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Metrics and convergence in moduli spaces of maps (*)

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Abstract. — We provide a general framework to study convergence properties of families of maps between manifolds which have distinct domains. For manifolds M and N where M is equipped with a volume form we consider families of maps in the collection $\{(\phi, B_\phi) \mid B_\phi \subset M, \phi\colon B_\phi \to N$ with B_ϕ, ϕ both measurable$\}$ and we define a distance function similar to a generalized L^1 distance on such a collection. We show that the resulting metric space is always complete. We then shift our focus to exploring the convergence properties of families of such maps.

Résumé. — Nous présentons un cadre général pour l'étude de la convergence des familles d'applications entre variétés dont les domaines de définition sont distincts. Étant données deux variétés M et N, M étant munie d'une forme volume, nous considérons des familles d'applications dans l'ensemble $\{(\phi, B_\phi) \mid B_\phi \subset M, \phi\colon B_\phi \to N$ avec B_ϕ, ϕ mesurable$\}$ et nous définissons une distance sur cet ensemble, de type distance L^1 généralisée. Nous démontrons que l'espace métrique ainsi obtenu est toujours complet. Nous nous concentrons ensuite sur l'étude des propriétés de convergence de telles familles d'applications.

1. Introduction

In [11] the authors show that if M and N are symplectic manifolds with $B_t \subset M$ for each $t \in (a,b)$ and

$$\{(\phi_t, B_t) \mid t \in (a,b) \text{ and } \phi_t\colon B_t \hookrightarrow N\}$$

is a smooth (see Definition 7.1) family of symplectic embeddings such that

(1) each B_t is open and simply connected;
(2) if $s < t$ then $\overline{B_t} \subset B_s$;
(3) for all $t, s \in (a,b)$ the set $\bigcup_{v\in[t,s]}\phi_v(B_v)$ is relatively compact in N,

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Article proposé par Gilles Carron.

then there exists a symplectic embedding

$$\phi_0\colon \bigcup_{t\in(a,b)} B_t \hookrightarrow N.$$

Given a family of embeddings with distinct domains which satisfy certain conditions not related to convergence, this result assures the existence of an embedding from the union of their domains, which takes the place of the limit. This motivates the study of families of maps with distinct domains in more general settings, and in particular the study of convergence properties of such families. In the present article we introduce a distance function on maps which do not necessarily have the same domain. The new distance function is related to the L^1 norm, and we establish results related to completeness and convergence which are analogous to those that hold in the case of the L^1 distance, though we also see that there are some important differences. See Remark 3.2 for a comparison to the L^1 distance.

With a metric defined on the space of such maps, we can ask the following question: given a collection of embeddings with distinct domains which does not converge, how much does each embedding need to be perturbed (with respect to the new distance) in order to produce a convergent collection? In particular we study when a family of embeddings which does not converge can be perturbed by an arbitrarily small amount to produce a convergent family. Note that in this case, unlike in the result of Pelayo–Vũ Ngọc above, we are more interested in the nature of the family of embeddings than the existence of such a limiting embedding.

The study of collections of maps between smooth manifolds, particularly of embeddings or diffeomorphisms, has recently attracted a lot of interest [1, 2, 13, 11, 12]. Defining a metric space structure on collections of such mappings allows one to study the topological properties of the collection and also to study deformations of the mappings, as is done in [14]. It is the goal of this paper to define a new distance function on collections of maps with distinct domains, which are subsets of the same manifold, and study the properties of the resultant metric space.

1.1. Outline of paper

In Section 2 we define the space of maps over which we will be working and the distance function. We state the main results of this paper in Section 3. In Section 4 we prove several properties of the distance function including some parts of Theorem A. In Section 5 we examine the convergence properties of the distance and prove part of Theorem B, and in Section 6 we prove the rest of Theorem A and Theorem B. Sections 7 and 8 are somewhat distinct from the rest of the paper; in these sections we pursue an application of the framework developed in the previous sections. In Section 7 we study families of embeddings which do not converge to an embedding and prove Theorem C. In our last section, Section 8, we comment on how the ideas from this paper can be used to further study such families and mention some other possibilities for applications of this distance.

2. The distance function

Considering families of maps with different domains is essential for applications, see for instance the work of Pelayo–Vũ Ngọc [11, 12]. Suppose that the maps are defined on subsets of a smooth manifold M with a volume form $\mathcal{V}$ and map to a complete Riemannian manifold N with natural distance d. By this we mean that if g is the Riemannian metric on N and $y_1, y_2 \in N$ then

$$d(y_1,y_2) = \inf\left\{ \int_0^1 \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t \,\middle|\, \begin{array}{l}\gamma : [0,1] \to N \text{ is piecewise } C^1 \text{ with} \\ \gamma(0) = y_1 \text{ and } \gamma(1) = y_2\end{array} \right\}.$$

We will soon see that the properties of the distance will not depend on the choice of metric g and it is known that any smooth manifold admits a complete Riemannian metric, so we are not making any assumptions on N. Throughout the paper by *metric* we will always mean a metric function on the space and if referring to a metric tensor we will always specify the Riemannian metric. Also, it is well known (see the Hopf–Rinow Theorem [6, Satz I]) that (N, g) is a geodesically complete Riemannian manifold if and only if (N, d) is a complete metric space, so throughout this paper we will call such a manifold complete without specifying. Let $\mu_{\mathcal{V}}$ be the measure on M induced by $\mathcal{V}$. That is, for any $A \subset M$ we have $\mu_{\mathcal{V}}(A) = \int_A \mathcal{V}$. Now we will define the set of maps we will be working with (shown in Figure 2.1).

Definition 2.1. — *Let*

$$\mathcal{M}(M,N) := \left\{ (\phi, B_\phi) \,\middle|\, \begin{array}{l} B_\phi \subset M \textit{ a nonempty measurable set and} \\ \phi : B_\phi \to N \textit{ a measurable function}\end{array} \right\}$$

which we denote by $\mathcal{M}$ when M and N are understood. We also occasionally write only ϕ where the associated domain is understood to be denoted by B_ϕ. Also let

$$\mathcal{F}(\mathcal{M}) = \left\{ \{(\phi_t, B_t)\}_{t\in(a,b)} \subset \mathcal{M} \,\middle|\, a, b \in \mathbb{R} \textit{ with } a < b \right\}.$$

For the remainder of the paper we will denote by $\mathcal{F}(\mathcal{S})$ the collection of one parameter families in a set $\mathcal{S}$ indexed by an open interval in $\mathbb{R}$.

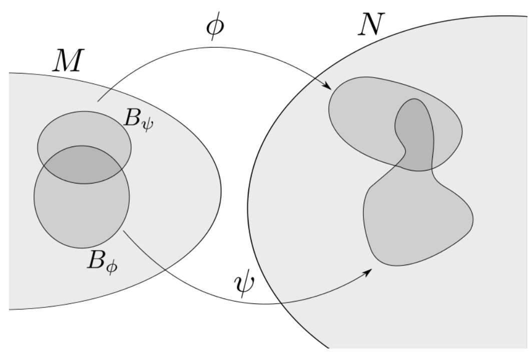

Figure 2.1. We will be considering maps from subsets of M to N.

Recall the symmetric difference of sets A and B is given by $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Definition 2.2. — *For* $(\phi, B_\phi), (\psi, B_\psi) \in \mathcal{M}$ *we define the* penalty function $p^d_{\phi\psi} : M \to [0,1]$ *by*

$$p^d_{\phi\psi}(x) = \begin{cases} 1 & \text{if } x \in B_\phi \triangle B_\psi; \\ \min\{1, d(\phi(x), \psi(x))\} & \text{if } x \in B_\phi \cap B_\psi; \\ 0 & \text{otherwise,} \end{cases} .$$

and we define

$$\mathcal{D}^d_M((\phi, B_\phi), (\psi, B_\psi)) = \int_M p^d_{\phi\psi}\, \mathrm{d}\mu_{\mathcal{V}}.$$

A reasonable first guess for the "distance" between two elements in $\mathcal{M}$ would be to integrate a penalty function over M. That is, we start with a function which assigns a penalty at each point in M depending on how different the mappings are at that point, and then compute the "distance" between the two mappings by adding up all of these penalties via integration. For each point in the symmetric difference, we know that one mapping acts on it while the other does not, so we assign it a maximum penalty of 1. For each point which is in the intersection of the domains, we simply find the distance between where each map sends the point, cut off to not exceed a maximum value of 1, and use this as the penalty.

Notice that we need the minimum in Definition 2.2 to make sure that any point on which both mappings act is not penalized more than the points which are only acted on by one mapping. It is worth noting that even though the choice of the constant 1 may seem arbitrary it is shown in Proposition 4.3 that any positive constant may be used instead and the induced distance will be strongly equivalent (see Definition 4.1). Also, if the Riemannian metric g is chosen so that the associated metric space (N, d) is complete (which can always be done [10, Theorem 1]) the choice of g will not change the properties of the induced metric.

However while $\mathcal{D}_M^d$ is the natural "distance" it turns out to not be a distance function on $\mathcal{M}$. There are two main problems. First, it is possible that $\mathcal{D}_M^d$ will evaluate to zero on two distinct elements of $\mathcal{M}$ and second it might be that $\mathcal{D}_M^d$ evaluates to infinity. The first problem is a common one and can be addressed in the standard way, by having $\mathcal{D}_M^d$ act on equivalence classes of maps, but the second problem will require a more delicate solution.

The problem of $\mathcal{D}_M^d$ evaluating to infinity is even worse than it seems. Suppose that $\phi_t(x) = (x, t)$ takes $\mathbb{R}$ into $\mathbb{R}^2$ for all $t \in (0, 1)$. Using the notation from above in this case we have that $M = B_{\phi_t} = \mathbb{R}$ for all $t \in (0, 1)$ and $N = \mathbb{R}^2$ with $d_{\mathbb{R}^2}$ the usual distance. Then ϕ_t has a pointwise limit of $\phi_0(x) := (x, 0)$ as $t \to 0$, but despite this we have that $\mathcal{D}_M^{d_{\mathbb{R}^2}}(\phi_t, \phi_0)$ is infinite for all $t \in (0, 1)$. This example shows that $\mathcal{D}_M^d$ is not always able to capture when a family of maps is converging. We are able to solve this problem by observing $\mathcal{D}_M^d$ restricted to various subsets of M.

Definition 2.3. — *We define $\mathcal{D}$ restricted to a measurable set $\mathcal{S} \subset M$ by*

$$\mathcal{D}_{\mathcal{S}}^d((\phi, B_\phi), (\psi, B_\psi)) = \int_S p_{\phi\psi}^d \, \mathrm{d}\mu_{\mathcal{V}}.$$

Figure 2.2 shows a good way to visualize computing $\mathcal{D}_{\mathcal{S}}^d$. Now each $\mathcal{D}_{\mathcal{S}}^d$ contains all of the information about $\mathcal{D}_M^d$ on the set $\mathcal{S}$ and, as long as $\mathcal{S}$ is chosen to be of finite volume, $\mathcal{D}_{\mathcal{S}}^d$ cannot evaluate to infinity. The problem now, of course, is that we no longer have just a single metric with information about all of M but instead have an infinite family of metrics which each have information about only one finite volume subset of M. We solve this last problem by recalling that any manifold admits a nested exhaustion by compact sets, which must each have finite volume. For the remaining portion of this paper by exhaustion we will always mean a countable nested exhaustion by finite volume sets. In the following definition we set up the framework for this paper. We write $\nu_{\{\mathcal{S}_n\}}$ in place of $\nu_{\{\mathcal{S}_n\}_{n=1}^\infty}$ and $\mathcal{D}_{\{\mathcal{S}_n\}}^d$ in place of $\mathcal{D}_{\{\mathcal{S}_n\}_{n=1}^\infty}^d$ for simplicity.

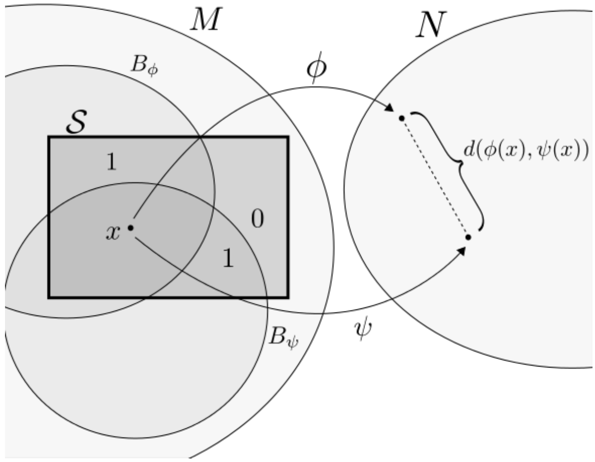

Figure 2.2. A graphic representing the values of $p^d_{\phi\psi}$ on $\mathcal{S} \subset M$.

Definition 2.4. — *Let M and N be manifolds with d a metric on N induced by a Riemannian metric.*

(1) *Let $\{\mathcal{S}_n\}_{n=1}^\infty$ be a exhaustion of M by nested finite volume sets and let $\nu_{\{\mathcal{S}_n\}}$ be the measure on M given by*

$$\nu_{\{\mathcal{S}_n\}}(A) = \sum_{n=1}^{\infty} 2^{-n} \frac{\mu_{\mathcal{V}}(A \cap S_n)}{\mu_{\mathcal{V}}(S_n)}$$

for $A \subset M$. Notice that $\nu_{\{\mathcal{S}_n\}}(M) = 1$ so $\nu_{\{\mathcal{S}_n\}}$ is a probability measure. Then define

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi,\psi) = \int_M p^d_{\phi\psi}\, \mathrm{d}\nu_{\{\mathcal{S}_n\}}.$$

(2) *If $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi,\psi) = 0$ for one choice of exhaustion then, by Corollary 4.7, it equals zero for all choices of exhaustion and complete metrics d, so in that case we write $\mathcal{D}(\phi,\psi) = 0$.*

(3) *Let*

$$\mathcal{M}^\sim(M,N) := \mathcal{M}(M,N)/\sim$$

where $(\phi, B_\phi) \sim (\psi, B_\psi)$ if and only if $\mathcal{D}(\phi,\psi) = 0$. As before we will frequently shorten this to $\mathcal{M}^\sim$ and we denote by $[\phi, B_\phi]$ the equivalence class of $(\phi, B_\phi) \in \mathcal{M}$.

There is an equivalent definition of $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ given in Proposition 4.4 which is used in some of the proofs in this paper and explicitly shows the relation between $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ and $\mathcal{D}^d_{\mathcal{S}}$.

3. Main results

3.1. Foundational results

Now we have enough notation to state our first result. Let M and N be manifolds and $\mathcal{V}$ a volume form on M.

Theorem A. — *For any choice of a metric d on N induced by a complete Riemannian metric and a countable exhaustion $\{\mathcal{S}_n\}_{n=1}^{\infty}$ of M by nested finite volume sets, the space $(\mathcal{M}^{\sim}, \mathcal{D}^d_{\{\mathcal{S}_n\}})$ is a complete metric space. Moreover, such a metric and exhaustion always exist and if d' and $\{\mathcal{S}'_n\}_{n=1}^{\infty}$ are other such choices then $\mathcal{D}^{d'}_{\{\mathcal{S}'_n\}}$ induces the same topology as $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ on $\mathcal{M}^{\sim}$.*

Theorem A follows from Proposition 4.8, Lemma 4.11, Lemma 6.3, and the fact that every manifold admits a complete Riemannian metric [10, Theorem 1]. In light of Theorem A we can now make the following definitions. Recall that $\mathcal{F}(\mathcal{M})$ denotes the collection of one-parameter families of $\mathcal{M}$ indexed by an interval $(a,b) \subset \mathbb{R}$.

Definition 3.1. — *Let $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a,b]$. Also let $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ and $\psi \in \mathcal{M}$.*

(1) *Let $\mathcal{S} \subset M$ be any subset. If $\lim_{t\to c} \mathcal{D}^d_{\mathcal{S}}(\phi_t, \psi) = 0$ we write*
$$\phi_t \xrightarrow{\mathcal{D}^d_{\mathcal{S}}} \psi \text{ as } t \to c.$$

(2) *If $\lim_{t\to c} \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \psi) = 0$ for one, and hence all, choices of $\{\mathcal{S}_n\}_{n=1}^{\infty}$ and d, we write*
$$\phi_t \xrightarrow{\mathcal{D}} \psi \text{ as } t \to c.$$

(3) *Since all metrics $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ generate the same topology on the set $\mathcal{M}^{\sim}$ we denote this set with such topology as $(\mathcal{M}^{\sim}, \mathcal{D})$.*

Thus $\mathcal{M}^{\sim}$ is a metric space with metric $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ for any choice of exhaustion and complete metric and the metric spaces for different choices of exhaustion are all equivalent topologically. Notice that all of the information about $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is contained in $\mathcal{D}^d_M$ if M is finite volume, and in this case we will only have to consider $\mathcal{D}^d_M$, see Remark 4.12.

Remark 3.2. — Recall that L^p spaces are collections of maps from a fixed measure set to $\mathbb{R}$. Since $\mathcal{M}$ is a collection of all maps between fixed manifolds we can see that in some sense $\mathcal{M}$ is a generalization of L^p spaces. The function $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is similar to the L^1 norm, but there are several differences. It is noteworthy that *any* measurable mapping from M to N is "integrable" with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$, in the sense that the distance between any two measurable mappings is finite. This is why $\mathcal{M}$ includes all measurable maps, while L^p includes only functions which satisfy a growth restriction. In Example 4.13 we work out a specific case which does not converge in L^p for any p but does converge with respect to the distance defined in this paper.

Now that there is a metric defined on $\mathcal{M}^\sim$ we can explore families in $\mathcal{F}(\mathcal{M}^\sim)$ which converge with respect to that metric. In Section 5 we study another type of convergence and we explore the connection between these two natural forms of convergence on $\mathcal{M}^\sim$. The limit inferior and limit superior of a family of sets are reviewed in Equations (5.1) and (5.2) in Section 5.

Definition 3.3. — *Let $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$. Let $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ and suppose there exists some measurable $B \subset M$ satisfying*

$$B \subset \left\{ x \in \varliminf_{t\to c} B_t \;\middle|\; \lim_{t\to c} \phi_t(x) \text{ exists} \right\}$$

and $\mu_{\mathcal{V}}\left(\varlimsup_{t\to c} B_t \setminus B\right) = 0$. This in particular requires that the domains converge as sets as is described in Definition 5.1. Then, with

$$\begin{aligned} \phi : B &\to N \\ x &\mapsto \lim_{t\to c} \phi_t(x). \end{aligned}$$

we say that $\{(\phi_t, B_t)\}_{t\in(a,b)}$ converges to (ϕ, B) almost everywhere pointwise as $t \to c$ *in $\mathcal{M}$ and we write $\phi_t \xrightarrow{a.e.} \phi$ as $t \to c$.*

Theorem B. — *Let $a, b, c \in \mathbb{R}$ such that $a < b$ and $c \in [a, b]$. Suppose $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is a family such that $(\phi_t, B_t) \in \mathcal{M}$ for $t \in (a, b)$ and let $(\phi, B) \in \mathcal{M}$. If $\phi_t \xrightarrow{a.e.} \phi$ as $t \to c$ then $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to c$. A partial converse also holds: if $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to c$ then there exists a sequence $\{t_i\}_{i\in\mathbb{N}} \subset (a, b)$ such that $\{(\phi_{t_i}, B_{t_i})\}_{i\in\mathbb{Z}_{>0}}$ converges to (ϕ, B) almost everywhere as $i \to \infty$.*

It is not true that convergence with respect to $\mathcal{D}$ is equivalent to almost everywhere convergence, see Example 4.13 and Remark 5.6. Theorem B is a combination of Lemma 5.5 and Corollary 6.2.

3.2. An application

The preceding results have laid out a framework to study the families of maps we are interested in, and there are many different directions one could head from this point. Since there is research already being done regarding the convergence properties of families of embeddings [11, 12] we will explore that field. As an application of Theorems A and B we will use $\mathcal{D}$ to study families of embeddings which do not converge to an embedding, such as the maps mentioned in the motivating theorem in Section 1, and quantify how far they are from converging. With this in mind we make the following definitions.

Definition 3.4. — *Define* $\mathrm{Emb}_{\subset}(M,N) \subset \mathcal{M}$ *to be those elements* $(\phi, B) \in \mathcal{M}$ *such that* $B \subset M$ *is a submanifold and* $\phi\colon B \hookrightarrow N$ *is an embedding. Also define* $\mathrm{Emb}^{\sim}_{\subset}(M,N) \subset \mathcal{M}^{\sim}$ *to be those equivalence classes* $[\phi, B] \in \mathcal{M}^{\sim}$ *such that* $[\phi, B]$ *contains an element of* $\mathrm{Emb}_{\subset}(M,N)$*. In Definition 7.1 we define the notion of a smooth family of maps. We say that a smooth family* $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathrm{Emb}_{\subset}(M,N))$ *has a* singular limit *if either*

(1) *the family does not converge in* $\mathcal{D}$ *as* $t \to a$*;*
(2) *there exists some* $\phi_0 \in \mathcal{M}$ *such that* $\phi_t \xrightarrow{\mathcal{D}} \phi_0$ *as* $t \to a$*, but* $[\phi_0] \notin \mathrm{Emb}^{\sim}_{\subset}(M,N)$*.*

If a smooth family has a singular limit, we are interested in perturbing that family to produce a family that converges.

Definition 3.5. — *Let* $a, b \in \mathbb{R}$ *with* $a < b$*,* $\varepsilon \geqslant 0$*, and* $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$*. We say that a smooth family* $\{(\widetilde{\phi_t}, \widetilde{B_t})\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ *is a* convergent ε-perturbation (with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$) *of* $\{(\phi_t, B_t)\}_{t\in(a,b)}$ *if*

(1) *there exists* $(\widetilde{\phi}, \widetilde{B}) \in \mathrm{Emb}_{\subset}(M,N)$ *such that* $\widetilde{\phi_t} \xrightarrow{a.e.} \widetilde{\phi}$ *as* $t \to a$*;*
(2) $B_t = \widetilde{B_t}$ *for all* $t \in (a,b)$ *and* $\underline{\lim}_{t\to a} \widetilde{B_t} \subset \widetilde{B}$*;*
(3) *for all* $t \in (a,b)$ *we have that* $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \widetilde{\phi_t}) \leqslant \varepsilon$*.*

The function

$$r^d_{\{\mathcal{S}_n\}} : \mathcal{F}(\mathcal{M}) \to [0, \infty]$$

takes a family in $\mathcal{F}(\mathcal{M})$ *to its* radius of convergence *given by*

$$r^d_{\{\mathcal{S}_n\}}\big(\{(\phi_t, B_t)\}_{t\in(a,b)}\big) := \inf\left\{ \varepsilon \geqslant 0 \;\middle|\; \begin{array}{l} \text{there exists a smooth convergent} \\ \varepsilon\text{-perturbation of } \{(\phi_t, B_t)\}_{t\in(a,b)} \end{array} \right\}.$$

In part 2 of Definition 3.5 we make a requirement on the domains. This is so that the singular points cannot simply be removed from the domain to form a convergent ε-perturbation. It is important to notice that, unlike many of the properties we have introduced so far, $r^d_{\{\mathcal{S}_n\}}$ does depend on the choice of d and $\{\mathcal{S}_n\}_{n=1}^\infty$. We are most interested in the $r^d_{\{\mathcal{S}_n\}} = 0$ case, where an arbitrarily small perturbation can cause the family to converge to an embedding. It is unknown if having zero radius of convergence is independent of the choice of parameters d and $\{\mathcal{S}_n\}_{n=1}^\infty$ (see Section 8.2).

It is natural to wonder whether a family can have radius of convergence zero but still not converge to any element of $\mathcal{M}$. The following Theorem addresses this.

Theorem C. — *Let $a, b \in \mathbb{R}$ with $a < b$, $\{(\phi_t, B_t)\}_{t\in(a,b)}$ be such that $(\phi_t, B_t) \in \mathcal{M}$ for each $t \in (a, b)$, and let $r^d_{\{\mathcal{S}_n\}}$ be the radius of convergence function associated to a complete Riemannian distance d on N and an exhaustion of finite volume nested sets $\{\mathcal{S}_n\}_{n=1}^\infty$ of M. Then the following hold:*

1. *if $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$ then there exists $(\phi, B) \in \mathcal{M}$ unique up to $\sim$ such that $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to a$;*
2. *Suppose $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is a smooth family of embeddings and there exists $T \in (a, b)$ such that $s < t < T$ implies $B_s \subset B_t$. In this case, if there exists $(\phi, B) \in \mathcal{M}$ such that $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to a$, then $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$*

In fact, a more general version of part (2) is true, which is stated in Lemma 7.3. Part (2) of Theorem C is a partial converse to part (1), under some additional conditions. This theorem is important in the study of families with $r^d_{\{\mathcal{S}_n\}} = 0$ because to characterize such families we may assume right away that there exists some limit ϕ_0 and study its properties in order to understand the family we started with. In the final section we explore some ideas about the open questions regarding $r^d_{\{\mathcal{S}_n\}}$ including restricting to embeddings with specific properties and considering a converse of Theorem C without the additional conditions.

4. Definitions and preliminaries

4.1. Basic properties of the distance

Let M be an orientable smooth manifold with volume form $\mathcal{V}$ and let N be a smooth Riemannian manifold with natural distance function d. Again let $\mu_{\mathcal{V}}$ be the measure on M induced by the volume form $\mathcal{V}$. In this section we will prove all but the completeness statement in Theorem A, which is postponed to Section 6. Recall the different notions of equivalent metrics. The use of these terms varies, but for this paper we will use the following conventions.

Definition 4.1. — *Let d_1 and d_2 be metrics on a set X. Then we say that d_1 and d_2 are:*

(1) topologically equivalent *if they induce the same topology on X;*
(2) weakly equivalent *if they induce the same topology on X and precisely the same collection of Cauchy sequences;*
(3) strongly equivalent *if there exist $c_1, c_2 > 0$ such that*

$$c_1 d_1 \leqslant d_2 \leqslant c_2 d_1.$$

Now we define the following function.

Definition 4.2. — *Let $(\phi, B_\phi), (\psi, B_\psi) \in \mathcal{M}$. For $\alpha > 0$ and a finite volume subset $\mathcal{S} \subset M$ define*

$$\mathcal{D}_{\mathcal{S}}^{d,\alpha}((\phi, B_\phi), (\psi, B_\psi)) = \int_{\mathcal{S}} p_{\phi\psi}^{d,\alpha} \, \mathrm{d}\mu_{\mathcal{V}}$$

where

$$p_{\phi\psi}^{d,\alpha}(x) = \begin{cases} \alpha & \text{if } x \in B_\phi \triangle B_\psi; \\ \min\{\alpha, d(\phi(x), \psi(x))\} & \text{if } x \in B_\phi \cap B_\psi; \\ 0 & \text{otherwise.} \end{cases}$$

In Definition 4.2 we have a family of functions depending on the choice of $\alpha > 0$, but in fact these will induce strongly equivalent metrics.

Proposition 4.3. — *Let $\mathcal{S}$ be a finite volume subset of M. If $\beta > \alpha > 0$ then*

$$\mathcal{D}_{\mathcal{S}}^{d,\alpha} \leqslant \mathcal{D}_{\mathcal{S}}^{d,\beta} \leqslant \frac{\beta}{\alpha} \mathcal{D}_{\mathcal{S}}^{d,\alpha}.$$

Proof. — This follows from the fact that if $0 < \alpha < \beta$ and $x \in \mathcal{S}$ we have

$$p_{\phi\psi}^{d,\alpha}(x) \leqslant p_{\phi\psi}^{d,\beta}(x) \leqslant \frac{\beta}{\alpha} p_{\phi\psi}^{d,\alpha}(x). \qquad \square$$

So Proposition 4.3 means that the choice of $\alpha > 0$ will not matter when we use $\mathcal{D}_{\mathcal{S}}^{d,\alpha}$ to define a metric, so henceforth we will assume that $\alpha = 1$. That is, for any finite volume subset $\mathcal{S} \subset M$ we have $\mathcal{D}_{\mathcal{S}}^{d}$ as defined in Definition 2.3. In the above proof we wrote out the definition of $\mathcal{D}_{\mathcal{S}}^{d}$ in a way which did not explicitly use the penalty function $p_{\phi\psi}^{d}$. We can now notice that there is an equivalent definition of $\mathcal{D}_{\mathcal{S}}^{d}$ which will be useful for several of the proofs.

Proposition 4.4. — *Let M and N be manifolds with a volume form $\mathcal{V}$ on M, d a distance on N induced by a Riemannian metric, $\mathcal{S} \subset M$ a compact subset, and $\{\mathcal{S}_n\}_{n=1}^{\infty}$ a nested exhaustion of M by finite volume sets. The function $\mathcal{D}^d_{\mathcal{S}}$ given in Definition 2.3 can be written*

$$\mathcal{D}^d_{\mathcal{S}}(\phi,\psi) = \int_{B_\phi \cap B_\psi \cap \mathcal{S}} \min\{1, d(\phi,\psi)\}\, \mathrm{d}\mu_{\mathcal{V}} + \mu_{\mathcal{V}}\big((B_\phi \vartriangle B_\psi)\cap \mathcal{S}\big).$$

and the function $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ from Definition 2.4 satisfies

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi,\psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{\mathcal{D}^d_{\mathcal{S}_n}(\phi,\psi)}{\mu_{\mathcal{V}}\left(\mathcal{S}_n\right)}.$$

This proposition has a trivial proof. Before the next Proposition we have a definition.

Definition 4.5. — *Suppose $a, b \in \mathbb{R}$ with $a < b$ and $c \in [a,b]$. For a set X and a function*

$$F : X \times X \to [0,\infty]$$

we say that a family $\{a_t\}_{t\in(a,b)} \subset X$ is Cauchy with respect to F as $t \to c$ *if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that $s,t \in (c-\delta, c+\delta) \cap (a,b)$ implies $F(a_t, a_s) < \varepsilon$.*

Below are several important properties of $\mathcal{D}^d_{\{\mathcal{S}_n\}}$, which is defined in Definition 2.4.

Proposition 4.6. — *Let $a, b \in \mathbb{R}$ with $a < b$, $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$, and $\phi, \psi \in \mathcal{M}$. Further suppose that d is a metric on N induced by a Riemannian metric and $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is an exhaustion of M by nested finite volume sets. The function $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ has the following properties.*

1. *$\{(\phi_t, B_t)\}_{t\in(a,b)}$ is Cauchy with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ as $t \to c$ if and only if it is Cauchy with respect to $\mathcal{D}^d_{\mathcal{S}}$ as $t \to c$ for all compact $\mathcal{S} \subset M$.*
2. *$\lim_{t\to c} \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \phi) = 0$ if and only if $\phi_t \xrightarrow{\mathcal{D}^d_{\mathcal{S}}} \phi$ as $t \to c$ for all compact $\mathcal{S} \subset M$.*
3. *$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi,\psi) = 0$ if and only if $\mathcal{D}^d_{\mathcal{S}}(\phi,\psi) = 0$ for all compact $\mathcal{S} \subset M$ if and only if $\mu_{\mathcal{V}}\big((B_\phi \vartriangle B_\psi)\cap \mathcal{S}\big) = 0$ for every compact $\mathcal{S} \subset M$ and $\phi = \psi$ almost everywhere on $B_\phi \cap B_\psi$.*

Proof. — Let $\varepsilon > 0$ and fix some compact subset $\mathcal{S} \subset M$. Then $\mathcal{S} \subset \bigcup_{n=1}^{\infty} \mathcal{S}_n = M$ and since $\mathcal{S}$ has finite volume and the $\mathcal{S}_n$ are nested we can find some $I \in \mathbb{N}$ such that $\mu_{\mathcal{V}}\left(\mathcal{S} \setminus \mathcal{S}_I\right) < \varepsilon$. This means that

$$\mathcal{D}^d_{\mathcal{S}} \leqslant \mathcal{D}^d_{\mathcal{S}_I} + \varepsilon.$$

Now that we have this fact we will prove the three properties.

(1). — It is sufficient to assume that $a = c = 0$ and $b = 1$. Suppose that $\{(\phi_t, B_t)\}_{t\in(0,1)}$ is Cauchy with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ as $t \to 0$ and fix some compact $\mathcal{S} \subset M$. Let $\varepsilon > 0$.

From the above fact we can find some $I \in \mathbb{N}$ such that $\mathcal{D}^d_{\mathcal{S}} \leqslant \mathcal{D}^d_{\mathcal{S}_I} + \frac{\varepsilon}{2}$. Now, since this family is Cauchy with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ we can find some $\delta \in (0,1)$ such that $s, t < \delta$ implies

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \phi_s) < \frac{\varepsilon}{2^{I+1}\mu_{\mathcal{V}}(\mathcal{S}_I)}.$$

Using the expression for $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ from Proposition 4.4 we have that

$$\sum_{n=1}^{\infty} 2^{-n} \frac{\mathcal{D}^d_{\mathcal{S}_n}(\phi_t, \phi_s)}{\mu_{\mathcal{V}}(\mathcal{S}_n)} < \frac{\varepsilon}{2^{I+1}\mu_{\mathcal{V}}(\mathcal{S}_I)}$$

which in particular means

$$2^{-I} \frac{\mathcal{D}^d_{\mathcal{S}_I}(\phi_t, \phi_s)}{\mu_{\mathcal{V}}(\mathcal{S}_I)} < \frac{\varepsilon}{2^{I+1}\mu_{\mathcal{V}}(\mathcal{S}_I)}$$

so $\mathcal{D}^d_{\mathcal{S}_I}(\phi_t, \psi_t) < \frac{\varepsilon}{2}$.

Finally, we have that for $s, t < \delta$

$$\mathcal{D}^d_{\mathcal{S}}(\phi_t, \phi_s) \leqslant \mathcal{D}^d_{\mathcal{S}_I}(\phi_t, \phi_s) + \frac{\varepsilon}{2} < \varepsilon.$$

The converse is easy and the proof of (2) is similar to the proof of (1).

(3). — Suppose $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) = 0$ and fix some compact $\mathcal{S} \subset M$. Notice that this means that $\mathcal{D}^d_{\mathcal{S}_n}(\phi, \psi) = 0$ for all n. For any $\varepsilon > 0$ from the fact above we know we can choose some I such that

$$\mathcal{D}^d_{\mathcal{S}}(\phi, \psi) \leqslant \mathcal{D}^d_{\mathcal{S}_I}(\phi, \psi) + \varepsilon = \varepsilon$$

so we may conclude that $\mathcal{D}^d_{\mathcal{S}}(\phi, \psi) = 0$.

Next, assume that $\mathcal{D}^d_{\mathcal{S}}(\phi, \psi) = 0$ for all compact $\mathcal{S} \subset M$, which means that $\mu_{\mathcal{V}}\big((B_\phi \triangle B_\psi) \cap \mathcal{S}\big) = 0$ because this is a term in $\mathcal{D}^d_{\mathcal{S}}$. Suppose that there is some set of positive measure in $B_\phi \cap B_\psi$ for which $\phi \neq \psi$. Then since manifolds are inner regular there exists some compact subset of positive measure K on which they are not equal. But this implies that $\mathcal{D}^d_K(\phi, \psi) \neq 0$.

If $\mu_{\mathcal{V}}\big((B_\phi \triangle B_\psi) \cap \mathcal{S}\big) = 0$ for every compact $\mathcal{S} \subset M$ and $\phi = \psi$ almost everywhere on $B_\phi \cap B_\psi$ it is clear that $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) = 0$. ☐

Corollary 4.7. — *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be an exhaustion of M and let d be a metric on N induced by a Riemannian metric. Suppose that (ϕ, B_ϕ), $(\psi, B_\psi) \in \mathcal{M}$ such that $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) = 0$. Then for any such parameters $\{\mathcal{S}'_n\}_{n=1}^{\infty}$ and d' we have that $\mathcal{D}^{d'}_{\{\mathcal{S}'_n\}}(\phi, \psi) = 0$ as well.*

Given the new information in Proposition 4.6 we can prove the following important Proposition.

Proposition 4.8. — *For any choice of an exhaustion of M by finite volume sets $\{\mathcal{S}_n\}_{n=1}^\infty$ we have that $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is well defined and is a distance function on $\mathcal{M}^\sim$. Also, if $\{\mathcal{S}'_n\}_{n=1}^\infty$ is another such choice of exhaustion then $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ and $\mathcal{D}^d_{\{\mathcal{S}'_n\}}$ are weakly equivalent metrics on $\mathcal{M}^\sim$.*

Proof. — Fix some $\{\mathcal{S}_n\}_{n=1}^\infty$ a compact exhaustion of M and let $\phi, \rho, \psi \in \mathcal{M}$. It is a straightforward exercise to show that

$$p^d_{\phi\psi}(x) \leqslant p^d_{\phi\rho}(x) + p^d_{\rho\psi}(x)$$

for each $x \in M$ and thus

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) \leqslant \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \rho) + \mathcal{D}^d_{\{\mathcal{S}_n\}}(\rho, \psi).$$

It should be noted that this inequality would not hold without the minimum in $p^d_{\phi\psi}$. From here we can see that if $\phi \sim \rho$ then

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) \leqslant \mathcal{D}^d_{\{\mathcal{S}_n\}}(\rho, \psi)$$

and similarly the opposite inequality is true as well. So

$$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \psi) = \mathcal{D}^d_{\{\mathcal{S}_n\}}(\rho, \psi)$$

and thus $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is well defined on $\mathcal{M}^\sim$.

Now $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is positive definite on $\mathcal{M}^\sim$ because it is positive on $\mathcal{M}$ and by definition $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi, \rho) = 0$ implies $\phi \sim \rho$. Since $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is well defined on $\mathcal{M}^\sim$ and satisfies the triangle inequality on $\mathcal{M}$ we know that it satisfies the triangle inequality on $\mathcal{M}^\sim$ and similarly we know that $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ is symmetric on $\mathcal{M}^\sim$.

Proposition 4.6 parts (1) and (2) characterize both convergent and Cauchy sequences of $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ in a way which is independent of the choice of $\{\mathcal{S}_n\}_{n=1}^\infty$. This means that different choices of $\{\mathcal{S}_n\}_{n=1}^\infty$ will produce weakly equivalent metrics $\mathcal{D}^d_{\{\mathcal{S}_n\}}$. □

4.2. Independence of Riemannian structure

We have seen that $\mathcal{M}^\sim$ is a metric space with metric $\mathcal{D}^d_{\{\mathcal{S}_n\}}$ for any choice of compact exhaustion and the metric spaces for different choices of exhaustion are all weakly equivalent. Now we will show that this construction is actually independent of the choice of Riemannian metric on N as well. For the remaining portion of the paper we will use $\|\cdot\|$ to denote the usual norm in $\mathbb{R}^k$ and $d_{\mathbb{R}^k}$ to denote the usual distance on $\mathbb{R}^k$.

Lemma 4.9. — *Fix any measurable finite volume subset $\mathcal{S} \subset M$ and let $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$. Now let $\{(\phi_t, B_t)\}_{t \in (a,b)} \in \mathcal{F}(\mathcal{M})$ and $(\phi, B) \in \mathcal{M}$. Suppose that $\phi_t \xrightarrow{\mathcal{D}^d_{\mathcal{S}}} \phi \in \mathcal{M}$ as $t \to c$ and $\mathcal{R} : B \cap \mathcal{S} \to (0, \infty)$ is any measurable function. Then*

$$\lim_{t \to c} \mu_{\mathcal{V}} \left(\{ x \in B_t \cap B \cap \mathcal{S} \mid d(\phi(x), \phi_t(x)) > \mathcal{R}(x) \} \right) = 0.$$

Proof. — It is sufficient to prove for $a = c = 0$ and $b = 1$. First, for $t \in (0, 1)$ let $C_t = \{ x \in B_t \cap B \cap \mathcal{S} \mid d(\phi(x), \phi_t(x)) > \mathcal{R}(x) \}$. Since $C_t \subset \mathcal{S}$ we notice that

$$\begin{aligned} \mathcal{D}^d_{\mathcal{S}}(\phi_t, \phi) &\geqslant \int_{C_t} \min\{1, d(\phi, \phi_t)\} \, \mathrm{d}\mu_{\mathcal{V}} \\ &\geqslant \int_{C_t} \min\{1, \mathcal{R}\} \, \mathrm{d}\mu_{\mathcal{V}}. \end{aligned}$$

Now for each $n \in \mathbb{N}$ let $D^n = \{ x \in B \cap \mathcal{S} \mid \mathcal{R}(x) > 2^{-n} \}$ and notice that

$$\begin{aligned} \int_{C_t} \min\{1, \mathcal{R}\} \, \mathrm{d}\mu_{\mathcal{V}} &\geqslant \int_{D^n \cap C_t} \min\{1, \mathcal{R}\} \, \mathrm{d}\mu_{\mathcal{V}} \\ &\geqslant 2^{-n} \cdot \mu_{\mathcal{V}} \left(D^n \cap C_t \right). \end{aligned}$$

Now combining the above facts we have that $\mathcal{D}^d_{\mathcal{S}}(\phi_t, \phi) \geqslant 2^{-n} \cdot \mu_{\mathcal{V}} \left(D^n \cap C_t \right)$ for any choice of $n \in \mathbb{N}$ so

$$\lim_{t \to 0} \mu_{\mathcal{V}} \left(D^n \cap C_t \right) = 0 \tag{4.1}$$

for all $n \in \mathbb{N}$.

Finally fix $\varepsilon > 0$. Since $\mathcal{R}(x) > 0$ for all $x \in B \cap \mathcal{S}$ we know that the collection $\{D^n\}_{n=1}^{\infty}$ covers $B \cap \mathcal{S}$. Since $B \cap \mathcal{S}$ has finite volume we know there exists some $N \in \mathbb{N}$ such that $\mu_{\mathcal{V}} \left((B \cap \mathcal{S}) \setminus D^N \right) < \frac{\varepsilon}{2}$. This implies that for all $t \in (0, 1)$ we have that $\mu_{\mathcal{V}} \left(C_t \setminus D^N \right) < \frac{\varepsilon}{2}$. By Equation (4.1) we conclude that we can choose some T such that $t < T$ implies that $\mu_{\mathcal{V}} \left(C_t \cap D^N \right) < \frac{\varepsilon}{2}$. Now for $t < T$ we have that $\mu_{\mathcal{V}} \left(C_t \right) = \mu_{\mathcal{V}} \left(C_t \setminus D^N \right) + \mu_{\mathcal{V}} \left(C_t \cap D^N \right) < \varepsilon$. □

Now we show that any choice of continuous metric on N will produce a weakly equivalent metric on $\mathcal{M}^{\sim}$.

Lemma 4.10. — *Suppose that d_1 and d_2 are topologically equivalent metrics on N each induced by a Riemannian metric and let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be any exhaustion of M by finite volume sets. Then $\mathcal{D}^{d_1}_{\{\mathcal{S}_n\}}$ and $\mathcal{D}^{d_2}_{\{\mathcal{S}_n\}}$ are topologically equivalent metrics on $\mathcal{M}^{\sim}$.*

Proof. — Fix finite volume $\mathcal{S} \subset M$. If we show $\mathcal{D}^{d_1}_{\mathcal{S}}$ and $\mathcal{D}^{d_2}_{\mathcal{S}}$ are topologically equivalent then we have proved the lemma by Proposition 4.6. It is sufficient to show that the same families indexed by $(0, 1)$ converge so suppose $\{(\phi_t, B_t)\}_{t\in(0,1)} \in \mathcal{F}(\mathcal{M})$ and $(\phi_0, B_0) \in \mathcal{M}$ such that $\phi_t \xrightarrow{\mathcal{D}^{d_1}_{\mathcal{S}}} \phi_0$ as $t \to 0$ and we will show that $\phi_t \xrightarrow{\mathcal{D}^{d_2}_{\mathcal{S}}} \phi_0$ as $t \to 0$. Fix $\varepsilon > 0$ and without loss of generality assume that $\varepsilon < \mu_{\mathcal{V}}(\mathcal{S})$. Let

$$C_t^2 = \left\{ x \in B_t \cap B_0 \cap \mathcal{S} \;\middle|\; d_2(\phi_0(x), \phi_t(x)) > \frac{\varepsilon}{3\mu_{\mathcal{V}}(\mathcal{S})} \right\}.$$

Let $b^i_{y_0}(r) = \{y \in N \mid d_i(y, y_0) < r\}$ for $i = 1, 2$. Since d_1 and d_2 are weakly equivalent metrics for each $y \in N$ there exists some radius $r_y > 0$ such that the ball with respect to d_1 of radius r_y centered at y is a subset of the ball with respect to d_2 of radius $\frac{\varepsilon}{3\mu_{\mathcal{V}}(\mathcal{S})}$ centered at y. In fact, the weak equivalence of the metrics also implies that there exists a continuous (with respect to the induced topology) function $\psi : N \to (0, \infty)$ such that $b^1_y(\psi(y)) \subset b^2_y(\frac{\varepsilon}{3\mu_{\mathcal{V}}(\mathcal{S})})$ for all $y \in N$. Thus, the function $\mathcal{R} : B_0 \cap \mathcal{S} \to (0, \infty)$ given by $\mathcal{R} = \psi \circ \phi_0$ is measurable and satisfies

$$b^1_{\phi_0(x)}\big(\mathcal{R}(x)\big) \subset b^2_{\phi_0(x)}\left(\frac{\varepsilon}{3\mu_{\mathcal{V}}(\mathcal{S})}\right) \text{ for all } x \in B_0 \cap \mathcal{S}. \tag{4.2}$$

Define $C_t^1 = \{x \in B_t \cap B_0 \cap \mathcal{S} \mid d_1(\phi_0(x), \phi_t(x)) > \mathcal{R}(x)\}$ and notice that Equation (4.2) implies that $C_t^2 \subset C_t^1$. By Lemma 4.9 since $\phi_t \xrightarrow{\mathcal{D}^{d_1}_{\mathcal{S}}} \phi_0$ as $t \to 0$ and $\mathcal{R}$ is measurable we know that $\lim_{t\to 0} \mu_{\mathcal{V}}(C_t^1) = 0$ and so we can conclude that

$$\lim_{t\to 0} \mu_{\mathcal{V}}\left(C_t^2\right) = 0.$$

Now we can find some $T \in (0, 1)$ such that if $t < T$ then $\mu_{\mathcal{V}}(C_t^2) < \frac{\varepsilon}{3}$ and also $\mu_{\mathcal{V}}\big((B_t \vartriangle B_0) \cap \mathcal{S}\big) < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
\mathcal{D}^{d_2}_{\mathcal{S}}(\phi_t, \phi_0) &= \int_{B_t \cap B_0 \cap \mathcal{S}} \min\{1, d_2(\phi_t, \phi_0)\}\, \mathrm{d}\mu_{\mathcal{V}} + \mu_{\mathcal{V}}\big((B_t \vartriangle B_0) \cap \mathcal{S}\big) \\
&\leqslant \int_{(B_t \cap B_0 \cap \mathcal{S}) \setminus C_t^2} \min\{1, d_2(\phi_t, \phi_0)\}\, \mathrm{d}\mu_{\mathcal{V}} \\
&\qquad + \int_{C_t^2} \min\{1, d_2(\phi_t, \phi_0)\}\, \mathrm{d}\mu_{\mathcal{V}} + \mu_{\mathcal{V}}\big((B_t \vartriangle B_0) \cap \mathcal{S}\big) \\
&\leqslant \int_{\mathcal{S}} \frac{\varepsilon}{3\mu_{\mathcal{V}}(\mathcal{S})}\, \mathrm{d}\mu_{\mathcal{V}} + \mu_{\mathcal{V}}\left(C_t^2\right) + \mu_{\mathcal{V}}\big((B_t \vartriangle B_0) \cap \mathcal{S}\big) \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$ □

We conclude this section with the following lemma.

Lemma 4.11. — *Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be a nested exhaustion of M by finite volume sets and suppose that d_1 and d_2 are metrics on N induced by smooth Riemannian metrics. Then $\mathcal{D}^{d_1}_{\{\mathcal{S}_n\}}$ and $\mathcal{D}^{d_2}_{\{\mathcal{S}_n\}}$ are topologically equivalent metrics on $\mathcal{M}^{\sim}$.*

Proof. — Both d_1 and d_2 are continuous with respect to the given topology on N. This means that they are topologically equivalent metrics and so by Lemma 4.10 the result follows. □

Remark 4.12. — If M is finite volume, such as in the case that M is compact, then there is an obvious preferred choice to make when choosing the exhaustion, namely simply $\{M\}$ itself. In such a case we will always use

$$\mathcal{D}^d_M(\phi,\psi) = \int_M p^d_{\phi\psi}\,\mathrm{d}\mu_{\mathcal{V}} = \int_{B_\phi\cap B_\psi} \min\{1, d(\phi,\psi)\}\,\mathrm{d}\mu_{\mathcal{V}} + \mu_{\mathcal{V}}\left(B_\phi \vartriangle B_\psi\right).$$

There are also no choices now when defining convergent ε-perturbations or the radius of convergence except for the choice of metric on N.

4.3. A representative example

To conclude Section 4 we work out an important example which will be referenced throughout the paper.

Example 4.13. — Let $\Phi_{m,k} : (0,1) \to \mathbb{R}$ by

$$\Phi_{m,k}(x) = m \cdot \chi_{\left(\frac{k}{m},\frac{k+1}{m}\right)}(x)$$

(shown in Figure 4.1) for $k, m \in \mathbb{N}$ with $k < m$ where $\chi_{\mathcal{S}}$ is the indicator function for the set $\mathcal{S} \subset (0,1)$. We can see that

$$\int_{(0,1)} \Phi_{m,k} = 1$$

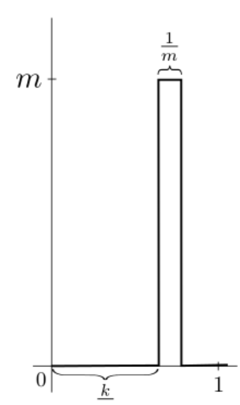

Figure 4.1. An image of $\Phi_{m,k}$.

for all possible values of k and m. We will use these functions to construct an example which is similar to the "traveling wave" example that is common in introductory analysis [4] except that our example changes height so it always integrates to 1.

Consider the sequence

$$\phi_1 = \Phi_{0,1},\ \phi_2 = \Phi_{0,2},\ \phi_3 = \Phi_{1,2},\ \phi_4 = \Phi_{0,3},$$
$$\phi_5 = \Phi_{1,3},\ \phi_6 = \Phi_{2,3},\ \phi_7 = \Phi_{0,4},\ \ldots$$

(as shown in Figure 4.2) and let $\phi_0 \colon (0,1) \to \mathbb{R}$ by

$$\phi_0(x) = 0 \text{ for all } x \in (0,1).$$

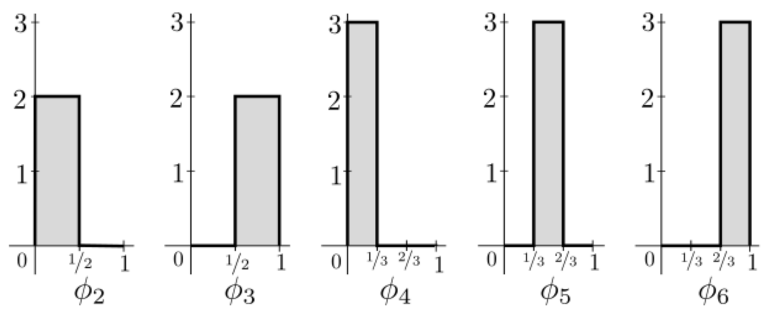

Figure 4.2. A few terms of $\{\phi_n\}$. It can be seen that each integrates to 1 and the "traveling waves" pass over every point infinitely many times, so pointwise convergence is impossible.

Notice that this sequence does not converge pointwise to ϕ_0 for any point $x \in (0,1)$. Also notice that since the integral of any element in this sequence is 1 we can conclude that this sequence does not converge in L^1 (or L^p for any $p \in [1,\infty]$) either (as is mentioned in Remark 3.2), but it will converge with respect to $\mathcal{D}$. This is because the measure of values in the domain which get sent to a number other than zero is becoming arbitrarily small, so we can conclude that

$$\lim_{n\to\infty} \mathcal{D}^{d_{\mathbb{R}}}_{(0,1)}(\phi_n, \phi_0) = 0.$$

This example shows a case in which we have a family which does not behave well pointwise almost everywhere or with respect to the L^p norm, but which does behave well with respect to $\mathcal{D}$.

Of course, if we replace the indicator function with a bump function we can produce a sequence of smooth functions which has the same essential properties as these functions. In fact, for this example we have considered a sequence of functions instead of a continuous family of functions because it made it easier to describe the sequence, but we could easily extend this sequence to a smooth (see Definition 7.1) family of smooth embeddings of $(0,1)$ into $(0,1)\times\mathbb{R}$ indexed by $t\in(0,1)$ which has the same properties.

5. Almost everywhere convergence and $\mathcal{D}$

We already have a definition of convergence in distance, so in this section we will define and explore the properties of a way in which these maps can converge pointwise almost everywhere. To talk about convergence of a family in $\mathcal{F}(\mathcal{M})$ we must have both the domains and the mappings converge. First, we will describe the convergence of the domains.

Let $a,b,c\in\mathbb{R}$ with $a<b$ and $c\in[a,b]$. Now let $\{B_t\subset M\}_{t\in(a,b)}$ be a collection of measurable subsets of M. Recall the limit inferior and limit superior of a family of sets, given by

$$\underline{\lim}_{t\to c}(B_t) := \bigcup_{\delta\in(0,1)}\left(\bigcap_{\substack{t\in(a,b),\\ |t-c|<\delta}} B_t\right) \tag{5.1}$$

and

$$\overline{\lim}_{t\to c}(B_t) := \bigcap_{\delta\in(0,1)}\left(\bigcup_{\substack{t\in(a,b),\\ |t-c|<\delta}} B_t\right) \tag{5.2}$$

respectively. So the limit inferior of the family is the collection of all points which are eventually in every B_t as $t\to c$ and the limit superior is the collection of all points which are not eventually outside of every B_t. Clearly it can be seen that $\underline{\lim}(B_t)\subset\overline{\lim}(B_t)$. We say that the family converges if these two sets only differ by a set of measure zero. That is,

Definition 5.1. — *Let $a,b,c\in\mathbb{R}$ with $a<b$ and $c\in[a,b]$ and let $\{(\phi_t,B_t)\}_{t\in(a,b)}\in\mathcal{F}(\mathcal{M})$. If*

$$\mu_{\mathcal{V}}\left\{\overline{\lim}_{t\to c}(B_t)\setminus\underline{\lim}_{t\to c}(B_t)\right\}=0$$

we say that the collection of sets $\{B_t\}_{t\in(a,b)}$ converges to $\underline{\lim}_{t\to c}(B_t)$ as *$t\to c$ or $\{(\phi_t,B_t)\}_{t\in(a,b)}$ has* converging domains as $t\to c$*. Furthermore, if $\{[\phi_t,B_t]\}_{t\in(a,b)}\in\mathcal{F}(\mathcal{M}^\sim)$ is such that $\{B_t\}_{t\in(a,b)}$ converges for one choice of representative we say it has* converging domains*.*

Remark 5.2. — Notice that any nested family of subsets will converge by this definition. For $a, b \in \mathbb{R}$ with $a < b$ let $\{B_t\}_{t\in(a,b)}$ be a family of subsets such that for $s, t \in (a, b)$ we have that $s < t$ implies $B_t \subset B_s$. Then

$$\underline{\lim}_{t\to a} B_t = \overline{\lim}_{t\to a} B_t = \bigcup_{t\in(a,b)} B_t.$$

Remark 5.3. — Notice that if $\{[\phi_t, B_t]\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M}^\sim)$ has converging domains as $t \to c$, for $a, b, c \in \mathbb{R}, a < b, c \in [a, b]$, then we can always choose some collection of representatives $\{(\phi'_t, B'_t) \in [\phi_t, B_t]\}_{t\in(a,b)}$ such that $\underline{\lim}\, B'_t = \overline{\lim}\, B'_t$ where both limits are taken as $t \to c$.

Now that we understand the convergence of domains we are prepared to describe almost everywhere convergence in $\mathcal{M}$. Let $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$. Notice that if $x \in \underline{\lim}_{t\to c}(B_t)$ then there exists some $\delta > 0$ such that if $t \in (a, b)$ and $|t - c| < \delta$ then $x \in B_t$. This means that $\phi_t(x)$ exists for such t so we may ask if $\{\phi_t(x)\}_{t\in(a,b)\cap(c-\delta,c+\delta)}$ converges as a family of points in N as $t \to c$. If it does converge then we have a limit

$$\lim_{t\to c} \phi_t(x)$$

and thus we arrive at Definition 3.3.

Remark 5.4. — Here it is important to notice that the limit (ϕ, B) from Definition 3.3 is not unique in $\mathcal{M}$ but by Corollary 5.8 we know it does represent a unique element in $\mathcal{M}^\sim$. Furthermore, given $\{[\phi_t, B_t]\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M}^\sim)$ we can create a family in $\mathcal{F}(\mathcal{M})$ by making a choice of representative for each $t \in (a, b)$. If a choice exists such that the resulting family in $\mathcal{F}(\mathcal{M})$ converges than we say that $\{[\phi_t, B_t]\}_{t\in(a,b)}$ converges almost everywhere pointwise. Corollary 5.7 shows that any limit computed in this way gives the same element of $\mathcal{M}^\sim$. In such a case we would write $[\phi_t] \xrightarrow{\text{a.e.}} [\phi_0]$ as $t \to c$. Note that the existence of one choice of representatives which converges does not guarantee that all choices will converge.

We are now ready to prove one direction of Theorem B.

Lemma 5.5. — *Let $a, b, c \in \mathbb{R}$ such that $a < b$ and $c \in [a, b]$. Suppose $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is a family such that $(\phi_t, B_t) \in \mathcal{M}$ for $t \in (a, b)$ and let $(\phi, B) \in \mathcal{M}$. If $\phi_t \xrightarrow{a.e.} \phi$ as $t \to c$ then $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to c$.*

Proof. — It is sufficient to consider families indexed by $(0, 1)$ and limits as $t \to 0$. Let $\{\mathcal{S}_n\}_{n=1}^{\infty}$ be a nested exhaustion of M by finite volume sets, $(\phi, B) \in \mathcal{M}$, and $\{(\phi_t, B_t)\}_{t\in(0,1)} \in \mathcal{F}(\mathcal{M})$ such that $\phi_t \xrightarrow{\text{a.e.}} \phi$ as $t \to 0$. For the duration of this proof let $\underline{\lim}(B_t)$ denote $\underline{\lim}_{t\to 0}(B_t)$ and $\overline{\lim}\, B_t$ denote $\overline{\lim}_{t\to 0} B_t$.

Recall that for $x \in B$ we have that $x \in \underline{\lim}\, B_t$ and $\phi_t(x) \to \phi(x)$ as $t \to 0$ by Definition 3.3. Thus

$$\begin{aligned}\lim_{t\to 0} p^d_{\phi_t\phi}(x) &= \lim_{t\to 0} \min\{1, d(\phi_t(x), \phi(x))\}\\ &= \min\left\{1, d\left(\lim_{t\to 0} \phi_t(x), \phi(x)\right)\right\} = 0.\end{aligned}$$

Also notice that for any $x \in M \setminus \overline{\lim}\, B_t$ we know that $x \notin B$ and also for small enough t we know $x \notin B_t$. That is, there exists some $T \in (0,1)$ such that $t < T$ implies that $x \notin B_t$ so for such t we have that $x \notin B \cup B_t$. This means that for $t < T$ we have that $p^d_{\phi_t\phi}(x) = 0$. Thus

$$\lim_{t\to 0} p^d_{\phi_t\phi}(x) = 0$$

for any $x \in M \setminus \overline{\lim}\, B$ as well. Every $x \in \mathcal{S}$ must either

(1) be in B or $M \setminus \overline{\lim}\, B_t$ and thus satisfy $\lim p^d_{\phi_t\phi}(x) = 0$ as $t \to 0$;
(2) be in $\overline{\lim}\, B_t \setminus B_0$, which is a set of measure zero.

This means that $p^d_{\phi_t\phi} \to 0$ as $t \to 0$ pointwise almost everywhere. Also notice that each $p^d_{\phi_t\phi}$ is bounded by the constant function 1, which is integrable on M because $\nu_{\{\mathcal{S}_n\}}(M) = 1$. These two facts allow us to invoke the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{t\to 0} \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \phi) = \lim_{t\to 0} \int_M p^d_{\phi_t\phi}\, \mathrm{d}\nu_{\{\mathcal{S}_n\}} = \int_M \lim_{t\to 0} p^d_{\phi_t\phi}\, \mathrm{d}\nu_{\{\mathcal{S}_n\}} = 0. \qquad \square$$

Remark 5.6. — Notice that the converse of Lemma 5.5 does not hold in general. We know because of Example 4.13 in which the family converges in $\mathcal{D}$ but not pointwise almost everywhere.

The following two results are a consequence of Lemma 5.5 and the fact that $(\mathcal{M}^\sim, \mathcal{D}^d_{\{\mathcal{S}_n\}})$ is a metric space.

Corollary 5.7. — *Almost everywhere pointwise limits of families in $\mathcal{F}(\mathcal{M}^\sim)$ are unique in $\mathcal{M}^\sim$. That is, let $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a,b]$. Now suppose $\{[\phi_t, B_t]\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M}^\sim)$, $(\phi^1_t, B^1_t), (\phi^2_t, B^2_t) \in [\phi_t, B_t]$ for $t \in (a,b)$, and $(\phi^1, B^1), (\phi^2, B^2) \in \mathcal{M}$ such that $(\phi^i_t, B^i_t) \xrightarrow{a.e.} (\phi^i, B^i)$ as $t \to c$ for $i = 1, 2$. Then $[\phi^1, B^1] = [\phi^2, B^2]$ in $\mathcal{M}^\sim$.*

Proof. — Let $\{[\phi_t, B_t]\}_{t\in(a,b)}$, (ϕ^1_t, B^1_t), (ϕ^2_t, B^2_t), (ϕ^1, B^1), and (ϕ^2, B^2) be as in the statement of the Corollary. Thus for any choice of a nested exhaustion of M by finite volume sets $\{\mathcal{S}_n\}_{n=1}^\infty$, a complete metric d on N which is induced by a Riemannian metric, and $t \in (a,b)$ we have that

$$0 \leqslant \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi^1, \phi^2) \leqslant \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi^1, \phi^1_t) + \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi^1_t, \phi^2_t) + \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi^2_t, \phi^2).$$

The middle term on the right side is zero because $(\phi_t^1, B_t^1) \sim (\phi_t^2, B_t^2)$ and the remaining terms both approach zero as $t \to c$ because $(\phi_t^i, B_t^i) \xrightarrow{\mathcal{D}} (\phi^i, B^i) \in \mathcal{M}^\sim$ as $t \to c$ by Lemma 5.5. $\square$

Corollary 5.8. — *Almost everywhere pointwise limits of families in $\mathcal{F}(\mathcal{M})$ are unique up to $\sim$. That is, suppose that $a, b, c \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$ and further suppose that $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ and $\phi, \phi' \in \mathcal{M}$. If $\phi_t \xrightarrow{a.e.} \phi$ and $\phi_t \xrightarrow{a.e.} \phi'$ then $\phi \sim \phi'$.*

6. Completeness of $\mathcal{M}^\sim$

In this section we prove the remaining parts of Theorem A and Theorem B. Recall that

$L^1_{\mathrm{loc}}(M) = \{f \colon M \to \mathbb{R} \text{ measurable} \mid f|_K \in L^1(K) \text{ for all compact } K \subset M\}.$

It is well-known that $L^1_{\mathrm{loc}}(M)$ is a complete metrizable space, with one choice of possible metric given by

$$d_{\mathrm{loc}}(f, g) = \sum_{k\in\mathbb{N}} 2^{-k} \int_{S_k} \min\{1, |f(x) - g(x)|\} \,\mathrm{d}\mu_{\mathcal{V}},$$

see for instance [4].

Lemma 6.1. — *Suppose that (N, d) is complete and $\{(\phi_\ell, B_\ell)\}_{\ell\in\mathbb{N}} \subset \mathcal{M}^\sim$ is Cauchy with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$. Then there exists a subsequence which converges almost everywhere.*

Proof. — For $\ell, \ell' \in \mathbb{N}$ let χ_{B_ℓ},$\chi_{B_{\ell'}}$ denote the characteristic function for B_ℓ, $B_{\ell'}$ respectively. For any compact $K \subset M$ we have

$$\begin{aligned}\mathcal{D}^d_{\{\mathcal{S}_n\}}((\phi_\ell, B_\ell), (\phi_{\ell'}, B_{\ell'})) &\geqslant \int_K \left|\chi_{B_\ell} - \chi_{B_{\ell'}}\right| \mathrm{d}\nu_{\{\mathcal{S}_n\}} \\ &\geqslant a_K \int_K \left|\chi_{B_\ell} - \chi_{B_{\ell'}}\right| \mathrm{d}\mu_{\mathcal{V}}\end{aligned}$$

where a_K is a constant which depends on K, and thus we see that $\{\chi_{B_\ell}\}_{\ell\in\mathbb{N}}$ is Cauchy in $L^1_{\mathrm{loc}}(M)$. Thus, since $L^1_{\mathrm{loc}}(M)$ is complete there exists some measurable set $B_\infty \subset M$ such that $\lim_{\ell\to\infty} \chi_{B_\ell} = \chi_{B_\infty}$ in $L^1_{\mathrm{loc}}(M)$ which is unique up to measure zero. That is, for every compact $K \subset M$ we have that

$$\lim_{\ell\to\infty} \int_K |\chi_{B_\ell} - \chi_{B_\infty}| \,\mathrm{d}\mu_{\mathcal{V}} = 0.$$

Furthermore, we claim that B_∞ can be chosen such that there exists a subsequence $\{\chi_{B_{\ell_k}}\}_{k\in\mathbb{N}}$ such that

$$\lim_{\ell\to\infty} \chi_{B_\ell}(x) = \chi_{B_\infty}(x)$$

for almost every $x \in M$. Indeed, letting

$$B_{\infty,m} := \{x \in M \mid x \in B_{\ell_k} \text{ for all } k \geqslant m\}$$

we may take $B_\infty = \cup_m B_{\infty,m}$.

Since (N,d) a complete the space $(N,\hat{d})$, where $\hat{d} = \min\{1,d\}$, is also complete. Now, we proceed as in the proof that L^1 is complete. For any $K \subset M$ compact and $m \in \mathbb{N}$ we can assume, possibly by passing to a smaller subsequence, that $\int_{K\cap B_{\infty,m}} \hat{d}(\phi_{\ell_k}(x), \phi_{\ell_{k+1}}(x)) \mathrm{d}\mu_{\mathcal{V}}(x) < 2^{-k}$ for each $k \geqslant m$ and thus

$$\sum_{k\geqslant m} \left(\int_{K\cap B_{\infty,m}} \hat{d}(\phi_{\ell_k}(x), \phi_{\ell_{k+1}}(x)) \mathrm{d}\mu_{\mathcal{V}}(x) \right) < \infty. \tag{6.1}$$

For each $x \in B_\infty$ there exists some m such that $x \in B_{\infty,m}$ and by Equation (6.1) for almost all such x the sequence $\{\phi_{\ell_k}(x)\}_{k\geqslant m}$ of points in N is Cauchy with respect to $\hat{d}$ and thus converges. Let B denote the collection of all such x and define $\phi(x) = \lim_{k\to\infty} \phi_{\ell_k}(x)$ for each $x \in B$. Thus, the subsequence converges to (ϕ, B). ☐

This implies the following, which is a portion of Theorem B.

Corollary 6.2. — *If $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to c$ then there exists $t_i \in (a,b)$ for $i \in \mathbb{Z}_{>0}$ such that $\{(\phi_{t_i}, B_{t_i})\}_{i\in\mathbb{Z}_{>0}}$ converges to ϕ almost everywhere as $t \to c$.*

Now we have the tools to prove the last part of Theorem A.

Lemma 6.3. — *Suppose that $\{\mathcal{S}_n\}_{n=1}^\infty$ is a nested exhaustion of M by finite measure sets and that d is a metric on N induced by a Riemannian metric. Then $\left(\mathcal{M}^\sim, \mathcal{D}^d_{\{\mathcal{S}_n\}}\right)$ is complete if and only if (N,d) is complete.*

Proof. — It is easy to see that if (N,d) is not complete then $\mathcal{M}^\sim$ is not complete. Consider a sequence of constant functions $\{\phi_t : M \to N\}_{t\in(0,1)}$ such that $\phi_t(x) = y_t$ where y_t is a Cauchy family in N which does not converge.

Now suppose that (N,d) is complete and suppose that $\{(\phi_\ell, B_\ell)\}_{\ell\in\mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{M}^\sim, \mathcal{D}^d_{\{\mathcal{S}_n\}}\right)$. It is sufficient to prove that $\{(\phi_\ell, B_\ell)\}_{\ell\in\mathbb{N}}$ converges in $\left(\mathcal{M}^\sim, \mathcal{D}^d_{\{\mathcal{S}_n\}}\right)$, and for this it is sufficient to find a subsequence which converges.

By Lemma 6.1 there exists a subsequence $\{(\phi_{\ell_k}, B_{\ell_k})\}_{k\in\mathbb{N}}$ and $(\phi, B) \in \mathcal{M}^\sim$ such that $(\phi_{\ell_k}, B_{\ell_k}) \to (\phi, B)$ almost everywhere and thus, by Lemma 5.5, we see that $(\phi_{\ell_k}, B_{\ell_k})$ converges with respect to $\mathcal{D}^d_{\{\mathcal{S}_n\}}$, as desired. ☐

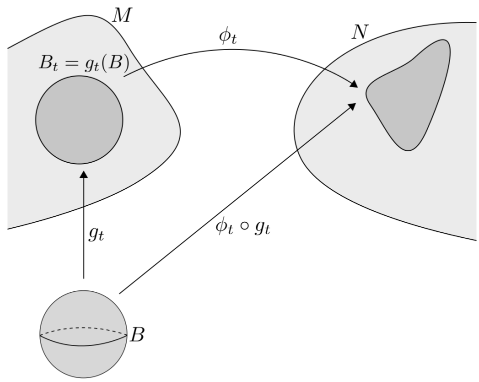

Figure 7.1. A figure of the relevant maps when defining a smooth family.

7. Families with singular limits

We will be considering one parameter families of mappings in $\mathcal{F}(\mathcal{M}^{\sim})$. For this type of family we can adapt the definition of smoothness from [11] which is visualized in Figure 7.1.

Definition 7.1 ([11]). — *Let $a, b \in \mathbb{R}$ with $a < b$. We say that a family of smooth maps $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ is* smooth *if:*

(1) *each element of $\{B_t\}$ is a submanifold of M;*
(2) *there exists a smooth manifold B and a smooth map $g : (a, b) \times B \to M$ such that*
 (a) *the mapping $g_t : x \mapsto g(t, x)$ is a smooth immersion;*
 (b) *for each $t \in (a, b)$ we have $g_t(B) = B_t$.*
(3) *the map $(t, x) \mapsto \phi_t \circ g_t(x)$ is smooth.*

Despite the choice of terminology, it is unknown if this sense of smoothness implies that the family is continuous with respect to the topology on $\mathcal{M}$.

Recall the function $r^d_{\{\mathcal{S}_n\}} : \mathcal{F}(\mathcal{M}) \to [0, \infty]$ from Definition 3.5. This function quantifies how far a family is from converging by measuring how much each embedding must be changed in order to create a new family which does converge. It is straightforward to show that r is surjective.

Proposition 7.2. — *For any $q \in [0, \infty]$ there exists some choice of manifolds M and N, an exhaustion $\{\mathcal{S}_n\}_{n=1}^{\infty}$ of M, a distance d induced by a complete Riemannian metric on N, $a, b \in \mathbb{R}$ such that $a < b$, and a smooth family $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ for which $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = q$.*

Proof. — From the existence of families of embeddings which do converge we know that 0 is in the image of r. Also, notice that if $\phi_t : (0,1) \to \mathbb{R}$, $\phi_t(x) = \left(\frac{1}{t}\right)\sin(\frac{1}{t})$ then $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = \infty$.

Pick some $q \in (0,1)$ and let $\phi_t : (0,3q) \to \mathbb{R}$ for $t \in (0,1)$ via

$$\phi_t(x) = \frac{x}{9q} + \frac{1}{3}\sin(\tfrac{1}{t}).$$

So in this case $B_t = (0,3q)$ for all t, $a = 0$, $b = 1$, $M = (0,3q)$ with the usual measure inherited from $\mathbb{R}$, and $N = \mathbb{R}$ with the usual distance. Since M is finite throughout this example let $\mathcal{D} := \mathcal{D}^d_{\{M\}}$ and $r := r^d_{\{M\}}$ where d is the standard distance on $\mathbb{R}$. Notice that if we perturbed this family to converge to some limit which did not have $(0,3q)$ as its domain we could change the domain of the limit to $(0,3q)$ and have a smaller perturbation. So we can assume that the domain of the limit is $(0,3q)$. Suppose that we wanted to change this family so it converged to some map $\phi_0 : (0,3q) \to \mathbb{R}$. We can see that the ϕ_t oscillate to the left and right, so let $\phi_L(x) = \frac{x}{9q} - \frac{1}{3}$ and $\phi_R(x) = \frac{x}{9q} + \frac{1}{3}$. Now let

$$l_n = \frac{2}{(4n+1)\pi} \text{ and } r_n = \frac{2}{(4n+3)\pi}$$

so that $\phi_{l_n} = \phi_L$ and $\phi_{r_n} = \phi_R$ for all $n \in \mathbb{N}$. Notice $d(\phi_L(x), \phi_R(x)) = \frac{2}{3}$ for all $x \in (0,3q)$ so

$$d(\phi_L(x), \phi_0(x)) + d(\phi_0(x), \phi_R(x)) \geqslant \tfrac{2}{3}.$$

Clearly this implies that

$$\min\{1, d(\phi_L(x), \phi_0(x))\} + \min\{1, d(\phi_0(x), \phi_R(x))\} \geqslant \tfrac{2}{3}$$

and so integrating each side over $(0,3q)$ gives $\mathcal{D}(\phi_L, \phi_0) + \mathcal{D}(\phi_0, \phi_R) \geqslant 2q$ so one of the two terms must be greater than or equal to q. Without loss of generality suppose that $\mathcal{D}(\phi_L, \phi_0) \geqslant q$. In such a case choose any $\varepsilon > 0$ and find some $T \in (0,1)$ such that $t < T$ implies $\mathcal{D}(\widetilde{\phi}_t, \phi_0) < \varepsilon$ where $\{\widetilde{\phi}_t\}$ is any family which converges to ϕ_0. Then pick some $n \in \mathbb{N}$ such that $l_n < T$ and let $t = l_n$. Now

$$\mathcal{D}(\phi_t, \widetilde{\phi}_t) + \mathcal{D}(\widetilde{\phi}_t, \phi_0) \geqslant \mathcal{D}(\phi_t, \phi_0)$$

so $\mathcal{D}(\phi_t, \widetilde{\phi}_t) \geqslant q - \varepsilon$ for all $\varepsilon > 0$. This allows us to conclude that $r(\{(\phi_t, B_t)\}) \geqslant q$.

Now let $\widetilde{\phi}_t : (0,3q) \to \mathbb{R}$ with $\widetilde{\phi}_t(x) = \frac{x}{9q}$ be a family of maps which is clearly smooth and has limit $\phi_0(x) = \frac{x}{9q}$. Now notice

$$\mathcal{D}(\phi_t, \widetilde{\phi}_t) = \int_{(0,3q)} \min\{1, d(\phi_t, \widetilde{\phi}_t)\}\, \mathrm{d}\mu_{\mathcal{V}} = q\left|\sin(\tfrac{1}{t})\right| \leqslant q$$

and it is important to notice that $\mathcal{D}(\phi_t, \widetilde{\phi_t}) = q$ is achieved infinitely often. Thus we know that $r(\{(\phi_t, B_t)\}) \leqslant q$ so in fact we know that $r(\{(\phi_t, B_t)\}) = q$. □

Lemma 7.3. — *Suppose that $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ and for all $\varepsilon > 0$ there exists family $\{(\widetilde{\phi_t}, \widetilde{B_t})\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ such that*

(1) *there exists $(\widetilde{\phi}, \widetilde{B}) \in \mathcal{M}$ such that $\widetilde{\phi}_t \xrightarrow{a.e.} \widetilde{\phi}$ as $t \to a$;*
(2) *$B_t = \widetilde{B_t}$ for all $t \in (a,b)$ and $\underline{\lim}_{t\to c} \widetilde{B_t} \subset \widetilde{B}$;*
(3) *$\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \widetilde{\phi_t}) \leqslant \varepsilon$ for all $t \in (a,b)$.*

Then there exists $(\phi, B) \in \mathcal{M}$ unique up to $\sim$ such that $\phi_t \xrightarrow{\mathcal{D}} \phi$ as $t \to a$

Proof. — Fix some compact $\mathcal{S} \subset M$ and we will show that $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is Cauchy with respect to $\mathcal{D}_{\mathcal{S}}$. Fix $\delta > 0$. Let $\varepsilon = \frac{\delta}{4}$ and let $\{(\widetilde{\phi^\varepsilon_t}, \widetilde{B^\varepsilon_t})\}$ be the family assumed to exist in the statement of the lemma.

From Theorem B and item (1) in the statement of the lemma we know that $\widetilde{\phi^\varepsilon_t} \xrightarrow{\mathcal{D}^d_{\mathcal{S}}} \widetilde{\phi^\varepsilon_0}$ as $t \to a$ so we can choose some $T \in (a,b)$ such that $t < T$ implies $\mathcal{D}_{\mathcal{S}}(\widetilde{\phi^\varepsilon_t}, \widetilde{\phi^\varepsilon}) < \varepsilon$. Finally, we can conclude that for any $t, s < T$ we have that

$$\mathcal{D}_{\mathcal{S}}(\phi_t, \phi_s) \leqslant \mathcal{D}_{\mathcal{S}}(\phi_t, \widetilde{\phi^\varepsilon_t}) + \mathcal{D}_{\mathcal{S}}(\widetilde{\phi^\varepsilon_t}, \widetilde{\phi^\varepsilon}) + \mathcal{D}_{\mathcal{S}}(\widetilde{\phi^\varepsilon}, \widetilde{\phi^\varepsilon_s}) + \mathcal{D}_{\mathcal{S}}(\widetilde{\phi^\varepsilon_s}, \phi_s) < 4\varepsilon = \delta.$$

This means that $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is Cauchy as $t \to a$ for each $\mathcal{D}_{\mathcal{S}}$ so by Proposition 4.6 we know that it is Cauchy with respect to $\mathcal{D}$ as $t \to a$. The result follows from the fact that $(\mathcal{M}^\sim, \mathcal{D})$ is complete by Theorem A. □

Now we are prepared to prove Theorem C.

Proof of Theorem C. — Part (1) follows immediately from Lemma 7.3 because the conditions of Theorem C part (1) are a special case of the conditions of Lemma 7.3.

Now we will show part (2). Suppose that the domains satisfy the required property for $T \in (a,b)$, $\{(\phi_t, B_t)\}_{t\in(a,b)}$ is a smooth family of embeddings, and that $\phi_t \xrightarrow{\mathcal{D}} \phi_0$ as $t \to a$. Fix $\varepsilon > 0$ and find some $T_1 \in (a, T)$ such that $s, t < T_1$ implies that $\mathcal{D}(\phi_t, \phi_s) < \varepsilon$. Now let $\mathcal{B} : (a,b) \to [0,1]$ be a smooth bump function such that $\mathcal{B}(t) = 0$ for $t \geqslant T_1$ and $b(t) = 1$ for $t < \frac{T_1+a}{2}$. Now define $f : (a,b) \to [\frac{(T_1+a)}{2}, b)$ via

$$f(t) = \big(1 - \mathcal{B}(t)\big)t + \mathcal{B}(t)\frac{T_1 + a}{2}.$$

Finally let $\widetilde{\phi}_t = \phi_{f(t)}|_{B_t}$ and notice that $\{(\widetilde{\phi}_t, B_t)\}$ is a smooth family satisfying $\widetilde{\phi}_t \xrightarrow{\text{a.e.}} \phi_{\frac{T_1}{2}}$ as $t \to a$. By the choice of T_1 we can see that for all

$t \in (a, b)$ we have $\mathcal{D}(\phi_t, \widetilde{\phi_t}) < \varepsilon$. Also, because of the requirement on the domains we know that $B_t \subset B_{f(t)}$ and thus $\widetilde{\phi_t} : B_t \to N$ is defined on all of B_t. ☐

Remark 7.4. — It is natural to wonder if $r(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$ implies the family must in fact converge pointwise almost everywhere in $\mathcal{M}^{\sim}$. The answer to this question is no; again consider Example 4.13. The functions in Example 4.13 converge in $\mathcal{D}$ and all have the same domain so we know that $r^d_{\{\mathcal{S}_n\}} = 0$ for these functions, but we also know that they do not converge pointwise almost everywhere.

8. Final remarks

8.1. Approaches to prove a converse to Theorem C

Now we have set up all of the machinery to begin to explore the converse of Theorem C in the case that the domains are not restricted to shrink or stabilize eventually. That is, we will outline some potential avenues to answer the following question.

Question 8.1. — *Is it true that $\{(\phi_t, B_t)\}_{t\in(a,b)} \xrightarrow{\mathcal{D}} \phi_0$ implies that $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$?*

There are two approaches in the general case: we can attempt to extend embeddings or we can smooth singular limits by understanding the singularities locally. For the following two subsections assume $\{(\phi_t, B_t)\}_{t\in(a,b)} \xrightarrow{\mathcal{D}} \phi_0$.

8.1.1. Extending embeddings to remove singularities

To get an ε-perturbation of $\{(\phi_t, B_t)\}_{t\in(a,b)}$ we choose some $T \in (a, b)$ such that $s, t < T$ implies that $\mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \phi_s) < \varepsilon$. Then, just as in the proof of Theorem C, we must smoothly change the family so that $t < \frac{T+a}{2}$ implies that $\widetilde{\phi_t} = \phi_{\frac{T+a}{2}}$. If $\underline{\lim}\, B_t \not\subset B_{\frac{T}{2}}$ then this will not define an embedding with domain all of B_0, so this embedding would have to be extended. Thus, this question comes down to asking when an embedding of some subset of M can be extended to a larger domain in M. Extending embeddings or smooth maps has been of independent interest for many years (i.e. the Tietze Extension Theorem [4, Theorem 4.16], the Whitney Extension Theorem [16, Theorem I], and the Extension Lemma [7, Lemma 2.27]). For a collection of more recent work in extension problems see [3].

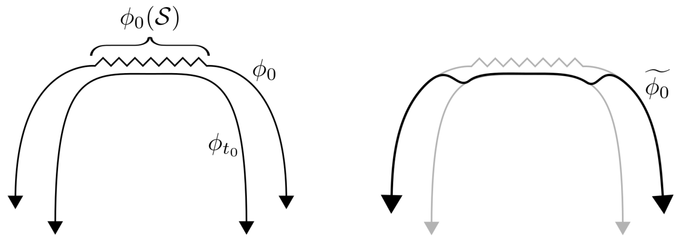

Figure 8.1. The strategy is to connect the embedding ϕ_{t_0} with the map ϕ which is an embedding away from $\mathcal{S}$. In this way we are able to avoid the singular part of ϕ while only changing it slightly on a small set.

8.1.2. Removing singularities locally

Suppose there exists a closed set $\mathcal{S} \subset B$ such that $\phi_0|_{B\setminus\mathcal{S}} : B \setminus \mathcal{S} \hookrightarrow N$ is an embedding and there exists some $T \in (a, b)$ such that $t < T$ implies $\mathcal{S} \subset B_t$. Then for some neighborhood U of $\mathcal{S}$ we can define $\widetilde{\phi_0}$ in U to be equal to ϕ_{t_0} for some small $t_0 \in (a, b)$ and define $\widetilde{\phi_0}$ outside of a larger neighborhood $V \supset U$ to be equal to ϕ_0. A schematic of this idea is shown in Figure 8.1. The difficulty is connecting the portion of $\widetilde{\phi_0}$ defined in U with the portion defined outside V; it is well known that partition of unity type arguments can be used to smoothly transition between two smooth maps [7] but in this case we must also preserve the embedding structure.

8.2. Implications of a positive answer to Question 8.1

If the answer to Question 8.1 were yes, then there are several implications. First, we will have a new characterization of families with $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}) = 0$, namely these are exactly the families which converge in $\mathcal{D}$. Second, there is then an easy proof that $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}) = 0$ does not depend on the choices of $\{\mathcal{S}_n\}_{n=1}^\infty$ and d. The proof is the following:

Let $\{\mathcal{S}_n\}_{n=1}^\infty$, $\{\mathcal{S}'_n\}_{n=1}^\infty$, d, and d' be choices of finite exhaustion and metric. Suppose that $a, b \in \mathbb{R}$ with $a < b$ and $\{(\phi_t, B_t)\}_{t\in(a,b)} \in \mathcal{F}(\mathcal{M})$ is a smooth family such that $r^d_{\{\mathcal{S}_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$. Then by Theorem C we know that $\lim \mathcal{D}^d_{\{\mathcal{S}_n\}}(\phi_t, \phi) = 0$ as $t \to a$ for some $\phi \in \mathcal{M}$. By Theorem A this means that $\lim \mathcal{D}^{d'}_{\{\mathcal{S}'_n\}}(\phi_t, \phi) = 0$ as $t \to a$ and thus by the assumed positive answer to Question 8.1 we know that $r^{d'}_{\{\mathcal{S}'_n\}}(\{(\phi_t, B_t)\}_{t\in(a,b)}) = 0$.

8.3. Further questions

In [14] the authors produce a metric on the space of toric integrable systems. An integrable system is a $2n$-dimensional symplectic manifold (M, ω) along with a map $F \colon M \to \mathbb{R}^n$ such that the components of F Poisson commute and are independent almost everywhere. The metric defined in the present paper could potentially be used to define a metric on more general spaces of integrable systems, as long as those systems could be viewed as subsets of the same manifold.

It would be interesting to study Question 8.1 restricted to a specific type of embedding. For example, thinking back to the original motivation from Section 1, one could consider whether this is true for the collection of symplectic embeddings where the original smooth family $\{(\phi_t, B_t)\}_{t\in(a,b)}$ consists exclusively of symplectic embeddings and the perturbed family $\{(\widetilde{\phi}_t, \widetilde{B}_t)_{t\in(a,b)}\}$ from the definition of the radius of convergence is also required to be symplectic. Resolving singular points of symplectic manifolds is related to this in spirit and has been studied extensively such as in [8]. Symplectic manifolds have been shown to admit a high degree of flexibility (see for example Moser's Theorem [9] or Darboux's Theorem [15]) although Gromov's nonsqueezing theorem [5] represents a level of rigidity that symplectic embeddings do need to respect. One could also consider the case of isometric embeddings of Riemannian manifolds, even in the case of $\mathcal{M}(\mathbb{R}, \mathbb{R}^2)$. Clearly studying further types of embeddings would be enlightening as it would allow us to gain a greater understanding of the rigidity of these structures. Indeed, it is the purpose of this paper to create a foundation off of which many types of families of embeddings may be studied.

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