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Mapping properties of the Hilbert and Fubini–Study maps in Kähler geometry ^(*)

YOSHINORI HASHIMOTO ⁽¹⁾

ABSTRACT. — Suppose that we have a compact Kähler manifold X with a very ample line bundle \mathcal{L} . We prove that any positive definite hermitian form on the space $H^0(X, \mathcal{L})$ of holomorphic sections can be written as an L^2 -inner product with respect to an appropriate hermitian metric on \mathcal{L} . We apply this result to show that the Fubini–Study map, which associates a hermitian metric on \mathcal{L} to a hermitian form on $H^0(X, \mathcal{L})$, is injective.

RÉSUMÉ. — Soit X une variété compacte kählérienne avec un fibré en droites \mathcal{L} qui est très ample. Nous prouvons que toute forme hermitienne définie positive sur $H^0(X, \mathcal{L})$ peut être écrite comme produit scalaire L^2 associé à une métrique hermitienne sur \mathcal{L} . Nous appliquons ce résultat pour montrer que l’application de Fubini–Study, des formes hermitiennes sur $H^0(X, \mathcal{L})$ vers les métriques hermitiennes sur \mathcal{L} , est injective.

1. Introduction and statement of the result

Let (X, \mathcal{L}) be a polarised Kähler manifold of complex dimension n . We have

- $\mathcal{H}(X, \mathcal{L}) :=$ the set of all positively curved hermitian metrics on \mathcal{L} ,
- $\mathcal{B} :=$ the set of all positive definite hermitian forms on $H^0(X, \mathcal{L})$,

where $\mathcal{H}(X, \mathcal{L})$ is infinite dimensional and \mathcal{B} is finite dimensional.

We can define the following two maps;

- the *Hilbert map* $\text{Hilb} : \mathcal{H}(X, \mathcal{L}) \rightarrow \mathcal{B}$ defined by the L^2 -inner product of $h \in \mathcal{H}(X, \mathcal{L})$,

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- the *Fubini–Study map* $FS : \mathcal{B} \rightarrow \mathcal{H}(X, \mathcal{L})$ defined as the pullback of the Fubini–Study metric on the projective space $\mathbb{P}(H^0(X, \mathcal{L})^*)$.

The result that we prove in this paper is the following.

THEOREM 1.1. — *Suppose that \mathcal{L} is very ample. Then Hilb is surjective and FS is injective.*

Since Hilb is a map from an infinite dimensional manifold to a finite dimensional manifold, it seems natural to speculate that it is surjective. Similarly, it also seems natural to speculate that FS is injective. Indeed, these statements seem to be widely believed among the experts in the field. However, the proof of these facts do not seem to be explicitly written in the literature previously, to the best of the author’s knowledge; in fact, as we shall see, the proof that we give involves application of the degree of continuous maps and has connection to the Aubin–Yau theorem [2, 13]. We shall provide the proof of these “folklore” statements in this paper.

For FS , we shall in fact prove a stronger quantitative result for the injectivity (cf. Lemma 3.1), which was applied in [9] to find a point in \mathcal{B} that is close to the minimum of the modified balancing energy. Generalisations to several variants of the Hilb map (cf. Proposition 2.17) will also be discussed in Section 2.4.

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2. Surjectivity of Hilb

2.1. Preliminary definitions

Suppose that \mathcal{L} is very ample. We shall first define the maps Hilb and FS more precisely as follows (cf. [6]).

DEFINITION 2.1. — *The Hilbert map $\text{Hilb} : \mathcal{H}(X, \mathcal{L}) \rightarrow \mathcal{B}$ is defined by*

$$\text{Hilb}(h) := \int_X h(\cdot, \cdot) \omega_h^n.$$

The Fubini–Study map $FS : \mathcal{B} \rightarrow \mathcal{H}(X, \mathcal{L})$ is defined by the equation

$$\sum_{i=1}^N |s_i|_{FS(H)}^2 = 1 \tag{2.1}$$

where $\{s_i\}_i$ is an H -orthonormal basis for $H^0(X, \mathcal{L})$ and $N := \dim_{\mathbb{C}} H^0(X, \mathcal{L})$.

Remark 2.2. — The above definition is slightly different from the ones that are often used in the literature by a factor of scaling, in the following sense. Often we consider the tensor power $\mathcal{L}^{\otimes k}$, and the usual definition of Hilb concerns a hermitian metric h on \mathcal{L} (rather than $\mathcal{L}^{\otimes k}$), with $\text{Hilb}(h) := \frac{N_k}{V} \int_X h^{\otimes k}(\cdot, \cdot) \frac{\omega_h^n}{n!}$ defining a hermitian form on the vector space $H^0(X, \mathcal{L}^{\otimes k})$ of dimension N_k , with $V := \int_X c_1(L)^n/n!$. For a hermitian form H on $H^0(X, \mathcal{L}^{\otimes k})$, $FS(H)$ is a hermitian metric on \mathcal{L} satisfying $\sum_i |s_i|_{FS(H)^{\otimes k}}^2 = 1$.

As far as Theorem 1.1 is concerned, these differing conventions will not cause any problem, and we shall use the one given above in this paper not to be concerned with extra exponent of k or the constant N_k . For the quantitative result Lemma 3.1, however, this does require some straightforward modification which is mentioned in Remark 3.2. We shall consider $\mathcal{L}^{\otimes k}$ also in Section 2.4.

We also recall (and slightly modify) some definitions made by Bourguignon–Li–Yau [5, Section 2] that we shall use in the proof of surjectivity of Hilb in Section 2.3. Since \mathcal{L} is very ample we have the Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*) \xrightarrow{\sim} \mathbb{P}^{N-1}$. First of all pick homogeneous coordinates $\{Z_i\}_i$ on \mathbb{P}^{N-1} ; all matrices appearing in what follows will be with respect to this basis $\{Z_i\}_i$, unless otherwise specified. This then defines a hermitian metric $\tilde{h} := \tilde{h}_{FS(\text{Id})}$ on $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ and the Fubini–Study metric $\omega_{\tilde{FS}(\text{Id})}$ on \mathbb{P}^{N-1} , where Id stands for the identity matrix. We write $d\mu_Z$ for the volume form on \mathbb{P}^{N-1} defined by $\omega_{\tilde{FS}(\text{Id})}$.

Suppose that we pick $B \in GL(N, \mathbb{C})$ and change the basis from $\{Z_i\}_i$ to $BZ := \{Z'_i\}_i$ (where $Z'_i := \sum_j B_{ij}Z_j$). This defines a new Fubini–study metric $\omega_{\widetilde{FS}(H)}$ associated to the hermitian metric $\widetilde{h}_{FS(H)}$, where $H := \overline{(B^{-1})^t}B^{-1}$; note that H has BZ as its orthonormal basis. We also write $d\mu_{BZ}$ for the volume form corresponding to $\omega_{\widetilde{FS}(H)}$.

DEFINITION 2.3. — *We define two spaces of hermitian matrices as follows:*

- $\mathcal{J}^\circ := \{N \times N \text{ positive definite hermitian matrices with operator norm } 1\}$,
- $\mathcal{H}^\circ := \{N \times N \text{ positive definite hermitian matrices with trace } 1\}$.

Since $B \in \mathcal{J}^\circ$ is positive definite hermitian, the operator norm $\|B\|_{op}$ of B being 1 is equivalent to the largest eigenvalue of B being 1 (all eigenvalues of B are real and positive). Observe also that any $N \times N$ positive definite hermitian matrix B' can be divided by a positive real constant $\alpha_1 := \|B'\|_{op}$ (resp. $\alpha_2 := \text{tr}(B')$) to be an element of \mathcal{J}° (resp. \mathcal{H}°).

In other words, \mathcal{J}° and \mathcal{H}° just denote two different scalings of the set of $N \times N$ positive definite hermitian matrices, and hence they are diffeomorphic under the map $\mathcal{J}^\circ \ni B \mapsto B/\text{tr}(B) \in \mathcal{H}^\circ$. Artificial as it may seem, it will be useful in Section 2.3 to fix the scalings as above.

Note $\dim_{\mathbb{R}} \mathcal{J}^\circ = \dim_{\mathbb{R}} \mathcal{H}^\circ = N^2 - 1$ and that \mathcal{J}° and \mathcal{H}° can be identified with connected bounded open subsets in \mathbb{R}^{N^2-1} . We endow them with the Euclidean topology inherited from \mathbb{R}^{N^2-1} .

With respect to this topology, we can compactify \mathcal{J}° (resp. \mathcal{H}°) to \mathcal{J} (resp. \mathcal{H}) by adding a topological boundary $\partial\mathcal{J}$ (resp. $\partial\mathcal{H}$) defined as

- $\partial\mathcal{J} := \{N \times N \text{ positive semi-definite hermitian matrices with operator norm } 1 \text{ and rank } \leq N - 1\}$,
- $\partial\mathcal{H} := \{N \times N \text{ positive semi-definite hermitian matrices with trace } 1 \text{ and rank } \leq N - 1\}$.

2.2. Degree of continuous maps

We summarise some well-known results on the degree of continuous maps between domains in Euclidean spaces. The reader is referred to [1, Chapter 12] for more details. Let Ω be a bounded open subset in \mathbb{R}^m and $\bar{\Omega}$ be its closure in \mathbb{R}^m with respect to the Euclidean topology. We also write $\partial\Omega := \bar{\Omega} \setminus \Omega$ for the boundary.

We first consider a C^1 -map $\phi \in C^1(\bar{\Omega}, \mathbb{R}^m)$. Writing componentwise in terms of \mathbb{R}^m , we can write $\phi = (\phi_1, \dots, \phi_m)$. The *Jacobian* of ϕ at x is defined as $J_\phi(x) := \det \left(\frac{\partial \phi_i}{\partial x_j}(x) \right)$. Recall that $x \in \bar{\Omega}$ is called a *critical point* of $\phi \in C^1(\bar{\Omega}, \mathbb{R}^m)$ if $J_\phi(x) = 0$. If x is a critical point of ϕ , $\phi(x)$ is called a *critical value*. Suppose that $\phi \in C^1(\bar{\Omega}, \mathbb{R}^m)$, $p \notin \phi(\partial\Omega)$, and p is not a critical value of ϕ . The *degree* of ϕ at p relative to Ω is defined by

$$\deg(\phi, \Omega, p) := \sum_{x \in \phi^{-1}(p)} \text{sign } J_\phi(x). \tag{2.2}$$

Note $\deg(\phi, \Omega, p) = 0$ if $\phi^{-1}(p) = \emptyset$.

If p is a critical value of ϕ , we define $\deg(\phi, \Omega, p)$ by slightly perturbing p (cf. [1, Definition 12.4]). Moreover, for a continuous map $Q \in C^0(\bar{\Omega}, \mathbb{R}^m)$, we take a “perturbation” $\phi_{Q,p} \in C^1(\bar{\Omega}, \mathbb{R}^m)$ of Q to define $\deg(Q, \Omega, p)$ as $\deg(\phi_{Q,p}, \Omega, p)$ (cf. [1, Definition 12.5]). These are well-defined, since they do not depend on the perturbations chosen [1, Theorems 12.3–12.9].

An important result is the homotopy invariance of the degree.

THEOREM 2.4 (cf. [1, Theorem 12.11]). — *Let $Q_0, Q_1 \in C^0(\bar{\Omega}, \mathbb{R}^m)$ and Q_t be a homotopy in $C^0(\bar{\Omega}, \mathbb{R}^m)$ between Q_0 and Q_1 . Let $p \in \mathbb{R}^m$ such that $p \notin Q_t(\partial\Omega)$ for all $t \in [0, 1]$. Then $\deg(Q_0, \Omega, p) = \deg(Q_1, \Omega, p)$.*

Another important result is that nontriviality of degree “detects” surjectivity.

THEOREM 2.5 (cf. [1, Theorem 12.10]). — *Let $Q \in C^0(\bar{\Omega}, \mathbb{R}^m)$ and $p \notin Q(\partial\Omega)$. If $\deg(Q, \Omega, p) \neq 0$ then there exists $x \in \Omega$ such that $Q(x) = p$.*

In what follows, we shall take $\Omega = \mathcal{J}^\circ$ and $\bar{\Omega} = \mathcal{J}$, and define a map $Q_X \in C^0(\bar{\Omega}, \mathbb{R}^{N^2-1})$ with $Q_X(\bar{\Omega}) \subset \mathcal{H}$ and $Q_X(\partial\Omega) \subset \partial\mathcal{H}$ (note that this implies $Q_X : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ is proper). We shall apply the above theorems to claim surjectivity of Q_X , which in turn proves the surjectivity of Hilb.

2.3. Proof of surjectivity of Hilb

We now prove the first part of Theorem 1.1, i.e. surjectivity of Hilb. After several technical definitions and lemmas, the main surjectivity result will be proved in Proposition 2.15. The main line of the argument presented below is similar to Bourguignon–Li–Yau [5, Section 2].

DEFINITION 2.6. — We define a continuous map $Q_{\mathbb{P}} : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ as

$$Q_{\mathbb{P}}(B)_{ij} := \left(\int_{\mathbb{P}^{N-1}} \frac{\sum_l |Z_l|_{\hbar}^2}{\sum_l |\sum_m B_{lm} Z_m|_{\hbar}^2} d\mu_{BZ} \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |\sum_m B_{lm} Z_m|_{\hbar}^2} d\mu_{BZ},$$

where $Q_{\mathbb{P}}(B)_{ij}$ stands for the (i, j) -th entry of the matrix $Q_{\mathbb{P}}(B) \in \mathcal{H}^\circ$.

Writing $\xi_B : \mathbb{P}^{N-1} \xrightarrow{\sim} \mathbb{P}^{N-1}$ for the biholomorphic map induced from the linear action of $B \in \mathcal{J}^\circ$ on \mathbb{P}^{N-1} , we note

$$\begin{aligned} Q_{\mathbb{P}}(\text{Id})_{ij} &= \left(\int_{\mathbb{P}^{N-1}} d\mu_Z \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |Z_l|_{\hbar}^2} d\mu_Z \\ &= \left(\int_{\mathbb{P}^{N-1}} (\xi_B^* d\mu_Z) \right)^{-1} \int_{\mathbb{P}^{N-1}} \xi_B^* \left(\frac{\tilde{h}(Z_i, Z_j)}{\sum_l |Z_l|_{\hbar}^2} \right) (\xi_B^* d\mu_Z) \\ &= \left(\int_{\mathbb{P}^{N-1}} d\mu_{BZ} \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\sum_{l,m} \tilde{h}(B_{il} Z_l, B_{jm} Z_m)}{\sum_l |\sum_m B_{lm} Z_m|_{\hbar}^2} d\mu_{BZ} \end{aligned}$$

and hence, recalling $\text{tr}(Q_{\mathbb{P}}(B)) = 1$ and writing B^t for the transpose of B , we get

$$Q_{\mathbb{P}}(B) = \frac{(B^t)^{-1} Q_{\mathbb{P}}(\text{Id}) (B^t)^{-1}}{\text{tr}((B^t)^{-1} Q_{\mathbb{P}}(\text{Id}) (B^t)^{-1})}, \quad (2.3)$$

by recalling $B = B^*$ which implies $B^t = \bar{B}$.

We have the following lemma.

LEMMA 2.7. — $Q_{\mathbb{P}}(\text{Id})$ is a constant multiple of the identity matrix.

Proof. — We define a local coordinate system (z_1, \dots, z_{N-1}) on a Zariski open subset \mathcal{U} of \mathbb{P}^{N-1} defined by $Z_N \neq 0$, so that $z_i = Z_i/Z_N$. We write z_i in polar coordinates as $z_i = r_i e^{\sqrt{-1}\theta_i}$.

Assuming $i, j \neq N$, we get

$$\frac{\tilde{h}(Z_i, Z_j)}{\sum_{l=1}^N |Z_l|_{\hbar}^2} = \frac{\bar{z}_i z_j}{1 + \sum_{l=1}^{N-1} |z_l|^2} = \frac{r_i r_j e^{\sqrt{-1}(\theta_j - \theta_i)}}{1 + \sum_{l=1}^{N-1} r_l^2}.$$

Recall also that in this coordinate system, the Fubini–Study volume form $d\mu_Z$ can be written as

$$d\mu_Z = \frac{1}{\left(1 + \sum_{l=1}^{N-1} r_l^2\right)^N} \prod_{i=1}^{N-1} \left(\frac{r_i}{\pi} dr_i \wedge d\theta_i \right).$$

Thus, for $i, j \neq N$, we get the integral (since \mathcal{U} is Zariski open in \mathbb{P}^{N-1}) as

$$\begin{aligned} Q_{\mathbb{P}(\text{Id})_{ij}} &= \int_{\mathcal{U}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |Z_l|_{\tilde{h}}^2} d\mu_Z \\ &= \int_0^{2\pi} \int_0^{2\pi} e^{\sqrt{-1}(\theta_j - \theta_i)} d\theta_i d\theta_j \left(\int_0^\infty \cdots \int_0^\infty \frac{(2\pi)^{N-3} r_i r_j}{(1 + \sum_{l=1}^{N-1} r_l^2)^{N+1}} \prod_{m=1}^{N-1} \frac{r_m dr_m}{\pi} \right), \end{aligned}$$

which is non-zero if and only if $i = j$ because of the integral in θ_i and θ_j .

Exactly the same reasoning shows that $Q_{\mathbb{P}(\text{Id})_{iN}} = Q_{\mathbb{P}(\text{Id})_{Nj}} = 0$ for $i, j \neq N$; we only need to observe that, in this case, we have

$$\frac{\tilde{h}(Z_i, Z_N)}{\sum_{l=1}^N |Z_l|_{\tilde{h}}^2} = \frac{\bar{z}_i}{1 + \sum_{l=1}^{N-1} |z_l|^2} = \frac{r_i e^{-\sqrt{-1}\theta_i}}{1 + \sum_{l=1}^{N-1} r_l^2},$$

which becomes zero under the integration in θ_i , as we did above.

Thus the only nonzero entries of $Q_{\mathbb{P}}$ are the diagonal ones. On the other hand, we observe that the symmetry in exchanging the variables Z_1, \dots, Z_N imply that the diagonal entries of $Q_{\mathbb{P}}$ must be all equal. Thus $Q_{\mathbb{P}}$ must be a constant multiple of the identity. \square

LEMMA 2.8. — $Q_{\mathbb{P}}$ defines a diffeomorphism between \mathcal{J}° and \mathcal{H}° .

Proof. — By Lemma 2.7 we have $Q_{\mathbb{P}}(B) = (B^t)^{-2}/\text{tr}((B^t)^{-2})$. By recalling the general fact that a positive definite hermitian matrix A has a unique positive square root $A^{1/2}$, it easily follows that $Q_{\mathbb{P}}$ is bijective.

Since $Q_{\mathbb{P}}$ is smooth and the map $A \mapsto A^{1/2}$ is smooth as long as A is positive definite hermitian, we see that $Q_{\mathbb{P}} : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ is a diffeomorphism. \square

LEMMA 2.9. — $Q_{\mathbb{P}} : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ extends continuously to the boundary of \mathcal{J} , and sends elements of $\partial\mathcal{J}$ into the ones in $\partial\mathcal{H}$, i.e. $Q_{\mathbb{P}} \in C^0(\mathcal{J}, \mathcal{H})$ with $Q_{\mathbb{P}}(\partial\mathcal{J}) \subset \partial\mathcal{H}$.

Proof. — Suppose that we are given a sequence $\{B_\nu\}_\nu \subset \mathcal{J}^\circ$ converging to a point B_∞ in $\partial\mathcal{J}$ (in the Euclidean topology, as remarked at the end of Section 2.1) as $\nu \rightarrow \infty$. Let $\lambda_\nu > 0$ be the smallest eigenvalue of B_ν . Since $B_\nu \rightarrow \partial\mathcal{J}$ as $\nu \rightarrow \infty$ and the operator norm of B_ν is fixed to be 1, we have $\lambda_\nu \rightarrow 0$.

Observe, by homogeneity, that we have

$$\begin{aligned} Q_{\mathbb{P}}(B_\nu) &= \frac{(B_\nu^t)^{-1} Q_{\mathbb{P}}(\text{Id})(B_\nu^t)^{-1}}{\text{tr}((B_\nu^t)^{-1} Q_{\mathbb{P}}(\text{Id})(B_\nu^t)^{-1})} \\ &= \frac{\lambda_\nu (B_\nu^t)^{-1} Q_{\mathbb{P}}(\text{Id}) \lambda_\nu (B_\nu^t)^{-1}}{\text{tr}(\lambda_\nu (B_\nu^t)^{-1} Q_{\mathbb{P}}(\text{Id}) \lambda_\nu (B_\nu^t)^{-1})}. \end{aligned}$$

We shall prove in Lemma 2.10 that $(\lambda_\nu^{-1}B_\nu)^{-1} = \lambda_\nu B_\nu^{-1}$ converges to a well-defined positive semidefinite hermitian matrix β_∞ that is strictly semidefinite (i.e. with $\text{rk } 1 \leq \text{rk}(\beta_\infty) \leq N - 1$). Assuming this result, the above equation means that $(\lambda_\nu^{-1}B_\nu^t)^{-1}Q_{\mathbb{P}}(\text{Id})(\lambda_\nu^{-1}B_\nu^t)^{-1}$ (the numerator of $Q_{\mathbb{P}}(B_\nu)$) converges to a well-defined positive semidefinite hermitian matrix as $\nu \rightarrow \infty$, which is strictly semidefinite since $\beta_\infty = \lim_{\nu \rightarrow \infty} (\lambda_\nu^{-1}B_\nu)^{-1}$ is.

Thus we see that $Q_{\mathbb{P}}(B_\nu)$ converges to a well-defined positive semidefinite hermitian matrix as $\nu \rightarrow \infty$. Hence we get

$$Q_{\mathbb{P}}\left(\lim_{\nu \rightarrow \infty} B_\nu\right) = \frac{\beta_\infty Q_{\mathbb{P}}(\text{Id})\beta_\infty}{\text{tr}(\beta_\infty Q_{\mathbb{P}}(\text{Id})\beta_\infty)} = \lim_{\nu \rightarrow \infty} Q_{\mathbb{P}}(B_\nu).$$

Moreover, since β_∞ is strictly semidefinite, we see that the limit is an element in $\partial\mathcal{H}$, i.e. $Q_{\mathbb{P}}(\partial\mathcal{J}) \subset \partial\mathcal{H}$.

We finally note that any element in $\partial\mathcal{J}$ can be realised as the limit of a sequence $\{B_\nu\}_\nu \subset \mathcal{J}^\circ$, which establishes the continuous extension $Q_{\mathbb{P}} \in C^0(\mathcal{J}, \mathcal{H})$ that we claimed, granted Lemma 2.10 that we prove below. \square

LEMMA 2.10. — *Suppose that we are given a sequence $\{B_\nu\}_\nu \subset \mathcal{J}^\circ$ converging to a point B_∞ in $\partial\mathcal{J}$. Let $\lambda_\nu > 0$ be the smallest eigenvalue of B_ν . Then $(\lambda_\nu^{-1}B_\nu)^{-1} = \lambda_\nu B_\nu^{-1}$ converges to a well-defined positive semidefinite hermitian matrix β_∞ which is strictly semidefinite, i.e. its rank satisfies $1 \leq \text{rk}(\beta_\infty) \leq N - 1$.*

Proof. — Since $B_\nu \rightarrow B_\infty$ as $\nu \rightarrow \infty$, we see that each eigenvalue (ordered, counted with multiplicities) of $\lambda_\nu B_\nu^{-1}$ converges. Now, since the unitary group $U(N)$ is compact, we may pick a subsequence $\{\nu(i)\}_i \subset \{\nu\}_\nu$ such that unitarily diagonalising bases for $B_{\nu(i)}$'s converge; more precisely, this means the existence of a convergent sequence of bases $\{Z_{1,i}\}_i$ (identified with a convergent sequence in $U(N)$) such that $B_{\nu(i)}$ is diagonal with respect to $Z_{1,i}$, whose entries are the eigenvalues of $B_{\nu(i)}$. For this subsequence $\{\nu(i)\}_i$ we have a well-defined limit $\beta_{1,\infty} := \lim_{i \rightarrow \infty} \lambda_{\nu(i)} B_{\nu(i)}^{-1}$, since both the eigenvalues and the diagonalising bases for $\lambda_{\nu(i)} B_{\nu(i)}^{-1}$ converge. Suppose that we take another such subsequence $\{\nu(j)\}_j$ with the limit $\beta_{2,\infty}$, where each $B_{\nu(j)}$ is diagonal with respect to a basis $Z_{2,j}$, whose entries are the eigenvalues of $B_{\nu(j)}$.

We shall assume without loss of generality that eigenvalues are always ordered and counted with multiplicities, when a matrix is (unitarily) diagonalised.

Let $u_{i,j} \in U(N)$ be a change of basis matrix from $Z_{1,i}$ to $Z_{2,j}$ with the hermitian conjugate $u_{i,j}^*$, which implies that $u_{i,j}^* \lambda_{\nu(j)} B_{\nu(j)}^{-1} u_{i,j}$ is diagonal with respect to the basis $Z_{1,i}$. A crucially important point is that $\{u_{i,j}\}_{i,j}$

converges as $i, j \rightarrow \infty$, since $\{Z_{1,i}\}_i$ and $\{Z_{2,j}\}_j$ converge. Thus, by taking the limit $i \rightarrow \infty$ and $j \rightarrow \infty$, we see that $\beta_{1,\infty} = \widehat{u}^* \beta_{2,\infty} \widehat{u}$ for a well-defined unitary matrix $\widehat{u} := u_{\infty,\infty} \in U(N)$.

Our aim is to prove $\beta_{1,\infty} = \beta_{2,\infty}$. We now take the basis $Z_{1,i}$ as above, which unitarily diagonalises $B_{\nu(i)}$. Recall also that $B_{\nu(j)}$ is unitarily diagonalised by another basis $Z_{2,j}$, where two bases $Z_{1,i}$ and $Z_{2,j}$ are related by $u_{i,j} \in U(N)$ as $Z_{2,j} = u_{i,j} \cdot Z_{1,i}$. Thus, $B_{\nu(i)}$ and $u_{i,j}^* B_{\nu(j)} u_{i,j}$ are both diagonal with respect to $Z_{1,i}$, whose entries are their eigenvalues (ordered and counted with multiplicities). Since the eigenvalues of $\{B_\nu\}_\nu$ converge, for any $\epsilon > 0$ there exists $\nu' \in \mathbb{N}$ such that

$$\|B_{\nu(i)} - u_{i,j}^* B_{\nu(j)} u_{i,j}\|_{op} < \epsilon$$

holds for all $\nu(i), \nu(j) > \nu'$. Note on the other hand

$$B_{\nu(i)} - u_{i,j}^* B_{\nu(j)} u_{i,j} = B_{\nu(i)} - u_{i,j}^* B_{\nu(i)} u_{i,j} + u_{i,j}^* (B_{\nu(i)} - B_{\nu(j)}) u_{i,j}.$$

Since $\{B_\nu\}_\nu$ itself converges, for any $\epsilon > 0$ there exists $\nu' \in \mathbb{N}$ such that

$$\begin{aligned} & \|B_{\nu(i)} - u_{i,j}^* B_{\nu(i)} u_{i,j}\|_{op} \\ & \leq \|B_{\nu(i)} - u_{i,j}^* B_{\nu(j)} u_{i,j}\|_{op} + \|u_{i,j}^* (B_{\nu(i)} - B_{\nu(j)}) u_{i,j}\|_{op} < 2\epsilon \end{aligned}$$

holds for all $\nu(i), \nu(j) > \nu'$, by also noting that unitary matrices preserve the operator norm. Since $u_{i,j}$ converges to $\widehat{u} \in U(N)$ as $i, j \rightarrow \infty$, we take this limit in the above to conclude $B_{1,\infty} = \widehat{u}^* B_{1,\infty} \widehat{u}$ with respect to the basis $Z_{1,\infty}$, where $B_{1,\infty} := \lim_{i \rightarrow \infty} B_{\nu(i)}$. Recall that $B_{\nu(i)}$ is diagonal with respect to $Z_{1,i}$ and $B_{1,\infty}$ is diagonal with respect to $Z_{1,\infty}$, by definition. Hence \widehat{u} can be written as a block diagonal matrix with respect to $Z_{1,\infty}$ as $\widehat{u} = \text{diag}(\widehat{u}_1, \dots, \widehat{u}_m)$, where each \widehat{u}_l corresponds to the unitary matrix acting on the l -th eigenspace (including $\ker B_{1,\infty}$) of $B_{1,\infty}$. Recall that $\beta_{1,\infty} := \lim_{i \rightarrow \infty} \lambda_{\nu(i)} B_{\nu(i)}^{-1}$ is diagonal with respect to the basis $Z_{1,\infty}$ and has the same eigenspace decomposition as $B_{1,\infty}$. In particular, $\widehat{u} \beta_{1,\infty} \widehat{u}^* = \beta_{1,\infty}$. Thus, combined with our hypothesis $\beta_{1,\infty} = \widehat{u}^* \beta_{2,\infty} \widehat{u}$, we get $\beta_{1,\infty} = \beta_{2,\infty}$ as claimed.

We thus conclude that the limit $\beta_\infty := \lim_{\nu \rightarrow \infty} \lambda_\nu B_\nu^{-1}$ is well-defined, independently of the subsequence chosen. This is positive semidefinite hermitian since it is a limit of positive definite hermitian matrices, and has rank $1 \leq \text{rk}(\beta_\infty) \leq N - 1$ since λ_ν is the minimum eigenvalue of B_ν that tends to zero as $\nu \rightarrow \infty$. \square

Recall the Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*) \xrightarrow{\sim} \mathbb{P}^{N-1}$ and the volume form $d\mu_{BZ}$ on \mathbb{P}^{N-1} , defined with respect to $B \in GL(N, \mathbb{C})$ and homogeneous coordinates $\{Z_i\}_i$ as given in Section 2.1.

Now suppose that we write $\iota_X^*(d\mu_{BZ})$ for the volume form on $\iota(X) \subset \mathbb{P}^{N-1}$, defined by the Fubini–Study metric associated to the hermitian matrix

H that has BZ as its orthonormal basis (in the notation of Section 2.1). Analogously to Definition 2.6, we define the following.

DEFINITION 2.11. — We define a continuous map $Q_X : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ as

$$Q_X(B)_{ij} := \left(\int_{\mathbb{P}^{N-1}} \frac{\sum_l |Z_l|^2_{\tilde{h}}}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} \iota_X^*(d\mu_{BZ}) \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} \iota_X^*(d\mu_{BZ}).$$

Remark 2.12. — Recalling (2.1) in Definition 2.1, we observe that the above can be written as $Q_X(B)_{ij} = \text{Hilb}(FS(H))(Z_i, Z_j) / \text{tr}(\text{Hilb}(FS(H)))$, with $H = \overline{(B^{-1})^t} B^{-1}$.

As in Lemma 2.9, we need to show that Q_X extends continuously to the boundary, i.e. $Q_X \in C^0(\mathcal{J}, \mathcal{H})$.

Recall the biholomorphic map $\xi_B : \mathbb{P}^{N-1} \xrightarrow{\sim} \mathbb{P}^{N-1}$ induced from the linear action of $B \in \mathcal{J}^\circ$ on \mathbb{P}^{N-1} , which moves $\iota(X) \subset \mathbb{P}^{N-1}$ to $\xi_B \circ \iota(X) \subset \mathbb{P}^{N-1}$. Recall also that $\iota_X^*(d\mu_{BZ})$ is, as a measure on X , equal to $\iota^*(\omega_{FS(H)}^n)$. We then have

$$\begin{aligned} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} \iota_X^*(d\mu_{BZ}) &= \int_{\iota(X)} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} \omega_{FS(H)}^n \\ &= \sum_{r,s} (B^*)_{ri}^{-1} B_{js}^{-1} \sum_{p,q} \int_{\iota(X)} \frac{\tilde{h}(B_{rp} Z_p, B_{sq} Z_q)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} \omega_{FS(H)}^n \\ &= \sum_{r,s} B_{ri}^{-1} B_{js}^{-1} \int_{\xi_B \circ \iota(X)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|^2_{\tilde{h}}} \omega_{FS(H)}^n, \end{aligned}$$

since $(\xi_B \circ \iota)^*(Z_i) = \sum_p B_{ip} \iota^*(Z_p)$. Writing $\Phi(B)$ for the matrix defined by

$$\Phi(B)_{rs} := \int_{\xi_B \circ \iota(X)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|^2_{\tilde{h}}} \omega_{FS(H)}^n,$$

we have

$$Q_X(B) = \frac{(B^t)^{-1} \Phi(B) (B^t)^{-1}}{\text{tr}((B^t)^{-1} \Phi(B) (B^t)^{-1})}, \quad (2.4)$$

analogously to (2.3).

The following two lemmas are of crucial importance in the proof of the main surjectivity result (Proposition 2.15).

LEMMA 2.13. — Q_X continuously extends to the boundary of \mathcal{J} , and sends elements in $\partial\mathcal{J}$ to the ones in $\partial\mathcal{H}$, i.e. $Q_X \in C^0(\mathcal{J}, \mathcal{H})$ with $Q_X(\partial\mathcal{J}) \subset \partial\mathcal{H}$.

Proof. — Suppose that $\{B_\nu\}_\nu$ is any sequence in \mathcal{J}° converging to a point B_∞ in $\partial\mathcal{J}$, in the Euclidean topology induced from \mathcal{J}° , $\mathcal{H}^\circ \subset \mathbb{R}^{N^2-1}$ (as in Section 2.1). Each B_ν induces a holomorphic (in fact biholomorphic) map $\xi_\nu : \mathbb{P}^{N-1} \rightarrow \mathbb{P}^{N-1}$ given by the linear action of B_ν on \mathbb{P}^{N-1} . In particular, for each B_ν we have an embedded variety $\xi_\nu \circ \iota(X) \subset \mathbb{P}^{N-1}$.

We now consider the Hilbert scheme $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ of subvarieties in \mathbb{P}^{N-1} of dimension n and with the Hilbert polynomial

$$P(m) = \sum_{i=1}^n (-1)^i \dim H^i(X, \mathcal{L}^{\otimes m}).$$

Let $\{p_\nu\}_\nu$ be the sequence of closed points in $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ defined by the embedded varieties $\{\xi_\nu \circ \iota(X)\}_\nu \hookrightarrow \mathbb{P}^{N-1}$ (or equivalently, the homogeneous ideals that define these embedded varieties). Since $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ is a projective scheme [8], it is proper and hence there exists a convergent subsequence $\{p_{\nu(i)}\}_i$ in $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ with the limit $p_{\infty,1}$, say (where the convergence is in terms of the analytic topology). Now choose another convergent subsequence $\{p_{\nu(j)}\}_j \subset \text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ with the limit $p_{\infty,2}$. Assume $p_{\infty,1} \neq p_{\infty,2}$. Then, since $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ is separated, by taking $\nu(i)$ and $\nu(j)$ to be sufficiently large, we can find open subsets U_1 and U_2 (in the analytic topology) in $\text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ such that $U_1 \cap U_2 = \emptyset$, $\{p_{\nu(i)}\}_i \subset U_1$, and $\{p_{\nu(j)}\}_j \subset U_2$. However, fixing a set of homogeneous polynomials that define $\iota(X) \hookrightarrow \mathbb{P}^{N-1}$ (i.e. generators of the homogeneous ideal that define $\iota(X) \subset \mathbb{P}^{N-1}$), $B_\nu \rightarrow B_\infty$ (as $\nu \rightarrow \infty$) implies that the homogeneous polynomials defining $p_\nu := \xi_\nu \circ \iota(X)$ must converge (in the sense that their coefficients converge). This contradicts $U_1 \cap U_2 = \emptyset$ with $\{p_{\nu(i)}\}_i \subset U_1$, $\{p_{\nu(j)}\}_j \subset U_2$, which was the definition of U_1 and U_2 . Hence we get $p_{\infty,1} = p_{\infty,2}$, i.e. the sequence $\{p_\nu\}_\nu$ has a well-defined limit $p_\infty \in \text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$.

Now we take the Chow scheme $\text{Chow}_{\mathbb{P}^{N-1}}(n, d)$ of algebraic cycles in \mathbb{P}^{N-1} with dimension n and degree $d := \deg \mathcal{L}$, and recall that there exists a universal morphism of schemes $f_{HC} : \text{Hilb}_{\mathbb{P}^{N-1}}(n, P) \rightarrow \text{Chow}_{\mathbb{P}^{N-1}}(n, d)$, called the Hilbert–Chow morphism, where $f_{HC}(p)$ is the algebraic cycle defined by $p \in \text{Hilb}_{\mathbb{P}^{N-1}}(n, P)$ (cf. [11, Section 5.4]). We define the limit cycle $\lim_{\nu \rightarrow \infty} \xi_\nu \circ \iota(X)$ to be $f_{HC}(p_\infty)$. Since f_{HC} is a morphism of schemes, we have $f_{HC}(p_\infty) = \lim_{\nu \rightarrow \infty} f_{HC}(p_\nu)$ in $\text{Chow}_{\mathbb{P}^{N-1}}(n, d)$, by continuity of f_{HC} in the analytic topology.

By the definition of the Hilbert scheme limit p_∞ , we have

$$\Phi\left(\lim_{\nu \rightarrow \infty} B_\nu\right)_{rs} = \int_{f_{HC}(p_\infty)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|_{\tilde{h}}^2} \omega_{\widetilde{FS}(H)}^n = \int_{\lim_{\nu \rightarrow \infty} \xi_{\nu, \alpha}(X)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|_{\tilde{h}}^2} \omega_{\widetilde{FS}(H)}^n.$$

On the other hand, we also have

$$\begin{aligned} \int_{f_{HC}(p_\infty)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|_{\tilde{h}}^2} \omega_{\widetilde{FS}(H)}^n &= \lim_{\nu \rightarrow \infty} \int_{f_{HC}(p_\nu)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|_{\tilde{h}}^2} \omega_{\widetilde{FS}(H)}^n \\ &= \lim_{\nu \rightarrow \infty} \int_{\xi_{\nu, \alpha}(X)} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|_{\tilde{h}}^2} \omega_{\widetilde{FS}(H)}^n = \lim_{\nu \rightarrow \infty} \Phi(B_\nu), \end{aligned}$$

by $f_{HC}(p_\infty) = \lim_{\nu \rightarrow \infty} f_{HC}(p_\nu)$, which follows from the continuity of f_{HC} . We thus get

$$\Phi\left(\lim_{\nu \rightarrow \infty} B_\nu\right) = \lim_{\nu \rightarrow \infty} \Phi(B_\nu). \quad (2.5)$$

Observe that this is positive semidefinite hermitian, as it is a limit of positive definite hermitian matrices.

We now prove that the limit $Q_X(\lim_{\nu \rightarrow \infty} B_\nu)$ is a well-defined element in $\partial\mathcal{H}$. As we did in the proof of Lemma 2.9, we write λ_ν for the smallest eigenvalue of B_ν and observe, by homogeneity, that

$$\begin{aligned} Q_X(B_\nu) &= \frac{(B_\nu^t)^{-1} \Phi(B_\nu) (B_\nu^t)^{-1}}{\text{tr}((B_\nu^t)^{-1} \Phi(B_\nu) (B_\nu^t)^{-1})}, \\ &= \frac{\lambda_\nu (B_\nu^t)^{-1} \Phi(B_\nu) \lambda_\nu (B_\nu^t)^{-1}}{\text{tr}(\lambda_\nu (B_\nu^t)^{-1} \Phi(B_\nu) \lambda_\nu (B_\nu^t)^{-1})}. \end{aligned}$$

Thus, writing $\beta_\infty = \lim_{\nu \rightarrow \infty} \lambda_\nu (B_\nu^t)^{-1}$, which is strictly positive semidefinite (as we saw in Lemma 2.10), we get

$$Q_X\left(\lim_{\nu \rightarrow \infty} B_\nu\right) = \frac{\beta_\infty \Phi(\lim_{\nu \rightarrow \infty} B_\nu) \beta_\infty}{\text{tr}(\beta_\infty \Phi(\lim_{\nu \rightarrow \infty} B_\nu) \beta_\infty)}.$$

Combining this equality with (2.5), we get

$$Q_X\left(\lim_{\nu \rightarrow \infty} B_\nu\right) = \frac{\beta_\infty \lim_{\nu \rightarrow \infty} \Phi(B_\nu) \beta_\infty}{\text{tr}(\beta_\infty \lim_{\nu \rightarrow \infty} \Phi(B_\nu) \beta_\infty)} = \lim_{\nu \rightarrow \infty} Q_X(B_\nu).$$

Moreover, since β_∞ is strictly positive semidefinite (i.e. positive semidefinite with $\text{rank } 1 \leq \text{rk}(\beta_\infty) \leq N-1$) and $\lim_{\nu \rightarrow \infty} \Phi(B_\nu)$ is positive semidefinite, we see that $Q_X(\lim_{\nu \rightarrow \infty} B_\nu)$ is a strictly positive semidefinite hermitian matrix, and hence is an element in $\partial\mathcal{H}$.

Since any element in $\partial\mathcal{J}$ can be realised as a limit of a sequence $\{B_\nu\}_\nu \subset \mathcal{J}^\circ$, Q_X continuously extends to the boundary of \mathcal{J} , and sends elements in $\partial\mathcal{J}$ to the ones in $\partial\mathcal{H}$. \square

We now define a 1-parameter family of continuous maps $Q_t : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$, $t \in [0, 1]$, by

$$Q_t(B) := tQ_X(B) + (1 - t)Q_{\mathbb{P}}(B).$$

LEMMA 2.14. — Q_t continuously extends to the boundary of \mathcal{J} , and sends elements in $\partial\mathcal{J}$ to the ones in $\partial\mathcal{H}$ for all $t \in [0, 1]$, i.e. $Q_t \in C^0(\mathcal{J}, \mathcal{H})$ with $Q_t(\partial\mathcal{J}) \subset \partial\mathcal{H}$ for all $t \in [0, 1]$.

Proof. — We argue similarly to the proof of Lemma 2.13. Observe first that we can write

$$Q_t(B) = (B^t)^{-1} \left(\frac{t\Phi(B)}{\operatorname{tr}((B^t)^{-1}\Phi(B)(B^t)^{-1})} + \frac{(1-t)Q_{\mathbb{P}}(\operatorname{Id})}{\operatorname{tr}((B^t)^{-1}Q_{\mathbb{P}}(B^t)^{-1})} \right) (B^t)^{-1},$$

by recalling (2.3) and (2.4). Given a sequence $\{B_\nu\}_\nu \subset \mathcal{J}^\circ$ converging to $B_\infty \in \partial\mathcal{J}$ as before, we get (again by homogeneity)

$$Q_t(B_\nu) = \lambda_\nu(B^t)^{-1} \left(\frac{t\Phi(B_\nu)}{\operatorname{tr}(\lambda_\nu(B_\nu^t)^{-1}\Phi(B_\nu)\lambda_\nu(B_\nu^t)^{-1})} + \frac{(1-t)Q_{\mathbb{P}}(\operatorname{Id})}{\operatorname{tr}(\lambda_\nu(B_\nu^t)^{-1}Q_{\mathbb{P}}(\operatorname{Id})\lambda_\nu(B_\nu^t)^{-1})} \right) \lambda_\nu(B_\nu^t)^{-1},$$

where λ_ν is the smallest eigenvalue of B_ν , as in the proof of Lemma 2.9 or 2.13. Thus, we get

$$\begin{aligned} & Q_t \left(\lim_{\nu \rightarrow \infty} B_\nu \right) \\ &= \beta_\infty \left(\frac{t \lim_{\nu \rightarrow \infty} \Phi(B_\nu)}{\operatorname{tr}(\beta_\infty \lim_{\nu \rightarrow \infty} \Phi(B_\nu)\beta_\infty)} + \frac{(1-t)Q_{\mathbb{P}}(\operatorname{Id})}{\operatorname{tr}(\beta_\infty Q_{\mathbb{P}}(\operatorname{Id})\beta_\infty)} \right) \beta_\infty, \quad (2.6) \\ &= \lim_{\nu \rightarrow \infty} Q_t(B_\nu), \end{aligned}$$

where $\beta_\infty = \lim_{\nu \rightarrow \infty} \lambda_\nu(B_\nu^t)^{-1}$ as in Lemma 2.10, and we used (2.5).

Since the terms in the bracket of (2.6) is positive semidefinite and β_∞ is strictly positive semidefinite (i.e. positive semidefinite with rank $1 \leq \operatorname{rk}(\beta_\infty) \leq N - 1$), we conclude that $Q_t(\lim_{\nu \rightarrow \infty} B_\nu)$ is a strictly positive semidefinite hermitian matrix for all $t \in [0, 1]$, which completes the proof. \square

PROPOSITION 2.15. — *Suppose that \mathcal{L} is very ample. Then*

$$\operatorname{Hilb} : \mathcal{H}(X, \mathcal{L}) \rightarrow \mathcal{B}$$

is surjective.

Proof. — We now recall $Q_{\mathbb{P}}$ as defined in (2.3). By Lemmas 2.8 and 2.9, $Q_{\mathbb{P}}$ defines a diffeomorphism between \mathcal{J}° and \mathcal{H}° which continuously extends to a map $\mathcal{J} \rightarrow \mathcal{H}$ with $Q_{\mathbb{P}}(\partial\mathcal{J}) \subset \partial\mathcal{H}$. Since $Q_{\mathbb{P}} : \mathcal{J}^\circ \rightarrow \mathcal{H}^\circ$ is a diffeomorphism, we have

$$\deg(Q_{\mathbb{P}}, \mathcal{J}^\circ, p) \neq 0 \quad (2.7)$$

for all $p \in \mathcal{H}^\circ$ by (2.2); note $\mathcal{H}^\circ \cap Q_{\mathbb{P}}(\partial\mathcal{J}) = \emptyset$.

We now pick an arbitrary $p \in \mathcal{H}^\circ$ and compute the degree $\deg(Q_X, \mathcal{J}^\circ, p)$. Since Lemma 2.14 shows $Q_t(\partial\mathcal{J}) \subset \partial\mathcal{H}$ for all $t \in [0, 1]$, we can apply Theorem 2.4 to prove

$$\deg(Q_X, \mathcal{J}^\circ, p) = \deg(Q_{\mathbb{P}}, \mathcal{J}^\circ, p) \neq 0,$$

by also recalling (2.7). Since $p \in \mathcal{H}^\circ$ and $\mathcal{H}^\circ \cap Q_X(\partial\mathcal{J}) = \emptyset$ by Lemma 2.13, we can apply Theorem 2.5 to prove that there exists $x \in \mathcal{J}^\circ$ such that $Q_X(x) = p$. We thus conclude that Q_X is surjective.

Finally, we recall that $\iota_X^*(d\mu_{BZ}) = \iota^*(\omega_{FS(H)}^n)$ is equal to $\omega_{FS(H)}^n$ as a volume form on X . Note also that, writing h for $\iota^*\tilde{h} = \iota^*\tilde{h}_{FS(\text{Id})}$, we can re-write the definition of $Q_X(B)$ (cf. Definition 2.11) as

$$Q_X(B)_{ij} = \left(\int_X \frac{\sum_l |s_l|_h^2}{\sum_l |\sum_m B_{lm} s_m|_h^2} \omega_{FS(H)}^n \right)^{-1} \int_X \frac{h(s_i, s_j)}{\sum_l |\sum_m B_{lm} s_m|_h^2} \omega_{FS(H)}^n. \quad (2.8)$$

where we wrote $s_i := \iota^*Z_i$. Recalling Definition 2.1 (and also Remark 2.12), we observe that this can be re-written as

$$Q_X(B)_{ij} = \left(\int_X \sum_l |s_l|_{FS(H)}^2 \omega_{FS(H)}^n \right)^{-1} \int_X FS(H)(s_i, s_j) \omega_{FS(H)}^n,$$

which is a (positive) constant multiple of $\text{Hilb}(FS(H))(s_i, s_j)$. Thus, the surjectivity of Q_X that we proved implies the following consequence: fixing a basis $\{s_i\}_i$ for $H^0(X, \mathcal{L})$, for any positive definite hermitian matrix G there exists $H \in \mathcal{B}$ and a constant α such that $\text{Hilb}(FS(H))(s_i, s_j) = e^\alpha G_{ij}$, or equivalently

$$\text{Hilb}(e^{-\alpha} FS(H))(s_i, s_j) = G_{ij}.$$

Since α is a constant, $e^{-\alpha} FS(H)$ is positively curved (with the curvature $\omega_{FS(H)} > 0$) and hence defines an element of $\mathcal{H}(X, \mathcal{L})$. Thus, the above establishes the required statement that Hilb is surjective. \square

Remark 2.16. — We provide another point of view regarding the equation (2.8). Fixing a hermitian metric $h \in \mathcal{H}(X, \mathcal{L})$, we observe that there exists $\beta \in C^\infty(X, \mathbb{R})$ such that $\omega_{FS(H)}^n = e^\beta \omega_h^n$. Writing (2.8) in terms of h , the surjectivity of Q_X can be regarded as ensuring the existence of $\phi \in C^\infty(X, \mathbb{R})$ such that

$$\int_X e^{\beta+\phi} h(s_i, s_j) \omega_h^n = G_{ij}, \quad (2.9)$$

for any given positive definite hermitian matrix G_{ij} .

The left hand side of the above equation is a priori not a Hilb of a hermitian metric, because $e^{\beta+\phi}h$ is used for the metric on \mathcal{L} whereas ω_h is used for the volume form. However, we can always find a function $f \in C^\infty(X, \mathbb{R})$, such that $e^{-f}h$ is positively curved and $\text{Hilb}(e^{-f}h)(s_i, s_j) = \int_X e^{\beta+\phi}h(s_i, s_j)\omega_h^n$, for any β and ϕ ; indeed, for this purpose, it is sufficient to solve for f the following nonlinear PDE:

$$\left(\omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f\right)^n = e^{f+\beta+\phi}\omega_h^n,$$

which is solvable by the Aubin–Yau theorem (cf. [2] and [13, Theorem 4, p. 383]).

Thus, to prove the surjectivity of Hilb, it suffices to find *some* $\beta, \phi \in C^\infty(X, \mathbb{R})$ that satisfy (2.9) with respect to the fixed $h \in \mathcal{H}(X, \mathcal{L})$, which is weaker than the surjectivity of Hilb itself.

2.4. Variants of the Hilbert map

We now recall that there are several variants of the Hilbert map that also appear in the literature [3, 4, 7, 10, 12]. We define the Hilb $_\nu$ map

$$\text{Hilb}_\nu(h) := \int_X h(\cdot, \cdot) d\nu$$

where the volume form $d\nu$ is one of the following.

- (1) $d\nu$ is a *fixed* volume form on X ; an example of this is when X is Calabi–Yau, in which case we can use the holomorphic volume form $\Omega \in H^0(X, K_X)$ to define $d\nu := \Omega \wedge \bar{\Omega}$,
- (2) $d\nu$ is *anticanonical*; a hermitian metric h on $-K_X$ defines a volume form $d\nu^{ac}(h)$, where we note $d\nu^{ac}(e^{-\varphi}h) = e^{-\varphi}d\nu^{ac}(h)$,
- (3) $d\nu$ is *canonical*; a hermitian metric h on K_X defines a dual metric on $-K_X$, which defines a volume form $d\nu^c(h)$ with $d\nu^c(e^{-\varphi}h) = e^\varphi d\nu^c(h)$.

We prove the following analogue of Proposition 2.15. As we can see from the statement below, we need to consider the higher tensor power $\mathcal{L}^{\otimes k}$ and the vector space $H^0(X, \mathcal{L}^{\otimes k})$ of dimension N_k .

PROPOSITION 2.17. — *Let Hilb $_\nu$ be defined by one of the three volume forms as defined above, and let $\mathcal{L}^{\otimes k}$ be a very ample line bundle. Then any positive definite hermitian matrix on $H^0(X, \mathcal{L}^{\otimes k})$ can be realised as*

$$\text{Hilb}_\nu(\tilde{h}^{\otimes k}) := \int_X \tilde{h}^{\otimes k}(\cdot, \cdot) d\nu$$

for some hermitian metric \tilde{h} on \mathcal{L} if

- (1) \mathcal{L} is any ample line bundle and $d\nu$ is any fixed volume form,
- (2) $\mathcal{L} = -K_X$ is ample with the anticanonical volume form $d\nu^{ac}(\tilde{h})$,
- (3) $\mathcal{L} = K_X$ is ample with the canonical volume form $d\nu^c(\tilde{h})$.

Proof. — Fixing a basis $\{s_i\}_i$ and a hermitian metric $h^{\otimes k}$ on $\mathcal{L}^{\otimes k}$, (2.9) implies that for any positive definite hermitian matrix G_{ij} there exists $\varphi \in C^\infty(X, \mathbb{R})$ such that

$$\int_X e^\varphi h^{\otimes k}(s_i, s_j) \omega_h^n = G_{ij},$$

(we used ω_h^n for the volume form instead of $\omega_h^{n \otimes k}$, but the difference is just a constant multiple which we can absorb in φ).

Observe that for each of the three choices $d\nu$, $d\nu^{ac}(h)$, $d\nu^c(h)$ of the volume form $d\nu$, there exists a function $\phi \in C^\infty(X, \mathbb{R})$ such that

$$\omega_h^n = e^\phi d\nu,$$

and hence the claim follows from the following;

- (1) $d\nu$ fixed: take $\tilde{h} := \exp\left(\frac{1}{k}(\varphi + \phi)\right)h$ so that \tilde{h} satisfies $\text{Hilb}_\nu(\tilde{h})(s_i, s_j) = G_{ij}$;
- (2) $d\nu$ anticanonical: take $\tilde{h} := \exp\left(\frac{1}{k+1}(\varphi + \phi)\right)h$ so that $\text{Hilb}_\nu(\tilde{h})(s_i, s_j) = G_{ij}$ with $d\nu = d\nu^{ac}(\tilde{h})$;
- (3) $d\nu$ canonical: take $\tilde{h} := \exp\left(\frac{1}{k-1}(\varphi + \phi)\right)h$ so that $\text{Hilb}_\nu(\tilde{h})(s_i, s_j) = G_{ij}$ with $d\nu = d\nu^c(\tilde{h})$. \square

Remark 2.18. — Unlike the case of the usual Hilb as treated in Proposition 2.15, the above proof does not show that \tilde{h} has positive curvature; the associated curvature form $\omega_{\tilde{h}}$ may not be a Kähler metric.

3. Injectivity of FS

We establish the following “quantitative injectivity” to prove the second part of Theorem 1.1.

LEMMA 3.1. — *Suppose that \mathcal{L} is very ample, and that $H, H' \in \mathcal{B}$ satisfy*

$$FS(H) = (1 + f)FS(H')$$

with $\sup_X |f| \leq \epsilon$ for $\epsilon \geq 0$ satisfying $N^{\frac{3}{2}}\epsilon \leq 1/4$.

Then we have $\|H - H'\|_{op} \leq 4N^{\frac{3}{2}}\epsilon$, where $\|\cdot\|_{op}$ is the operator norm, i.e. the maximum of the moduli of the eigenvalues. In particular, considering the case $\epsilon = 0$, we see that FS is injective.

Proof. — We now pick an H -orthonormal basis $\{s_i\}_i$ and represent H (resp. H') as a matrix H_{ij} (resp. H'_{ij}) with respect to the basis $\{s_i\}_i$. H_{ij} is the identity matrix, and replacing $\{s_i\}_i$ by an H -unitarily equivalent basis if necessary, we may further assume $H'_{ij} = \text{diag}(d_1^2, \dots, d_N^2)$ for some $d_i > 0$. Recall that the equation (2.1) implies that we can write $FS(H') = e^{-\varphi}FS(H)$ with $\varphi = \log(\sum_{i=1}^N d_i^{-2}|s_i|_{FS(H)}^2)$. Thus the equation $FS(H) = (1+f)FS(H')$ implies $1+f = \sum_i d_i^{-2}|s_i|_{FS(H)}^2$, and hence, by recalling (2.1),

$$(1+f) \sum_i |s_i|_h^2 = \sum_i d_i^{-2}|s_i|_h^2, \tag{3.1}$$

with respect to *any* hermitian metric h on \mathcal{L} , by noting that we may multiply both sides of (3.1) by any strictly positive function e^φ . We now fix this basis $\{s_i\}_i$, and the operator norm $\|\cdot\|_{op}$ or the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ used in this proof will all be computed with respect to this basis.

We now choose N hermitian metrics h_1, \dots, h_N on \mathcal{L} as follows. Recall now that, by Proposition 2.15, for any N -tuple of strictly positive numbers $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ there exists $\phi_{\vec{\lambda}} \in C^\infty(X, \mathbb{R})$ such that the hermitian metric $h' := \exp(\phi_{\vec{\lambda}})h$ satisfies $\int_X |s_i|_{h'}^2 \omega_{h'}^n = \lambda_i$. We thus take

$$\vec{\lambda}_i = (\delta, \dots, \delta, 1, \delta, \dots, \delta)$$

with 1 in the i -th place, where $0 < \delta \ll 1$ is chosen to be small enough so that the matrix defined as

$$\Lambda := \begin{pmatrix} \vec{\lambda}_1 \\ \vdots \\ \vec{\lambda}_N \end{pmatrix}$$

satisfies $\|\Lambda\|_{op} \leq 2$ and $\|\Lambda^{-1}\|_{op} \leq 2$. (How small δ must be depends on N , but this will not concern us.)

We now choose $\phi_i \in C^\infty(X, \mathbb{R})$ appropriately (cf. Proposition 2.15) so that $h_i := \exp(\phi_i)h$ satisfies

$$\vec{\lambda}_i = \left(\int_X |s_1|_{h_i}^2 \omega_{h_i}^n, \int_X |s_2|_{h_i}^2 \omega_{h_i}^n, \dots, \int_X |s_N|_{h_i}^2 \omega_{h_i}^n \right).$$

Then, multiplying both sides of (3.1) by $\exp(\phi_i)$ and integrating over X with respect to the measure $\omega_{h_i}^n/n!$, we get the following system of linear equations

$$(\Lambda + F) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \Lambda \begin{pmatrix} d_1^{-2} \\ \vdots \\ d_N^{-2} \end{pmatrix},$$

where F is a matrix defined by

$$F_{ij} := \int_X f |s_j|_{h_i}^2 \omega_{h_i}^n$$

whose max norm (i.e. the maximum of the moduli of its entries) satisfies $\|F\|_{max} \leq \sup_X |f| \leq \epsilon$ since the modulus of each entry of Λ is at most 1. We thus get

$$\begin{pmatrix} d_1^{-2} - 1 \\ \vdots \\ d_N^{-2} - 1 \end{pmatrix} = \Lambda^{-1} F \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus, noting $\|\Lambda^{-1} F\|_{op} \leq \|\Lambda^{-1}\|_{op} \|F\|_{op} \leq 2\|F\|_{HS} \leq 2N\|F\|_{max} \leq 2N\epsilon$, we get

$$|d_i^{-2} - 1| \leq \sqrt{\sum_i |d_i^{-2} - 1|^2} \leq 2N^{1+\frac{1}{2}}\epsilon.$$

Thus we get $1 - 2N^{\frac{3}{2}}\epsilon \leq d_i^{-2} \leq 1 + 2N^{\frac{3}{2}}\epsilon$, and by the assumption $N^{\frac{3}{2}}\epsilon \leq 1/4$ we have

$$1 - 4N^{\frac{3}{2}}\epsilon < 1 - \frac{2N^{\frac{3}{2}}\epsilon}{1 + 2N^{\frac{3}{2}}\epsilon} \leq d_i^2 \leq 1 + \frac{2N^{\frac{3}{2}}\epsilon}{1 - 2N^{\frac{3}{2}}\epsilon} < 1 + 4N^{\frac{3}{2}}\epsilon$$

as required. □

Remark 3.2. — We recast Lemma 3.1 in the case we use the usual scaling convention for the Fubini–Study map (cf. Remark 2.2). Suppose that $\mathcal{L}^{\otimes k}$ is very ample and write \mathcal{B}_k for the space of hermitian forms on the vector space $H^0(X, \mathcal{L}^{\otimes k})$ of dimension N_k . The statement for this convention is as follows: if $H, H' \in \mathcal{B}_k$ satisfy $FS(H)^{\otimes k} = (1 + f)FS(H')^{\otimes k}$ with $\sup_X |f| \leq \epsilon$ for $0 < N_k^{\frac{3}{2}}\epsilon \leq 1/4$, then we have $\|H - H'\|_{op} \leq 4N_k^{\frac{3}{2}}\epsilon$.

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