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Symmetric powers of Severi–Brauer varieties [(*)]

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Abstract. — We classify products of symmetric powers of a Severi–Brauer variety, up to stable birational equivalence. The description also includes Grassmannians, flag varieties and moduli spaces of genus 0 stable maps.

Résumé. — Nous classons les produits de puissances symétriques d'une variété de Severi–Brauer, à équivalence birationnelle stable près. Notre classification concerne aussi les grassmanniennes, les variétés de drapeaux et les espaces de modules d'applications stables de genre 0.

1. Introduction

Let P be a *Severi–Brauer variety* over a field k. That is,

$$P_{k^s} := P \times_{\operatorname{Spec} k} \operatorname{Spec} k^s \cong \mathbb{P}^{\dim P},$$

where k^s denotes a separable closure of k; see Definition 2.1 for related notions and basic properties of Severi–Brauer varieties.

There are several ways to associate other varieties to P. These include

- the Grassmannians $\operatorname{Grass}(\mathbb{P}^{m-1}, P)$,
- the flag varieties $\operatorname{Flag}(\mathbb{P}^{m_1-1}, \dots, \mathbb{P}^{m_r-1}, P)$,
- the symmetric powers $\operatorname{Sym}^m(P)$ and
- the moduli spaces $\bar{M}_0(P, d)$ of genus 0 stable maps of degree d to P;

see Definitions 2.2–2.5. While all these varieties are geometrically rational, they are usually not rational over the ground field and it is an interesting problem to understand their birational properties over k. The results of this note are partly weaker than birational classification, since we describe

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only the stable birational equivalence classes, but partly stronger, since we also describe the products of these varieties. Birational equivalence of two varieties is denoted by $X \overset{\text{bir}}{\sim} Y$ and stable birational equivalence by $X \overset{\text{stab}}{\sim} Y$. The latter holds iff $X \times \mathbb{P}^m \overset{\text{bir}}{\sim} Y \times \mathbb{P}^n$ for some $n, m \in \mathbb{N}$.

Thus let $\mathrm{MSym}(P)$ denote the multiplicative monoid generated by stable birational equivalence classes of Grassmannians, flag varieties, symmetric powers of P and the moduli spaces $\bar{M}_0(P, d)$. We show that $\mathrm{MSym}(P)$ is finite, identify its elements and also the multiplication rules.

THEOREM 1.1. — *Let P be a Severi–Brauer variety of index $i(P)$ (see Definition 2.1 (2)) over a field k. Then*

(1) $\mathrm{MSym}(P) = \{\mathrm{Grass}(\mathbb{P}^{d-1}, P) : d \mid i(P)\}$ *and products are given by*
(2) $\mathrm{Grass}(\mathbb{P}^{d-1}, P) \times \mathrm{Grass}(\mathbb{P}^{e-1}, P) \overset{\text{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^{(d,e)-1}, P)$, *where (d, e) is the greatest common divisor. The identity is* $\mathrm{Grass}(\mathbb{P}^{i(P)-1}, P) \overset{\text{stab}}{\sim} \mathbb{P}^0$.

The class of a flag variety is given by the rule

(3) $\mathrm{Flag}(\mathbb{P}^{m_1-1}, \dots, \mathbb{P}^{m_r-1}, P) \overset{\text{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^{e-1}, P)$ *where $e := (m_1, \dots, m_r, i(P))$. In particular,* $\mathrm{Grass}(\mathbb{P}^{d-1}, P) \overset{\text{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^{e-1}, P)$ *where $e = (d, i(P))$.*

The class of a symmetric power is given by the rules

(4) $\mathrm{Sym}^d(P) \overset{\text{stab}}{\sim} \mathrm{Sym}^{(d, i(P))}(P)$ *for every $d \geqslant 0$,*
(5) $\mathrm{Sym}^d(P) \overset{\text{bir}}{\sim} \mathrm{Grass}(\mathbb{P}^{d-1}, P) \times \mathbb{P}^{d(d-1)}$ *for $d \leqslant n + 1$ and*
(6) $\mathrm{Sym}^d(P) \times \mathrm{Sym}^e(P) \overset{\text{stab}}{\sim} \mathrm{Sym}^{(d,e)}(P)$.

The class of $\bar{M}_0(P, d)$ is determined by the parity of d.

(7) $\bar{M}_0(P, 2e) \overset{\text{stab}}{\sim} P$ *except when $\dim P = e = 1$ in which case $\bar{M}_0(P, 2) \cong \mathbb{P}^2$.*
(8) $\bar{M}_0(P, 2e + 1) \overset{\text{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^1, P)$. *Note that $\mathrm{Grass}(\mathbb{P}^1, P)$ is rational iff $i(P) \in \{1, 2\}$ and stably birational to P iff $i(P)$ is odd.*

Remark 1.2. — Several key special cases of these results have been known. [18] shows that $\mathrm{Sym}^{\dim P+1}(P)$ is rational and the statement 1.1 (3) on flag varieties is proved in [17, 4.2].

The most natural description seems to be in terms of symmetric powers, so we start with them. The relationship with Grassmannians and flag varieties is easy to establish. The moduli spaces $\bar{M}_0(P, d)$ end up birationally the simplest but understanding them is more subtle.

The case $\dim P = e = 1$ is quite exceptional for the moduli space of genus 0 stable maps. $\bar{M}_0(P, 2)$ aims to classify double covers of P ramified at 2 points. The coarse moduli space is $\mathrm{Sym}^2(P) \cong \mathbb{P}^2$. However, if $P \not\cong \mathbb{P}^1$ then there are no such double covers defined over k. The problem is that every double cover has an order 2 automorphism. In all other cases, a dense open subset of $\bar{M}_0(P, d)$ parametrizes maps without automorphisms, even embeddings if $\dim P \geqslant 3$.

It is possible that the stable birational equivalences in Theorem 1.1 can be replaced by birational equivalences. For instance, it is possible that

$$\mathrm{Sym}^d(P) \overset{\mathrm{bir}}{\sim} \mathrm{Grass}(\mathbb{P}^{(d, i(P))-1}, P) \times \mathbb{P}^m \quad \text{for suitable } m,$$

but I do not even know how to show that

$$\mathrm{Sym}^{n+2}(P) \overset{\mathrm{bir}}{\sim} P \times \mathbb{P}^{n(n+1)} \quad \text{for } n = \dim P.$$

Several steps in the proof naturally give only stable birational equivalences and the difference between stable birational equivalence and birational equivalence is not even understood for Severi–Brauer varieties.

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2. Basic definitions

Definition 2.1 (Severi–Brauer varieties I). — *Let k be a field with separable closure k^s. A k-scheme P is called a* Severi–Brauer variety *if $P_{k^s} := P \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k^s \cong \mathbb{P}^n$ for some n. The following basic results go back to Severi and Châtelet, see* [7, Chap. 5] *or* [15] *for modern treatments and references.*

(1) *P is* trivial, *that is $P \cong \mathbb{P}^n$, iff $P(k) \neq \emptyset$.*
(2) *The* index *of P is the gcd of the degrees of all 0-cycles on P; it is denoted by $i(P)$. Its value divides* $\dim P + 1$ *and P contains a reduced, effective 0-cycle Z of degree $i(P)$. Thus P has a k' point for some separable field extension k'/k of degree $i(P)$.*
(3) *A subscheme $L \subset P$ is called* twisted linear *if L_{k^s} is a linear subspace of $P_{k^s} \cong \mathbb{P}^n$. Thus L is also a Severi–Brauer variety. The minimal twisted linear subvarieties have dimension $i(P)-1$ and they are isomorphic to each other; call this isomorphism class $P^{\min}$.*

(4) *Given* $P^{\min}$ *and* $r \geqslant 1$ *there is a unique (up-to isomorphism) Severi–Brauer variety* P_r *of dimension* $r(\dim P^{\min}+1)-1$ *such that* $(P_r)^{\min} \cong P^{\min}$.

(5) $P_r \overset{\text{bir}}{\sim} P^{\min} \times \mathbb{P}^m$ *for* $m=(r-1)(\dim P^{\min}+1)$.

(6) *Two Severi–Brauer varieties* P_1, P_2 *are* Brauer equivalent, *denoted by* $P_1 \sim P_2$, *iff* $P_1^{\min} \cong P_2^{\min}$. *This holds iff the smaller dimensional one is isomorphic to a twisted linear subvariety of the other.*

Definition 2.2 (Grassmannians and flag varieties). — *Let* P *be a Severi–Brauer variety of dimension* n. *Fix natural numbers* $0 < m_1 < \cdots < m_r < n$. *The* flag variety $\mathrm{Flag}(\mathbb{P}^{m_1-1}, \ldots, \mathbb{P}^{m_r-1}, P)$ *is the* k*-scheme that represents the functor that associates to a scheme* S *the set of all nested subschemes*

$$L_1 \subset \cdots \subset L_r \subset P_S$$

where, for every i, *the projection* $L_i \to S$ *is flat and its geometric fibers are linear subspaces of* P *of dimension* m_i. *For* $r=1$ *we set* $\mathrm{Grass}(\mathbb{P}^{m_1-1}, P) := \mathrm{Flag}(\mathbb{P}^{m_1-1}, P)$.

The flag varieties $\mathrm{Flag}(\mathbb{P}^{m_1-1}, \ldots, \mathbb{P}^{m_r-1}, \mathbb{P}^n)$ *are rational; in fact they can be written as finite, disjoint unions of affine spaces; see* [9, Chap. XIV] *or* [3, 19].

Definition 2.3 (Symmetric powers). — *Let* X *be a quasi-projective* k*-scheme. Its* n*th* symmetric power *is the* k*-scheme* $\mathrm{Sym}^n X := X^n/S_n$, *the quotient of the* n*th power* X^n *by the action of the symmetric group* S_n *permuting the coordinates.*

It is easy to see that $\mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$. *For* $n, r \geqslant 2$ *the symmetric powers* $\mathrm{Sym}^n \mathbb{P}^r$ *are singular (hence not isomorphic to a projective space) but they are rational; see* [20] *for a short proof.*

The following result says that stable birational equivalence is preserved by symmetric powers.

Lemma 2.4. — *Let* U *be a positive dimensional, geometrically irreducible* k*-variety. Then* $\mathrm{Sym}^m(U \times \mathbb{P}^r) \overset{\text{bir}}{\sim} \mathrm{Sym}^m(U) \times \mathbb{P}^{rm}$.

Therefore, if U, V *are positive dimensional, geometrically irreducible* k*-varieties such that* $U \overset{\text{stab}}{\sim} V$ *then* $\mathrm{Sym}^m(U) \overset{\text{stab}}{\sim} \mathrm{Sym}^m(V)$ *for every* m.

Proof. — There is a natural projection map $\mathrm{Sym}^m(U \times \mathbb{P}^r) \to \mathrm{Sym}^m(U)$. We claim that its generic fiber F_{gen} is rational. To construct it, set $L := k(\mathrm{Sym}^m(U))$ and $K := k(\mathrm{Sym}^{m-1}(U) \times U)$. Here we think of $\mathrm{Sym}^m(U)$ as U^m/S_m and $\mathrm{Sym}^{m-1}(U) \times U$ as U^m/S_{m-1} where $S_{m-1} \subset S_m$ are the permutations that fix the last factor. Thus K/L is a degree m field extension and $F_{gen} \overset{\text{bir}}{\sim} \Re_{K/L}(\mathbb{P}^r)$, the Weil restriction of $\mathbb{P}^r$ from K to L. Thus F_{gen} is rational. □

Definition 2.5 (Spaces of maps). — *Let X, Y be k-schemes and L_X, L_Y line bundles on them. Let* $\mathrm{Mor}((X, L_X), (Y, L_Y))$ *denote the k-scheme that represents the functor that associates to a scheme S the set of all morphisms $\phi : X_S \to Y_S$ such that $\phi^*(L_Y)_S \cong (L_X)_S$. See* [13, Sec. I.1] *for details where the notation* $\mathrm{Hom}(\,\cdot\,,\cdot\,)$ *is used instead.*

If $X \cong \mathbb{P}^1$ and we fix an ample line bundle L_Y on Y then $\mathrm{Aut}(\mathbb{P}^1)$ *acts on the space of maps by precomposition and*

$$M_0^\circ(Y, d) := \mathrm{Mor}((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)), (Y, L_Y))/\,\mathrm{Aut}(\mathbb{P}^1)$$

is called the open moduli space of genus 0 stable maps of degree d to Y.

We refer to [6] *for a general introduction to stable maps and the definition of the "true" moduli space $\bar{M}_g(Y, d)$ of genus g stable maps of degree d.*

For $m \geqslant 2$ the open moduli space $M_0^\circ(\mathbb{P}^m, d)$ is dense in the "true" moduli space $\bar{M}_0(\mathbb{P}^m, d)$; cf. [2, Sec. 4]. *So we do not lose any birational information by working with $M_0^\circ(\mathbb{P}^m, d)$.*

It is easy to see (we in fact prove this in Lemma 4.7) that the moduli spaces $\bar{M}_0(\mathbb{P}^m, d)$ are unirational. They are even rational but this is harder to prove; see [11, 8, 12].

3. Symmetric powers

A key step in understanding symmetric powers is the following.

Theorem 3.1 ([18]). — *Let P be a Severi–Brauer variety of dimension n. Then* $\mathrm{Sym}^{n+1}(P)$ *is rational.*

The following is a geometric proof. The Euler number of $\mathbb{P}^n$ is $n+1$, thus a general section of the tangent bundle $T_{\mathbb{P}^n}$ vanishes at $n+1$ points. For any Severi–Brauer variety this gives a rational map $\pi : H^0(P, T_P) \dashrightarrow \mathrm{Sym}^{n+1}(P)$.

Let $Z \subset P$ be a reduced 0-cycle of degree $n+1$ whose linear span equals P. Then $\pi^{-1}(Z)$ is the linear space $H^0(P, T_P(-Z)) \subset H^0(P, T_P)$ of dimension n. Let $V \subset H^0(P, T_P)$ be a general affine-linear subspace of codimension n. Then $\pi|_V : V \dashrightarrow \mathrm{Sym}^{n+1}(P)$ is birational.

Corollary 3.2. — *Let P be a Severi–Brauer variety of index $i(P)$. Then* $\mathrm{Sym}^d(P)$ *is stably rational iff $i(P) \mid d$.*

Proof. — If d is not divisible by $i(P)$ then $\mathrm{Sym}^d(P)(k) = \emptyset$ by 2.1(2), hence $\mathrm{Sym}^d(P)$ is not stably rational.

To see the converse, assume that $i(P) \mid d$. By 2.1(4)–(5) P is stably birational to a Severi–Brauer variety P' of dimension $d-1$. Furthermore, $\mathrm{Sym}^d(P')$ is rational by Theorem 3.1 and it is stably birational to $\mathrm{Sym}^d(P)$ by Lemma 2.4. □

Proof of Theorem 1.1(1)–(6). — The easiest is 1.1(5). Given $d \leqslant n+1$ points in linearly general position, they span a linear subspace of dimension $d-1$. This gives a natural map $\pi : \mathrm{Sym}^d(P) \dashrightarrow \mathrm{Grass}(\mathbb{P}^{d-1}, P)$. (A priori, this map is defined only over k^s. However, it is invariant under $\mathrm{Gal}(k^s/k)$, thus π is defined over k. This can be seen by applying Weil's lemma on the field of definition of a subscheme to the graph of π; see [21, I.7.Lem. 2] or [16, Sec. 3.4] for proofs.)

Let K be the function field of $\mathrm{Grass}(\mathbb{P}^{d-1}, P)$ and $L_K \subset P_K$ the linear subspace corresponding to the generic point. Thus L_K is a Severi–Brauer subvariety of dimension $d-1$. The generic fiber of π is $\mathrm{Sym}^d(L_K)$ which is rational by Theorem 3.1. Thus $\mathrm{Sym}^d(P) \overset{\mathrm{bir}}{\sim} \mathrm{Grass}(\mathbb{P}^{d-1}, P) \times \mathbb{P}^{d(d-1)}$.

Next we show the stable birational equivalences

$$\begin{aligned}\mathrm{Sym}^d(P) \times \mathrm{Sym}^e(P) \times \mathbb{P}^{n(d,e)} &\overset{\mathrm{stab}}{\sim} \mathrm{Sym}^d(P) \times \mathrm{Sym}^e(P) \times \mathrm{Sym}^{(d,e)}(P) \\ &\overset{\mathrm{stab}}{\sim} \mathbb{P}^{nd} \times \mathbb{P}^{ne} \times \mathrm{Sym}^{(d,e)}(P).\end{aligned}$$

First let K be the function field of $\mathrm{Sym}^d(P) \times \mathrm{Sym}^e(P)$. Then P_K has 0-cycles of degrees d and e, thus it also has a 0-cycle of degree (d,e). Thus $\mathrm{Sym}^{(d,e)}(P_K)$ is stably rational by Corollary 3.2, proving the first part.

Similarly, let L be the function field of $\mathrm{Sym}^{(d,e)}(P)$. Then $\mathrm{Sym}^d(P_L)$ and $\mathrm{Sym}^e(P_L)$ are stably rational by Lemma 3.2, proving the second part. This implies 1.1(6).

Using this for $e = i(P)$ gives that

$$\mathrm{Sym}^d(P) \times \mathrm{Sym}^{i(P)}(P) \overset{\mathrm{stab}}{\sim} \mathrm{Sym}^{(d,i(P))}(P).$$

Since $\mathrm{Sym}^{i(P)}(P)$ is stably rational by Corollary 3.2, we get 1.1(4) and, together with 1.1(5), it implies 1.1(2).

Next we show, following [17, 4.2], that, with $e := (m_1, \dots, m_r, i(P))$,

$$\begin{aligned}\mathrm{Flag}(\mathbb{P}^{m_1-1}, \dots, \mathbb{P}^{m_r-1}, P) &\overset{\mathrm{stab}}{\sim} \mathrm{Flag}(\mathbb{P}^{m_1-1}, \dots, \mathbb{P}^{m_r-1}, P) \times \mathrm{Grass}(\mathbb{P}^{e-1}, P) \\ &\overset{\mathrm{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^{e-1}, P).\end{aligned}$$

Let L be the function field of $\mathrm{Flag}(\mathbb{P}^{m_1-1}, \ldots, \mathbb{P}^{m_r-1}, P)$. Then P_L has twisted linear subspaces of dimensions $m_1 - 1, \ldots, m_r - 1$, hence 0-cycles of degrees $m_1, \ldots, m_r$. Therefore $i(P_L) \mid e := (m_1, \ldots, m_r, i(P))$ and $\mathrm{Sym}^e(P_L)$ is stably rational by Lemma 3.2. Thus $\mathrm{Grass}(\mathbb{P}^{e-1}, P_L)$ is stably rational by 1.1 (5).

Conversely, let K be the function field of $\mathrm{Grass}(\mathbb{P}^{e-1}, P)$. We use induction on r to show that $\mathrm{Flag}(\mathbb{P}^{m_1-1}, \ldots, \mathbb{P}^{m_r-1}, P_K)$ is stably rational. Since $e \mid m_1$, there is a twisted-linear subvariety $R_K \subset P_K$ of codimension m_1. Then the map

$$(L_1 \subset L_2 \subset \cdots \subset L_r) \longmapsto (L_2 \cap R_K \subset \cdots \subset L_r \cap R_K) \times (L_1)$$

determines a birational equivalence

$$\begin{aligned}&\mathrm{Flag}(\mathbb{P}^{m_1-1}, \ldots, \mathbb{P}^{m_r-1}, P_K)\\ &\qquad\qquad \overset{\mathrm{bir}}{\sim} \mathrm{Flag}(\mathbb{P}^{m'_2-1}, \ldots, \mathbb{P}^{m'_r-1}, R_K) \times \mathrm{Grass}(\mathbb{P}^{m_1-1}, P_K)\end{aligned}$$

where $m'_i := m_i - m_1$. By induction $\mathrm{Flag}(\mathbb{P}^{m'_2-1}, \ldots, \mathbb{P}^{m'_r-1}, R_K)$ is stably rational and so is $\mathrm{Grass}(\mathbb{P}^{m_1-1}, P_K)$ by Lemma 3.2 and 1.1 (5). This shows 1.1 (3).

We have proved that every flag variety of P is stably birational to a symmetric power $\mathrm{Sym}^d(P)$ for some $d \mid i(P)$. Next we show that these $\mathrm{Sym}^d(P)$ are not stably birational to each other.

Let $d < e$ be different divisors of $i(P)$. There is thus a prime p such that $d = p^a d', e = p^c e'$ where $a < c$ and d', e' are not divisible by p. Let p^b be the largest p-power dividing $i(P)$.

By 2.1 (2) P has a k' point for some field extension k'/k of degree $i(P)$. Let k''/k be the Galois closure of k'/k and K the invariant subfield of a p-Sylow subgroup of $\mathrm{Gal}(k''/k)$. Set $K' = k'K$. Note that p does not divide $\deg(K/k)$ and $\deg(K'/K) = p^b$, hence $i(P_K) = p^b$.

Although K'/K need not be Galois, the Galois group of its Galois closure is a p-group, hence nilpotent. Thus there is a subextension $K' \supset L \supset K$ of degree p^{b-a}. It is enough to show that $\mathrm{Sym}^d(P_L)$ and $\mathrm{Sym}^e(P_L)$ are not stably birational over L. By 1.1 (4),

$$\mathrm{Sym}^d(P_L) \overset{\mathrm{stab}}{\sim} \mathrm{Sym}^{p^a}(P_L) \quad \text{and} \quad \mathrm{Sym}^e(P_L) \overset{\mathrm{stab}}{\sim} \mathrm{Sym}^{p^c}(P_L).$$

Note that P_L has a point in K' and $\deg(K'/L) = p^a$, hence $i(P_L) = p^a$ and so $\mathrm{Sym}^{p^a}(P_L)$ is stably rational by Corollary 3.2. By contrast $\mathrm{Sym}^{p^c}(P_L)$ does not have any L-points. Indeed, an L-point on $\mathrm{Sym}^{p^c}(P_L)$ would mean a 0-cycle of degree p^c on P_L hence a 0-cycle of degree $p^{b-a}p^c = p^{b-a+c}$ on

P_K. This is impossible since $i(P_K)=p^b$ and $b-a+c<b$. Thus $\mathrm{Sym}^{p^a}(P_L)$ and $\mathrm{Sym}^{p^c}(P_L)$ are not stably birational. $\square$

4. Moduli of Severi–Brauer subvarieties

We need some results on twisted line bundles and maps between Severi–Brauer varieties; see [7, Chap. 5] or [15] for proofs and further references.

DEFINITION 4.1 (Twisted line bundles). — *Let X be a geometrically normal, proper k-variety. A* twisted line bundle *of X is a line bundle L on X_{k^s} such that $L^\sigma \cong L$ for every $\sigma \in \mathrm{Gal}(k^s/k)$. For example, if P is a Severi–Brauer variety then $\mathcal{O}_{P_{k^s}}(r)$ is a twisted line bundle for every r. Let $|L|$ denote the irreducible component of the Hilbert scheme (or Chow variety) of X parametrizing subschemes $H \subset X$ such that H_{k^s} is in the linear system $|L_{k^s}|$. (See* [13, Chap. I.] *for the Hilbert scheme or the Chow variety.) This is clearly a Severi–Brauer variety. There is a natural map $\iota_L : X \dashrightarrow |L|^\vee$ given by $x \mapsto \{H : H \ni x\}$.*

The dual *of a Severi–Brauer variety P is defined as $P^\vee := |\mathcal{O}_{P_{k^s}}(1)|$.*

Let $\phi : X \dashrightarrow Y$ be a map between geometrically normal, proper varieties and L_Y a twisted line bundle on Y. Assume that either ϕ is a morphism or X is smooth. Then ϕ^*L_Y is a twisted line bundle on X and $|\phi^*L_Y| \sim |L_Y|$.

Let X,Y be geometrically normal, proper varieties and L_X, L_Y twisted line bundles on them. Definition 2.5 extends to give $\mathrm{Mor}((X,L_X),(Y,L_Y))$*, the moduli space of all maps $\phi : X \dashrightarrow Y$ such that $\phi^*L_Y \cong L_X$. For Severi–Brauer varieties we write*

$$\mathrm{Mor}_d(Q,P) := \mathrm{Mor}((Q,\mathcal{O}_Q(d)),(P,\mathcal{O}_P(1))).$$

Composing with ι_L gives an isomorphism

$$\mathrm{Mor}((X,L),(P,\mathcal{O}_P(1))) \cong \mathrm{Mor}_1(|L|^\vee, P).$$

RESULTS 4.2 (Severi–Brauer varieties II). — *Let P,Q be Severi–Brauer varieties. We use the following results; see* [7] *or* [15] *for proofs and further references.*

(1) *Their product is defined as $|\mathcal{O}_{P^\vee\times Q^\vee}(1,1)| \cong |\mathcal{O}_{P\times Q}(1,1)|^\vee$. I denote this by $P\otimes Q$. It is better to think of this as defined on Brauer equivalence classes. The set of Brauer equivalence classes forms the* Brauer group *with product $P\otimes Q$, identity $\mathbb{P}^0 \sim \mathbb{P}^m$ and inverse $P \mapsto P^\vee$. The group is torsion; the order of P is called the* period *of P and denoted by* $\mathrm{per}(P)$*. The period divides the index.*

(2) $\mathrm{Mor}_1(Q,P) \sim Q^\vee \otimes P$. *The natural map is* $\phi \mapsto \{(x,H) : \phi(x) \in H\} \in |\mathcal{O}_{Q\times P^\vee}(1,1)|$.
(3) *If* $|L|$ *is non-empty then* $|L^m| \sim |L|^{\otimes m}$; *this comes from identifying the symmetric power of a vector space* V *with the subspace of symmetric tensors in* $V^{\otimes m}$.
(4) $\mathrm{Mor}_d(Q,P) \sim (Q^\vee)^{\otimes d} \otimes P$; *this follows from the previous two claims.*

Putting together 4.2 (1) and 4.2 (4) we obtain the following.

Corollary 4.3. — *Let* $P \sim P', Q \sim Q'$ *be Severi–Brauer subvarieties and* $d, d' \in \mathbb{N}$ *such that* $d \equiv d' \mod \mathrm{per}(Q)$. *Then*

$$\mathrm{Mor}_d(Q,P) \sim \mathrm{Mor}_{d'}(Q',P').$$

Proof. — This follows from the chain of Brauer equivalences

$$\mathrm{Mor}_d(Q,P) \sim (Q^\vee)^{\otimes d} \otimes P \sim ({Q'}^\vee)^{\otimes d'} \otimes P' \sim \mathrm{Mor}_{d'}(Q',P'). \qquad \square$$

We next define the spaces of Severi–Brauer subvarieties of a Severi–Brauer variety. That is, given a Severi–Brauer variety P we look at the subset of the Chow variety $\mathrm{Chow}(P)$ parametrizing subvarieties $X \subset P$ whose normalization $\bar{X}$ is a Severi–Brauer variety. For technical reasons it is better to work with $\bar{X} \to P$.

Definition 4.4. — *Fix integers* $0 \leqslant m \leqslant n$, $1 \leqslant d$ *and a Severi–Brauer variety* P *of dimension* n. *Let* $M^\circ_{\mathbb{P}^m}(P,d)$ *denote the moduli space parametrizing morphisms* $\phi : Q \to P$ *satisfying the following assumptions.*

(1) Q *is a Severi–Brauer variety of dimension* m.
(2) $\phi^*\mathcal{O}_P(1) \cong \mathcal{O}_Q(d)$.
(3) *Either* $m < n$ *and* $\phi : Q \to \phi(Q)$ *is birational or* $m = n$ *and every automorphism of the triple* $(\phi : Q \to P)$ *that is the identity on* P *is also the identity on* Q.
(4) *Two such morphisms* $\phi_i : Q_i \to P$ *are identified if there is an isomorphism* $\tau : Q_1 \cong Q_2$ *such that* $\phi_1 = \phi_2 \circ \tau$.

The spaces $M^\circ_{\mathbb{P}^m}(P,d)$ *are quasi-projective. They should be thought of as open subschemes of the projective moduli spaces of stable maps* $\bar{M}_{\mathbb{P}^m}(P,d)$ [1]. *Since we are interested in their birational properties, these compactifications are not important to us.*

(Comment on the notation. The moduli space of stable maps from a genus g *curve to* Y *is usually denoted by* $M_g(Y,\beta)$, *where* β *is the homology class of the image. If* $Y = \mathbb{P}^n$ *then* β *is usually replaced by* $\deg\beta$. *Thus* $M^\circ_{\mathbb{P}^m}(P,d)$ *follows mostly the stable maps convention, except that the degree of* $\phi(Q)$ *is* d^m.*)*

Note that if $\phi : Q \to \phi(Q)$ is birational then every automorphism of $\phi : Q \to P$ that is the identity on P is also the identity on Q. This is why the most naive way of identifying two maps is adequate in (4):

$$(\phi_1 : Q_1 \to P)_{\bar{k}} \cong (\phi_2 : Q_2 \to P)_{\bar{k}} \Leftrightarrow (\phi_1 : Q_1 \to P) \cong (\phi_2 : Q_2 \to P).$$

(As we discussed in Remark 1.2, failure of this is one of the problems with $\bar{M}_0(P,2)$ if $\dim P = 1$.)

If $d = 1$ then we get $M^\circ_{\mathbb{P}^m}(P,1) = \operatorname{Grass}(\mathbb{P}^m, P)$ and if $m = 1$ then the $M^\circ_{\mathbb{P}^1}(P,d)$ are open subschemes of the space of genus 0 stable maps $\bar{M}_0(P,d)$.

These moduli spaces are related to the spaces of maps from Definition 4.1:

$$M^\circ_{\mathbb{P}^m}(P,d) \stackrel{\text{bir}}{\sim} \operatorname{Mor}_d(\mathbb{P}^m, P)/\operatorname{Aut}(\mathbb{P}^m).$$

The resulting map $\Pi : \operatorname{Mor}_d(\mathbb{P}^m, P) \dashrightarrow M^\circ_{\mathbb{P}^m}(P,d)$ is not a product, not even birationally. Indeed the fiber of Π over a given $\phi : Q \to P$ is the space of isomorphisms $\operatorname{Isom}(\mathbb{P}^m, Q)$. This is a principal homogeneous space under $\operatorname{Aut}(\mathbb{P}^m)$ but it is not isomorphic to $\operatorname{Aut}(\mathbb{P}^m)$ unless Q is trivial since a k-point of $\operatorname{Isom}(\mathbb{P}^m, Q)$ is exactly an isomorphism $\mathbb{P}^m \cong Q$.

Our aim is to understand the spaces $M^\circ_{\mathbb{P}^m}(P,d)$ for arbitrary ground fields. This is achieved only for $m = 1$ but we have the following general periodicity property.

Theorem 4.5. — *Let $P \sim P'$ be Brauer equivalent Severi–Brauer varieties of dimensions n, n'. Fix $0 \leqslant m \leqslant \min\{n, n'\}$ and $1 \leqslant d, d'$. Assume that $d \equiv d' \mod (m+1)$. Then*

$$M^\circ_{\mathbb{P}^m}(P,d) \stackrel{\text{stab}}{\sim} M^\circ_{\mathbb{P}^m}(P',d').$$

Proof. — The idea is similar to the "no-name method" explained in [4, Sec. 4], where it is attributed to Bogomolov and Lenstra.

Let $\operatorname{Isom}_{\mathbb{P}^m}(d, P, d', P')$ denote the scheme parametrizing triples

$$\{(\phi : Q \to P); (\phi' : Q' \to P'); \tau\}$$

where $(\phi : Q \to P) \in M^\circ_{\mathbb{P}^m}(P,d)$, $(\phi' : Q' \to P') \in M^\circ_{\mathbb{P}^m}(P',d')$, and $\tau : Q \to Q'$ is an isomorphism. (No further assumptions on ϕ and $\phi' \circ \tau$.) We prove that

$$M^\circ_{\mathbb{P}^m}(P,d) \stackrel{\text{stab}}{\sim} \operatorname{Isom}_{\mathbb{P}^m}(d, P, d', P') \stackrel{\text{stab}}{\sim} M^\circ_{\mathbb{P}^m}(P',d'),$$

using the natural projections

$$\pi : \operatorname{Isom}_{\mathbb{P}^m}(d, P, d', P') \longrightarrow M^\circ_{\mathbb{P}^m}(P,d)$$
$$\text{and} \quad \pi' : \operatorname{Isom}_{\mathbb{P}^m}(d, P, d', P') \longrightarrow M^\circ_{\mathbb{P}^m}(P',d').$$

It is sufficient to show that their generic fibers are rational. The roles of d, d' are symmetrical, thus it is enough to consider $\pi : \mathrm{Isom}_{\mathbb{P}^m}(d, P, d', P') \to M^{\circ}_{\mathbb{P}^m}(P, d)$.

Note that the fiber of π over $(\phi : Q \to P)$ consists of pairs $\{(\phi' : Q' \to P'); \tau\}$ where $(\phi' : Q' \to P') \in M^{\circ}_{\mathbb{P}^m}(P', d')$ and $\tau : Q \to Q'$ is an isomorphism. Specifying such a pair is the same as giving $(\phi \circ \tau : Q \to P') \in \mathrm{Mor}_{d'}(Q, P')$. We have thus proved that the fiber of π over $(\phi : Q \to P)$ is isomorphic to $\mathrm{Mor}_{d'}(Q, P')$.

We assumed that $d \equiv d' \mod (\dim Q + 1)$, thus

$$d \equiv d' \mod \mathrm{per}(Q). \tag{4.1}$$

Therefore $\mathrm{Mor}_{d'}(Q, P') \sim \mathrm{Mor}_d(Q, P)$ by Corollary 4.3.

Let now K be the function field of $M^{\circ}_{\mathbb{P}^m}(P, d)$ and $\phi_K : Q_K \to P_K$ the corresponding map. Then ϕ_K gives a K-point of $\mathrm{Mor}_d(Q, P)$. Thus $\mathrm{Mor}_{d'}(Q, P')$ also has a K-point hence it is rational by (4.1).

This shows that

$$M^{\circ}_{\mathbb{P}^m}(P, d) \overset{\mathrm{stab}}{\sim} \mathrm{Isom}_{\mathbb{P}^m}(d, P, d', P')$$

$$\text{and } \mathrm{Isom}_{\mathbb{P}^m}(d, P, d', P') \overset{\mathrm{stab}}{\sim} M^{\circ}_{\mathbb{P}^m}(P', d')$$

follows by interchanging P and P'. □

Remark 4.6. — Note that the assumption $d \equiv d' \mod (m + 1)$ is used only through its consequence $d \equiv d' \mod \mathrm{per}(Q)$ in (4.1).

There are a few more cases when one can guarantee (4.1). For example, assume that d, d' and $\mathrm{per}(P)$ are all relatively prime to $m + 1$. If there is a map $Q \to P$ of degree d then $\mathrm{per}(Q)$ divides $d \cdot \mathrm{per}(P)$. Since $\mathrm{per}(Q)$ also divides $\dim Q + 1$, Q is in fact trivial. Using this observation for $d' = 1$ we obtain that

$$M^{\circ}_{\mathbb{P}^m}(P, d) \overset{\mathrm{stab}}{\sim} M^{\circ}_{\mathbb{P}^m}(P', 1) \cong \mathrm{Grass}(\mathbb{P}^m, P') \quad \text{if } (m + 1, d \cdot \mathrm{per}(P)) = 1.$$

As a consequence of Theorem 4.5, in order to describe the stable birational types of $M^{\circ}_{\mathbb{P}^m}(P, d)$, it is sufficient to understand $M^{\circ}_{\mathbb{P}^m}(P, d)$ for $d \leqslant m + 1$. There are two cases for which the answer is easy to derive.

Lemma 4.7. — *Let P be a Severi–Brauer variety. Then*

(1) $M^{\circ}_{\mathbb{P}^m}(P, d) \overset{\mathrm{stab}}{\sim} \mathrm{Grass}(\mathbb{P}^m, P)$ *if* $d \equiv 1 \mod (m + 1)$.
(2) $M^{\circ}_{\mathbb{P}^m}(P, d) \overset{\mathrm{stab}}{\sim} P$ *if* $d \equiv 0 \mod (m + 1)$ *and* $(m + 1) \mid 420$.

Proof. — If $d \equiv 1 \mod (m + 1)$ then $M^{\circ}_{\mathbb{P}^m}(P, d) \overset{\mathrm{stab}}{\sim} M^{\circ}_{\mathbb{P}^m}(P, 1)$ by Theorem 4.5 and, essentially by definition, $M^{\circ}_{\mathbb{P}^m}(P, 1) = \mathrm{Grass}(\mathbb{P}^m, P)$.

For the second claim we check the stable birational isomorphisms

$$M^{\circ}_{\mathbb{P}^m}(P, m+1) \overset{\text{stab}}{\sim} M^{\circ}_{\mathbb{P}^m}(P, m+1) \times P \overset{\text{stab}}{\sim} P.$$

First let K be the function field of $M^{\circ}_{\mathbb{P}^m}(P, m+1)$. We need to show that P_K is trivial. By assumption, there is a K-map $\phi_K : Q_K \to P_K$ for some Severi–Brauer variety Q_K of dimension m. By 4.2 (4) this corresponds to a K-point of $(Q_K^{\vee})^{\otimes m+1} \otimes P_K$. By 4.2 (1) $Q_K^{\otimes m+1}$ is trivial and so is P_K.

For the second part, let L be the function field of P. Then P_L is trivial, hence

$$\begin{aligned} &M^{\circ}_{\mathbb{P}^m}(P_L, m+1) \\ &\qquad \cong M^{\circ}_{\mathbb{P}^m}(\mathbb{P}^n_L, m+1) \overset{\text{bir}}{\sim} \mathbb{P}(H^0(\mathbb{P}^m_L, \mathcal{O}_{\mathbb{P}^m}(m+1))^{n+1}) / \operatorname{PGL}_{m+1}. \end{aligned}$$

It is conjectured that this quotient if always stably rational, but this seems to be known only when $(m+1) \mid 420$; see [5, p. 316] and the references there. $\square$

Next we give a geometric proof of Lemma 4.7.2 for $m = 1$. That is, we show that $\bar{M}_0(P, 2) \overset{\text{bir}}{\sim} P$ for $\dim P \geqslant 2$.

Results 4.8 (Conics in Severi–Brauer varieties). — *We compute, in two different ways, the space T parametrizing triples (C, ℓ_1, ℓ_2) where $C \subset P$ is a conic and the ℓ_i are secant lines of C.*

Forgetting the lines gives a map to $\bar{M}_0(P, 2)$. Let C_K be the conic corresponding to the generic point of $\bar{M}_0(P, 2)$. The linear span of C_K is a 2-dimensional Severi–Brauer variety that contains a conic, hence isomorphic to $\mathbb{P}^2_K$. A secant line of C_K is determined by a point in the dual $(\mathbb{P}^2_K)^{\vee} \cong \mathbb{P}^2_K$. Thus $T \overset{\text{bir}}{\sim} \bar{M}_0(P, 2) \times \mathbb{P}^4$.

Generically the secants lines ℓ_i meet at a unique point; this gives a map $T \dashrightarrow P$. Given $p \in P$, the fiber is obtained by first picking 2 points in $\mathbb{P}(T_p P) \cong \mathbb{P}^{n-1}_{k(p)}$. Once we have 2 lines, they determine a plane $\langle \ell_1, \ell_2 \rangle$ and the 5-dimensional linear system $|\ell_1 + \ell_2|$ on the plane $\langle \ell_1, \ell_2 \rangle$ gives the conics. Thus $T \overset{\text{bir}}{\sim} P \times \mathbb{P}^{2\dim P + 3}$ and hence $\bar{M}_0(P, 2) \overset{\text{stab}}{\sim} P$.

Proof of Theorem 1.1 (7)–(8). — If $d = 2e$ is even then

$$\bar{M}_0(P, 2e) \overset{\text{bir}}{\sim} M_{\mathbb{P}^1}(P, 2e) \overset{\text{stab}}{\sim} M_{\mathbb{P}^1}(P, 2) \overset{\text{bir}}{\sim} \bar{M}_0(P, 2),$$

where the birational equivalences are by definition and the stable birational equivalence holds by Theorem 4.5. Next $\bar{M}_0(P, 2) \overset{\text{stab}}{\sim} P$ follows either from Lemma 4.7 (2) or from Paragraph 4.8. This gives 1.1 (7).

Similarly, if $d = 2e + 1$ is odd then 1.1 (8) follows from

$$\bar{M}_0(P, 2e+1) \stackrel{\text{bir}}{\sim} M_{\mathbb{P}^1}(P, 2e+1) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^1}(P, 1) = \mathrm{Grass}(\mathbb{P}^1, P). \qquad \square$$

Remark 4.9. — So far we have worked with a fixed Severi–Brauer variety P, but it would be interesting to understand how the $\mathrm{MSym}(P)$ for different Severi–Brauer varieties interact with each other.

For example, assume that P, Q are Severi–Brauer varieties such that $\mathrm{index}(P)$ and $\mathrm{index}(Q)$ are relatively prime. We claim that $\mathrm{Sym}^d(P)$ and $\mathrm{Sym}^e(Q)$ are stably birational to each other iff they are both stably rational.

To see this, assume that $\mathrm{Sym}^e(Q)$ is not stably rational. By Corollary 3.2 this holds iff $\mathrm{index}(Q) \nmid e$. Set $K := k(P)$. Since $\mathrm{index}(P)$ and $\mathrm{index}(Q)$ are relatively prime, $\mathrm{index}(Q_K) = \mathrm{index}(Q)$, thus $\mathrm{Sym}^e(Q_K)$ is not stably rational by Corollary 3.2. By contrast, P_K is trivial hence $\mathrm{Sym}^d(P_K)$ is stably rational; even rational by [20]. Thus $\mathrm{Sym}^d(P)$ and $\mathrm{Sym}^e(Q)$ are not stably birational to each other.

See also [14, 10] for related questions.

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