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Some Hardy-type estimates in realized homogeneous Besov and Triebel–Lizorkin spaces

Some Hardy-type estimates in realized homogeneous Besov and Triebel–Lizorkin spaces [(*)]

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Abstract. — We prove that the realized homogeneous Besov $\dot{\tilde{B}}^{s_0}_{p_0,p}(\mathbb{R}^n)$ and Triebel–Lizorkin $\dot{\tilde{F}}^{s_0}_{p_0,p}(\mathbb{R}^n)$ spaces are continuously embedded in the quasi-Banach weighted Lebesgue spaces $L_p(\mathbb{R}^n; |x|^{-p(s_0-n/p_0+n/p)}\mathrm{d}x)$ for $p_0 < p$ and $(n/p_0 - n)_+ < s_0 < n/p_0$.

Résumé. — Nous monterons que les espaces homogènes réalisés de Besov $\dot{\tilde{B}}^{s_0}_{p_0,p}(\mathbb{R}^n)$ et de Triebel–Lizorkin $\dot{\tilde{F}}^{s_0}_{p_0,p}(\mathbb{R}^n)$ s'injectent continûment dans les espaces quasi-Banach de Lebesgue à poids $L_p(\mathbb{R}^n; |x|^{-p(s_0-n/p_0+n/p)}\mathrm{d}x)$ pour $p_0 < p$ et $(n/p_0 - n)_+ < s_0 < n/p_0$.

1. Introduction and main results

The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and the homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach, defined as spaces of distributions modulo polynomials (abbreviated in the sequel by B and F, respectively) in the sense that $\|f\|_{\dot{B}^s_{p,q}} = \|f\|_{\dot{F}^s_{p,q}} = 0$ if and only if f is a polynomial on $\mathbb{R}^n$. For this reason, we cannot, e.g., identify $\dot{F}^0_{p,2}(\mathbb{R}^n)$ with $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$ since for any nonzero polynomial f it holds $\|f\|_p = +\infty$ while $\|[f]_\mathcal{P}\|_{\dot{F}^0_{p,2}} = 0$, see [14, Prop. 5.2] (here and throughout this paper, $[f]_\mathcal{P}$ denotes the equivalence class of the tempered distribution f modulo polynomials). However using the notion of *realization*, cf. [2], we obtain the realized spaces of $\dot{B}^s_{p,q}(\mathbb{R}^n)$

and $\dot{F}^s_{p,q}(\mathbb{R}^n)$, denoted by $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\widetilde{F}}^s_{p,q}(\mathbb{R}^n)$, and endowed by the same quasi-norms of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$, respectively. We will use the notation $\dot{A}^s_{p,q}(\mathbb{R}^n)$ for either $\dot{B}^s_{p,q}(\mathbb{R}^n)$ or $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n)$ for either $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$ or $\dot{\widetilde{F}}^s_{p,q}(\mathbb{R}^n)$, if there is no need to distinguish them. The spaces $\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n)$ are subsets of the tempered distributions space $\mathcal{S}'(\mathbb{R}^n)$, where their definitions depend on the polynomials of degree less than a parameter ν which is given by G. Bourdaud in, e.g., [4] (the value of ν depends only on the 4-tuple (n, s, p, q)). Also, for their definitions we need the notion of distributions vanishing at infinity that we will recall.

DEFINITION 1.1. — *We say that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishes at the infinity in the weak sense if $\lim_{\lambda \to 0} f\left(\lambda^{-1}(\cdot)\right) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. The set of all such distributions is denoted by $\widetilde{C}_0$.*

In Subsection 2.2 below, we give some properties of $\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n)$ and some examples of distributions in $\widetilde{C}_0$. Note that some interesting properties of these spaces can be found in, e.g., [2, 4, 14, 25]. For instance we have:

- $\dot{\widetilde{F}}^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ $(1 < p < \infty)$; here $\dot{\widetilde{F}}^0_{p,2}(\mathbb{R}^n)$ is defined by all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $[f]_{\mathcal{P}} \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ and $f \in \widetilde{C}_0$, see [14],

- $\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n)$ is continuously embedded in the weighted space $L_\infty(\mathbb{R}^n; |x|^{n/p-s})$, if either $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - n/p \in \mathbb{N}$ and $0 < q \leqslant 1$ $(0 < p \leqslant 1$ in case of the F-space); here $\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n)$ is defined by all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $[f]_{\mathcal{P}} \in \dot{A}^s_{p,q}(\mathbb{R}^n)$, $f \in C^{\nu-1}(\mathbb{R}^n)$, $f^{(\alpha)}(0) = 0$ for $|\alpha| < \nu$ and $f^{(\alpha)} \in \widetilde{C}_0$ for $|\alpha| = \nu$, where $\nu := [s - n/p] + 1$ if $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and $\nu := s - n/p$ if $s - n/p \in \mathbb{N}$ and $0 < q \leqslant 1$ $(0 < p \leqslant 1$ in case of the F-space), where $[s - n/p]$ is the integer part of $s - n/p$, see [14].

In the sense of the last point, we want to show some inequalities of Hardy-type in the particular case $s < n/p$, where under this restriction the realized spaces are defined as:

$$\dot{\widetilde{A}}^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ [f]_{\mathcal{P}} \in \dot{A}^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad f \in \widetilde{C}_0 \right\}, \qquad (1.1)$$

with $\|f\|_{\dot{\widetilde{A}}^s_{p,q}} := \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}$. Then we essentially prove the following statement:

THEOREM 1.2. — *Let $0 < p < \infty$. Let $s \in \mathbb{R}$ be such that*

$$\left(\frac{n}{p} - n\right)_+ < s < \frac{n}{p}.$$

Then there exists a constant $c > 0$ such that the inequality

$$\left(\int_{\mathbb{R}^n} |x|^{-sp} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant c \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} \tag{1.2}$$

holds for all $f \in \dot{\tilde{B}}^s_{p,p}(\mathbb{R}^n)$.

Our proof will be limited to the case $p \leqslant 1$, in which we will use some properties in homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as a composition of functions and a pointwise multiplication. We will also use an approximation with the help of smooth functions. We will divided the proof into two parts with respect to $s \geqslant 1$ and $s < 1$. Recall that the Hardy inequality on Slobodeckij spaces $B^s_{p,p}(\mathbb{R}^n)$ with $0 < s < 1$ has a certain history, e.g., Grisvard [10, Lem. 4.1], Lions and Magenes [12, §10], Triebel [19, 3.2.6], Schmeisser and Triebel [16, §4.3].

However, the part of the proof when $p > 1$ will be omitted since it is proved in [25, Prop. 7], also, we can easily obtain this case by applying the inequality for weighted Lebesgue spaces of E. M. Stein and G. Weiss [18, Thm. B*] and an approximation with smooth functions. On the other hand, using the embedding properties of $\dot{A}^s_{p,q}(\mathbb{R}^n)$, we have an immediate consequence of Theorem 1.2:

COROLLARY 1.3. — *Let $0 < p_0 < p < \infty$. Let $s_0 \in \mathbb{R}$ be such that*

$$\left(\frac{n}{p_0} - n \right)_+ < s_0 < \frac{n}{p_0} \quad and \quad \beta := s_0 - \frac{n}{p_0} + \frac{n}{p} \geqslant 0 \,.$$

(1) *Then there exists a constant $c > 0$ such that it holds*

$$\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant c \|[f]_{\mathcal{P}}\|_{\dot{B}^{s_0}_{p_0,p}}, \quad \forall \, f \in \dot{\tilde{B}}^{s_0}_{p_0,p}(\mathbb{R}^n) \,. \tag{1.3}$$

(2) *Then there exists a constant $c > 0$ such that it holds*

$$\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant c \|[f]_{\mathcal{P}}\|_{\dot{F}^{s_0}_{p_0,\infty}}, \quad \forall \, f \in \dot{\tilde{F}}^{s_0}_{p_0,\infty}(\mathbb{R}^n) \,. \tag{1.4}$$

Remark 1.4. — Owing to the embedding proprieties again (see Subsection 2.1 below) the previous corollary holds also if we replace in (1.3) the space $\dot{\tilde{B}}^{s_0}_{p_0,p}(\mathbb{R}^n)$ by $\dot{\tilde{B}}^{s_0}_{p_0,q}(\mathbb{R}^n)$ with $0 < q < p$. Similarly in (1.4), that $\dot{\tilde{F}}^{s_0}_{p_0,\infty}(\mathbb{R}^n)$ can be replaced by $\dot{\tilde{F}}^{s_0}_{p_0,q}(\mathbb{R}^n)$ with $0 < q \leqslant \infty$.

Remark 1.5. — Observe that Theorem 1.2 cannot be true with only functions in the homogeneous spaces $\dot{B}^s_{p,p}(\mathbb{R}^n)$. In other words, the inequality (1.2) fails if the condition $f \in \dot{\tilde{B}}^s_{p,p}(\mathbb{R}^n)$ is replaced by $[f]_{\mathcal{P}} \in \dot{B}^s_{p,p}(\mathbb{R}^n)$. Indeed, in that case we take the function $f(x) := 1$ ($\forall \, x \in \mathbb{R}^n$) (or any

nonzero polynomial), then the left-hand side of (1.2) is $+\infty$, while $\|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} = 0$, then we have a contradiction. Similarly for the inequalities (1.3) and (1.4). In this context, we note that the same observation holds for the spaces:

(1) $\dot{B}^{s,m}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,m}_{p,q}(\mathbb{R}^n)$, which are defined by differences Δ^m_h, see Definition 3.3 and Remark 3.5 below.

(2) $\dot{B}^{s,m,W}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,m,W}_{p,q}(\mathbb{R}^n)$, which are defined by the Gauss–Weierstrass semi-group of the heat equation, see Definition 3.4 below or, e.g., [23, p. 59], see also Remark 3.5 below.

In the other direction, using Theorem 1.2 and Corollary 1.3 we easily obtain their counterparts of the inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. Namely:

THEOREM 1.6. — *Let p_0, p, s_0 and β be real numbers given as in Corollary 1.3.*

(1) *Then there exists a constant $c > 0$ such that it holds*

$$\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant c \|[f]_{\mathcal{P}}\|_{\dot{B}^{s_0}_{p_0,p}}, \quad \forall\, f \in B^{s_0}_{p_0,p}(\mathbb{R}^n). \qquad (1.5)$$

(2) *Then there exists a constant $c > 0$ such that it holds*

$$\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant c \|[f]_{\mathcal{P}}\|_{\dot{F}^{s_0}_{p_0,\infty}}, \quad \forall\, f \in F^{s_0}_{p_0,\infty}(\mathbb{R}^n). \qquad (1.6)$$

Remark 1.7. — By Theorem 1.2 and Corollary 1.3 we have the previous results given in [25, Prop. 7] and [26, Prop. 14] for $p \geqslant 1$. They also cover the assertion in [22, §16.3, p. 238].

Notation and plan of the paper

All function spaces occurring in this work are defined on Euclidean space $\mathbb{R}^n$ ($n = 1, 2, \dots$). We omit $\mathbb{R}^n$ in notations. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of integers and $\mathbb{R}$ the set of real numbers. For $a \in \mathbb{R}$ we put $a_+ := \max(0, a)$. The symbol $\hookrightarrow$ indicates a continuous embedding. If $p \in\,]0, \infty]$, $\|f\|_p$ denotes the quasi-norm of functions f in L_p. If $|\cdot|^r f \in L_p$ we write $\||\cdot|^r f\|_p := \||x|^r f\|_p$. The symbol $\mathcal{S}$ denotes the Schwartz space, and $\mathcal{S}'$ its topological dual. For a function $f \in L_1$, the Fourier transform is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x \qquad (\forall\, \xi \in \mathbb{R}^n).$$

The operator $\mathcal{F}$ can be extended to the whole space $\mathcal{S}'$ in the usual way.

We denote by $\mathcal{P}_\infty$ the set of all polynomials on $\mathbb{R}^n$. The symbol $\mathcal{S}_\infty$ will be used for the set of all $\varphi \in \mathcal{S}$ such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_\infty$, and $\mathcal{S}'_\infty$ denotes its topological dual, which can identified to the quotient space $\mathcal{S}'/\mathcal{P}_\infty$. The mapping which takes any $[f]_\mathcal{P}$ to the restriction of f to $\mathcal{S}_\infty$ turns out to be an isomorphism from $\mathcal{S}'/\mathcal{P}_\infty$ onto $\mathcal{S}'_\infty$.

Throughout this work, we fix a cut-off function denoted by ρ, a radial C^∞ function, such that $0 \leqslant \rho \leqslant 1$, $\rho(\xi) = 1$ if $|\xi| \leqslant 1$ and $\rho(\xi) = 0$ if $|\xi| \geqslant 3/2$. We put $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$ for all $\xi \in \mathbb{R}^n$. Then γ is supported by the compact annulus $1/2 \leqslant |\xi| \leqslant 3/2$, and it holds

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1 \qquad (\forall\, \xi \in \mathbb{R}^n \setminus \{0\})\,, \tag{1.7}$$

$$\rho(2^{-k}\xi) + \sum_{j>k} \gamma(2^{-j}\xi) = 1 \qquad (\forall\, \xi \in \mathbb{R}^n,\, \forall\, k \in \mathbb{Z})\,. \tag{1.8}$$

For any $j \in \mathbb{Z}$, we introduce the pseudodifferential operators $S_j := \rho(2^{-j}D)$ and $Q_j := \gamma(2^{-j}D)$ by means of the formulas $\widehat{S_j f}(\xi) = \rho(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{Q_j f}(\xi) = \gamma(2^{-j}\xi)\widehat{f}(\xi)$. It is clear that S_j is defined on $\mathcal{S}'$ and that Q_j is defined on $\mathcal{S}'_\infty$ since $Q_j f = 0$ ($\forall\, j \in \mathbb{Z}$) if and only if f is a polynomial. We make use of the following convention:

If $f \in \mathcal{S}'_\infty$ we set $Q_j f := Q_j f_1$ for all f_1 satisfying $[f_1]_\mathcal{P} = f$.

Finally, the symbols $c, c_1, \dots$ denote positive constants which depend only on the fixed parameters $n, s, p, q, \dots$

This work is organized as follows. In Section 2 we collect definitions and basic properties of the homogeneous spaces and their realized versions. Section 3 is devoted to the proofs and some remarks.

2. Preparations

2.1. The Besov and Triebel–Lizorkin spaces

The Littlewood–Paley decomposition approach is the basic theory to define the Besov and Triebel–Lizorkin spaces, for this reason we recall the convergence property of a such decomposition of a tempered distribution. That is, using the partitions of unity (1.7) and (1.8), if $f \in \mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively), then $f = \sum_{j \in \mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively). Similarly, if $f \in \mathcal{S}$ ($\mathcal{S}'$, respectively) and $k \in \mathbb{Z}$, then $f = S_k f + \sum_{j>k} Q_j f$ in $\mathcal{S}$ ($\mathcal{S}'$, respectively).

DEFINITION 2.1. — *Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.*

(i) *Let $0 < p \leqslant \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ satisfying*

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} (2^{sj} \|Q_j f\|_p)^q \right)^{1/q} < \infty \, .$$

(ii) *Let $0 < p < \infty$. The homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ satisfying*

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |Q_j f|)^q \right)^{1/q} \right\|_p < \infty \, .$$

The spaces $\dot{A}^s_{p,q}$ are quasi-Banach for the above defined quasi-seminorms, they do not depend on the chosen function ρ, cf. [9, Rem. 2.6] and [20, Rem. 5.1.3/2]. Both spaces contain continuously $\mathcal{S}_\infty$ and are embedded continuously in $\mathcal{S}'_\infty$, and

(i) $\dot{B}^s_{p,\min(p,q)} \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\max(p,q)}$, $\dot{A}^s_{p,q_1} \hookrightarrow \dot{A}^s_{p,q_2}$ if $q_1 < q_2$,

(ii) for $0 < q, r \leqslant \infty$, $s_1 > s_2$ and $0 < p_1 < p_2 < \infty$ such that $s_1 - n/p_1 = s_2 - n/p_2$ it holds $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q} \hookrightarrow \dot{B}^{s_2-n/p_2}_{\infty,q}$, $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}$ and $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}$,

(iii) $\|f(\lambda(\cdot))\|_{\dot{A}^s_{p,q}} \equiv \lambda^{s-n/p} \|f\|_{\dot{A}^s_{p,q}}$ for all $f \in \dot{A}^s_{p,q}$ and all $\lambda > 0$.

We also have *the estimates of Nikol'skij-type*, see, e.g., [6, Prop. 4], [13, Prop. 3.4] and [14, Props. 2.15, 2.17].

PROPOSITION 2.2. — *Let $s \in \mathbb{R}$ and $p, q \in]0, \infty]$ (with $p < \infty$ in case of the F-space). Let a, b be real numbers such that $0 < a < b$. Let $(u_j)_{j \in \mathbb{Z}}$ be a sequence in $\mathcal{S}'$ such that $\widehat{u_j}$ is supported by the compact annulus $a2^j \leqslant |\xi| \leqslant b2^j$ and $A := (\sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q)^{1/q} < \infty$ ($A := \|(\sum_{j \in \mathbb{Z}} (2^{js} |u_j|)^q)^{1/q}\|_p < \infty$ in case of the F-space).*

(i) *Then the series $\sum_{j \in \mathbb{Z}} u_j$ converges in $\mathcal{S}'_\infty$ to a limit u which belongs to $\dot{A}^s_{p,q}$ and satisfies $\|u\|_{\dot{A}^s_{p,q}} \leqslant cA$; the constant c depends only on n, s, p, q, a and b.*

(ii) *If in addition $s > (n/p - n)_+$ $\left(\text{with } s > \left(\frac{n}{\min(p,q)} - n \right)_+ \text{ in case of} \right.$ the F-space$)$, the same conclusion holds for $a = 0$.*

The Fatou property given below is useful. The proof in [8, Thm. 2.6/1] for the case of inhomogeneous spaces $A^s_{p,q}$ can be extended easily to homogeneous ones. See also [5, Prop. 14], [6, Prop. 7] and [13, Prop. 3.13].

PROPOSITION 2.3. — *Let $s \in \mathbb{R}$ and $p, q \in\ ]0, \infty]$ (with $p < \infty$ in case of the F-space). Let $f \in \mathcal{S}'_\infty$. If there exists a bounded sequence $(u_k)_{k \in \mathbb{N}_0}$ in $\dot{A}^s_{p,q}$ such that $\lim_{k \to \infty} u_k = f$ in $\mathcal{S}'_\infty$, then*

$$f \in \dot{A}^s_{p,q} \quad and \quad \|f\|_{\dot{A}^s_{p,q}} \leqslant c \liminf_{k \to \infty} \|u_k\|_{\dot{A}^s_{p,q}}.$$

We refer to, e.g., [4, 7, 9, 11] for other properties of $\dot{A}^s_{p,q}$, as characterizations, equivalent quasi-norms, embeddings..etc.

We now give the definition of the inhomogeneous Besov and Triebel–Lizorkin spaces.

DEFINITION 2.4. — *Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.*

(i) *Let $0 < p \leqslant \infty$. The Besov space $B^s_{p,q}$ is the set of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} := \|S_0 f\|_p + \left(\sum_{j \geqslant 1} (2^{sj} \|Q_j f\|_p)^q \right)^{1/q} < \infty.$$

(ii) *Let $0 < p < \infty$. The Triebel–Lizorkin space $F^s_{p,q}$ is the set of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{F^s_{p,q}} := \|S_0 f\|_p + \left\| \left(\sum_{j \geqslant 1} (2^{sj} |Q_j f|)^q \right)^{1/q} \right\|_p < \infty.$$

In connection with the homogeneous spaces $\dot{A}^s_{p,q}$, we have the following assertion which is proved in, e.g., [21, Thm. 2.3.3, p. 98].

PROPOSITION 2.5. — *Let $0 < p, q \leqslant \infty$ (with $p < \infty$ in case of the F-space). Let s be a real such that $s > (n/p - n)_+$. Then $A^s_{p,q}$ is the set of all $f \in L_p$ such that $[f]_\mathcal{P} \in \dot{A}^s_{p,q}$. Moreover the expression $\|f\|_p + \|[f]_\mathcal{P}\|_{\dot{A}^s_{p,q}}$ defines an equivalent quasi-norm in $A^s_{p,q}$.*

2.2. The realized spaces

In this section we assume that $s < n/p$ and begin by the definition of a realization.

DEFINITION 2.6. — *Let E be a vector subspace of $\mathcal{S}'_\infty$ endowed with a quasi-norm which renders continuous the embedding $E \hookrightarrow \mathcal{S}'_\infty$. A realization of E in $\mathcal{S}'$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'$ such that $[\sigma(f)]_\mathcal{P} = f$ for all $f \in E$. The image set $\sigma(E)$ is called the realized space of E with respect to σ.*

In [4, §4] there exists an explication about realizations and their properties as the commutation with translations and dilations. Thus, we recall an example of a realization given by the Littlewood–Paley decomposition. For all $f \in \dot{A}_{p,q}^s$, the series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in $\mathcal{S}'$ to a limit denoted by $\sigma(f)$ which belongs to $\widetilde{C}_0$, see, e.g., [14, Prop. 2.15], ($\widetilde{C}_0$ is defined in Definition 1.1). The mapping $\sigma : \dot{A}_{p,q}^s \to \mathcal{S}'$ defined in such a way is a translation and dilation commuting realization of $\dot{A}_{p,q}^s$ into $\mathcal{S}'$. Such a realization of $\dot{A}_{p,q}^s$ is *unique*, see, e.g., [4, Thms. 4.1, 4.2]. Also by [14, Thm. 1.2], we have

$$\sigma(\dot{A}_{p,q}^s) = \dot{\widetilde{A}}_{p,q}^s, \qquad (2.1)$$

where the space $\dot{\widetilde{A}}_{p,q}^s$ is defined in (1.1). To prove (2.1), we take $g \in \dot{\widetilde{A}}_{p,q}^s$, i.e., $[g]_{\mathcal{P}} \in \dot{A}_{p,q}^s$, then g and $\sigma([g]_{\mathcal{P}})$ differ by a polynomial, and both the functions g and $\sigma([g]_{\mathcal{P}})$ belong to $\widetilde{C}_0$, we conclude that they coincide since it is easy to check that

$$\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\},$$

see, e.g., [2, p. 46]. Now, it is clear that without reference to the Littlewood–Paley decomposition, $\dot{\widetilde{A}}_{p,q}^s$ is well-defined and with $\|\cdot\|_{\dot{A}_{p,q}^s}$ is a quasi-Banach space in $\mathcal{S}'$.

We finish this subsection by some examples of functions in $\widetilde{C}_0$ and $\dot{\widetilde{A}}_{p,q}^s$.

Examples 2.7. — (i) If $f \in L_p$ with $1 \leqslant p < \infty$ then $f \in \widetilde{C}_0$.

(ii) Distributional derivatives of bounded functions belong to $\widetilde{C}_0$.

(iii) If $f \in \widetilde{C}_0$ then $f^{(\alpha)} \in \widetilde{C}_0$.

(iv) Let $f(x) := |x|^d$ with $-n < d \notin 2\mathbb{N}_0$. The function f is not a polynomial, locally integrable and $[f]_{\mathcal{P}} \in \dot{B}_{p,\infty}^{d+n/p}$. Indeed, $\widehat{f}(\lambda\xi) = \lambda^{-n-d}\widehat{f}(\xi)$ in $\mathcal{S}'$ for all $\lambda > 0$, and $\widehat{f}$ is a radiale function, that is to say we can find a function g defined on $\mathbb{R}^+$ such that $\widehat{f}(\xi) = g(|\xi|)$; as $d \notin 2\mathbb{N}_0$ then there exists a constant $c_0 \neq 0$ ($c_0 = g(1)$) such that $\widehat{f}(\xi) = c_0|\xi|^{-n-d}$. This implies $\mathcal{F}(Q_k f)(\xi) = c_0|\xi|^{-n-d}\gamma(2^{-k}\xi)$ on $\mathbb{R}^n \setminus \{0\}$; we define $\psi \in \mathcal{S}_\infty$ by $\widehat{\psi}(\xi) := c_0|\xi|^{-n-d}\gamma(\xi)$ and obtain $\|Q_k f\|_p = 2^{-k(d+n/p)}\|\psi\|_p$. Now, if $-n < d < 0$ we deduce that $f \in \dot{\widetilde{B}}_{p,\infty}^{d+n/p}$.

2.3. Some properties of the realized spaces

We recall some properties of $\dot{\widetilde{B}}_{p,q}^s$ which will be useful when we deal with the proof of Theorem 1.2. We limit ourselves to the case of the B-spaces since all assertions given below are valid also for the F-spaces.

PROPOSITION 2.8. — *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $(n/p - n)_+ < s < n/p$. Then the continuously embedding $\dot{\tilde{B}}^s_{p,q} \hookrightarrow L^{\mathrm{loc}}_1$ holds.*

Proof. — See, e.g., [17]. □

PROPOSITION 2.9. — *If $0 < p, q < \infty$ and $0 < s < n/p$, then $\mathcal{S}_\infty$ is a dense subspace of $\dot{\tilde{B}}^s_{p,q}$.*

Proof. — The proof given in [1, Prop. 12] in case $\dot{\tilde{F}}^s_{p,2}$ can be easily extended to obtain the case $\dot{\tilde{B}}^s_{p,q}$ for any q. See also [4, Prop. 3.11] for the same idea. □

PROPOSITION 2.10. — *Let either $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ or $0 < p < 1$ and $0 < q < \infty$. Let $0 < s < n/p$. Then the space $\dot{\tilde{B}}^s_{p,q}$ satisfies the Fatou property: there exists a constant $c > 0$ such that every a bounded sequence $(f_k)_{k \in \mathbb{N}_0}$ in $\dot{\tilde{B}}^s_{p,q}$ admits a subsequence $(f_{k_j})_{j \in \mathbb{N}_0}$ such that $f := \lim_{j \to \infty} f_{k_j}$ exists in $\mathcal{S}'$ and*

$$\|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leqslant c \liminf_{j \to \infty} \|[f_{k_j}]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}.$$

Proof. — The case $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ is given in [3, Thm. 3.1], and the same proof can be extended to the case $0 < p < 1$ and $0 < q < \infty$, since for $0 < p < 1$ it holds that $\dot{\tilde{B}}^s_{p,q} = (\dot{b}^{-s-n+n/p}_{\infty,\infty})'$ if $0 < q \leqslant 1$ and $\dot{\tilde{B}}^s_{p,q} = (\dot{b}^{-s-n+n/p}_{\infty,q'})'$ if $q > 1$, where $\dot{b}^s_{p,q}$ denotes the closure of $\mathcal{S}_\infty$ into $\dot{\tilde{B}}^s_{p,q}$ and $q' := q/(q-1)$. □

Remark 2.11. — Some of the results, known for homogeneous Besov spaces, do not extend to realized spaces in a trivial way. For instance, the Riesz operator $\mathcal{I}_r$, which is defined by $\widehat{\mathcal{I}_r f}(\xi) := |\xi|^{-r} \hat{f}(\xi)$ where $r \in \mathbb{R}$, takes $\dot{B}^s_{p,q}$ isomorphically into $\dot{B}^{s+r}_{p,q}$, however $\mathcal{I}_r : \dot{\tilde{B}}^s_{p,q} \to \dot{\tilde{B}}^{s+r}_{p,q}$ is an open question.

3. Proofs and some remarks

Proof of Theorem 1.2. —

Step 1: the case $1 < p < \infty$ and $0 < s < n/p$. — See [25].

Step 2: the case $0 < p \leqslant 1$. — We first prove (1.2) with functions f in $\mathcal{S}_\infty$, where for simplicity we will split our investigation into two substeps given with respect to cases $s < 1$ and $s \geqslant 1$. In a third substep, we will show the assertion with functions f in $\dot{\tilde{B}}^s_{p,p}$.

Substep 2.1: the case $n/p - n < s < 1$. — Let $f \in \mathcal{S}_\infty$. Let us choose a number η such that $\eta < p$. We write $\||x|^{-s}f\|_p = \||x|^{-s\eta}|f|^\eta\|_{p/\eta}^{1/\eta}$. Then the desired estimate (i.e., (1.2) with $f \in \mathcal{S}_\infty$) follows easily by Step 1 and the following assertion concerning composition of functions in the inhomogeneous Besov spaces:

LEMMA 3.1. — *Let $0 < u, v \leqslant \infty$ and $(n/u - n)_+ < r < 1$. Then there exists a constant $c > 0$ such that the inequality*

$$\|[|g|^\eta]_{\mathcal{P}}\|_{\dot{B}^{r\eta}_{u/\eta, v/\eta}} \leqslant c\|[g]_{\mathcal{P}}\|_{\dot{B}^r_{u,v}}^\eta$$

holds, for all $g \in B^r_{u,v}$ and all $0 < \eta < 1$.

Proof of Lemma 3.1. — In [15, Thm. 5.4.4, p. 365], with the assumptions of this lemma on u, v, r, η and g, it has been proved that $\||g|^\eta\|_{B^{r\eta}_{u/\eta, v/\eta}} \leqslant c\|g\|_{B^r_{u,v}}^\eta$. Now, by Proposition 2.5 it holds

$$\|[|g|^\eta]_{\mathcal{P}}\|_{\dot{B}^{r\eta}_{u/\eta, v/\eta}} \leqslant c\Big(\|g\|_u + \|[g]_{\mathcal{P}}\|_{\dot{B}^r_{u,v}}\Big)^\eta, \quad \forall\, g \in B^r_{u,v}. \qquad (3.1)$$

In (3.1), we replace g by $g(\lambda(\cdot))$ for any $\lambda > 0$. Using homogeneity properties of $L_u, \dot{B}^r_{u,v}$ and $\dot{B}^{r\eta}_{u/\eta, v/\eta}$, dividing by $\lambda^{r\eta - n\eta/u}$ and letting $\lambda \to \infty$, we obtain the result. $\qquad\square$

Substep 2.2: the case $1 \leqslant s < n/p$. — Assume that $f \in \mathcal{S}_\infty$. Let p_1 and δ be such that

$$1 < p_1 < \infty \quad \text{and} \quad 0 < \delta := \frac{p}{p_1} < 1\,.$$

Let also s_1 be a parameter defined by

$$0 < s_1 < \delta\,, \tag{3.2}$$

and will be fixed at the end of the computation. Then we write

$$\||x|^{-s}f\|_p = \||x|^{-s_1}\,|x|^{-s\delta + s_1}|f|^\delta\|_{p_1}^{1/\delta}\,,$$

and apply both Step 1 with s_1, p_1 and Lemma 3.1 with $g(x) := |x|^{-s+s_1/\delta}f(x)$, $r := s_1/\delta, u = v := p$ and $\eta := \delta$, we obtain

$$\||x|^{-s}f\|_p \leqslant c_1\|[|x|^{-s\delta+s_1}|f|^\delta]_{\mathcal{P}}\|_{\dot{B}^{s_1}_{p_1, p_1}}^{1/\delta} \leqslant c_2\|[|x|^{-s+s_1/\delta}f]_{\mathcal{P}}\|_{\dot{B}^{s_1/\delta}_{p,p}}\,. \tag{3.3}$$

We continue by using the following lemma concerning pointwise multipliers in the homogeneous Besov spaces.

LEMMA 3.2. — *Let $0 < u < \infty$ and $0 < v \leqslant \infty$. Let $r, t \in \mathbb{R}$ be such that $r < n/u$ and $t > 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\|[\varphi g]_{\mathcal{P}}\|_{\dot{B}^{r-t}_{u,v}} \leqslant c\|[g]_{\mathcal{P}}\|_{\dot{B}^{n/u-t}_{u,\infty}}\|[\varphi]_{\mathcal{P}}\|_{\dot{B}^r_{u,v}}$$

holds, for all $\varphi \in \mathcal{S}_\infty$ and all $g \in \mathcal{S}'$ satisfying $[g]_{\mathcal{P}} \in \dot{B}^{n/u-t}_{u,\infty}$.

We need to introduce a family of locally integrable functions $g_d(x) := |x|^{-d}$, $(\forall\, x \in \mathbb{R}^n, 0 < d < n)$; d will be chosen later on. We have $g_d \in \tilde{B}_{p,\infty}^{n/p-d}$, which has been observed before. We also need to introduce a parameter N such that

$$N \in \mathbb{N} \quad \text{and} \quad \frac{s_1}{\delta} + Nd < \frac{n}{p}. \tag{3.4}$$

By applying N-times Lemma 3.2 to the last term in (3.3), with $g := g_d$, $\varphi(x) := |x|^{-s+s_1/\delta+d} f(x)$, $u = v := p$, $t := d$ and $r := r_j = s_1/\delta + jd$ $(j = 1, 2, \ldots, N)$, we deduce

$$\||x|^{-s} f\|_p \leqslant c_1 \|[g_d]_{\mathcal{P}}\|_{\dot{B}_{p,\infty}^{n/p-d}} \|[|x|^{-s+s_1/\delta+d} f]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s_1/\delta+d}}$$

$$\leqslant c_2 \|[g_d]_{\mathcal{P}}\|_{\dot{B}_{p,\infty}^{n/p-d}}^2 \|[|x|^{-s+s_1/\delta+2d} f]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s_1/\delta+2d}}$$

$$\vdots$$

$$\leqslant c_N \|[g_d]_{\mathcal{P}}\|_{\dot{B}_{p,\infty}^{n/p-d}}^N \|[|x|^{-s+s_1/\delta+Nd} f]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s_1/\delta+Nd}}.$$

We again apply Lemma 3.2 with $g(x) := |x|^{-s+s_1/\delta+Nd}$ and $\varphi := f$, then we need to satisfy the following condition:

$$0 < s - \frac{s_1}{\delta} - Nd < n, \tag{3.5}$$

and obtain

$$\||x|^{-s} f\|_p \leqslant c \|[g_d]_{\mathcal{P}}\|_{\dot{B}_{p,\infty}^{n/p-d}}^N \|[g]_{\mathcal{P}}\|_{\dot{B}_{p,\infty}^{n/p-s+s_1/\delta+Nd}} \|[f]_{\mathcal{P}}\|_{\dot{B}_{p,p}^s}.$$

Finally the inequality (1.2) follows with functions $f \in \mathcal{S}_\infty$. Now for $p \in\,]0, 1]$ and $s \in [1, n/p[$, we need to select $p_1 \in\,]1, \infty[$, $s_1 \in\,]0, p/p_1[$, $d \in\,]0, n[$ and $N \in \mathbb{N}$ such that (3.4) and (3.5) are satisfied (recall that $\delta := p/p_1$). The inequality (3.4) is guaranteed by $s < n/p$ and (3.5). Then taking (3.2) into account, it suffices to see (3.5) which is equivalent to

$$s - n < \frac{s_1}{\delta} + Nd < s.$$

To discuss this inequality we distinguish two cases: $s > n$ and $s \leqslant n$. If $s \leqslant n$, then the left-hand side is trivially satisfied and the right-hand side follows from $s_1/\delta < 1 \leqslant s$ and Nd small. If $s > n$, then we choose Nd such that

$$s - n - \frac{s_1}{\delta} < Nd < s - \frac{s_1}{\delta}.$$

Observe, that the right-hand side is strictly positive. So we may choose $Nd > 0$ in any case.

Proof of Lemma 3.2. — Let us take $\varphi \in \mathcal{S}_\infty$ and $g \in \mathcal{S}'$ such that $[g]_{\mathcal{P}} \in \dot{B}_{u,\infty}^{n/u-t}$. By an Abel transform we have

$$\sum_{k=-j}^{j}(S_k\varphi)(Q_k g) + \sum_{k=-j}^{j-1}(S_k g)(Q_{k+1}\varphi)$$
$$= (S_j\varphi)(S_j g) - (S_{-j}\varphi)(S_{-j-1}g), \quad \forall\, j > 0. \quad (3.6)$$

On the one hand, by [15, Lem. 4.2.1/1, p. 144], since φ is infinitely differentiable, bounded with all its derivatives and $g \in \mathcal{S}'$, we obtain

$$\lim_{j\to\infty}(S_j\varphi)(S_j g) = \varphi g \quad \text{in} \quad \mathcal{S}'. \quad (3.7)$$

On the other hand, we prove that

$$\lim_{j\to\infty}(S_{-j}\varphi)(S_{-j-1}g) = 0 \quad \text{in} \quad L_\infty. \quad (3.8)$$

Indeed, we can write $S_{-j}g = \sum_{k\leqslant -j}Q_k g$, cf. the beginning of Subsection 2.1, and by the embedding $\dot{B}_{u,\infty}^{n/u-t} \hookrightarrow \dot{B}_{\infty,\infty}^{-t}$ we get

$$\|S_{-j-1}g\|_\infty \leqslant \|[g]_{\mathcal{P}}\|_{\dot{B}_{\infty,\infty}^{-t}} \sum_{k\leqslant -j} 2^{kt} \leqslant c2^{-jt}\|[g]_{\mathcal{P}}\|_{\dot{B}_{\infty,\infty}^{-t}} \quad (\forall\, j \in \mathbb{Z}),$$

which yields

$$\|(S_{-j}\varphi)(S_{-j-1}g)\|_\infty \leqslant c2^{-jt}\|\varphi\|_\infty\|[g]_{\mathcal{P}}\|_{\dot{B}_{\infty,\infty}^{-t}},$$

where the positive constant c is independent of φ, g and j. The assertion (3.8) is proved. Now (3.7) and (3.8), and by taking $j \to \infty$ in (3.6) we obtain the convergence in $\mathcal{S}'$. Consequently, we have

$$\lim_{j\to\infty}\left(\sum_{k=-j}^{j}(S_k\varphi)(Q_k g) + \sum_{k=-j}^{j-1}(S_k g)(Q_{k+1}\varphi)\right) = [\varphi g]_{\mathcal{P}} \quad \text{in} \quad \mathcal{S}_\infty'.$$

For brevity, we put $A_j := \sum_{k=-j}^{j}(S_k\varphi)(Q_k g)$ and $B_j := \sum_{k=-j}^{j-1}(S_k g)(Q_{k+1}\varphi)$, and let us define a sequence $(u_j)_{j\in\mathbb{N}_0}$ by $u_j := A_j + B_j$. From above we have $\lim_{j\to\infty} u_j = [\varphi g]_{\mathcal{P}}$ in $\mathcal{S}_\infty'$. Also, the sequence $(u_j)_{j\in\mathbb{N}_0}$ is bounded in $\dot{B}_{u,v}^{r-t}$. Indeed, the functions $\mathcal{F}[(S_k\varphi)(Q_k g)]$ and $\mathcal{F}[(S_k g)(Q_{k+1}\varphi)]$ are supported by the ball $|\xi| \leqslant 3 \cdot 2^k$, then we can apply Proposition 2.2 and obtain

$$\|[A_j]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r-t}} \leqslant c_1\left(\sum_{k\in\mathbb{Z}}\left(2^{(r-t)k}\|S_k\varphi\|_\infty\|Q_k g\|_u\right)^v\right)^{1/v}$$

$$\leqslant c_2\|[g]_{\mathcal{P}}\|_{\dot{B}_{u,\infty}^{n/u-t}}\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j\leqslant k}2^{(r-n/u)(k-j)}\,2^{rj}\|Q_j\varphi\|_u\right)^v\right)^{1/v};$$

the constants c_1, c_2 are independent of φ, g and j, see again Proposition 2.2. Now, using the convolution inequality, see, e.g., [24, Lem. 3.8], we have

$$\|[A_j]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r-t}} \leqslant c\|[g]_{\mathcal{P}}\|_{\dot{B}_{u,\infty}^{n/u-t}}\|[\varphi]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r}}.$$

Also, since $\|S_k g\|_\infty \leqslant c2^{kt}\|[g]_{\mathcal{P}}\|_{\dot{B}_{u,\infty}^{n/u-t}}$ for all $k \in \mathbb{Z}$ (recall the assumption $t > 0$), then

$$\|[B_j]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r-t}} \leqslant c_1 \left(\sum_{k \in \mathbb{Z}} \left(2^{(r-t)k}\|S_k g\|_\infty\|Q_{k+1}\varphi\|_u \right)^v \right)^{1/v}$$
$$\leqslant c_2\|[g]_{\mathcal{P}}\|_{\dot{B}_{u,\infty}^{n/u-t}}\|[\varphi]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r}} \quad (\forall\, j \in \mathbb{N}_0).$$

Hence $\|[u_j]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r-t}}$ is bounded by $c\|[g]_{\mathcal{P}}\|_{\dot{B}_{u,\infty}^{n/u-t}}\|[\varphi]_{\mathcal{P}}\|_{\dot{B}_{u,v}^{r}}$ uniformly with respect to all $j \in \mathbb{N}_0$. Then we finish by applying Proposition 2.3. $\qquad\square$

Substep 2.3. — Summarizing, we have shown

$$\||x|^{-s}f\|_p \leqslant c\|[f]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s}} \qquad (\forall\, f \in \mathcal{S}_\infty). \tag{3.9}$$

Assume now $f \in \widetilde{\dot{B}}_{p,p}^{s}$. We take a sequence $(f_l)_{l \in \mathbb{N}_0}$ in $\mathcal{S}_\infty$ such that $f_l \to f$ in $\widetilde{\dot{B}}_{p,p}^{s}$. Then, in the one hand using (3.9) with f_l, it holds

$$\||x|^{-s}f_l\|_p \leqslant c\|[f_l]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s}} \qquad (\forall\, l \in \mathbb{N}_0). \tag{3.10}$$

On the other hand, by Proposition 2.8 we get $f_l \to f$ in L_1^{loc}. Then, classically there exits a subsequence $(f_{l_k})_{k \in \mathbb{N}_0}$ of $(f_l)_{l \in \mathbb{N}_0}$ such that

$$\|[f_{l_{k+1}}]_{\mathcal{P}} - [f_{l_k}]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s}} \leqslant 2^{-k} \quad \text{and} \quad f_{l_k} \to f \text{ almost everywhere.}$$

Let Ω be a compact set in $\mathbb{R}^n$. There exists a constant $c = c(\Omega) > 0$ such that, if we fix an integer k and apply the Fatou's lemma to the sequence $(|f_{l_{k+j}} - f_{l_k}|)_{j \in \mathbb{N}_0}$ on Ω, we obtain

$$\int_\Omega |f(x) - f_{l_k}(x)|\mathrm{d}x \leqslant c \liminf_{j \to \infty} \int_\Omega |f_{l_{k+j}}(x) - f_{l_k}(x)|\mathrm{d}x.$$

Again by Proposition 2.8, we get

$$\int_\Omega |f_{l_{k+j}}(x) - f_{l_k}(x)|\mathrm{d}x \leqslant c_1\|[f_{l_{k+j}}]_{\mathcal{P}} - [f_{l_k}]_{\mathcal{P}}\|_{\dot{B}_{p,p}^{s}} \leqslant c_2 \sum_{m=0}^{j-1} 2^{-k-m} \leqslant c_3 2^{-k},$$

with constants c_1, c_2, c_3 are independent of j and k. Since Ω is arbitrary, then $\lim_{k \to \infty} f_{l_k} = f$ in the sense of distributions. The sequence $(f_{l_k})_{k \in \mathbb{N}_0}$ is a Cauchy in $\widetilde{\dot{B}}_{p,p}^{s}$, then by the embedding $\widetilde{\dot{B}}_{p,p}^{s} \hookrightarrow \mathcal{S}'$ we obtain $\widetilde{f} := \lim_{k \to \infty} f_{l_k}$ in $\mathcal{S}'$, a fortiori in the distribution functions space $\mathcal{D}'$. It follows that $\widetilde{f} = f$.

Applying Proposition 2.10 to the sequence $(f_{l_{k+j}} - f_{l_k})_{j\in\mathbb{N}_0}$ (k is fixed in $\mathbb{N}_0$), and arguing as above, we obtain

$$\|[f]_{\mathcal{P}} - [f_{l_k}]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} \leqslant c_4 \liminf_{j\to\infty} \|[f_{l_{k+j}}]_{\mathcal{P}} - [f_{l_k}]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} \leqslant c_5 2^{-k},$$

the constants c_4 and c_5 are independent of k. Now, using (3.10) with the subsequence $(f_{l_k})_{k\in\mathbb{N}_0}$, then, for an arbitrary $\varepsilon > 0$, there exists an integer $k_\varepsilon \in \mathbb{N}_0$ such that it holds

$$\||x|^{-s} f_{l_k}\|_p \leqslant c_6 \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} + c_5 2^{-k} \leqslant c_6 \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} + \varepsilon \qquad (\forall\, k \geqslant k_\varepsilon). \quad (3.11)$$

On the other hand, since $0 < p \leqslant 1$, then the elementary inequality

$$\left| |f_{l_k}(x)|^p - |f(x)|^p \right| \leqslant \left| f_{l_k}(x) - f(x) \right|^p \qquad (\forall\, x \in \mathbb{R}^n)$$

gives pointwise convergence of $|f_{l_k}|^p$ to $|f|^p$ almost everywhere.

Thus, by applying the Fatou's lemma in (3.11) to the sequence $(|f_{l_k}|^p)_{k\geqslant k_\varepsilon}$ with the measure $|x|^{-sp}\mathrm{d}x$, it follows $\||x|^{-s}f\|_p \leqslant c_6\|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,p}} + \varepsilon$ for all $f \in \dot{\tilde{B}}^s_{p,p}$. By arbitrariness of ε, we deduce the estimate (1.2). The proof of Theorem 1.2 is complete. $\qquad\square$

Proof of Corollary 1.3. — First observe that $(n/p_0 - n)_+ < s_0 < n/p_0$ implies $(n/p - n)_+ < \beta < n/p$. This guarantees the application of Theorem 1.2 in the two cases. Now, inequalities (1.3) and (1.4) follow immediately by using the embeddings $\dot{B}^{s_0}_{p_0,p} \hookrightarrow \dot{B}^\beta_{p,p}$ and $\dot{F}^{s_0}_{p_0,\infty} \hookrightarrow \dot{B}^\beta_{p,p}$, respectively. $\quad\square$

Proof of Theorem 1.6. — With assumptions, we first note that $A^s_{p,q} \hookrightarrow \widetilde{C}_0$. Indeed, if $1 \leqslant p < \infty$, the assertion follows by $L_p \hookrightarrow \widetilde{C}_0$, however, if $0 < p < 1$, then $A^s_{p,q} \hookrightarrow L_1$ since $s > n/p-n$, and again we conclude by $L_1 \hookrightarrow \widetilde{C}_0$. Now, inequalities (1.5) and (1.6) are consequences of Proposition 2.5 and Corollary 1.3. $\qquad\square$

We turn to Remark 1.5 and first introduce two notations. For any arbitrary function f and $h \in \mathbb{R}^n$ we set $\Delta_h^1 f := f(\cdot + h) - f$ and $\Delta_h^m f := \Delta_h^1(\Delta_h^{m-1}f)$ where $m = 2, 3, \ldots$ We denote by W_t the semi-group of the heat equation defined on $\mathcal{S}'$ by $\widehat{W_t f}(\xi) = \mathrm{e}^{-t|\xi|^2}\widehat{f}(\xi)$, see, e.g., [20, p. 183].

DEFINITION 3.3. — *Let $0 < q \leqslant \infty$. Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$ be such that $0 < s < m$.*

(i) *Let $0 < p \leqslant \infty$. We denote by $\dot{B}^{s,m}_{p,q}$ the space of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{B}^{s,m}_{p,q}} := \left(\int_{\mathbb{R}^n} \left(|h|^{-s}\|\Delta_h^m f\|_p\right)^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} < \infty.$$

(ii) *Let $0 < p < \infty$. We denote by $\dot{F}_{p,q}^{s,m}$ the space of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{F}_{p,q}^{s,m}} := \left\| \left(\int_{\mathbb{R}^n} \left(|h|^{-s} |\Delta_h^m f| \right)^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} \right\|_p < \infty \,.$$

DEFINITION 3.4. — *Let $0 < q \leqslant \infty$. Let $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$ be such that $s < 2m$.*

(i) *Let $0 < p \leqslant \infty$. We denote by $\dot{B}_{p,q}^{s,m,W}$ the space of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{B}_{p,q}^{s,m,W}} := \left(\int_0^\infty \left(t^{m-s/2} \|\partial_t^m W_t f\|_p \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \,.$$

(ii) *Let $0 < p < \infty$. We denote by $\dot{F}_{p,q}^{s,m,W}$ the space of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{F}_{p,q}^{s,m,W}} := \left\| \left(\int_0^\infty \left(t^{m-s/2} |\partial_t^m W_t f| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right\|_p < \infty \,.$$

Remarks 3.5. — (1) Clearly that $\Delta_h^m f = 0$ if and only if f is a polynomial of degree less than m, and $\partial_t^m W_t f = W_t \Delta^m f = 0$ (where Δ is the Laplacian in $\mathbb{R}^n$) if, e.g., f is a polynomial of degree less than $2m$; in that case, we have $\|f\|_{\dot{B}_{p,q}^{s,m}} = \|f\|_{\dot{B}_{p,q}^{s,m,W}} = 0$. Similar to the spaces $\dot{F}_{p,q}^{s,m}$ and $\dot{F}_{p,q}^{s,m,W}$.

(2) We can prove that $\| \cdot \|_{\dot{B}_{p,q}^{s,m}}$ and $\| \cdot \|_{\dot{F}_{p,q}^{s,m}}$ are equivalent quasi-norms of $\tilde{\dot{B}}_{p,q}^s$ and $\tilde{\dot{F}}_{p,q}^s$ if $(n/p-n)_+ < s < m$ and $\frac{n}{\min(p,q)} < s < m$, respectively. Thus in the right-hand side of (1.2)–(1.4), it is possible to use the quasi-seminorms $\|[f]_{\mathcal{P}}\|_{\dot{B}_{p,q}^{s,m}}$ or $\|[f]_{\mathcal{P}}\|_{\dot{F}_{p,q}^{s,m}}$ instead of $\|[f]_{\mathcal{P}}\|_{\dot{A}_{p,q}^s}$ with the needed assumptions on the parameters n, s, p and q.

(3) It seems interesting to extend the point 2. to $\dot{B}_{p,q}^{s,m,W}$ and $\dot{F}_{p,q}^{s,m,W}$.

Acknowledgments

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