

ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

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Tome XXIV, n° 5 (2015), p. 1261-1292.

http://afst.cedram.org/item?id=AFST_2015_6_24_5_1261_0

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Groups of PL homeomorphisms of cubes

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Dédié à Michel Boileau sur son soixantième anniversaire. Santé!

RÉSUMÉ. — Nous étudions les propriétés algébriques de groupes des homéomorphismes PL ou lisses des cubes unitaires dans toutes les dimensions, point par point fixe sur la frontière, et des groupes plus généralement PL ou lisses agissant sur les variétés et qui fixe ponctuellement une sous-variété de codimension 1 (resp. codimension 2), et montrent que ces groupes sont localement indicables (resp. de circulairement ordonnable). Nous donnons également de nombreux exemples de groupes intéressants qui peuvent agir, et discutons de certaines autres contraintes algébriques que ces groupes doivent satisfaire, y compris le fait qu'un groupe d'homéomorphismes PL de la n -cube (ponctuelle fixe sur la frontière) ne contient pas d'éléments qui soient plus qu'exponentiellement distordus.

ABSTRACT. — We study algebraic properties of groups of PL or smooth homeomorphisms of unit cubes in any dimension, fixed pointwise on the boundary, and more generally PL or smooth groups acting on manifolds and fixing pointwise a submanifold of codimension 1 (resp. codimension 2), and show that such groups are locally indicable (resp. circularly orderable). We also give many examples of interesting groups that can act, and discuss some other algebraic constraints that such groups must satisfy, including the fact that a group of PL homeomorphisms of the n -cube (fixed pointwise on the boundary) contains no elements that are more than exponentially distorted.

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1. Introduction

We are concerned in this paper with algebraic properties of the group of PL homeomorphisms of a PL manifold, fixed on some PL submanifold (usually of codimension 1 or 2, for instance the boundary) and some of its subgroups (usually those preserving some structure). The most important case is the group of PL homeomorphisms of I^n fixed pointwise on ∂I^n ; hence these are “groups of PL homeomorphisms of the (n -)cube”.

The *algebraic* study of transformation groups (often in low dimension, or preserving some extra structure such as a symplectic or complex structure) has recently seen a lot of activity; however, much of this activity has been confined to the smooth category. It is striking that many of these results can be transplanted to the PL category. This interest is further strengthened by the possibility of working in the PL category over (real) algebraic rings or fields (this possibility has already been exploited in dimension 1, in the groups F and T of Richard Thompson).

Many theorems we prove have analogs in the (C^1) smooth category, and we usually give proofs of such theorems for comparison where they are not already available in the literature.

1.1. Statement of results

The main algebraic properties of our groups that we establish are *left orderability* (in fact, local indicability) and *controlled distortion*. These are algebraic properties which at the same time are readily compared with geometric or topological properties of a group action. For example, the property of left orderability for a countable group is *equivalent* to the existence of a faithful action on $(I, \partial I)$ by homeomorphisms. As another example, control of (algebraic) distortion has recently been used by Hurtado [18] to prove strong rigidity results for homomorphisms between various (smooth) transformation groups.

In § 2 we state for the convenience of the reader standard definitions and results from the theory of left-orderable groups.

In § 3 we begin our analysis of PL groups of homeomorphisms of manifolds, fixed pointwise on a codimension 1 submanifold. If M is a PL manifold and K a submanifold, we denote by $\text{PL}_+(M, K)$ the group of orientation-preserving PL homeomorphisms of M fixed pointwise on K . We similarly denote by $\text{Diff}_+(M, K)$ and $\text{Homeo}_+(M, K)$ the groups of diffeomorphisms (resp. homeomorphisms) of a smooth (resp. topological) manifold M , fixed pointwise on a smooth (resp. topological) submanifold K of codimension 1.

The first main theorem of this section is:

PL LOCALLY INDICABLE THEOREM 3.4. — *Let M be an n dimensional connected PL manifold, and let K be a nonempty closed PL submanifold of codimension at most 1. Then the group $\text{PL}_+(M, K)$ is left orderable; in fact, it is locally indicable.*

We give some basic constructions of interesting groups of PL homeomorphisms of cubes (of dimension at least 2) fixed pointwise on the boundary, including examples of free subgroups, groups with infinite dimensional spaces of (dynamically defined) quasimorphisms, and right-angled Artin groups; in fact, the method of Funar, together with a recent result of Kim and Koberda shows that every RAAG embeds in $\text{PL}(I^2, \partial I^2)$.

We conclude this section by studying distortion in $\text{PL}(I^m, \partial I^m)$, and prove

PL DISTORTION COROLLARY 3.30. — *Every element of $\text{PL}(I^m, \partial I^m) - \text{Id}$ is at most exponentially distorted.*

This lets us easily construct explicit examples of locally indicable groups which are not isomorphic to subgroups of $\text{PL}(I^m, \partial I^m)$ for any m (for example, iterated HNN extensions).

In § 4 we prove Theorem 4.3, the analog of Theorem 3.4 for groups of C^1 smooth diffeomorphisms. The main purpose of this section is to compare and contrast the methods of proof in the PL and smooth categories.

In § 5 we sharpen our focus to the question of bi-orderability (i.e. orders which are invariant under both left and right multiplication), for transformation groups acting on cubes in various dimensions and with differing degrees of analytic control. In dimensions bigger than 1, none of the groups we study are bi-orderable. In dimension 1, the groups $\text{PL}(I, \partial I)$ and $\text{Diff}^\omega(I, \partial I)$ are bi-orderable.

In § 6 we consider groups acting on manifolds and fixing pointwise submanifolds (informally, “knots”) of codimension two. Here we are able to bootstrap our results from previous sections to show the following:

PL CIRCULARLY ORDERABLE THEOREM 6.11. — *Let M be an n dimensional connected PL orientable manifold, and let K be a nonempty $n - 2$ dimensional closed submanifold. Then the group $\text{PL}_+(M, K)$ is circularly orderable.*

An analog for C^1 diffeomorphisms is also proved. An interesting application is to the case that M is a 3-manifold, and K is a hyperbolic knot (i.e. a knot with a hyperbolic complement). In this case, if $G_0(M, K)$ denotes the group of PL homeomorphisms (or C^1 diffeomorphisms) isotopic to the identity and taking K to itself (but *not* necessarily fixing it pointwise) then $G_0(M, K)$ is left-orderable. This depends on theorems of Hatcher and Ivanov on the topology of $G_0(M, K)$ as a topological group.

Finally, in § 7 we briefly discuss groups of homeomorphisms with no analytic restrictions. Our main point is how little is known in this generality in dimension > 1 ; in particular, it is not even known if the group $\text{Homeo}(I^2, \partial I^2)$ is left-orderable. We also make the observation that the groups $\text{Homeo}(I^n, \partial I^n)$ are torsion-free for all n ; this follows immediately from Smith theory, but the result does not seem to be well-known to people working on left-orderable groups, so we believe it is useful to include an argument here.

Acknowledgement. — We would like to thank Andrés Navas and Amie Wilkinson for some helpful discussions about this material. We are also very grateful to the anonymous referee who pointed us to several useful references. Danny Calegari was supported by NSF grant DMS 1005246. Dale Rolfsen was supported by a grant from the Canadian Natural Sciences and Engineering Research Council.

2. Left orderable groups

This section contains a very brief summary of some standard results in the theory of left orderable groups. These results are collected here for the convenience of the reader. For proofs, see e.g. [7], Chapter 2.

DEFINITION 2.1 (Left orderable). *A group G is left orderable (usually abbreviated to LO) if there is a total order \prec on G so that for all $f, g, h \in G$ the relation $g \prec h$ holds if and only if $fg \prec fh$ holds.*

LEMMA 2.2. — *If there is a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$ and both K and H are left orderable, then G is left orderable.*

LEMMA 2.3. — *A group is left orderable if and only if every finitely generated subgroup is left orderable.*

LEMMA 2.4. — *A countable group G is left orderable if and only if it is isomorphic to a subgroup of $\text{Homeo}(I, \partial I)$.*

DEFINITION 2.5 (Locally indicable). — *A group G is locally indicable if for every finitely generated nontrivial subgroup H of G there is a surjective homomorphism from H to \mathbb{Z} .*

THEOREM 2.6 (Burns-Hale, [6]). — *Every locally indicable group is left orderable.*

3. PL group actions on cubes

3.1. Definitions

We assume the reader is familiar with the concept of a PL homeomorphism between compact polyhedra in \mathbb{R}^n , namely one for which the domain can be subdivided into *finitely many* linear simplices so that the restriction of the homeomorphism to each simplex is an affine linear homeomorphism to its image. We say that the simplices in the domain are *linear for f* . A PL *manifold* is one with charts modeled on \mathbb{R}^n and transition functions which are the restrictions of PL homeomorphisms.

We define the *dimension* of a (linear) polyhedron to be the maximum of the dimensions of the simplices making it up, in any decomposition into simplices.

Notation 3.1. — Let M be a PL manifold (possibly with boundary) and let K be a closed PL submanifold of M . We denote by $\text{PL}(M, K)$ the group of PL self-homeomorphisms of M which are fixed pointwise on K . If M is orientable, we denote by $\text{PL}_+(M, K)$ the subgroup of orientation-preserving PL self-homeomorphisms.

The main example of interest is the following:

Notation 3.2. — We denote by I^n the cube $[-1, 1]^n$ in Euclidean space with its standard PL structure. Note with this convention that 0 is a point in the *interior* of I^n . We denote the boundary of I^n by ∂I^n .

So we denote by $\text{PL}(I^n, \partial I^n)$ the group of PL self-homeomorphisms of the unit cube I^n in \mathbb{R}^n which are fixed pointwise on the boundary. If ω denotes the standard (Lebesgue) volume form on I^n , we denote by $\text{PL}_\omega(I^n, \partial I^n)$ the subgroup of $\text{PL}(I^n, \partial I^n)$ preserving ω .

3.2. Transformation groups as discrete groups

Let G be a transformation group — i.e. a group of homeomorphisms of some topological space X . It is often useful to endow G with a topology compatible with the action; for instance, the *compact-open topology*. If we denote G^δ as the same group but with the discrete topology, the identity

homomorphism $G^\delta \rightarrow G$ is a continuous map of topological groups, and induces maps on cohomology $H^*(BG; R) \rightarrow H^*(BG^\delta; R) = H^*(K(G, 1); R)$ for any coefficient module R . Thus one interesting source of algebraic invariants of the discrete group G arise by thinking of its homotopy type as a topological group.

However, the groups of most interest to us in this paper are not very interesting as (homotopy types of) topological spaces:

PROPOSITION 3.3 (Alexander trick). — *The groups $\text{Homeo}(I^n, \partial I^n)$, $\text{PL}(I^n, \partial I^n)$, $\text{PL}_\omega(I^n, \partial I^n)$ are all contractible in the compact-open topology.*

Proof. — Given $f : I^n \rightarrow I^n$ fixed pointwise on ∂I^n , define $f_t : I^n \rightarrow I^n$ by

$$f_t(x) := \begin{cases} tf(x/t) & \text{if } 0 \leq \|x\| \leq t \\ x & \text{if } t \leq \|x\| \leq 1 \end{cases}$$

where $\|\cdot\|$ denotes the L^∞ norm on I^n . Then $f_0 = f$ and $f_1 = \text{Id}$, and the assignment $f \rightarrow f_t$ defines a deformation retraction of any one of the groups in question to the identity. \square

3.3. Left orderability

In this section we prove the left orderability of certain groups of PL homeomorphisms.

The purpose of this section is to develop the tools to prove the following theorem:

THEOREM 3.4 (Locally indicable). — *Let M be an n dimensional connected PL manifold, and let K be a nonempty closed PL submanifold of codimension at most 1. Then the group $\text{PL}_+(M, K)$ is left orderable; in fact, it is locally indicable.*

An important special case is $M = I^n$ and $K = \partial I^n$.

First we introduce some notation and structure. Recall that if X is a closed subset of Y , the *frontier* of X in Y is the intersection of X with closure of $Y - X$.

Notation 3.5. — *Let $f \in \text{PL}(M, K)$. The fixed point set of f is denoted $\text{fix}(f)$ and the frontier of $\text{fix}(f)$ is denoted $\text{fro}(f)$. Similarly, if G is a subgroup of $\text{PL}(M, K)$, the fixed point set of G is denoted $\text{fix}(G)$. The frontier of $\text{fix}(G)$ is denoted $\text{fro}(G)$.*

LEMMA 3.6. — Let M be an n dimensional connected PL manifold, and let K be a nonempty $n - 1$ dimensional closed PL submanifold. Let G be a nontrivial finitely generated subgroup of $\text{PL}(M, K)$.

1. $\text{fix}(G)$ is a closed polyhedron of dimension at least $n - 1$.
2. $\text{fro}(G)$ is a polyhedron of dimension $n - 1$.
3. If g_1, \dots, g_m is a finite generating set for G , then $\text{fro}(G) \subset \cup_i \text{fro}(g_i)$.

Proof. — Let g_1, \dots, g_m be a finite generating set for G .

(1) We have $\text{fix}(G) = \cap_i \text{fix}(g_i)$. Since each $\text{fix}(g_i)$ is a closed polyhedron containing K , so is $\text{fix}(G)$.

(2) If $\text{fix}(G)$ has no interior, then its complement is open and dense, so $\text{fro}(G) = \text{fix}(G)$. Otherwise, $\text{fix}(G)$ has some interior, and $\text{fro}(G)$ separates some point in the interior from some point in $M - \text{fix}(G)$. Since M is connected, $\text{fro}(G)$ has dimension at least $n - 1$.

(3) Every point p in $\text{fro}(G)$ is in $\text{fix}(g_i)$ for all i , and for some i there are points arbitrarily near p moved nontrivially by g_i . Thus $p \in \text{fro}(g_i)$ for this i . □

DEFINITION 3.7. — Let G be a subgroup of $\text{PL}(M, K)$. The action of G is semilinear at a point p if there is some codimension 1 plane π through p and a convex open neighborhood U of p so that the restriction of G is linear on both components of $U - (\pi \cap U)$. We call π the dividing plane.

Note with this definition that a linear action at a point is semilinear. Note also that if G is semilinear at a point p but not linear there, the dividing plane is unique.

LEMMA 3.8. — Let M be an n dimensional connected PL manifold, and let K be a nonempty $n - 1$ dimensional closed PL submanifold. Let G be a finitely generated subgroup of $\text{PL}(M, K)$. Then for every $n - 1$ dimensional linear simplex σ in $\text{fro}(G)$, there is an open and dense subset of σ where G is semilinear, and the dividing plane is tangent to σ .

Proof. — Fix a finite generating set g_1, \dots, g_m for G , and recall from Lemma 3.6 bullet (3) that $\text{fro}(G) \subset \cup_i \text{fro}(g_i)$. Let σ be an $n - 1$ dimensional simplex in $\text{fro}(G)$, and suppose $\sigma \cap \text{fro}(g_i)$ has full dimension. Let $\sigma_i = \sigma \cap \text{fro}(g_i)$. Associated to g_i there is a decomposition of M into linear simplices; away from the $n - 2$ skeleton of this decomposition, g_i acts semilinearly. So g_i acts semilinearly on the complement of an $n - 2$ dimensional

polyhedral subset b_i of σ_i . For each point $p \in \sigma_i - b_i$ the action of g_i on the tangent plane to σ_i is the identity (since σ_i is fixed by g_i) and therefore we can take the dividing plane to be equal to this tangent plane. For, either the action at p is semilinear but not linear (in which case any codimension 1 plane on which the action is linear is the unique dividing plane), or the action at p is linear, in which case any codimension 1 plane can be taking to be the dividing plane. Thus the dividing plane of g_i is tangent to σ_i along $\sigma_i - b_i$. Furthermore, since $\sigma - \sigma_i$ is in $\text{fix}(g_i)$ but not $\text{fro}(g_i)$, the element g_i acts trivially in a neighborhood of each point of $\sigma - \sigma_i$.

It follows that each g_i acts semilinearly at each point of $\sigma - \cup_i b_i$ with dividing plane tangent to σ , and therefore all of G acts semilinearly at each point of $\sigma - \cup_i b_i$ with dividing plane tangent to σ . \square

LEMMA 3.9. — *The (pointwise) stabilizer in $\text{GL}(n, \mathbb{R})$ of an m dimensional subspace π of \mathbb{R}^n is isomorphic to $\mathbb{R}^{m(n-m)} \rtimes \text{GL}(m, \mathbb{R})$ and the subgroup preserving orientation is isomorphic to $\mathbb{R}^{m(n-m)} \rtimes \text{GL}^+(m, \mathbb{R})$.*

Proof. — The subgroup of $\text{GL}(n, \mathbb{R})$ fixing π can be conjugated into the set of matrices of the form $\begin{pmatrix} \text{Id} & V \\ 0 & A \end{pmatrix}$ where Id is the identity in $\text{GL}(n - m, \mathbb{R})$, where V is an arbitrary $(n - m) \times m$ matrix, and where $A \in \text{GL}(m, \mathbb{R})$. The proof follows. \square

Example 3.10 (Codimension 1 stabilizer). — In the special case $m = n - 1$ the stabilizer is isomorphic to $\mathbb{R}^{n-1} \rtimes \mathbb{R}^*$, and the orientation preserving subgroup is isomorphic to $\mathbb{R}^{n-1} \rtimes \mathbb{R}^+$ which is an extension of the locally indicable group \mathbb{R}^+ by the locally indicable group \mathbb{R}^{n-1} , and is therefore locally indicable.

We are now ready to give the proof of Theorem 3.4:

Proof. — By Theorem 2.6 it suffices to show that every nontrivial finitely generated subgroup of $\text{PL}_+(M, K)$ surjects onto \mathbb{Z} . Let G be such a subgroup, and let p be a point in a codimension 1 simplex in $\text{fro}(G)$ where G acts semilinearly. Since p is in $\text{fro}(G)$, some g_i acts nontrivially on one side of $\text{fro}(G)$, so there is a *nontrivial* homomorphism from G to $\mathbb{R}^{n-1} \rtimes \mathbb{R}^+$ which is locally indicable, and therefore G surjects onto \mathbb{Z} . Since G was arbitrary, the theorem is proved. \square

3.4. Free subgroups of $\text{PL}(I^n, \partial I^n)$

In this section we discuss algebraic properties of the groups $\text{PL}(I^n, \partial I^n)$ and their subgroups. The case $n = 1$ is well-studied, and one knows the following striking theorem:

THEOREM 3.11 (Brin-Squier, [5]). — *The group $\text{PL}(I, \partial I)$ obeys no law, but does not contain a nonabelian free subgroup.*

On the other hand, we will shortly see that the group $\text{PL}(I^n, \partial I^n)$ contains nonabelian free subgroups for all $n > 1$.

LEMMA 3.12. — *The group $\text{PL}(I^2, \partial I^2)$ contains nonabelian free subgroups.*

Proof. — For any $M \in \text{GL}(2, \mathbb{R})$ there is some g_M in $\text{PL}(I^2, \partial I^2)$ which fixes 0, which is linear near 0, and which satisfies $dg_M(0) = M$. Let G be the group generated by the g_M . Then G fixes 0 and is linear there, so there is a surjective homomorphism $G \rightarrow \text{GL}(2, \mathbb{R})$. If F is a nonabelian free subgroup of $\text{GL}(2, \mathbb{R})$, there is a section from F to G whose image is a free subgroup of G . \square

In fact we will shortly see (Theorem 3.23) that every right-angled Artin group embeds in $\text{PL}(I^2, \partial I^2)$.

LEMMA 3.13. — *For all n there is an injective homomorphism*

$$S : \text{PL}(I^n, \partial I^n) \rightarrow \text{PL}(I^{n+1}, \partial I^{n+1})$$

called the suspension homomorphism.

Proof. — We include I^n into I^{n+1} as the subset with last coordinate equal to 0. The *bipyramid* P^{n+1} is the convex hull of I^n and the two points $(0, 0, \dots, 0, \pm 1)$. For every simplex σ in I^n there are two simplices $S^\pm \sigma$ of one dimension higher, obtained by coning σ to $(0, 0, \dots, 0, \pm 1)$. If f is a PL homeomorphism of I^n fixed on the boundary, and σ is a simplex in I^n linear for f , then there is a unique PL homeomorphism Sf of P^{n+1} which takes each $S^\pm \sigma$ linearly to $S^\pm f(\sigma)$. The map $f \rightarrow Sf$ defines an injective homomorphism $\text{PL}(I^n, \partial I^n) \rightarrow \text{PL}(P^{n+1}, \partial P^{n+1})$. Then any element of $\text{PL}(P^{n+1}, \partial P^{n+1})$ can be extended by the identity on $I^{n+1} - P^{n+1}$ to an element of $\text{PL}(I^{n+1}, \partial I^{n+1})$. \square

COROLLARY 3.14. — *The group $\text{PL}(I^n, \partial I^n)$ contains a nonabelian free subgroup for all $n > 1$.*

Proof. — By Lemma 3.12, Lemma 3.13 and induction. \square

Actually, we will see many more constructions of free subgroups of $\text{PL}(I^n, \partial I^n)$ in the sequel.

3.5. Area preserving subgroups

We are interested in the following subgroup of $\text{PL}(I^2, \partial I^2)$.

DEFINITION 3.15. — *Let ω denote the (standard) area form on I^2 . Let $\text{PL}_\omega(I^2, \partial I^2)$ denote the subgroup of $\text{PL}(I^2, \partial I^2)$ consisting of transformations which preserve ω .*

The group $\text{PL}_\omega(I^2, \partial I^2)$ contains many interesting subgroups, and has a rich algebraic structure. We give some indications of this.

Example 3.16 (Dehn twist). — Figure 1 depicts an area-preserving PL homeomorphism of a square, which preserves the foliation by concentric squares, and (for suitable choices of edge lengths) whose 12th power is a Dehn twist (in fact, this transformation is contained in a 1-parameter subgroup of $\text{PL}_\omega(I^2, \partial I^2)$ consisting of powers of a Dehn twist at discrete times).

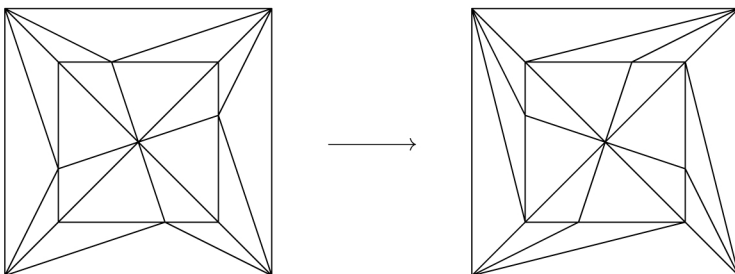


Figure 1. — A 12th root of a PL Dehn twist.

The following theorem is due to Gratzka [15]; in fact he proved the analogous statement for *piecewise linear symplectic* automorphisms of I^{2n} .

THEOREM 3.17 (Gratzka). — *The group $\text{PL}_\omega(I^2, \partial I^2)$ is dense in $\text{Diff}_\omega(I^2, \partial I^2)$, in the C^0 topology.*

Gratzka's theorem has the following application: it can be used to certify that $\text{PL}_\omega(I^2, \partial I^2)$ admits an infinite dimensional space of nontrivial *quasimorphisms* (i.e. quasimorphisms which are not homomorphisms). By contrast, no subgroup of $\text{PL}(I, \partial I)$ admits a nontrivial quasimorphism, by [8]. For an introduction to the theory of quasimorphisms and its relation to stable commutator length, see [9].

DEFINITION 3.18. — Let G be a group. A homogeneous quasimorphism on G is a function $\phi : G \rightarrow \mathbb{R}$ satisfying the following properties:

1. (homogeneity) for any $g \in G$ and any $n \in \mathbb{Z}$ we have $\phi(g^n) = n\phi(g)$; and
2. (quasimorphism) there is a least non-negative number $D(\phi)$ (called the defect) so that for all $g, h \in G$ there is an inequality

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D(\phi)$$

The (real vector) space of homogeneous quasimorphisms on G is denoted $Q(G)$.

A function satisfying the second condition but not the first is said to be (simply) a *quasimorphism*. If $\phi : G \rightarrow \mathbb{R}$ is any quasimorphism, the function $\bar{\phi} : G \rightarrow \mathbb{R}$ defined by $\bar{\phi}(g) = \lim_{n \rightarrow \infty} \phi(g^n)/n$ is a homogeneous quasimorphism, and satisfies $\bar{\phi} - \phi \leq D(\phi)$. This operation is called *homogenization* of quasimorphisms.

A homogeneous quasimorphism has defect 0 if and only if it is a homomorphism to G . Thus D descends to a *norm* on the quotient space $Q(G)/H^1(G; \mathbb{R})$. It is a fact that Q/H^1 with the defect norm is a *Banach space*.

It is known that Q/H^1 vanishes on any subgroup G of $\text{PL}(I, \partial I)$. By contrast, we show that the subgroup $\text{PL}_\omega(I^2, \partial I^2)$ admits an infinite dimensional Q/H^1 .

The proof is by explicit construction, and based on a general method due to Gambaudo-Ghys [14]. The construction is as follows. First, fix some n and let B_n denote the braid group on n strands, and fix n distinct points x_1^0, \dots, x_n^0 in the interior of I^2 . Let $\mu : B_n \rightarrow \mathbb{R}$ be any function.

Now, since $\text{Homeo}(I^2, \partial I^2)$ is path-connected and simply-connected, for any $g \in \text{PL}_\omega(I^2, \partial I^2)$ there is a unique homotopy class of isotopy g_t from g to Id .

For any (generic) n -tuple of distinct points x_1, \dots, x_n in the interior of I^2 , and any $g \in \text{PL}_\omega(I^2, \partial I^2)$, let $\gamma(g; x_1, \dots, x_n) \in B_n$ be the braid obtained by first moving the x_i^0 in a straight line to the x_i , then composing with the isotopy g_t from x_i to $g(x_i)$, then finally moving the $g(x_i)$ in a straight line back to the x_i^0 (note that γ does not depend on the choice of g_t , since $\text{Homeo}(I^2, \partial I^2)$ is simply-connected).

Now we define

$$\Phi_\mu(g) = \int_{I^2 \times \dots \times I^2} \mu(\gamma(g; x_1, \dots, x_n)) d\omega(x_1) d\omega(x_2) \cdots d\omega(x_n)$$

where the integral is taken over the subset of the product of n copies of (the interior of) I^2 consisting of distinct n -tuples of points where γ is well-defined.

LEMMA 3.19. — *Suppose μ is a quasimorphism on B_n . Then Φ_μ is a quasimorphism on $\text{PL}_\omega(I^2, \partial I^2)$*

Proof. — Changing the vector x^0 to a new vector y^0 changes γ by multiplication by one of finitely many elements of B_n ; since μ is a quasimorphism, this changes Φ_μ by a bounded amount.

For any two $g, h \in \text{PL}_\omega(I^2, \partial I^2)$ and generic x_1, \dots, x_n we have

$$\gamma(gh; x_1, \dots, x_n) = \gamma(h; x_1, \dots, x_n) \gamma(g; h(x_1), \dots, h(x_n))$$

Now integrate over $I^2 \times \dots \times I^2$ and use the fact that μ is a quasimorphism to see that Φ_μ is a quasimorphism. \square

The homogenization $\overline{\Phi}_\mu$ is a homogeneous quasimorphism, and one may check that it is nontrivial whenever $\overline{\mu}$ is nontrivial on B_n . In fact, it is easiest to check this for the special case of homogeneous quasimorphisms on B_n that vanish on reducible elements. Such quasimorphisms are plentiful; for example, any “counting quasimorphism” arising from a weakly properly discontinuous action of B_n on a hyperbolic simplicial complex. See [9], § 3.6 for more details.

LEMMA 3.20. — *Let $\overline{\mu}$ be nontrivial on B_n and vanish on all reducible elements. Then $\overline{\Phi}_\mu$ is nontrivial on $\text{PL}_\omega(I^2, \partial I^2)$.*

Proof. — For $1 \leq i \leq n$ let R_i be the rectangle with lower left corner $(\frac{i-1}{n} + \epsilon, \epsilon)$ and upper right corner $(\frac{i}{n} - \epsilon, 1 - \epsilon)$. For any braid $b \in B_n$ we can build an area-preserving homeomorphism ϕ_b which permutes the R_i , taking each R_i to its image by a translation, and which performs the conjugacy class of the braid b on each n tuple of points of the form $(x, x + 1, \dots, x + n - 1)$ for $x \in R_1$.

If we are systematic about the way we extend ϕ_b to $I^2 - \cup_i R_i$ (i.e. by decomposing ϕ_b into a product of standard “elementary” moves, corresponding to the factorization of b into elementary braids) we can ensure that ϕ_b satisfies an estimate of the form $|\mu(\gamma(\phi_b; x_1, \dots, x_n))| \leq C \cdot |b| \cdot D(\mu)$ where $|b|$ denotes the word length of b , where $D(\mu)$ denotes the defect of

μ , and where C is some constant depending only on n , but *not* on ϵ . It is straightforward to build such an area-preserving homeomorphism; to see that it can be approximated by a PL area-preserving homeomorphism with similar properties, we appeal to Gratzka's Theorem 3.17.

For any n -tuple $x := (x_1, \dots, x_n)$ where the x_i are all contained in distinct R_j , the powers of ϕ_b on x are conjugates of the powers of b . For n -tuples where two x_i are in the same R_j , the powers of ϕ_b on x are reducible. Thus we can estimate

$$\overline{\Phi}_\mu(\phi_b) = (n!/n^n)\overline{\mu}(b) + O(\epsilon)$$

Taking b to be a braid on which $\overline{\mu}$ is nonzero, and taking $\epsilon \rightarrow 0$ we obtain the desired result. \square

In particular, $Q(\text{PL}_\omega(I^2, \partial I^2))$ is infinite dimensional. This should be contrasted with the 1-dimensional case, where it is shown in [8] that for *any* subgroup G of $\text{PL}(I, \partial I)$ the natural map $H^1(G) \rightarrow Q(G)$ is surjective.

3.6. Right-angled Artin groups

Recall that a *right-angled Artin group* (hereafter a RAAG) associated to a (finite simplicial) graph Γ is the group with one generator for each vertex of Γ , and with the relation that two generators commute if the corresponding vertices are joined by an edge, and with no other relations. We denote this group by $A(\Gamma)$

Funaru [13] discovered a powerful method to embed certain RAAGs in transformation groups. To describe Funaru's method, it is convenient to work with the complement graph Γ^c , which has the same vertex set as Γ , and which has an edge between two distinct vertices if and only if Γ does *not* have an edge between these vertices. Thus: two vertices of Γ^c are joined by an edge if and only if the corresponding generators of the RAAG do not commute.

Let S be a surface, and for each vertex i of Γ^c , let γ_i be an embedded circle in S , chosen with the following properties:

1. the various γ_i intersect transversely; and
2. two circles γ_i and γ_j intersect if and only if the corresponding vertices of Γ^c are joined by an edge.

We say in this case that the pattern of intersection of the γ_i *realizes* Γ^c .

THEOREM 3.21 (Funar [13]). — *With notation as above, let τ_i denote a Dehn twist in a sufficiently small tubular neighborhood of γ_i . Then for sufficiently big N , the group generated by the τ_i^N is isomorphic to $A(\Gamma)$ by an isomorphism in which the τ_i^N become the “standard” generators.*

Kim and Koberda [21] applied Funar’s technique to show that any finitely-generated RAAG embeds in $\text{Diff}_\omega^\infty(D^2, \partial D^2)$. A similar argument applies to the PL case, as we now outline. Although not every finite graph embeds in the disk, the following theorem saves the day.

THEOREM 3.22 (Kim-Koberda [21]). — *For each finite graph Γ there exists a finite tree T such that $A(\Gamma)$ is isomorphic with a subgroup of $A(T^c)$.*

In fact Kim-Koberda show that $A(\Gamma)$ embeds quasi-isometrically in $A(T^c)$.

THEOREM 3.23. — *For any $m \geq 2$, every (finitely generated) RAAG embeds in $\text{PL}_\omega(I^m, \partial I^m)$.*

Proof. — By the suspension trick (Lemma 3.13), it suffices to consider the case $m = 2$. Given Γ , apply Theorem 3.22 to find a tree so that $A(T^c)$ contains $A(\Gamma)$ as a subgroup. Since trees are planar, one can find PL curves γ_i in I^2 whose pattern of intersection realizes T . Then consider area-preserving PL Dehn twists about these curves and apply Theorem 3.21 to see that $A(T^c)$, and hence $A(\Gamma)$, embeds in $\text{PL}_\omega(I^2, \partial I^2)$. \square

COROLLARY 3.24. — *For any $m \geq 2$, if M is a hyperbolic 3-manifold, some finite-index subgroup of $\pi_1(M)$ embeds in $\text{PL}(I^m, \partial I^m)$.*

Proof. — Agol’s proof of Wise’s conjecture [1] implies that $\pi_1(M)$ virtually embeds in some RAAG. Now apply Theorem 3.23. \square

3.7. Distortion

Theorem 3.4 says that every subgroup of $\text{PL}(I^m, \partial I^m)$ is locally indicable. So far, for $m \geq 2$ we have exhibited no other obstruction to embedding a locally indicable group in $\text{PL}(I^m, \partial I^m)$. In this subsection we give some more powerful obstructions based on the notion of *distortion*.

DEFINITION 3.25 (Distortion). — *Let G be a group with a finite generating set S , and let $|\cdot|_S$ denote word length in G with respect to S . An element $g \in G$ is distorted if*

$$\lim_{n \rightarrow \infty} \frac{|g^n|_S}{n} = 0$$

More precisely, we say g has polynomial/exponential/superexponential (etc.) distortion if

$$\phi_g(m) := \min\{n \text{ such that } |g^n|_S \geq m\}$$

grows polynomially in m (with degree > 1), exponentially, superexponentially, etc.

If G is an infinitely generated group, $g \in G$ is distorted if there is a finitely generated subgroup H of G containing g so that g is distorted in H . It has polynomial/exponential/superexponential (etc.) distortion in G if it has such distortion in some H .

Note with this definition that torsion elements are (infinitely) distorted; some authors prefer the convention that distortion elements should have infinite order. For an introduction to the use of distortion in dynamics, see e.g. [12], [10] or [23].

Remark 3.26. — We remark that other kinds of distortion can be usefully studied in dynamics; for instance, if G is a group, a finitely generated subgroup H is *distorted* in G if there is a finitely generated subgroup $K \subset G$ containing H so that the inclusion of H into K is not a quasi-isometric embedding (with respect to their respective word metrics). The kind of distortion we consider here is the restriction to the case that H is \mathbb{Z} . With this more flexible definition, many interesting examples of distortion can be obtained, even in $\text{PL}(I, \partial I)$. For example, Wladis [26] shows that Thompson's group of dyadic rational PL homeomorphisms of the interval is exponentially distorted in the (closely related) Stein-Thompson groups.

The simplest way to show that an element g is undistorted in a group G , or to bound the amount of distortion, is to find a subadditive, non-negative function on G whose restriction to powers of g grows at a definite rate. We give *two* examples of such functions on groups of the form $\text{PL}(I^m, \partial I^m)$, which show that elements can be at most exponentially distorted.

DEFINITION 3.27 (Matrix norm). — For $M \in \text{GL}(m, \mathbb{R})$ define

$$D(M) = \sup_{i,j} |M_{ij}|$$

where the M_{ij} are the matrix entries of M . If $g \in \text{PL}_+(I^m, \partial I^m)$, let Σ_i be simplices of I^m (of full dimension) such that the restriction of g to each Σ_i is affine, and let $M_i(g) \in \text{GL}(m, \mathbb{R})$ be the linear part of g on Σ_i . Then define

$$D(g) = \sup_i D(M_i(g))$$

The next lemma relates the quantity D to word length in finitely generated subgroups of $\text{PL}(I^m, \partial I^m)$.

LEMMA 3.28. — *Let G be a finitely generated subgroup of $\text{PL}(I^m, \partial I^m)$, and let S be a finite generating set. There is an inequality*

$$D(g) \leq (m \cdot \max_{s \in S} D(s))^{|g|_S}$$

for any $g \in G$, where $|g|_S$ denotes word length of g with respect to the given generating set.

Proof. — If M and N are elements of $\text{GL}(m, \mathbb{R})$, then $D(MN) \leq mD(M)D(N)$. If $g, h \in \text{PL}(I^m, \partial I^m)$ are arbitrary, with linear parts $M_i(g)$ and $N_j(h)$ with respect to subdivisions of I^m into simplices Δ_i, Δ'_j then the linear part of gh is everywhere equal to some $M_i(g)N_j(h)$; thus $D(gh) \leq mD(g)D(h)$. From this the lemma follows. \square

Thus the logarithm of D is (up to an additive constant) a subadditive function on $\text{PL}(I^m, \partial I^m)$. On the other hand, it turns out that under powers, $D(g^n)$ must grow at least *linearly* with n :

PROPOSITION 3.29. — *Let $g \in \text{PL}(I^m, \partial I^m) - \text{Id}$. Then there is an inequality*

$$D(g^n) \geq nC$$

for some positive C depending on g .

Proof. — Let σ be a simplex of codimension 1 in the frontier of $\text{fix}(g)$. After subdividing σ if necessary, we can find a simplex Δ on which g is affine and nontrivial, in such a way that σ is a face of Δ . Then for any point p in the interior of σ , and for any n , the interior of $g^n(\Delta) \cap \Delta$ contains points arbitrarily close to p . If M is the linear part of g on Δ , then g^n has linear part M^n on some nonempty open subset. Since M fixes a codimension 1 subset, either M has an eigenvalue of absolute value $\neq 1$ (in which case $D(g^n)$ grows exponentially) or M is a shear, in which case $D(g^n)$ grows linearly. \square

From Lemma 3.28 and Proposition 3.29 we immediately obtain the following corollary:

COROLLARY 3.30. — *Every element of $\text{PL}(I^m, \partial I^m) - \text{Id}$ is at most exponentially distorted.*

We can use this corollary to give many examples of locally indicable groups which do not admit an injective homomorphism to any $\text{PL}(I^m, \partial I^m)$.

Example 3.31. — Let G be the iterated HNN extension given by the presentation

$$G := \langle a, b, c \mid bab^{-1} = a^2, cbc^{-1} = b^2 \rangle$$

Evidently, b is exponentially distorted, and a is super-exponentially distorted. On the other hand, G is locally indicable. To see this, first observe that there is a surjection from G onto the Baumslag-Solitar group $BS(1, 2) := \langle b, c \mid cbc^{-1} = b^2 \rangle$ sending $b \rightarrow b$, $c \rightarrow c$ and $a \rightarrow 1$. The group $BS(1, 2)$ is locally indicable, and the kernel is a free product of copies of the locally indicable group $\mathbb{Z}[\frac{1}{2}]$, and therefore G is locally indicable as claimed.

In fact the very same group that appears in Example 3.31 has been shown not to embed in $\text{Diff}_+^1(I)$; see [3], Thm. 1.12.

It is interesting that the combinatorial structure of a PL map naturally gives rise to another subadditive function on $\text{PL}(I^m, \partial I^m)$ which can also be used to control distortion.

DEFINITION 3.32 (Triangle number). — *Let $g \in \text{PL}(I^m, \partial I^m)$. The triangle number of g , denoted $\Delta(g)$, is the least number of PL simplices into which I^m can be subdivided so that the restriction of g to each simplex is linear.*

Note that $\Delta(g^{-1}) = \Delta(g)$. For, if τ_g is a triangulation such that g is linear on each simplex, then the image $g(\tau_g)$ is a triangulation with the same number of simplices, such that g^{-1} is linear on each simplex.

The next lemma relates triangle number to word length in finitely generated subgroups of $\text{PL}(I^m, \partial I^m)$.

LEMMA 3.33. — *Let G be a finitely generated subgroup of $\text{PL}(I^m, \partial I^m)$, and let S be a finite generating set. There is a constant C (depending only on the dimension m) and an inequality*

$$\Delta(g) \leq (C \cdot \max_{s \in S} \Delta(s))^{|g|_S}$$

Proof. — For each generator $s \in S$, let's let τ_s be a triangulation of I^m by $\Delta(s)$ simplices such that the restriction of s^{-1} to each simplex is linear. Now let σ be any PL simplex in I^m . For each simplex τ of τ_s the intersection $\tau \cap \sigma$ is convex and cut out by at most $2m + 2$ hyperplanes (since each simplex individually is cut out by $m + 1$ hyperplanes). Thus it is one of finitely many combinatorial types (depending only on the dimension m) and there is a constant C such that this intersection may be subdivided into at most C linear simplices.

It follows that if $w \in G$ is arbitrary, and τ_w is a triangulation of I^m by $\Delta(w)$ simplices such that the restriction of w to each simplex is linear, then by subdividing the intersections of simplices in τ_s and τ_w , we obtain a triangulation of I^m , which by construction can be pulled back by s^{-1} to a triangulation with at most $C\Delta(s)\Delta(w)$ simplices, and such that the product ws is linear on each simplex. The lemma follows by induction. \square

The case of dimension 1 is much simpler. Triangle number counts simply the number of breakpoints (plus 1), and can grow at most linearly:

LEMMA 3.34. — *Let G be a finitely generated subgroup of $\text{PL}(I, \partial I)$, and let S be a finite generating set. There is an inequality*

$$\Delta(g) - 1 \leq \max_{s \in S} (\Delta(s) - 1) \cdot |g|_S$$

Proof. — Let $w \in G$ have breakpoints at p_i and let s have breakpoints at q_j . Then each breakpoint of ws is of the form q_j or $s^{-1}(p_i)$. \square

On the other hand, at least in dimension 2, triangle number must grow at least linearly under powers:

PROPOSITION 3.35. — *Let $g \in \text{PL}(I^2, \partial I^2) - \text{Id}$. Then there is an inequality*

$$\Delta(g^n) \geq n$$

Proof. — Consider the action of g at a vertex in the frontier of $\text{fix}(g)$. The projectivization of this action is a piecewise projective action on the interval. We claim that there is a vertex $v \in \text{fro}(g)$ and a subinterval J of the projective tangent bundle at v which is fixed by g , where g acts (topologically) conjugate to a translation, and such that g has a break point in the interior of J . Assuming this claim, the proof proceeds as follows. Suppose (for the sake of argument) that g moves points on this interval in the positive direction, and suppose furthermore that there is at least one breakpoint in this interval. Let p be the uppermost breakpoint. Denote by $[p, J^+]$ the positive subinterval of J bounded (on one side) by p . Then the points $g^i(p)$ for $0 < i < n$ are all distinct and contained in the open interval (p, J^+) where the action of g is smooth; thus by induction, g^n is singular on J at these points, so $\Delta(g^n) \geq n$.

We now prove the claim. Let γ be a component of the frontier of $\text{fix}(g)$, so that γ is an immersed polyhedral circle. We further subdivide γ if necessary, adding vertices at points where g is not locally linear. Denote the edges of γ in cyclic order by e_i (indices taken cyclically), and orient these edges so they point in the positive direction around γ (with respect to the

orientation it inherits as a boundary component of the region P that g does not fix pointwise). Let v_i be the initial vertex of e_i . Subdivide P near v_i into polyhedra on which the element g acts linearly, and let e_i^+ and e_i^- be the extremal edges of this subdivision which point into the interior of P , where the choice of \pm is such that the tangents to the edges e_i, e_i^-, e_i^+ at v_i are positively circularly ordered in the unit tangent circle at v_i . We think of γ now as a map $\gamma : S^1 \rightarrow I^2$ and pull back the unit tangent bundle over the image to a circle bundle U over γ . Let $V \subset U$ be the subset of tangents pointing locally into the interior of P ; thus V is an (open) interval bundle over S^1 . The argument function lifts locally (in γ) to a real valued function on each fiber, in such a way that holonomy around γ in the positive direction increases the argument by 2π (this is just a restatement of the fact that the winding number of an embedded loop is 1 around the region it bounds). Since the (real-valued) argument of e_i^- in each fiber is at most equal to the argument of e_i^+ , it follows that there must be some index i for which the argument of e_{i+1}^+ is *strictly* bigger than that of e_i^- , as measured in the bundle V restricted to the contractible subset $e_i \subset \gamma$ (where the argument is globally well-defined). If P_i is the region of P on which g is linear which is bounded on three (consecutive) sides by e_i^-, e_i, e_{i+1}^+ , then e_i^- and e_{i+1}^+ can be extended to bound a triangle T in such a way that $T \cap P_i$ contains a neighborhood of e_i in $I^2 - \text{fix}(g)$. See Figure 2.

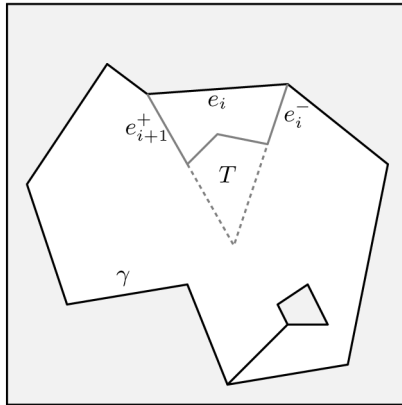


Figure 2. — The (dashed) triangle T associated to the region P_i .

A linear map of the plane which fixes the edges of a triangle is trivial; but since the interior of P_i is not in $\text{fix}(g)$, it follows that g must move at least one of the edges e_i^- and e_{i+1}^+ off itself; the projective action of g at the corresponding vertex must therefore move some breakpoint off itself, proving the claim, and therefore the proposition. \square

The argument of Proposition 3.35 actually shows $\Delta(g^n) \geq n$ for any $g \in \text{PL}(I, \partial I)$, since if p is an uppermost breakpoint of g , the points $g^i(p)$ are all breakpoints of g^n for $0 < i < n$. Thus by Lemma 3.34 we conclude that every nontrivial element is undistorted in $\text{PL}(I, \partial I)$.

QUESTION 3.36. — *Is there a nontrivial distorted element in $\text{PL}(I^2, \partial I^2)$? How about $\text{PL}(I^m, \partial I^m)$ for any $m \geq 2$?*

QUESTION 3.37. — *Does the argument of Proposition 3.35 generalize to dimensions $n \geq 3$?*

Remark 3.38. — We learned from the referee that it is an open question whether the group $\text{PL}(S^1)$ contains distorted elements. The referee points out that such an element (if it exists) must be (topologically) conjugate to a rotation. The referee further remarks that Avila [2] has shown that every rotation is (arbitrarily) distorted in the group of C^∞ diffeomorphisms of S^1 .

4. C^1 group actions

In this section, for the sake of completeness and to demonstrate the similarity and difference of methods, we prove the analog of Theorem 3.4 for groups acting smoothly on manifolds, fixing a codimension 1 submanifold pointwise.

4.1. Definitions

Notation 4.1. — *Let M be a smooth manifold (possibly with smooth boundary) and let K be a closed smooth submanifold of M . We denote by $\text{Diff}^1(M, K)$ the group of C^1 self-diffeomorphisms of M which are fixed pointwise on K . If M is orientable, we denote by $\text{Diff}_+^1(M, K)$ the subgroup of orientation-preserving C^1 self-diffeomorphisms.*

The main example of interest is the smooth closed unit ball D^n in \mathbb{R}^n , and its boundary $\partial D^n = S^{n-1}$.

One of the most important theorems about groups of C^1 diffeomorphisms is the *Thurston Stability Theorem*:

THEOREM 4.2 (Thurston Stability Theorem, [25]). — *Let G be a group of germs of C^1 diffeomorphisms of \mathbb{R}^n fixing 0. Let $\rho : G \rightarrow \text{GL}(n, \mathbb{R})$ be the linear part of G at 0; i.e. the representation sending $g \rightarrow dg|_0$, and let K be the kernel of ρ . Then K is locally indicable.*

4.2. Left orderability

The purpose of this section is to prove the following C^1 LO Theorem:

THEOREM 4.3 (C^1 LO). — *Let M be an n dimensional connected smooth manifold, and let K be a nonempty $n - 1$ dimensional closed submanifold. Then the group $\text{Diff}_+^1(M, K)$ is left orderable; in fact it is locally indicable.*

An important special case is $M = D^n$ and $K = \partial D^n = S^{n-1}$.

DEFINITION 4.4 (Fixed rank). — *Let $f \in \text{Diff}^1(M, K)$. The fixed rank of f at a point p is defined to be -1 if f does not fix p , and otherwise to be the dimension of the subspace of T_p fixed by $df|_p$. Similarly, if G is a subgroup of $\text{Diff}^1(M, K)$, the fixed rank of G at a point p is -1 if p is not in $\text{fix}(G)$, and otherwise is the dimension of the subspace of T_p fixed by $\rho_p : G \rightarrow \text{GL}(n, \mathbb{R})$ sending $g \rightarrow dg|_p$ for all $g \in G$.*

Recall that a (real-valued) function f on a topological space is *upper semi-continuous* if for every point p and every $\epsilon > 0$ there is a neighborhood U of p so that $f(x) \leq f(p) + \epsilon$ for all $x \in U$. In other words, the value of f can “jump up” at a limit, but not down. If f is upper semi-continuous, then for any t the set where $f \geq t$ is closed.

LEMMA 4.5. — *Let G be a subgroup of $\text{Diff}^1(M, K)$. Then the fixed rank of G is upper semi-continuous on M . Consequently the subset where the fixed rank is $\geq m$ is closed, for any m .*

Proof. — Let $p_i \rightarrow p$ and suppose that G has fixed rank $\geq m$ on all p_i . If $m = -1$ there is nothing to prove. Otherwise, the p_i are all fixed by every $g \in G$ and there is a subspace $\pi_i \subset T_{p_i}$ of dimension m fixed by every $dg|_{p_i}$. By compactness of the Grassmannian of m -dimensional subspaces of \mathbb{R}^n , after passing to a subsequence we can assume that π_i converges to some m -dimensional subspace $\pi \subset T_p$. Observe that for every g we must have that $dg|_p$ fixes π , or else some $dg|_{p_i}$ would fail to fix π_i , since the action is C^1 . Thus the fixed rank of G at p is $\geq m$. \square

We can now give the proof of Theorem 4.3:

Proof. — Let G be a nontrivial finitely generated subgroup of $\text{Diff}_+^1(M, K)$, and let X be the subset of G where the fixed rank is $\geq n - 1$. Since G is arbitrary, it suffices to show that G admits a surjection to \mathbb{Z} .

By Lemma 4.5 the set X is closed. Furthermore, it is nonempty, since it includes K . Since G is nontrivial, some point of M is moved by some

element of G , and therefore the fixed rank is -1 at that point; since M is connected, it follows that the frontier $\text{fro}(X)$ is nonempty.

Let $p \in \text{fro}(X)$ be arbitrary, and let $\pi \subset T_p M$ be an $n - 1$ dimensional plane fixed by $dg|_p$ for all $g \in G$. Let $\rho : G \rightarrow \text{GL}(n, \mathbb{R})$ send $g \rightarrow dg|_p$. By Lemma 3.9 and Example 3.10 the image of ρ is locally indicable. If the image is nontrivial we are done. So suppose the image is trivial. Since p is in the frontier of X it must be in the frontier of $\text{fix}(G)$ (or else it would be in the interior of the set where the fixed rank is n) so the image of G in the group of germs of diffeomorphisms fixing p is nontrivial. So the Thurston stability theorem (i.e. Theorem 4.2) implies that G surjects to \mathbb{Z} . This completes the proof. \square

5. Bi-orderability

If a left-order \prec of a group G is also invariant under right-multiplication, we'll call it a *bi-order* and say that G is *bi-orderable*. It is easy to see that a left-order \prec is a bi-order if and only if its positive cone

$$P := \{g \in G \text{ such that } 1 \prec g\}$$

is invariant under conjugation; i.e. if and only if $g^{-1}Pg \subset P$ for all $g \in G$.

In this section we discuss bi-orderability for various subgroups of $\text{Homeo}(I^n, \partial I^n)$ in specific dimensions, and with various kinds of regularity.

5.1. Bi-orderability in general

First of all, it should be noted that the distinction between left orderability and bi-orderability is not vacuous:

Example 5.1. — If G is a group in which there is a nontrivial element g so that some product of conjugates of g is trivial, then G is not bi-orderable. For, in any left ordering, we may assume that $g \in P$ (up to reversing the order). So, for example, the Klein bottle group $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ is not bi-orderable, though it *is* locally indicable.

One may summarize the existence of LO groups which are not bi-orderable in the following proposition:

PROPOSITION 5.2. — $\text{Homeo}(I, \partial I)$ is not bi-orderable.

This is because $\text{Homeo}(I, \partial I)$ contains isomorphic copies of all countable LO groups, many of which are not bi-orderable. Suspending to higher dimensions, we conclude:

COROLLARY 5.3. — For all $n \geq 1$, $\text{Homeo}(I^n, \partial I^n)$ is not bi-orderable.

5.2. Bi-orderability in PL

The following was observed in a slightly different context by Chehata [11].

PROPOSITION 5.4. — $\text{PL}(I, \partial I)$ is bi-orderable.

One can take as positive cone P for a bi-ordering all PL functions $f : I \rightarrow I$ whose graph departs from the diagonal for the first time with slope greater than 1. In other words, if x_0 is the maximal element of $I = [-1, 1]$ such that $f(x) = x$ for all $x \in [-1, x_0]$, then for all sufficiently small $\epsilon > 0$ we have $f(x_0 + \epsilon) > x_0 + \epsilon$. Then P is closed under composition of functions, conjugation, and $\text{PL}(I, \partial I) = \{1\} \sqcup P \sqcup P^{-1}$, so it defines a bi-order by the recipe $g < h \iff g^{-1}h \in P$.

PROPOSITION 5.5. — $\text{PL}_\omega(I^2, \partial I^2)$ is not bi-orderable.

To see this, we consider two functions $f, g \in \text{PL}_\omega(I^2, \partial I^2)$ defined as follows. Let h denote the function illustrated in Figure 1. Recall that h^{12} is a Dehn twist, which is the identity on the inner square, as well as on ∂I^2 . Let $f := h^6$, so that f rotates the inner square by 180 degrees. Referring to Figure 3, define g to be the identity outside the little squares, which are strictly inside the inner square rotated by f . On the little square on the left, let g act as h , suitably scaled, and on the little square to the right let g act as h^{-1} . Noting that f interchanges the little squares, and that h commutes with 180 degree rotation, one checks that $f^{-1}gf = g^{-1}$.

Such an equation cannot hold (for g not the identity) in a bi-ordered group, as it would imply the contradiction that g is greater than the identity if and only if g^{-1} is greater than the identity (this is just the Klein bottle group from Example 5.1).

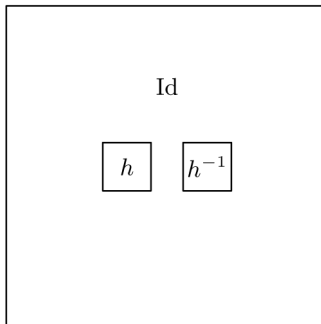


Figure 3. — Building the function $g \in \text{PL}_\omega(I^2, \partial I^2)$.

COROLLARY 5.6. — *For $n \geq 2$ none of the groups $\text{PL}_\omega(I^n, \partial I^n)$, $\text{PL}(I^n, \partial I^n)$ is bi-orderable.*

This follows from Lemma 3.13, noting that suspension also takes $\text{PL}_\omega(I^n, \partial I^n)$ isomorphically into $\text{PL}_\omega(I^{n+1}, \partial I^{n+1})$.

Recall Theorem 3.4: if M is a connected PL n -manifold and K is a nonempty closed PL $n - 1$ dimensional submanifold, then $\text{PL}(M, K)$ is LO. Noting that one may include $\text{PL}(I^n, \partial I^n)$ in $\text{PL}(M, K)$, by acting on a small n -ball in M , we see also

COROLLARY 5.7. — *For $n \geq 2$, $\text{PL}(M, K)$ is not bi-orderable.*

5.3. Bi-orderability in Diff

By an argument similar to Proposition 5.5 one can show

PROPOSITION 5.8. — *For $n \geq 2$ and for any $0 \leq p \leq \infty$, the group $\text{Diff}^p(I^n, \partial I^n)$ is not bi-orderable.*

Note that in every case bi-orderability is ruled out by the existence of a nontrivial element which is conjugate to its inverse.

For $n = 1$ and $p = 0, 1$ bi-orderability in $\text{Diff}^p(I, \partial I)$ is ruled out for the same algebraic reason:

Example 5.9 (C^1 example). — For $i \in \mathbb{Z}$ let x_i be a discrete, ordered subset of the interior of I accumulating at the endpoints like the harmonic series. Let f be a C^1 diffeomorphism whose fixed point set in the interior of I is exactly the union of the x_i , and which alternates between translating in the positive and negative direction on intervals (x_i, x_{i+1}) for i respectively even and odd. Let g take x_i to x_{i+1} and conjugate the action of f to f^{-1} . This construction can evidently be made C^1 in the interior. Moreover, if we arrange for f' to converge uniformly to 0 towards the endpoints, then the same is true of g , since g acts there almost like a linear map $[1 - 1/i, 1] \rightarrow [1 - 1/(i + 1), 1]$, whose derivatives converge uniformly to 0.

On the other hand, this construction cannot be made C^2 , by Kopell's Lemma, which says the following:

THEOREM 5.10 (Kopell's Lemma [22], Lemma 1). — *Let f and g be commuting elements of $\text{Diff}^2(I, \partial I)$. If g has no fixed point in the interior of I but f has a fixed point in the interior of I then $f = \text{Id}$.*

COROLLARY 5.11. — *In $\text{Diff}^2(I, \partial I)$ no nontrivial element is conjugate to its inverse.*

Proof. — If f is nontrivial and conjugate to its inverse, there are some subintervals where it acts as a positive translation, and some where it acts as a negative translation, and some point between the two is fixed. Let g conjugate f to f^{-1} . Applying Kopell's lemma to g and f^2 we see that $f^2 = \text{Id}$ so $f = \text{Id}$. \square

In particular, the Klein bottle group does not embed in $\text{Diff}^2(I, \partial I)$.

Remark 5.12. — We suspect that the group $\text{Diff}^p(I, \partial I)$ is not bi-orderable for any finite p , but have not been able to show this. In this remark we give a detailed construction of a C^p group action (for any fixed p) containing two specific elements which are C^1 conjugate; if the construction could be done in such a way that the elements are C^p conjugate, it would prove that $\text{Diff}^p(I, \partial I)$ is not bi-orderable, but we have not been able to show this.

We build a C^p diffeomorphism f of I as follows (for simplicity of formulae, we take here I to be the interval $[0, 1]$ instead of $[-1, 1]$ as throughout the rest of the paper). For $i < 0$ let $p_i = 2^i$ and for $i \geq 0$ let $p_i = 1 - 2^{-i-2}$. We build f fixing each p_i , and acting alternately as a positive and negative translation on complementary regions. We will insist that f is infinitely tangent to the identity at each p_i . For concreteness, on each of the intervals $[7/8, 15/16]$ and $[1/16, 1/8]$ we let f be the time 1 flow of a C^∞ vector field with no zeros in the interior of the intervals, and infinitely tangent to zero at the endpoints. The definition of f on the rest of I will be defined implicitly in what follows.

Now, we let g be a diffeomorphism which takes p_i to p_{i+1} for $i \geq 0$ and takes p_i to p_{i-1} for $i < 0$, and let g act as a dilation (i.e. with locally constant nonzero derivative) on open neighborhoods of 0 and 1. Evidently we can choose such a g to be C^∞ . Finally, we arrange the dynamics of f on the complementary intervals so that g conjugates f to f^{-k} outside the interval $[1/4, 7/8]$ for some big k , depending on p (which we will specify shortly). Since f (on the intervals $[7/8, 15/16]$ and $[1/16, 1/8]$) is the time 1 flow of a vector field, taking an n th root multiplies each C^p norm by $O(1/n)$ when n is big. On the other hand, conjugating a diffeomorphism by dilation by $1/\lambda$ blows up the C^p norm by a factor of λ^{p-1} , so providing $k > 2^{p-1}$ the diffeomorphism f we build with this property will be C^∞ on $(0, 1)$, and C^p tangent to the identity at 0 and 1, and therefore is an element of $\text{Diff}^p(I, \partial I)$. By construction, the product $h := f^k g f g^{-1}$ is supported on the interval $[1/4, 7/8]$ which is subdivided into 3 intervals J_0, J_1, J_2 where h acts alternately as positive, negative, positive translation. If we are careful

in our choice of g , we may ensure that h is infinitely tangent to the identity at the endpoints of the J_i .

By a similar construction, we can choose a slightly different element j , whose germs at 0 and 1 agree with g , so that j conjugates f to f^{-k} outside the interval $[1/8, 3/4]$, and then we can arrange that $h' := f^k j f j^{-1}$ is supported on the interval $[1/8, 3/4]$ and acts alternately on a subdivision J'_0, J'_1, J'_2 as negative, positive, negative translation, and infinitely tangent to the identity at the endpoints.

Now, suppose for a suitable choice of g and j as above, we could find some e which conjugates h' to h^{-1} , so that we get a relation of the form

$$e(f^k j f j^{-1})e^{-1} f^k g f g^{-1} = \text{Id}$$

But this word is a product of conjugates of f , and therefore a relation of this kind would certify that $\text{Diff}^p(I, \partial I)$ is not bi-orderable. We suspect that such an e can be found (for some g and j), but leave this as an open problem.

In view of the construction outlined in Remark 5.12 it is a bit surprising that the group $\text{Diff}^\omega(I, \partial I)$ is bi-orderable:

PROPOSITION 5.13. — *The group $\text{Diff}^\omega(I, \partial I)$ is bi-orderable.*

Proof. — Define a bi-ordering on $\text{Diff}^\omega(I, \partial I)$ as follows. If the derivative $f'(0)$ is > 1 , put the element f in the positive cone. If $f'(0) = 1$, look at the Taylor expansion at 0. This is of the form

$$f(t) = t + a_2 t^2 + a_3 t^3 + \dots$$

Since f is real-analytic, either $f = \text{Id}$, or else there is some first index i such that $a_i \neq 0$. Then put f in the positive cone if $a_i > 0$.

One checks that this really does define a cone, and that for every non-trivial element f , either f is in the cone, or f^{-1} is in the cone. Finally, the cone is conjugation invariant.

More geometrically, f is in the positive cone if $f(t) > t$ for all *sufficiently small* $t > 0$. That this is conjugation invariant and a cone is obvious. That it is well-defined follows from the fact that a nontrivial real analytic diffeomorphism of I can't have infinitely many fixed points. \square

We remark Proposition 5.13 has already appeared in the literature; see [24], Example 3.5.

6. Groups fixing codimension 2 submanifolds

In this section we establish analogs of Theorem 3.4 and Theorem 4.3 for groups acting in a PL or smooth manner on manifolds, fixing a submanifold of codimension 2 pointwise. The conclusion is not that such groups are left orderable (they are not in general) but that they are *circularly orderable*.

6.1. Circularly ordered groups

We recall standard definitions and facts about circular orders; [7], Chapter 2 is a reference.

DEFINITION 6.1. — *Let Σ be a set and let $\Sigma^{(3)}$ denote the space of distinct, ordered triples in Σ . A circular order on Σ is a map $e : \Sigma^{(3)} \rightarrow \pm 1$ satisfying the cocycle condition*

$$e(s_1, s_2, s_3) - e(s_0, s_2, s_3) + e(s_0, s_1, s_3) - e(s_0, s_1, s_2) = 0$$

for all distinct quadruples s_0, s_1, s_2, s_3 in Σ .

A (left) circular order on a group G is a circular order on G which is invariant under left-multiplication; i.e. it satisfies

$$e(gg_1, gg_2, gg_3) = e(g_1, g_2, g_3)$$

for all g in G and all distinct triples $(g_1, g_2, g_3) \in G^{(3)}$.

Remark 6.2. — Note that e may be extended to all triples Σ^3 by defining e to be 0 on $\Sigma^3 - \Sigma^{(3)}$; this extension also satisfies the cocycle condition.

Notation 6.3. — A group which admits an invariant circular order is said to be *circularly orderable*, which we usually abbreviate by CO.

LEMMA 6.4. — *Let G be a group. Suppose every finitely generated subgroup of G is CO. Then G is CO.*

LEMMA 6.5. — *If there is a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$ where H is circularly orderable and K is left orderable, then G is circularly orderable.*

In particular, left orderable groups are circularly orderable.

LEMMA 6.6. — *A countable group is circularly orderable if and only if it is isomorphic to a subgroup of $\text{Homeo}_+(S^1)$. Conversely, every subgroup of $\text{Homeo}_+(S^1)$ is circularly orderable.*

LEMMA 6.7. — *Suppose G is circularly orderable. The (extended) function e is a \mathbb{Z} -valued 2-cocycle on G and thereby determines a class $[e] \in H^2(G; \mathbb{Z})$. If the class $[e] = 0$ then G is left orderable.*

Example 6.8. — The group $\mathrm{GL}^+(2, \mathbb{R})$ is an extension $0 \rightarrow \mathbb{R}^+ \rightarrow \mathrm{GL}^+(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R}) \rightarrow 0$. The group $\mathrm{SL}(2, \mathbb{R})$ acts faithfully and in an orientation-preserving way on the circle of linear rays through the origin, and is therefore circularly orderable, by Lemma 6.6. By Lemma 6.5 the group $\mathrm{GL}^+(2, \mathbb{R})$ is CO.

6.2. Knots

We would like to extend the results from § 3.3 and § 4.2 to pairs (M, K) where the codimension of K is 2. Motivated by the interesting example where $M = S^3$ and K is a circle, we use the informal term “knot” for K although we do not assume either that K is connected, or that it is homeomorphic to a sphere.

THEOREM 6.9 (C^1 Knots CO). — *Let M be an n dimensional connected smooth orientable manifold, and let K be a nonempty $n - 2$ dimensional closed submanifold. Then the group $\mathrm{Diff}_+^1(M, K)$ is circularly orderable.*

Proof. — Fix some point $k \in K$. The linear representation $\rho : \mathrm{Diff}_+^1(M, K) \rightarrow \mathrm{GL}(n, \mathbb{R})$ defined by $\rho(g) = dg|_k$ fixes a subspace π of dimension $n - 2$, and therefore by Lemma 3.9 has image in $\mathbb{R}^{2(n-2)} \rtimes \mathrm{GL}^+(2, \mathbb{R})$. The image group is CO by Lemma 6.5, since $\mathrm{GL}^+(2, \mathbb{R})$ is CO (by Example 6.8) and $\mathbb{R}^{2(n-2)}$ is locally indicable and therefore LO. Let K denote the kernel, and let G be a finitely generated subgroup of the kernel. If the germ of G at k is nontrivial, then G surjects onto \mathbb{Z} , by the Thurston stability theorem (i.e. Theorem 4.2). If the germ of G at k is trivial, then each of the finite generators g_i fixes an open neighborhood of k , and therefore $\mathrm{fix}(G)$ has nonempty interior. It follows that the subset X where the fixed rank of G is $\geq n - 1$ is nonempty, and since M is connected $\mathrm{fro}(X)$ is also nonempty. Therefore as in the proof of Theorem 4.3 we deduce that G surjects onto \mathbb{Z} . It follows that K is locally indicable and therefore LO, so $\mathrm{Diff}_+^1(M, K)$ is CO by Lemma 6.5. \square

Before proving the corresponding theorem for PL actions, we must analyze the local structure of a group of PL homeomorphisms at an arbitrary point on an m -dimensional fixed submanifold.

LEMMA 6.10. — *Let G be a countable group of germs at 0 of PL diffeomorphisms of \mathbb{R}^n fixing \mathbb{R}^m . There is a homomorphism from G to the group of piecewise projective automorphisms of \mathbb{RP}^{n-m-1} , and for every finitely*

generated subgroup H of the kernel, either H surjects to \mathbb{Z} or $\text{fix}(H)$ has nonempty interior arbitrarily close to 0. In particular, the kernel is locally indicable, and therefore LO.

Proof. — A linear automorphism of \mathbb{R}^n fixing \mathbb{R}^m acts linearly on the quotient $\mathbb{R}^n/\mathbb{R}^m$ and therefore projectively on \mathbb{RP}^{n-m-1} . For each element $g \in G$ we can pick a subdivision into linear simplices such that the restriction of g to each simplex is linear. The set of points on \mathbb{R}^m in the complement of the $m-1$ skeleton of this subdivision is open and dense, and therefore since G is countable, there is a point p on \mathbb{R}^m near 0 which is in the complement of the $m-1$ skeleton of the subdivision associated to every $g \in G$.

For every $g \in G$ and every simplex σ in the linear subdivision associated to g , there is a well-defined projective action of g on each simplex of the link of g . Putting these actions together for all $g \in G$ gives a piecewise projective action of G on \mathbb{RP}^{n-m-1} at p .

A finitely generated subgroup H of the kernel locally preserves the foliation by planes parallel to \mathbb{R}^m , and acts there by translation. If this action is trivial, $\text{fix}(H)$ has interior near p . \square

THEOREM 6.11 (PL Knots CO). — *Let M be an n dimensional connected PL orientable manifold, and let K be a nonempty $n-2$ dimensional closed submanifold. Then the group $\text{PL}_+(M, K)$ is circularly orderable.*

Proof. — Let G be a countable subgroup of $\text{PL}_+(M, K)$ and consider the action near a point p in an $n-2$ dimensional simplex in K . By Lemma 6.10 there is a homomorphism from the germ of G at p to the group of piecewise projective automorphisms of \mathbb{RP}^1 with LO kernel. Thus the germ of G at p is CO. Any finitely generated subgroup H of the kernel of the map from G to the germ at p has the property that $\text{fix}(H)$ has nonempty interior, and therefore H is LO (in fact, locally indicable) as in the proof of Theorem 3.4. Thus every countable subgroup of $\text{PL}_+(M, K)$ is CO, and therefore $\text{PL}_+(M, K)$ is CO by Lemma 6.4. \square

6.3. Hyperbolic knots

An interesting application is to groups acting on a 3-manifold stabilizing a knot. Explicitly, let's specialize to the case that M is an orientable 3-manifold, and K is a knot with hyperbolic complement. Let $G(M, K)$ denote the group of orientation preserving PL (resp. Diff^1) homeomorphisms of M that take K to itself by an orientation preserving homeomorphism, where we

do *not* assume K is fixed pointwise; and let $G_0(M, K)$ denote the subgroup of such transformations isotopic to the identity.

Since $M - K$ is Haken, as a topological group, the homotopy type of $G(M, K)$ is well-understood by the work of Hatcher and Ivanov. In fact, one has:

THEOREM 6.12 (Hatcher; Hatcher, Ivanov). — *As a topological group, $G_0(M, K)$ is contractible.*

The PL case is due independently to Hatcher and Ivanov; see e.g. [16] or [19]. As was well-known at the time, the smooth case follows from the PL case once one knows the Smale conjecture, proved by Hatcher [17].

COROLLARY 6.13. — *With notation as above, $G_0(M, K)$ is left orderable.*

Proof. — Let's abbreviate $G_0(M, K)$ by G in what follows. The image of G in $\text{Homeo}_+(K)$ is circularly orderable, and we have already shown that the kernel is circularly orderable. We would like to show that both the image and the kernel are actually LO. By Lemma 6.7 we need to show that the cohomology class $[e]$ associated to either circular order is trivial. Notice by construction that $[e]$ is an element of $H^2(BG; \mathbb{Z})$ where G is thought of as a *topological group*, and BG denotes the classifying space for principal G -bundles. But since $M - K$ is hyperbolic, G is contractible, and therefore $H^2(G; \mathbb{Z}) = 0$. \square

7. Groups of homeomorphisms

It is natural to wonder whether the groups $\text{Homeo}(M, K)$ are left- or circularly-ordered when K is of codimension 1 or 2, by analogy with the PL or smooth case. But here the situation is utterly mysterious, even in dimension 2. In fact, the following question still seems far beyond reach:

QUESTION 7.1. — *Is $\text{Homeo}(I^2, \partial I^2)$ left-orderable?*

It is challenging to obtain any restrictions on the subgroups of $\text{Homeo}(I^n, \partial I^n)$ at all, for $n \geq 2$. The purpose of this section is to point out that Smith theory shows that the groups $\text{Homeo}(I^n, \partial I^n)$ are torsion free. This fact seems to be well known to the experts in Smith theory and its generalizations, but not to people working on left-orderable groups, and therefore it seems worthwhile to give a proof (or rather to point out how the result follows immediately from results which are well-documented in the literature).

We use the following theorem, generalizing work of Smith, and due in this generality to Borel [4]:

THEOREM 7.2 (Smith, Borel). — *Let X be a finitistic space (e.g. a compact space) with the mod p homology of a point, and let G be a finite p -group acting on X by homeomorphisms. Then X^G (the fixed point set of G) has the mod p homology of a point.*

Here a *finitistic space* is one such that every covering has a finite dimensional refinement. Any compact space is finitistic, so I^n is certainly finitistic (and the main application of Smith theory is to manifolds and manifold-like spaces).

COROLLARY 7.3. — *The group $\text{Homeo}(I^n, \partial I^n)$ is torsion-free for all n .*

Proof. — If f is a nontrivial periodic homeomorphism of I^n fixed on ∂I^n , then some power of f has prime order p for some nontrivial p , so without loss of generality we may assume that f itself has order p . By Theorem 7.2 the fixed point set of f has the mod p homology of a point. But this fixed point set includes ∂I^n , which is homologically essential (with mod p coefficients) in the complement of any point in I^n . Thus all of I^n is fixed by f . \square

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