\mathbf{C}}$ preserving the compact surface $\overline{S_\kappa}$ and its open surface $S_\kappa \subset \overline{S_\kappa}$. The following lemma concludes the proof of what we called El'Huti's theorem.

Lemma 4.1. — *The group $W(S_\kappa)$ is the group generated by*

(1) *the group of permutations of the coordinates (x, y, z), and*
(2) *the changes of sign of pairs of coordinates (such as $(x, y, z) \mapsto (-x, -y, z)$).*

Proof. — Let f be a linear projective transformation preserving $S_\kappa \subset \overline{S_\kappa}$. Then, f preserves the triangle $\overline{S_\kappa} \setminus S_\kappa$, of equation $\{w = 0,\ xyz = 0\}$. Composing f by a permutation of the coordinates, we may assume that (i) f induces an affine transformation of the affine space $\mathbb{A}^3_{\mathbf{C}}$ and (ii) f fixes the three points $[1:0:0:0]$, $[0:1:0:0]$ and $[0:0:1:0]$ at infinity. Thus, f becomes an affine transformation whose linear part is diagonal, i.e. $f(x, y, z) = (\alpha x + a, \beta y + b, \gamma z + c)$ for some complex numbers α, β, γ, a, b, and c with $\alpha\beta\gamma \neq 0$. Now, if one writes that S_κ is invariant, and look at the quadratic terms xy, yz, and zx, one sees that $a = b = c = 0$; then, α, β, and γ are all equal to $+1$ or -1. □

4.2. Invariance of S_4

Reducible representations correspond to the surface S_4: both A and B preserve a one dimensional subspace of $\mathbf{C}^2$, so that A and B can be written simultaneously as upper triangular matrices; there commutator $ABA^{-1}B^{-1}$ is upper triangular, with 1's on the diagonal, and $\mathsf{tr}(ABA^{-1}B^{-1}) = 2$.

If ρ is a reducible representation of $\mathbf{F}$, so is $\varphi_*\rho$; thus Φ induces an automorphism of S_4. Since $\mathsf{GL}_2(\mathbf{Z})$ generates a subgroup of $\mathsf{Aut}(S_4)$ of finite index, we obtain the following lemma.

Lemma 4.2. — *The endomorphism Φ is an automorphism of the complex algebraic variety $\chi(\mathbf{F}, \mathsf{SL}_2) = \mathbb{A}^3$ that preserves S_4. It induces an automorphism of S_4. There is an integer $k > 0$ and an element ψ of $\mathsf{Out}(\mathbf{F})$ such that $\Phi^k = \Psi$ on S_4.*

Here Ψ denotes the automorphism of $\chi(\mathbf{F}, \mathsf{SL}_2)$ which is defined by ψ_*.

Remark 4.3. — This remark is not needed in the proof, but illustrates the nice geometry of S_4. One can "uniformize" S_4 by $\mathbf{C}^* \times \mathbf{C}^*$, as follows. Given a pair $(z_1, z_2) \in \mathbf{C}^* \times \mathbf{C}^*$, consider two upper triangular matrices A and B whose diagonal coefficients are respectively $(z_1, 1/z_1)$ and $(z_2, 1/z_2)$. Then,

$$(\mathsf{tr}(A), \mathsf{tr}(B), \mathsf{tr}(AB)) = (z_1 + 1/z_1, z_2 + 1/z_2, z_1 z_2 + 1/(z_1 z_2)). \tag{4.5}$$

Then,

- the map $\pi\colon (z_1, z_2) \mapsto (z_1 + 1/z_1, z_2 + 1/z_2, z_1 z_2 + 1/(z_1 z_2))$ is invariant under the involution $\eta(z_1, z_2) = (1/z_1, 1/z_2)$ of $\mathbf{C}^* \times \mathbf{C}^*$;
- the image $\pi(\mathbf{C}^* \times \mathbf{C}^*)$ is S_4;
- the projection $\pi : \mathbf{C}^* \times \mathbf{C}^* \to S_4$ realizes S_4 as the quotient $(\mathbf{C}^* \times \mathbf{C}^*)/\eta$;
- the four fixed points $(\pm 1, \pm 1)$ of η give rise to the four singularities of S_4.

The group $\mathsf{GL}_2(\mathbf{Z})$ acts by automorphisms on the algebraic group $\mathbf{C}^* \times \mathbf{C}^*$; in coordinates (z_1, z_2), this action is given by monomial transformations $(z_1, z_2) \mapsto (z_1^a z_2^b, z_1^c z_2^d)$. From El'Huti's theorem, or by a direct computation, one easily deduces that this copy of $\mathsf{GL}_2(\mathbf{Z})$ in $\mathsf{Aut}(S_4)$ coincides with the one given by $\mathsf{Out}(\mathbf{F})$ and has finite index in $\mathsf{Aut}(S_4)$. (see [2, §1.2] for details)

4.3. Invariance of E_4

We replace Φ by Φ^k and compose it with Ψ^{-1}; after such a modification Φ is the identity on S_4.

Our goal is to show that, under this extra hypothesis, Φ is the identity. For this, note that the equation

$$E_4(x, y, z) = x^2 + y^2 + z^2 - xyz - 4 \tag{4.6}$$

of S_4 is transformed by the automorphism $\Phi\colon \mathbb{A}^3 \to \mathbb{A}^3$ into another (reduced) equation of the same (hyper)surface. Thus, there is a non-zero constant α such that

$$E_4 \circ \Phi = \alpha E_4. \tag{4.7}$$

Lemma 4.4. — *The constant* α *is equal to* 1. *Hence,* E_4 *is* Φ*-invariant, and each of the surfaces* S_κ *is* Φ *invariant.*

Proof. — Since $E_4 \circ \Phi = \alpha E_4$, the level sets S_κ of E_4 are permuted by the automorphism Φ. Among them, exactly two are singular surfaces. The surface S_4, and the Markov surface S_0. Indeed, the differential of E_4 is $(2x - yz)\,dx + (2y - zx)\,dy + (2z - xy)\,dz$; if it vanishes, we obtain $x^2 = y^2 = z^2 = xyz/2 = \kappa$, and we deduce that $\kappa = 0$ or 4. Since S_4 is Φ-invariant, the singular surface S_0 (and its singularity at the origin) must also be Φ-invariant. This implies $\alpha E_4(0,0,0) = E_4(0,0,0)$ and $\alpha = 1$. ☐

Remark 4.5. — Instead of looking at singularities of the surfaces S_κ, we could have considered the subset F of $\chi(\mathbf{F}, \mathsf{SL}_2)$ given by irreducible representations with finite image. This set is Φ-invariant, and it is finite. Thus, there exists $\ell > 0$ such that Φ^ℓ fixes F pointwise. This implies $E_4 \circ \Phi^\ell = E_4$; looking at the different possibilities for the finite images, one can even deduce that $\ell = 1$

4.4. Conclusion

From Lemma 4.4 we get $E_4 \circ \Phi = E_4$. Thus, Φ is an automorphism of the complex affine space $\chi(\mathbf{F}, \mathsf{SL}_2)$ that preserves every level set S_κ of E_4. Fix such a constant κ, and consider the restriction of Φ to S_κ. This is an automorphism of S_κ and we denote by $\overline{\Phi}$ its extension, as a birational transformation, to the compactification $\overline{S_\kappa}$ of S_κ in $\mathbb{P}^3(\mathbf{C})$. The trace of $\overline{S_\kappa}$ at infinity is the triangle given by $xyz = 0$ (see Section 4.1). This triangle does not depend on κ, and one verifies that the action of $\overline{\Phi}$ at infinity does not depend on κ either: indeterminacy points, and exceptional curves are the same for all values of κ (see [1, §2.4 and 2.6]). But for $\kappa = 4$, we know that this action is just the identity map. Thus, $\overline{\Phi}$ is in fact an automorphism of $\overline{S_\kappa}$ for all values of κ. From Section 4.1, we know this automorphism Φ is a composition of a permutation of the coordinates (x, y, z) with a diagonal linear map whose diagonal coefficients are ± 1. Since Φ is the identity on S_4, we deduce that Φ is the identity.

Thus, we have shown that there is an automorphism ψ of $\mathbf{F}$ and a positive integer k such that $\Phi^k \circ \Psi^{-1}$ is the identity map. In other words, $\Phi^k = \Psi$ on $\chi(\mathbf{F}, \mathsf{SL}_2)$. To conclude, one needs to show that an endomorphism φ of $\mathbf{F}$ that induces the identity map on $\chi(\mathbf{F}, \mathsf{SL}_2)$ is in fact an inner automorphism of the group $\mathbf{F}$. To prove it, fix a faithful representation $\rho\colon \mathbf{F} \to \mathsf{SL}_2(\mathbf{C})$; its image is automatically Zariski dense in the complex algebraic group $\mathsf{SL}_2(\mathbf{C})$. For instance, take

$$\rho(a) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \tag{4.8}$$

for $z=2$ or even for a generic $z \in \mathbf{C}$ (see [6, §II.B.25]). Then, the fiber of ρ for the quotient map $Rep(\mathbf{F}, \mathsf{SL}_2(\mathbf{C})) \to \chi(\mathbf{F}, \mathsf{SL}_2(\mathbf{C}))$ is an orbit for the action of $\mathsf{SL}_2(\mathbf{C})$ by conjugation on

$$Rep(\mathbf{F}, \mathsf{SL}_2(\mathbf{C})) \simeq \mathsf{SL}_2(\mathbf{C}) \times \mathsf{SL}_2(\mathbf{C}). \tag{4.9}$$

Since φ induces the identity map on $\chi(\mathbf{F}, \mathsf{SL}_2)$, ρ and $\rho\circ\varphi$ are in the same conjugacy class, there is an element $c \in \mathbf{F}$ such that $\rho\circ\varphi(w) = \rho(cwc^{-1})$ for every $w \in \mathbf{F}$, and φ coincides with the conjugation by c because ρ is faithful.

Remark 4.6. — It may also be possible to conclude the proof by showing that φ preserves the conjugacy classes of $aba^{-1}b^{-1}$ and its inverse (because Φ preserves the polynomial function E_4). And this property is sufficient to imply that φ is an automorphism of $\mathbf{F}$.

5. Two open problems

Theorem A leaves many natural questions open. One may, for instance, replace the free group of rank 2 by a free group of rank $n > 1$ (or by fundamental groups of closed surfaces) and the group SL_2 by other algebraic groups. One may also replace the field $\mathbf{C}$ by other fields, for instance by $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{Q}_p$. Let us now state two open problems that concern $\chi(\mathbf{F}, \mathsf{SL}_2)$.

5.1. Real coefficients

The proof makes use of the fact that $\mathbf{C}$ is algebraically closed in order to get the equivalence "Φ *is a bijection if and only if it is an automorphism*". Let us replace $\mathbf{C}$ by the field $\mathbf{R}$ of real numbers, and simply assume that Φ is a bijection of the real part $\mathbb{A}^3(\mathbf{R})$ of the character variety. The difficulty is that there are algebraic bijections of $\mathbf{R}$ which are not isomorphisms, for instance $t \mapsto t + t^3$.

There are two parts in $\mathbb{A}^3(\mathbf{R})$, corresponding respectively to representations of $\mathbf{F}$ in $\mathsf{SL}_2(\mathbf{R})$ and in SU_2. There common boundary is the surface $S_4(\mathbf{R})$. These subsets are Φ-invariant; in particular, $S_4(\mathbf{R})$ is invariant, as a subset or $\mathbb{A}^3(\mathbf{R})$ (this does not imply that its equation E_4 is invariant). I haven't been able to use this invariance to prove that Φ is a bijection of $\mathbb{A}^3(\mathbf{R})$ if and only if φ is an automorphism of $\mathbf{F}$. Thus, Theorem A remains an open problem if one replaces the field $\mathbf{C}$ by $\mathbf{R}$.

5.2. Topological degree

A better result than Theorem A would be to compute the topological degree of $\Phi\colon \mathbb{A}^3(\mathbf{C}) \to \mathbb{A}^3(\mathbf{C})$ given by any injective endomorphism φ of F, or at least to estimate it from below. Theorem A just says that it cannot be equal to 1.

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