ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Tien Duc Luu

On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

Tome XXII, nº 3 (2013), p. 465-493.

<http://afst.cedram.org/item?id=AFST_2013_6_22_3_465_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXII, n° 3, 2013
pp. 465–493

On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

Tien Duc LUU[1]

Abstract. — We prove in this paper the Hölder regularity of Almgren minimal sets of dimension 3 in $\mathbb{R}^4$ around a $\mathbb{Y}$-point and the existence of a point of particular type of a Mumford-Shah minimal set in $\mathbb{R}^4$, which is very close to a $\mathbb{T}$. This will give a local description of minimal sets of dimension 3 in $\mathbb{R}^4$ around a singular point and a property of Mumford-Shah minimal sets in $\mathbb{R}^4$.

Résumé. — On prouve dans cet article la régularité Höldérienne pour les ensembles minimaux au sens d'Almgren de dimension 3 dans $\mathbb{R}^4$ autour d'un point de type $\mathbb{Y}$ et dans le cas d'un ensemble Mumford-Shah minimal dans $\mathbb{R}^4$ qui est très proche d'un $\mathbb{T}$, l'existence d'un point avec une densité particulière. Cela donne une description locale des ensembles minimaux de dimension 3 dans $\mathbb{R}^4$ autour d'un point singulier et une propriété des ensembles Mumford-Shah minimaux dans $\mathbb{R}^4$.

1. Introduction

In this paper we will prove two theorems. The first theorem is about local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$ and the second theorem is about the existence of a point of a particular type of a Mumford-Shah minimal set, which is close enough to a cone of type $\mathbb{T}$.

Let us give the list of notions that we will use in this paper.

(*) Reçu le 22/03/2012, accepté le 20/12/2012

(1) Bâtiment 430, Département de Mathématique, Université Paris Sud XI, 91405 Orsay
luutienduc@gmail.com

Article proposé par Gilles Carron.

H^d the d-dimensional Hausdorff mesure.

$\theta_A(x,r) = \frac{H^d(A\cap B(x,r))}{r^d}$, where $A \subset \mathbb{R}^n$ is a set of dimension d and $x \in A$.

$\theta_A(x) = \lim_{r\to 0} \theta_A(x,r)$, called the density of A at x, if the limit exists.

Local Hausdorff distance $d_{x,r}(E,F)$. Let $E, F \subset \mathbb{R}^n$ be closed sets which meet the ball $B(x,r)$. We define

$$d_{x,r}(E,F) = \frac{1}{r}[\sup\{\text{dist}(z,F); x \in E\cap B(x,r)\}+\sup\{\text{dist}(z,E); z \in F\cap B(x,r)\}].$$

Let $E, F \subset \mathbb{R}^n$ be closed sets and $H \subset \mathbb{R}^n$ be a compact set. We define

$$d_H(E,F) = \sup\{\text{dist}(x,F); x \in E \cap H\} + \sup\{\text{dist}(x,E); x \in F \cap H\}.$$

Convergence of a sequence of sets. Let $U \subset \mathbb{R}^n$ be an open set, $\{E_k\} \subset U, k \geqslant 1$, be a sequence of closed sets in U and $E \subset U$. We say that $\{E_k\}$ converges to E in U and we write $\lim_{k\to\infty} E_k = E$, if for each compact $H \subset U$, we have

$$\lim_{k\to\infty} d_H(E_k, E) = 0.$$

Blow-up limit. Let $E \subset \mathbb{R}^n$ be a closed set and $x \in E$. A blow-up limit F of E at x is defined as

$$F = \lim_{k\to\infty} \frac{E-x}{r_k},$$

where $\{r_k\}$ is any positive sequence such that $\lim_{k\to\infty} r_k = 0$ and the limit is taken in $\mathbb{R}^n$.

Now we give the definition of Almgren minimal sets of dimension d in $\mathbb{R}^n$.

Definition 1.1. — *Let E be a closed set in $\mathbb{R}^n$ and $d \leqslant n-1$ be an integer. An Almgren competitor (Al-competitor) of E is a closed set $F \subset \mathbb{R}^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz mapping such that $W_\varphi = \{x \in \mathbb{R}^n; \varphi(x) \neq x\}$ is bounded.*

An Al-minimal set of dimension d in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $H^d(E \cap B(0,R)) < +\infty$ for every $R > 0$ and

$$H^d(E \setminus F) \leqslant H^d(F \setminus E)$$

for every Al-competitor F of E.

Next, we give the definition of Mumford-Shah (MS) minimal sets in $\mathbb{R}^n$.

Definition 1.2. — *Let E be a closed set in $\mathbb{R}^n$. A Mumford-Shah competitor (also called MS-competitor) of E is a closed set $F \subset \mathbb{R}^n$ such that we can find $R > 0$ such that*

$$F \setminus B(0,R) = E \setminus B(0,R) \tag{1.2.1}$$

and F separates $y, z \in \mathbb{R}^n \setminus B(0,R)$ when y, z are separated by E.

A Mumford-Shah minimal (MS-minimal) set in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that

$$H^{n-1}(E \setminus F) \leqslant H^{n-1}(F \setminus E) \tag{1.2.2}$$

for any MS-competitor F of E.

Here, E separates y, z means that y and z lie in different connected components of $\mathbb{R}^n \setminus E$.

It is easy to show that any MS-minimal set in $\mathbb{R}^n$ is also an Al-minimal set of dimension $n-1$ in $\mathbb{R}^n$. Next, if E is an MS-minimal set in $\mathbb{R}^n$, then $E \times \mathbb{R}$ is also an MS-minimal set in $\mathbb{R}^n \times \mathbb{R}$, by exercice 16, p 537 of [5].

We give now the definition of minimal cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$, of dimension 2 and 3 in $\mathbb{R}^n$.

Definition 1.3. — *A two-dimensional minimal cone of type Y is just a two-dimensional affine plane in $\mathbb{R}^n$. A three-dimensional minimal cone of type $\mathbb{P}$ is a three-dimensional affine plane in $\mathbb{R}^n$.*

Let S be the union of three half-lines in $\mathbb{R}^2 \subset \mathbb{R}^n$ that start from the origin 0 and make angles 120° with each other at 0. A two-dimensional minimal cone of type $\mathbb{Y}$ is set of the form $Y' = j(S \times L)$, where L is a line passing through 0 and orthogonal to $\mathbb{R}^2$ and j is an isometry of $\mathbb{R}^n$. A three-dimensional minimal cone of type $\mathbb{Y}$ is a set of the form $Y = j(S \times P)$, where P is a plane of dimension 2 passing through 0 and orthogonal to $\mathbb{R}^2$ and j is an isometry of $\mathbb{R}^n$. We call $j(L)$ the spine of Y' and $j(P)$ the spine of Y.

Take a regular tetrahedron $R \subset \mathbb{R}^3 \subset \mathbb{R}^n$, centered at the origin 0, let K be the cone centered at 0 over the union of the 6 edges of R. A two-dimensional minimal cone of type $\mathbb{T}$ is of the form $j(K)$, a three-dimensional minimal cone of type $\mathbb{T}$ is a set of the form $T = j(K \times L)$, where L is the line passing through 0 and orthogonal to $\mathbb{R}^3$ and j is an isometry of $\mathbb{R}^n$. We call $j(L)$ the spine of T.

We denote by d_P, d_Y, d_T the densities at the origin of the 3-dimensional minimal cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$, respectively. It is clear that $d_P < d_Y < d_T$.

We can now define a Hölder ball for a set $E \subset \mathbb{R}^n$.

Definition 1.4. — *Let E be a closed set in $\mathbb{R}^n$. Suppose that $0 \in E$. We say that $B(0,r)$ is a Hölder ball of E, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$ with exponent $1+\alpha$, if there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and a cone Y of dimension 2 or 3, centered at the origin, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$, respectively, such that*

$$|f(x) - x| \leqslant \alpha r \text{ for } x \in B(0,r) \tag{1.4.1}$$

$$(1-\alpha)[\frac{|x-y|}{r}]^{(1+\alpha)} \leqslant \frac{|f(x)-f(y)|}{r} \leqslant (1+\alpha)[\frac{|x-y|}{r}]^{(1-\alpha)} \text{for } x, y \in B(0,r) \tag{1.4.2}$$

$$E \cap B(0,(1-\alpha)r) \subset f(Y \cap B(0,r)) \subset E \cap B(0,(1+\alpha)r). \tag{1.4.3}$$

For the sake of simplicity, we will say that E is Bi-Hölder equivalent to Y in $B(0,r)$, with exponent $1+\alpha$.

If in addition, our function f is of class $C^{1,\alpha}$, then we say that E is $C^{1,\alpha}$ equivalent to Y in the ball $B(0,r)$. Here, f is said to be of class $C^{1,\alpha}$ if f is differentiable and its differential is a Hölder continuous function, with exponent α.

J. Taylor in [11] has obtained the following theorem about local C^1-regularity of two-dimensional minimal sets in $\mathbb{R}^3$.

Theorem 1.5. [11]. — *Let E be a two-dimensional minimal set in $\mathbb{R}^3$ and $x \in E$. Then there exists a radius $r > 0$ such that in the ball $B(x,r)$, E is $C^{1,\alpha}$ equivalent to a minimal cone $Y(x,r)$ of dimension 2, of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. Here α is a universal positive constant.*

As we know, any two-dimensional minimal cone in $\mathbb{R}^3$ is automatically of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. This is a great avantage when we study two-dimensional minimal sets of dimension 2 in $\mathbb{R}^3$, because each blow-up limit at some point of a two-dimensional minimal set is a minimal cone of the same dimension. So we can approximate our minimal set by cones which we know the structure of.

The problem of two-dimensional minimal sets in $\mathbb{R}^n$ with $n > 3$ is more difficult. Here we don't know the list of two-dimensional minimal cones. But G. David gives in section 14 of [3] a description of two-dimensional minimal cones in $\mathbb{R}^n$. Thanks to this, he can prove the local Hölder regularity of two-dimensional minimal sets in $\mathbb{R}^n$.

THEOREM 1.6. [3]. — *Let E be a two-dimensional minimal set in $\mathbb{R}^n$ and $x \in E$. Then for each $\alpha > 0$, there exists a radius $r > 0$ such that in the ball $B(x,r)$, E is Hölder equivalent to a two-dimensional minimal cone $Y(x,r)$, with exponent α.*

The C^1 regularity of two-dimensional minimal sets in $\mathbb{R}^n$ needs more efforts. We have to prove that the local distance between E and a two-dimensional minimal cone in $B(x,r)$ is of order r^a, where a is a positive universal constant when r tends to 0. G. David in [4] shows the C^1 regularity of E locally around x, but he needs to add an additional condition, called "'full length"' to some blow-up limit of E in x.

THEOREM 1.7. [4]. — *Let E be a two-dimensional minimal set in the open set $U \subset \mathbb{R}^n$ and $x \in E$. We suppose that some blow-up limit of E at x is a full length minimal cone. Then there is a unique blow-up limit X of E at x, and $x + X$ is tangent to E at x. In addition, there is a radius $r_0 > 0$ such that E is $C^{1,\alpha}$ equivalent to $x + X$ in the ball $B(x, r_0)$, where $\alpha > 0$ is a universal constant.*

Let us say more about the "full length" condition for a two dimensional minimal cone F centered at the origin in $\mathbb{R}^n$. As in [3, Sect 14], the set $K = F \cap \partial B(0,1)$ is a finite union of great circles and arcs of great circles $\mathfrak{C}_j, j \in J$. The $\mathfrak{C}_j$ can only meet when they are arcs of great circles and only by sets of 3 and at a common endpoint. Now for each $\mathfrak{C}_j$ whose length is more than $\frac{9\pi}{10}$, we cut $\mathfrak{C}_j$ into 3 sub-arcs $\mathfrak{C}_{j,k}$ with the same length so that we have a decomposition of K into disjoint arcs of circles $\mathfrak{C}_{j,k}, (j,k) \in \tilde{J}$ with the same length and for each $\mathfrak{C}_{j,k}$, we have length$(\mathfrak{C}_{j,k}) \leqslant 9\pi/10$. The full lengh condition says that if we have another net of geodesics $K_1 = \cup_{(i,j)\in\tilde{J}} \mathfrak{C}^1_{j,k}$, for which the Hausdorff distance $d(\mathfrak{C}_{j,k}, \mathfrak{C}^1_{j,k}) \leqslant \eta$, where η is a small constant which depends only on n, and if $H^1(K_1) > H^1(K)$, then we can find a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = x$ out of the ball $B(0,1)$ and $f(B(0,1)) \subset B(0,1)$ such that $H^2(f(F_1) \cap B(0,1)) \leqslant H^2(F_1 \cap B(0,1)) - C[H^1(K_1) - H^1(K)]$. Here $C > 0$ is a constant and F_1 is the cone over K_1. See [4, Sect 2] for more details.

It happens that all two-dimensional minimal cones in $\mathbb{R}^3$ satisfy the full length condition. So the theorem of G. David is a generalization of the theorem of J. Taylor.

For minimal sets of dimension $\geqslant 3$, little is known. Almgren in [1] showed that if F is a three-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin and over a smooth surface in $\mathbb{S}^3$, the unit sphere of dimension 3, then E must be a 3-plane. Then J. Simon in [10] showed that this is true for hyper minimal cones in $\mathbb{R}^n$ with $n < 7$. That is, if F is a minimal cone of dimension $n-1$ in $\mathbb{R}^n$, centered at the origin and over a smooth surface in $\mathbb{S}^{n-1}$, then F must be an $n-1$ plane. There is no theorem yet about the regularity of minimal sets of dimension $\geqslant 3$ with singularities.

Our first theorem is to prove a local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$. But we don't know the list of three-dimensional minimal cones in $\mathbb{R}^4$ and we don't have a nice description of three-dimensional minimal cones as we have for two-dimensional minimal cones. So we shall restrict to some particular type of points, at which we can obtain some information about the blow-up limits.

Now let E be a three-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$. We want to show that E is Bi-Hölder equivalent to a three-dimensional minimal cone of type $\mathbb{P}$ or $\mathbb{Y}$ in the ball $B(x,r)$, for some radius $r > 0$. If $\theta_E(x) = d_P$, then W. Allard in [2] showed that there exists a radius $r > 0$ such that in the ball $B(x,r)$, E is C^1 equivalent to a 3-dimensional plane. We consider then the next possible density of E at x, so we suppose that $\theta_E(x) = d_Y$. Since every blow-up limit of E at x is a 3-dimensional minimal cone of type $\mathbb{Y}$, then for each $\epsilon > 0$, there exists a radius $r > 0$ and a 3-dimensional minimal cone $Y(x,r)$ of type $\mathbb{Y}$ such that

$$d_{x,r}(E, Y(x,r)) \leqslant \epsilon. \tag{$*$}$$

By using $(*)$ and the minimality of E, we shall be able to approximate E by 3-dimensional minimal cones of type $\mathbb{P}$ or $\mathbb{Y}$ at every point in $E \cap B(x, r/2)$ and at every scale $t \leqslant r/2$. We shall then use Theorem 1.1 in [6] to conclude that E is Bi-Hölder equivalent to a 3-dimensional minimal cone of type $\mathbb{Y}$ in the ball $B(x, r/2)$. Our first theorem is the following.

THEOREM 1. — *Let E be a 3-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$ such that $\theta_E(x) = d_Y$. Then for each $\alpha > 0$, we can find a radius $r > 0$, which depends also on x, such that $B(x,r)$ is a Hölder ball (see Def 1.4) of type $\mathbb{Y}$ of E, with exponent $1+\alpha$.*

Our second theorem concerns Mumford-Shah minimal sets in $\mathbb{R}^4$. In [3], G. David showed that there are only 3 types of Mumford-Shah minimal sets in $\mathbb{R}^3$, which are the cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$. The most difficult part is to show that if F is a Mumford-Shah minimal set in $\mathbb{R}^3$, which is close enough in $B(0,2)$ to a $\mathbb{T}$ centered at 0, then there must be a $\mathbb{T}$-point of F in $B(0,1)$. To prove this proposition, G. David used very nice techniques which involve the list of connected components. We want to obtain a similar result for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is close enough to a $\mathbb{T}$ of dimension 3. But we cannot obtain a result which is as good as in [3, 18.1]. The reason is that we don't know if there exists a minimal cone C of dimension 3 in $\mathbb{R}^4$, centered at 0, which satisfies $d_Y < \theta_C(0) < d_T$. Our second theorem is the following.

THEOREM 2.— *There exists an absolute constant $\epsilon > 0$ such that the following holds. Let E be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius, and T be a 3-dimensional minimal cone of type $\mathbb{T}$ centered at the origin such that*

$$d_{0,r}(E, T) \leqslant \epsilon.$$

Then in the ball $B(0, r)$, there is a point of E which is neither of type $\mathbb{P}$ nor $\mathbb{Y}$.

See Definition 2.5 for the definition of points of type $\mathbb{P}$ and $\mathbb{Y}$. We divide the paper into two parts. In the first part, we prove Theorem 1. In the second part, we prove Theorem 2.

I would like to thank Professor Guy David for many helpful discussions on this paper.

2. Hölder regularity near a point of type $\mathbb{Y}$ for a 3-dimensional minimal set in $\mathbb{R}^4$

In this section we prove Theorem 1. We start with the following lemma.

LEMMA 2.1.— *Let F be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin, and let $x \in F \cap \partial B(0, 1)$. Then each blow-up limit G of F at x is a 3-dimensional minimal cone G of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ and centered at 0. The type of G depends only on x and $\theta_E(x) = \theta_G(0)$.*

We define the type of x to be the type of G.

Proof.— We denote by $0x$ the line passing by 0 and x. Suppose that G is a blow-up limit of F at x. Then $G = \lim_{k\to\infty} \frac{F-x}{r_k}$ with $\lim_{k\to\infty} r_k = 0$. Let $y \in G$, we want to show that $y + 0x \subset G$. Setting $F_k = \frac{F-x}{r_k}$, as $\{F_k\}$ converges to G, we can find a sequence $y_k \in F_k$ such that $\{y_k\}_{k=1}^{\infty}$ converges to y. Setting $z_k = r_k y_k + x$, then $z_k \in F$ by definition of F_k, and z_k converges to x because r_k converges to 0. We fix $\lambda \in \mathbb{R}$ and we set $v_k = (1 + \lambda r_k)z_k$. Then $v_k \in F$ as F is a cone centered at 0. We have next that $w_k = r_k^{-1}(v_k - x) \in F_k$. On the other hand,

$$\begin{aligned} w_k &= r_k^{-1}((1+\lambda r_k)z_k - x) \\ &= r_k^{-1}((1+\lambda r_k)(r_k y_k + x) - x) \\ &= r_k^{-1}(r_k y_k + \lambda r_k^2 y_k + \lambda r_k x) \\ &= y_k + \lambda x + \lambda r_k y_k, \end{aligned}$$

we see that $\lim_{k\to\infty} w_k = y + \lambda x$. As $\{F_k\}$ converges to G, we see that $y+\lambda x \in G$. Call H the tangent plane to $\partial B(0,1)$ at x. Since for each $y \in G$ and $\lambda \in \mathbb{R}$, we have $y+\lambda x \in G$, we have that $G = G' \times Ox$, with $G' \subset G \cap H$. Next, as F is a minimal set and G is a blow-up limit of F at x, by [3, 7.31], G is a minimal cone centered at 0. But $G = G' \times 0x$, then by [3, 8.3], G' is a minimal cone in H, centered at x. Since H is a 3-plane, we must have that G' is a 2-dimensional minimal cone of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$ and then G is also a 3-dimensional minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. Next, as G is a blow-up limit of F at x, by [3, 7.31], we have $\theta_F(x) = \theta_G(0)$. □

We see from this lemma that for each $x \in F \setminus \{0\}$, where F is a 3-dimensional minimal cone in $\mathbb{R}^4$ centered at the origin,

$$\theta_F(x) \text{ can take only one of the three values } d_P, d_Y, d_T. \tag{1}$$

But we do not know the list of possible values of $\theta_F(0)$. However, the following lemma says that for this cone F, it is not possible that $d_P < \theta_F(0) < d_Y$.

Lemma 2.2. — *There does not exist a 3-dimensional minimal cone F in $\mathbb{R}^4$, centered at the origin such that $d_P < \theta_F(0) < d_Y$.*

Proof. — Suppose that there is a cone F as in the hypothesis and

$$d_P < \theta_F(0) < d_Y. \tag{2.2.1}$$

We first show that

$$\text{for each } x \in F \cap \partial B(0,1), \text{ we have } \theta_F(0) \geqslant \theta_F(x). \tag{2.2.2}$$

Indeed, since F is a minimal cone, for each $z \in F$, the function $\theta_F(z,t)$ is nondecreasing. So for $r > 0$, we have $\theta_F(x,r) \geqslant \theta_F(x)$, which means that $H^3(F \cap B(x,r))/r^3 \geqslant \theta_F(x)$. Since $B(x,r) \subset B(0,r+1)$, we obtain $H^3(F \cap B(x,r)) \leqslant H^3(F \cap B(0,r+1))$ and thus $H^3(F \cap B(0,r+1))/r^3 \geqslant \theta_F(x)$. We deduce that $(H^3(F \cap B(0,r+1))/(r+1)^3)((r+1)^3/r^3) \geqslant \theta_F(x)$. Since F is a cone centered at 0, $H^3(F \cap B(0,r+1))/(r+1)^3 = \theta_F(0)$ for each $r > 0$. We deduce then $\theta_F(0)((r+1)^3/r^3) \geqslant \theta_F(x)$ for each $r > 0$. We let $r \to +\infty$ and we obtain then $\theta_F(0) \geqslant \theta_F(x)$, which is (2.2.2).

Now (2.2.1) and (2.2.2) give us that $\theta_F(x) < d_Y$ for each $x \in F \cap \partial B(0,1)$. By (1), we have $\theta_F(x) = d_P$ for $x \in F \cap \partial B(0,1)$. So by [2, 8.1], there exists a neighborhood U_x of x in $\mathbb{R}^4$ such that $F \cap U_x$ is a 3-dimensional smooth manifold. We deduce that $F \cap \partial B(0,1)$ is a 2-dimensional smooth sub-manifold of $\partial B(0,1)$. By [1, Lemma 1], F is a 3-plane passing through 0. But this implies that $\theta_F(0) = d_P$, we obtain then a contradiction, Lemma 2.2 follows. □

Lemma 2.3. — *Let F be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin 0. If $\theta_F(0) = d_Y$, then F is a 3-dimensional cone of type $\mathbb{Y}$.*

Proof. — As in the argument for (2.2.2), we have that for each $x \in F \cap \partial B(0,1)$, $\theta_F(x) \leqslant \theta_F(0) = d_Y$. So $\theta_F(x)$ can only take one of the two values d_P or d_Y. If all $x \in F \cap \partial B(0,1)$ are of type $\mathbb{P}$, then by the same argument as above, F will be a 3-plane, and then $\theta_F(0) = d_P$, a contradiction. So there must be a point $y \in F \cap \partial B(0,1)$, such that $\theta_F(y) = d_Y$. By the same argument like above, $\theta_F(0)(r+1)^3/r^3 \geqslant \theta_F(y,r)$ for each $r > 0$. Letting $r \to \infty$ and noting that $\theta_F(y,r)$ is non-decreasing in r, we have $d_Y \geqslant \lim_{r\to\infty} \theta_F(y,r)$. But $\theta_F(y,r) \geqslant \theta_F(y) = d_Y$ for each $r > 0$, so we must have $\theta_F(y,r) = d_Y$ for $r > 0$. By [3, 6.2], F must be a cone centered at y. But we have also that F is a cone centered at 0. So F is of the form $F = F' \times 0y$, where F' is a cone in a 3-plane H passing through 0 and orthogonal to $0y$. Since F is a minimal cone, by [3, 8.3], F' is also a 2-dimensional minimal cone in H and centered at 0. So F' must be a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. Since $\theta_F(0) = d_Y$, we must have that F' is a 2-dimensional minimal cone of type $\mathbb{Y}$ and we deduce that F is a 3-dimensional minimal cone of type $\mathbb{Y}$. □

We can now consider 3-dimensional minimal sets in $\mathbb{R}^4$. We start with the following lemma.

Lemma 2.4. — *Let E be a 3-dimensional minimal set in $\mathbb{R}^4$. Then*

(i) There does not exist a point $z \in E$ such that $d_P < \theta_E(z) < d_Y$.

(ii) If $x \in E$ such that $\theta_E(x) = d_P$, then each blow-up limit of E at x is a 3-dimensional plane.

(iii) If $\theta_E(x) = d_Y$, then each blow-up limit of E at x is a 3-dimensional minimal cone of type $\mathbb{Y}$.

Proof. — The proof uses Lemmas 2.2 and 2.3. Take any point $z \in E$, let F be a blow-up limit of E at z. Then by [3, 7.31], F is a cone and $\theta_F(0) = \theta_E(x)$. By Lemma 2.2, it is not possible that $d_P < \theta_F(0) < d_Y$, which means that it is also not possible that $d_P < \theta_E(x) < d_P$, (i) follows.

If $x \in E$ such that $\theta_E(x) = d_P$, then any blow-up limit F of E at x satisfies $\theta_F(0) = \theta_E(x) = d_P$. By the same arguments as in Lemma 2.2, for each $y \in F \cap \partial B(0,1)$, $\theta_F(y) \leqslant \theta_F(0) = d_P$. We deduce that $\theta_F(y) = d_P$ for each $y \in F \cap \partial B(0,1)$, and then F will be a 3-dimensional minimal cone over a smooth sub-manifold of $\partial B(0,1)$. By [1, Lemma 1], F must be a 3-dimensional plane, (ii) follows.

If $x \in E$ such that $\theta_E(x) = d_Y$, then any blow-up limit F of E at x satisfies $\theta_F(0) = d_Y$. By Lemma 2.3, F must be a 3-dimensional minimal cone of type $\mathbb{Y}$, (iii) follows. □

Lemma 2.4 allows us to define the points of type $\mathbb{P}$ and $\mathbb{Y}$ of a 3-dimensional minimal set in $\mathbb{R}^4$.

Definition 2.5. — *Let E be a 3-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$. We call x a point of type $\mathbb{P}$ if $\theta_E(x) = d_P$. We call x a point of type $\mathbb{Y}$ if $\theta_E(x) = d_Y$.*

The following proposition says that if a 3-dimensional minimal set E is close enough to a 3-dimensional plane P in the ball $B(x, 2r)$, then E is Bi-Hölder equivalent to P in $B(x,r)$.

Proposition 2.6. — *For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds.*

Let E be a 3-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$. Let P be a 3-dimensional plane such that

$$d_{x,2^5 r}(E,P) \leqslant \epsilon. \tag{2.6.1}$$

Then E is Bi-Hölder equivalent to P in the ball $B(x,r)$, with Hölder exponent $1+\alpha$.

Proof. — Take any point $y \in B(x,r)$. Since $B(y,2^4 r) \subset B(x,2^5 r)$, we have

$$d_{y,2^4 r}(E,P) \leqslant 2 d_{x,2^5 r}(E,P) \leqslant 2\epsilon. \tag{2.6.2}$$

By [3, 16.43], for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.6.2) holds, then

$$\begin{aligned} H^3(E \cap B(y,2^3 r)) &\leqslant H^3(P \cap B(y,(1+\epsilon_1)2^4 r)) + \epsilon_1 r^3 \\ &\leqslant d_P (2^3 r)^3 + C\epsilon_1 r^3. \end{aligned} \tag{2.6.3}$$

Now (2.6.3) implies that $\theta_E(y, 2^3 r) \leqslant d_P + C\epsilon_1$. If ϵ_1 is small enough, then $\theta_E(y) \leqslant \theta_E(y,2^3 r) < d_Y$. We deduce that $\theta_E(y) = d_P$ and y is a $\mathbb{P}$ point.

Since $\theta_E(y,t)$ is a non-decreasing function in t, we have

$$0 \leqslant \theta_E(y,t) - \theta_E(y) \leqslant C\epsilon_1 \text{ for } 0 < t \leqslant 2^3 r. \tag{2.6.4}$$

By [3, 7.24], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.6.4) holds, then there exists a 3-dimensional minimal cone F, centered at y, such that

$$d_{y,t/2}(E,F) \leqslant \epsilon_2 \text{ for } 0 < t \leqslant 2^3 r, \tag{2.6.5}$$

and

$$|\theta_E(y,2^2r) - \theta_F(y,2^2r)| \leqslant \epsilon_2. \tag{2.6.7}$$

Since $d_P \leqslant \theta_E(y,2^2r) \leqslant d_P + C\epsilon_1$, we deduce from (2.6.7) that $\theta_F(y,2^2r) \leqslant d_P + C\epsilon_1 + \epsilon_2$. So if ϵ_1 and ϵ_2 are small enough, then $\theta_F(y,2^2r) < d_Y$. Which implies $\theta_F(y) < d_Y$. Since F is a minimal cone centered at y, we deduce that F must be a 3-dimensional plane, by the same arguments as in second part of Lemma 2.4.

Now we can conclude that for each $y \in E \cap B(x,r)$ and each $t \leqslant r$, there exists a 3-dimensional plane $P(y,t)$, which is F in (2.6.5), such that $d_{y,t}(E,P(y,t)) \leqslant \epsilon_2$. By [6,2.2], for each $\alpha > 0$, we can find $\epsilon_2 > 0$, and then $\epsilon > 0$, such that E is Bi-Hölder equivalent to a P in the ball $B(x,r)$. □

Proposition 2.7. — *For each $\eta > 0$, we can find $\epsilon > 0$ with the following properties. Let E be a minimal set of dimension 3 in $\mathbb{R}^4$ and Y be a 3-dimensional minimal cone of type $\mathbb{Y}$, centered at the origin. Suppose that $d_{0,1}(E,Y) \leqslant \epsilon$. Then in the ball $B(0,\eta)$, there must be a point $y \in E$, which is not of type $\mathbb{P}$.*

Proof. — Suppose that the lemma fails. Then each $z \in B(0,\eta)$ is of type $\mathbb{P}$. We note F_1, F_2, F_3 the three half-plane of dimension 3 which form Y and L the spine of $\mathbb{Y}$, which is a plane of dimension 2. Then $F_i, 1 \leqslant i \leqslant 3$ have common boundary L. Take $w_i \in F_i \cap \partial B(0,\eta/4), 1 \leqslant i \leqslant 3$, such that the distance $\mathrm{dist}(w_i, L) = \eta/4$. We see that the w_i lie in a 2-dimensional plane orthogonal to L. Since $d_{0,1}(E,Y) \leqslant \epsilon$, we have that for each $1 \leqslant i \leqslant 3$, there exists $z_i \in E$ such that $d(z_i,w_i) \leqslant \epsilon$. Now $d(z_i,0) \leqslant d(w_i,0) + \epsilon = \eta/4 + \epsilon < 3\eta/8$ and $\mathrm{dist}(z_i,L) \geqslant \mathrm{dist}(w_i,L) - \epsilon = \eta/4 - \epsilon > 3\eta/16$. So if ϵ is small enough, we have that for each $1 \leqslant i \leqslant 3$, the ball $B(z_i,\eta/8)$ does not meet L. As a consequence, Y coincide with F_i in the ball $B(z_i,\eta/8)$ for $1 \leqslant i \leqslant 3$. We have next

$$\begin{aligned} d_{z_i,\eta/8}(E,F_i) &= d_{z_i,\eta/8}(E,Y) \\ &\leqslant \frac{8}{\eta} d_{0,1}(E,Y) \\ &\leqslant \frac{8\epsilon}{\eta}. \end{aligned} \tag{2.7.1}$$

Take a very small constant $\alpha > 0$, say, 10^{-15}. Then by Proposition 2.6, we can find $\epsilon > 0$ such that if (2.7.1) holds, then

E is Bi-Hölder equivalent to F_i in the ball $B(z_i, \eta/2^8)$ for each $1 \leqslant i \leqslant 3$ with Hölder exponent $1 + \alpha$. (2.7.2)

Next, since we suppose that each $z \in B(0, \eta)$ is of type $\mathbb{P}$, we have that there exists a radius $r_z > 0$, such that

E is Bi-Hölder equivalent to a 3-dimensional plane in the ball $B(z, r_z)$, with exponent $1 + \alpha$. (2.7.3)

In the ball $B(0, \eta)$, we have $d_{0,\eta}(E, Y) \leqslant \frac{1}{\eta} d_{0,1}(E, Y) \leqslant \frac{\epsilon}{\eta}$. (2.7.4)

We can adapt the arguments in [3], section 17 to obtain that there does not exist a set E, which satisfies the conditions (2.7.2), (2.7.3) and (2.7.4). The idea is as follows, we construct a sequence of simple and closed curves $\gamma_0, \gamma_1, ..., \gamma_k$ such that $\gamma_k \cap E = \varnothing$ and γ_0 intersects E transversally at exactly 3 points in the ball $B(z_i, \eta/2^8)$. For each $0 \leqslant i \leqslant k-1$, γ_i intersects E transversally at a finite number of points and $|\gamma_i \cap E| - |\gamma_{i+1} \cap E|$ is even, here $|\gamma_i \cap E|$ denotes the number of intersections of γ_i with E. This is impossible since $|\gamma_0 \cap E| = 3$ and $|\gamma_k \cap E| = 0$. We obtain then a contradiction. Proposition 2.7 follows. □

LEMMA 2.8. — *For each $\delta > 0$, we can find $\epsilon > 0$ such that the following holds.*

Let F be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin. Suppose that $d_Y < \theta_F(0) < d_Y + \epsilon$. Then there exists a 3-dimensional minimal cone Y_F, of type $\mathbb{Y}$, centered at 0 such that $d_{0,1}(F, Y_F) \leqslant \delta$.

Proof. — Suppose that the lemma fails. Then there exists $\delta > 0$, such that we can find 3-dimensional minimal cones $F_1, ..., F_k, ...$ centered at 0, satisfying $d_Y \leqslant \theta_{F_i} \leqslant d_Y + 1/2^i$, and for any 3-dimensional minimal cone Y of type $\mathbb{Y}$, centered at 0, we have $d_{0,1}(Y, F_i) > \delta$.

Now we can find a sub-sequence $\{F_{j_k}\}_{k=1}^{\infty}$ of $\{F_i\}_{i=1}^{\infty}$ such that this subsequence converges to a closed set $G \subset \mathbb{R}^4$. By [3, 3.3], G is also a minimal set. Since each F_{i_k} is a cone centered at 0, G is also a cone centered at 0. So G is a 3-dimensional minimal cone centered at 0. By [3, 3.3], we have

$$H^3(G \cap B(0,1)) \leqslant \liminf_{k \to \infty} H^3(F_{j_k} \cap B(0,1)), \tag{2.8.1}$$

which implies that

$$\theta_G(0) \leqslant \liminf_{k\to\infty}(d_Y + 1/2^{j_k}) = d_Y. \tag{2.8.2}$$

By [3, 3.12], we have

$$H^3(G\cap\overline{B}(0,1)) \geqslant \limsup_{k\to\infty} H^3(F_{j_k}\cap\overline{B}(0,1)), \tag{2.8.3}$$

which implies that

$$\theta_G(0) \geqslant \limsup_{k\to\infty}(d_Y + 1/2^{j_k}) = d_Y. \tag{2.8.4}$$

From (2.8.2) and (2.8.4), we have that $\theta_G(0) = d_Y$. Then by Lemma 2.3, G must be a 3-dimensional minimal cone of type $\mathbb{Y}$, centered at 0. Since $\lim_{k\to\infty} F_{j,k} = G$, there is $k > 0$ such that $d_{0,1}(F_{j_k}, G) \leqslant \delta/2$, which is a contradiction. The lemma follows. □

The following lemma is similar to Lemma 2.8, but we consider minimal sets in general.

LEMMA 2.9. — *For each $\delta > 0$, we can find $\epsilon > 0$ such that the following holds.*

Suppose that E is a 3-dimensional minimal set in $\mathbb{R}^4$ and $0 \in E$. Suppose that

$$d_Y \leqslant \theta_E(0) \leqslant d_Y + \epsilon, \tag{2.9.1}$$

and

$$\theta_E(0,4) - \theta_E(0) \leqslant \epsilon. \tag{2.9.2}$$

Then there exists a 3-dimensional minimal cone Y_E, of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,1}(E, Y_E) \leqslant \delta.$$

Proof. — By [3, 7.24], for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.9.2) holds, then there is a 3-dimensional minimal cone F centered at the origin, such that

$$d_{0,2}(F, E) \leqslant \epsilon_1, \tag{2.9.3}$$

and

$$|\theta_F(0,2) - \theta_E(0,2)| \leqslant \epsilon_1. \tag{2.9.4}$$

Since E is minimal, $\theta_E(0,4) \geqslant \theta_E(0,2) \geqslant \theta_E(0)$. So from (2.9.1) and (2.9.2), we have that $d_Y \leqslant \theta_E(0,2) \leqslant d_Y + 2\epsilon$. With (2.9.4), we have

$$d_Y - \epsilon_1 \leqslant \theta_F(0,2) \leqslant d_Y + 2\epsilon + \epsilon_1. \tag{2.9.5}$$

Now if we choose ϵ_1 small enough, then $\theta_F(0) = \theta_F(0,2) \geqslant d_Y - \epsilon_1 > d_P$, so by Lemma 2.2, we have $\theta_F(0) \geqslant d_Y$. Thus

$$d_Y \leqslant \theta_F(0) \leqslant d_Y + 2\epsilon + \epsilon_1. \tag{2.9.6}$$

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.9.6) holds, then there is a 3-dimensional minimal cone Y_F of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,2}(F, Y_F) \leqslant \epsilon_3. \tag{2.9.7}$$

From (2.9.3) and (2.9.7) we have

$$d_{0,1}(E, Y_F) \leqslant 2(d_{0,2}(E, F) + d_{0,2}(F, Y_F)) \leqslant 2(\epsilon_1 + \epsilon_3). \tag{2.9.8}$$

Now for each $\delta > 0$, we choose $\epsilon > 0$ such that $2(\epsilon_1 + \epsilon_3) < \delta$, we set then $Y_E = Y_F$ and the lemma follows. $\square$

We are ready to prove Theorem 1.

Theorem 2.10. — *For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds.*

Let E be a 3-dimensional minimal set in $\mathbb{R}^4$, which contains the origin 0. Suppose that there exists a radius $r > 0$ such that

$$d_Y \leqslant \theta_E(0) \leqslant d_Y + \epsilon, \tag{2.10.1}$$

and

$$\theta_E(0, 2^{11}r) - \theta_E(0) \leqslant \epsilon. \tag{2.10.2}$$

Then E is Bi-Hölder equivalent to a 3-dimensional minimal cone Y of type $\mathbb{Y}$ and centered at 0 in the ball $B(0,r)$, with Hölder exponent $1+\alpha$.

Proof. — By Lemma 2.9, for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.10.1) and (2.10.2) hold, then there exists a 3-dimensional minimal cone Y, of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,2^9 r}(E, Y) \leqslant \epsilon_1. \tag{2.10.3}$$

We consider a point $y \in E \cap B(0,r)$. We set

$$E_Y = \{z \in E \cap \overline{B}(0,4r)\}\ z \text{ is not a } \mathbb{P}\text{-point}. \tag{2.10.4}$$

We note that E_Y is closed. Indeed, if z is an accumulation point of E_Y, then if z is a $\mathbb{P}$-point, then there exists a neighborhood V_z of z in E such that V_z has only points of type $\mathbb{P}$, as in the proof of Proposition 2.6, which is not possible. So z cannot be a $\mathbb{P}$-point and as a consequence, $z \in E_Y$.

Case 1, $y \in E_Y$.

Since y is not a $\mathbb{P}$-point, $\theta_E(x) \neq d_P$, then by Lemma 2.4, we have

$$\theta_E(y) \geqslant d_Y; \tag{2.10.5}$$

Next, $B(y, 2^8 r) \subset B(0, 2^9 r)$, by (2.10.3), we have

$$d_{y,2^8 r}(E, Y) \leqslant 2 d_{0,2^9 r}(E, Y) \leqslant 2\epsilon_1. \tag{2.10.6}$$

By [3, 16.43], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.10.6) holds, then

$$H^3(E \cap B(y, 2^7 r)) \leqslant H^3(Y \cap B(y, (1+\epsilon_2)2^7 r)) + \epsilon_2 r^3, \tag{2.10.7}$$

which, together with (2.10.5), imply

$$d_Y \leqslant \theta_E(y, 2^7 r) \leqslant d_Y + C\epsilon_2. \tag{2.10.8}$$

But E is a minimal set, so the function $\theta_E(y, .)$ is non-decreasing. So we have

$$d_Y \leqslant \theta_E(y, t) \leqslant d_Y + C\epsilon_2 \text{ for } 0 < t \leqslant 2^7 r. \tag{2.10.9}$$

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_2, \epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.10.5) and (2.10.8) hold, then there exists a 3-dimensional minimal cone $Y(y,t)$ of type $\mathbb{Y}$, centered at y, such that

$$d_{y,t}(E, Y(y,t)) \leqslant \epsilon_3 \text{ for } 0 < t \leqslant 2^5 r. \tag{2.10.10}$$

We note as above, for $y \in B(0, r)$ and $t \leqslant 2^5 r$, $Y(y,t)$ the cone of type $\mathbb{Y}$ that satisfies (2.10.10).

Case 2, y is a $\mathbb{P}$ point.

Let $d = \mathrm{dist}(y, E_Y) > 0$. Take a point $u \in E_Y$ such that $d(y,u) = d$. Since $z \in B(0,r)$ and $0 \in E_Y$, we have $d \leqslant d(0,y) \leqslant r$. We take the cone $Y(u, 2d)$ as in (2.10.10), then

$$d_{u,2d}(E, Y(u, 2d)) \leqslant \epsilon_3. \tag{2.10.11}$$

Call L the spine of $Y(u, 2d)$, then L is a 2-dimensional plane passing through u. We want to show that

$$\mathrm{dist}(y, L) \geqslant d/2. \tag{2.10.12}$$

Indeed, if (2.10.12) fails, then there exists $u' \in L$ such that $d(y,u') = \mathrm{dist}(y,L) < d/2$. So $d(u',u) \leqslant d(u',y) + d(y,u) \leqslant 3d/2$. As a consequence, $B(u',d/2) \subset B(u,2d)$. We have next

$$d_{u',d/2}(E, Y(u,2d)) \leqslant 4d_{u,2d}(E, Y(u,2d)) \leqslant 4\epsilon_3. \tag{2.10.13}$$

By Proposition 2.7, we can choose $\epsilon_3 > 0$ such that if (2.10.13) holds, then there is a point $u_1 \in E \cap B(u', d/1000)$, which is not of type $\mathbb{P}$. Next, $d(y,u_1) \leqslant d(y,u') + d(u',u_1) \leqslant d/2 + d/1000 < 3d/4$ and since $y \in B(0,r)$, $u' \in B(0, r+3d/4) \subset B(0,4r)$. As u' is not a $\mathbb{P}$-point, we have that $u' \in E_Y$. So we can find a point $u' \in E_Y$ for which $d(y,u') < d$, a contradiction. We have then (2.10.12).

Since $B(y,d/2) \subset B(u,2d)$, we have

$$d_{y,d/2}(E, Y(u,2d)) \leqslant 4d_{u,2d}(E, Y(u,2d)) \leqslant 4\epsilon_3. \tag{2.10.14}$$

By [3, 16.43], for each $\epsilon_4 > 0$, we can find $\epsilon_3 > 0$ such that if (2.10.14) holds, then

$$H^3(E \cap B(y,d/4)) \leqslant H^3(Y(u,2d) \cap B(y,(1+\epsilon_4)d/4) + \epsilon_4 d^3. \tag{2.10.15}$$

Now as $\mathrm{dist}(y,L) \geqslant d/2$, we see that $Y(u,2d)$ coincide with a 3-dimensional plane in the ball $B(y,(1+\epsilon_4)d/4)$. So $H^3(Y(u,2d) \cap B(y,(1+\epsilon_4)d/4) \leqslant d_P((1+\epsilon_4)d/4)^3$, together with (2.10.15), we obtain

$$\theta_E(y,d/4) \leqslant d_P + C\epsilon_4. \tag{2.10.16}$$

By the proof of Proposition 2.6, we have that for each $\epsilon_5 > 0$, we can find $\epsilon_4 > 0$ such that for each $t \leqslant d/8$, there exists a plane $P(y,t)$ of dimension 3 passing by y, such that

$$d_{y,t}(E, P(y,t)) \leqslant \epsilon_5. \tag{2.10.17}$$

For the case $d/8 \leqslant t \leqslant r$, we take the cone $Y(u,t+d)$ as in 2.10.10 which is possible since $t + d < 8r$. Since $B(y,t) \subset B(u,t+d)$, we have

$$d_{y,t}(E, Y(u,t+d)) \leqslant \frac{t+d}{t} d_{u,t+d}(E, Y(u,t+d)) \leqslant 10\epsilon_3. \tag{2.10.18}$$

From (2.10.10), (2.10.17) and (2.10.18) we conclude that, for each $y \in E \cap B(0,r)$ and $t \leqslant r$, there exists a 3-dimensional minimal cone $Z(y,t)$ of type $\mathbb{P}$ or $\mathbb{Y}$, such that $d_{y,t}(E, Z(y,t)) \leqslant \epsilon_6$, where $\epsilon_6 = \max\{\epsilon_5, 10\epsilon_3\}$. By [6,2.2], we conclude that for each $\alpha > 0$, we can find $\epsilon > 0$ such that if (2.10.1) and (2.10.2) hold, then E is Bi-Hölder equivalent to a 3-dimensional minimal cone of type Y, centered at 0 in the ball $B(x,r)$, with Hölder exponent $1+\alpha$. □

Now we see that Theorem 1 is a consequence of Theorem 2.10, since $\theta_E(x) = d_Y$ which lies between d_Y and $d_Y + \epsilon$ for any $\epsilon > 0$. Next, for each $\epsilon > 0$, since $\lim_{r\to 0} \theta_E(x,r) = \theta_E(x)$, so we can find $r > 0$ such that $\theta_E(x, 2^{11}r) \leqslant \theta_E(x)+\epsilon = d_Y+\epsilon$. We conclude that E is Bi-Hölder equivalent to a cone of type $\mathbb{Y}$ in the ball $B(x,r)$.

Corollary 2.11. — *For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds. Let E be a 3-dimensional minimal set in $\mathbb{R}^4$, $x \in E$, r be a radius > 0 and Y be a 3-dimensional minimal cone of type $\mathbb{Y}$, centered at x such that*

$$d_{x,2^{14}r}(E,Y) \leqslant \epsilon. \tag{2.11.1}$$

Then E is Bi-Hölder equivalent to Y in the ball $B(x,r)$, with Hölder exponent $1+\alpha$.

Proof. — By Proposition 2.7, we can find ϵ small enough such that there exists a point $y \in B(x, r/1000)$ which is not of type $\mathbb{P}$. So $\theta_E(y) \geqslant d_Y$. Since $B(y, 2^{12}r) \subset B(x, 2^{13}r)$, we have

$$d_{y,2^{13}r}(E,Y) \leqslant 2d_{x,2^{14}r}(E,Y) \leqslant 2\epsilon. \tag{2.11.2}$$

By [3, 16.43], for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.11.2) holds, then

$$H^3(E \cap B(y, 2^{12}r)) \leqslant H^3(Y \cap B(y, (1+\epsilon_1)2^{12}r)) + \epsilon_1 r^3, \tag{2.11.3}$$

which implies that

$$\theta_E(y, 2^{12}r) \leqslant d_Y + C\epsilon_1. \tag{2.11.4}$$

Now (2.11.4) together with the fact that $\theta_E(y) \geqslant d_Y$ are the conditions in the hypothesis of Theorem 2.10 with the couple $(x, 2r)$. Following the proof of the theorem, for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that for each $z \in B(y, 2r)$ and for each $t \leqslant 2r$, there is a 3-dimensional minimal cone $Z(z,t)$ of type $\mathbb{P}$ or $\mathbb{Y}$ such that $d_{z,t}(Z(z,t), E) \leqslant \epsilon_2$. Since $B(x,r) \subset B(y, 2r)$, the above holds for any $z \in B(x,r)$ and $t \leqslant r$. Now since $d_{x,r}(E,Y) \leqslant 2^{14}\epsilon \leqslant \epsilon_2$, we can apply [DDT,2.2] to conclude that for each $\alpha > 0$, we can find $\epsilon > 0$ such that if (2.11.1) holds, then E is Hölder equivalent to Y in $B(x,r)$, with Hölder exponent $1+\alpha$. □

By construction of the Bi-Hölder function in [6], we see that if E is Bi-Hölder equivalent to a Y of type $\mathbb{Y}$ in $B(x,r)$ by a function f, then f is a bijection of the spine of Y in $B(x, r/2)$ to the points of type non-$\mathbb{P}$ of E in a neighborhood of x. We have the remark.

Remark 2.12. — Let E be a 3-dimensional minimal set in $\mathbb{R}^4$, $x \in E$ and $r > 0$. Suppose that E is Bi-Hölder equivalent to a 3-dimensional minimal cone Y of type $\mathbb{Y}$ and centered at x in the ball $B(x,r)$. Note E_Y the set of the points of type non-$\mathbb{Y}$ of E in $B(x,r)$ and L the spine of Y. Then

$$E_Y \cap B(x, r/8) \subset f(L \cap B(x, r/4)) \subset E_Y \cap B(x, r/2). \tag{2.12.1}$$

3. Existence of a point of type non-$\mathbb{P}$ and non-$\mathbb{Y}$ for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is near a $\mathbb{T}$

Let us restate Theorem 2.

THEOREM 2. — *There exists an absolute constant $\epsilon > 0$ such that the following holds. Let E be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius and T be a 3-dimensional minimal cone of type $\mathbb{T}$ centered at the origin such that*

$$d_{0,r}(E, T) \leqslant \epsilon. \tag{2.1}$$

Then in the ball $B(0,r)$, there is a point which is neither of type $\mathbb{P}$ nor $\mathbb{Y}$ of E.

We will prove Theorem 2 by contradiction. By homothety, we may assume that $r = 2^{10}$. Suppose that (2.1) fails, that is

$$\textit{there are only points of type } \mathbb{P} \textit{ and } \mathbb{Y} \textit{ in } E \cap B(0, 2^{10}). \tag{2.2}$$

We fix a coordinate (x_1, x_2, x_3, x_4) of $\mathbb{R}^4$. Without loss of generality, we suppose that T is of the form $T = T' \times l$, where T' is a 2-dimensional minimal cone of type $\mathbb{T}$ which belong to a 3-dimensional plane P of equation $P = \{x_1, x_2, x_3, x_4\} : x_4 = 0$ and l the line of equation $x_1 = x_2 = x_3 = 0$. We call l the spine of T, which is also the set of $\mathbb{T}$-points of T. Let l_1, l_2, l_3, l_4 be the four axes of T'; then $L_i = l_i \times l, i = 1, ..., 4$ are the 2-faces of T. We see that $\cup_{i=1}^4 L_i \setminus l$ is the set of $\mathbb{Y}$-points of T. Finally, let $F_j, 1 \leqslant j \leqslant 6$ the faces of T' in P. Then $F_j \times l, 1 \leqslant j \leqslant 6$ are the 3-faces of T and $\cup_{j=1}^6 F_j$ minus the set of $\mathbb{Y}$-points and the set of $\mathbb{T}$-points of T is the set of $\mathbb{P}$-points of T. The proof of Theorem 2 requires several lemmas. We begin with a lemma about the connected components of $\overline{B}(0,2) \setminus E$.

LEMMA 3.1. — *Let $a_i, 1 \leqslant i \leqslant 4$ be the four points in $\partial B(0, 2^9) \cap P$ whose distances to T' are maximal. Set $V_i, 1 \leqslant i \leqslant 4$ the connected component of $\overline{B}(0, 2^{10}) \setminus E$ which contains a_i. Then we have $V_i \neq V_j$ for $1 \leqslant i \neq j \leqslant 4$.*

Proof. — Suppose that the lemma fails. Then there are $i \neq j$ such that $V_i = V_j$. Without loss of generality, we may assume that $V_1 = V_2 = V$. Now the point $a = (a_1 + a_2)/2$ belongs to a 3-face P_{12} of T and T coincide with P_{12} in $B(a, 2^8)$.

Since $d_{0,2^{10}}(E, T) \leqslant \epsilon$, we have

$$d_{a,2^8}(E, T) = d_{a,2^8}(E, P_{12}) \leqslant 4\epsilon. \tag{3.1.1}$$

By Proposition 2.6, for a constant τ very small, say, 10^{-25}, we can find $\epsilon > 0$ such that E is Bi-Hölder equivalent to P_{12} in the ball $B(a, 2^3)$, with Hölder exponent $1+\tau$. We note f this Hölder function; then f is a homeomorphism and

$$E \cap B(a, 4) \subset f(P_{12} \cap B(a, 8)) \subset E \cap B(a, 16), \tag{3.1.2}$$

and

$$|f(x) - x| \leqslant \tau \text{ for } x \in B(a, 16). \tag{3.1.3}$$

We want to show that

$$\text{if } z \in \partial B(a, 4) \setminus E, \text{ then } z \in V. \tag{3.1.4}$$

Indeed, set $z' = f^{-1}(z)$, then $z' \in B(a, 8)$ and as $z \notin E$, we have $z' \notin P_{12}$. Now the 3-plane P_{12} separate $\mathbb{R}^4$ into two half-spaces H_1 and H_2 which contain a_1 and a_2, respectively. Let $z_1 \in H_1$ and $z_2 \in H_2$ be two points in $\partial B(a, 4)$ whose distances to P_{12} are maximal. We see that a is the mid-point of the segment $[z_1, z_2]$ and this segment is orthogonal to P_{12}. Since z_1 and z_2 lie in two different half-spaces of $\mathbb{R}^4$ separated by P_{12}, one of the two segment $[z', z_1]$ and $[z', z_2]$ doesn't meet P_{12}. We suppose that is the case of $[z', z_1]$; then the curve $\gamma = f([z', z_1])$ doesn't meet E.

Next, it is clear that $\text{dist}(u, T) \geqslant 2$ for $u \in [a_1, f(z_1)]$ as $|f(z_1) - z_1| \leqslant \tau$. Since $d_{0,2^{10}}(E, T) \leqslant \epsilon$, the segment $[a_1, f(z_1)]$ doesn't meet E. Now the curve γ' which goes first from a_1 to $f(z_1)$ by the segment $[a_1, f(z_1)]$ and then from $f(z_1)$ to $f(z') = z$ by the curve γ is a curve in $B(0, 2^9)$ which joint a_1 to z and doesn't meet E. We deduce that $z \in V_1 = V$, which is (3.1.4).

Now we want to obtain a contradiction. We will construct an MS-competitor F for E whose Hausdorff measure in $B(0, 2^{10})$ is smaller than that of E in the same ball. We set

$$F = E \setminus B(a, 4). \tag{3.1.5}$$

It is clear that $F \setminus \overline{B}(0, 2^{10}) = E \setminus \overline{B}(0, 2^{10})$. We want to show that F is an MS-competitor for E. For this, we suppose that $x_1, x_2 \in \mathbb{R}^4 \setminus (\overline{B}(0, 2^{10}) \cup E)$ such that x_1, x_2 are separated by E. We want to show that they are also separated by F.

We proceed by contradiction. Suppose that

$$\text{there is a curve } \Gamma \subset \mathbb{R}^4 \text{ connecting } x_1 \text{ and } x_2 \text{ which doesn't meet } F. \tag{3.1.6}$$

Now if $\Gamma \cap \overline{B}(a,4) = \varnothing$, then Γ doesn't meet E. Next, as $F = E \setminus B(a,4)$, we have that x_1, x_2 are not separated by E, a contradiction. So we must have that Γ meets $\overline{B}(a,4)$. Let x_1' be the first point at which Γ meets $\overline{B}(a,4)$ and x_2' be the last point at which Γ meets $\overline{B}(a,4)$. Then it is clear that $x_1', x_2' \in \partial B(a,4)$. We note Γ_1 the sub-curve of Γ from x_1 to x_1' and Γ_2 the sub-curve of Γ from x_2' to x_2. Since Γ_1 and Γ_2 belong to the same connected component of F and Γ_1, Γ_2 don't meet $B(a,4)$ and $F = E \setminus B(a,4)$, we deduce that Γ_1 and Γ_2 belong to the same connected component of $\mathbb{R}^4 \setminus E$.

In addition, since $x_1', x_2' \in \partial B(a,4) \setminus E$, so by (3.1.4), they both belong to V and then we can connect x_1' and x_2' by a curve Γ_3 which doesn't meet E.

Now the curve Γ_4 which is the union of Γ_1, Γ_2 and Γ_3 is a curve that connects x_1 and x_2 and doesn't meet E. This is a contradiction, as we suppose that x_1 and x_2 are separated by E.

Now since $\operatorname{dist}(a,E) \leqslant 2^{10}\epsilon$, there is a point $a' \in E$ such that $d(a,a') \leqslant 2^{10}\epsilon$ and by consequence $B(a',2) \subset B(a,4)$. Next

$$\begin{aligned} H^3(F \cap B(0,2^{10})) &= H^3(E \cap B(0,2^{10}) \setminus B(a,4)) \\ &\leqslant H^3(E \cap B(0,2^{10}) \setminus B(a',2)) \\ &= H^3(E \cap B(0,2^{10})) - H^3(E \cap B(a',2)) \\ &\leqslant H^3(E \cap B(0,2^{10})) - C2^3 < H^3(E \cap B(0,2^{10})). \end{aligned} \tag{3.1.7}$$

Where the last line is obtained from the fact that E is Alhfors-regular (see [7]). Now (3.1.7) contradicts the hypothesis that E is MS-minimal, we thus obtain the lemma. □

If x is a point of type $\mathbb{P}$ or $\mathbb{Y}$ of E, then by Proposition 2.6 and Theorem 1, for $\tau = 10^{-25}$, for example, we can find a radius $r > 0$ and a Bi-Hölder mapping $\psi_x : B(x,2r) \to \mathbb{R}^4$, and a 3-dimensional minimal cone Y of type $\mathbb{P}$ or $\mathbb{Y}$, respectively, centered at x, such that

$$|\psi_x(z) - z| \leqslant \tau r \text{ for } z \in B(x,2r) \tag{2}$$

$$E \cap B(x,r) \subset \psi_x(Y \cap B(x,3r/2)) \subset E \cap B(x,2r). \tag{3}$$

By (2.2), there are only points of type $\mathbb{P}$ or $\mathbb{Y}$ of $E \cap \overline{B}(0, 2^{10})$. We set then

$$E_Y \text{ the set of } \mathbb{Y}\text{-points of } E \cap \overline{B}(0, 2^{10}). \tag{4}$$

It is clear that E_Y is closed by the proof of Theorem 2.10. If $x \in E_Y \cap B(0, 2^{10})$, then there exists $r_x > 0$ such that $B(x, r_x) \subset B(0, 2^{10})$ and a minimal cone Y_x of type $\mathbb{Y}$, centered at x, and a Hölder mapping $\psi_x : B(x, 2r_x) \to \mathbb{R}^4$ such that (2) and (3) hold for ψ_x and Y_x. Let L_x be the spine of Y_x, then L_x is a 2-plane passing through x. By Remark 2.12, there is a neighborhood U_x of x such that

$$E_Y \cap U_x = \psi_x(B(x, r_x) \cap L_x). \tag{5}$$

Now we take four points $d_i, 1 \leqslant i \leqslant 4$ such that 0 is the mid-point of the segments $[a_i, d_i], 1 \leqslant i \leqslant 4$, here a_i is as in Lemma 3.1. It is clear that $d_i \in T' \subset T$. In addition, $d_i \in L_i, 1 \leqslant i \leqslant 4$, where L_i are described just after the second statement of Theorem 2. Next, for $1 \leqslant i \leqslant 4$, we have $d_{d_i,4}(E, T) \leqslant 2^8 d_{0,2^{10}}(E, T) \leqslant 2^8\epsilon$. But in the ball $B(d_i, 4)$, T coincide with a cone Y_i of type $\mathbb{Y}$ whose spine is L_i. So $d_{d_i,4}(E, Y_i) \leqslant 2^8\epsilon$. By Corollary 2.11, for $\tau = 10^{-25}$, we can find $\epsilon > 0$ such that E is Bi-Hölder equivalent to Y_i in the ball $B(d_i, 2)$, with Hölder exponent $1 + \tau$. Call ψ_i this Hölder mapping, then by Remark 2.12

$$E_Y \cap B(d_i, 1) \subset \psi_i(L_i \cap B(d_i, 3/2)) \subset E_Y \cap B(d_i, 2) \tag{6}$$

and

$$|\psi_i(z) - z| \leqslant \tau \text{ for } z \in B(d_i, 2). \tag{7}$$

Setting

$$b_i = \psi_i(d_i), 1 \leqslant i \leqslant 4. \tag{8}$$

By (7), we have $d(d_i, b_i) \leqslant \tau$. We want to prove the following lemma.

Lemma 3.2. — *The point $b_1 \in E_Y$ can be connected to another point $b_i \in E_Y$, $i \neq 1$ by a curve $\gamma \subset E_Y \cap B(0, 3 \cdot 2^8)$.*

Proof. — Recall that ψ_i, b_i, d_i are the same as (6),(7),(8) above. In addition, for each $x \in E_Y \cap B(0, 2^{10})$, there are a radius r_x and a Bi-Höder mapping ψ_x, a minimal cone Y_x of type $\mathbb{Y}$, centered at x such that (2),(3), and (5) hold.

We proceed by contradiction. We denote by E_Y^1 the connected component of $E_Y \cap B(0, 2^{10})$ which contains b_1. Since in each ball $B(b_i, 2)$, E_Y is Hölder equivalent to a 2-plane, by (6), we deduce that each $z \in E_Y \cap B(b_i, 1)$

can be connected to b_i by a curve in E_Y. So if the lemma fails, that is E_Y^1 doesn't contain any $b_i, i \neq 1$, we must have

$$E_Y^1 \cap B(b_i, 1) = \varnothing \text{ for } i \neq 1. \tag{3.2.1}$$

Recall next that $T = T' \times l$, where T' is a 2-dimensional minimal cone of type $\mathbb{T}$ in the 3-plane P of equation $x_4 = 0$ and l is the line of equation $x_1 = x_2 = x_3 = 0$.

Now we construct a family of functions $f_t, 0 \leqslant t \leqslant 1$ from $\mathbb{R}^4$ to $\mathbb{R}^2$ by the formula

$$f_t(x) = (x_4, |x - td_2|^2 - ((1-t)2^9)^2), \tag{3.2.2}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $0 \leqslant t \leqslant 1$. If $x \in E_Y^1$, then

$$|f_1(x)| \geqslant |x - d_2| \geqslant 1/2, \tag{3.2.3}$$

by (3.2.1) and the fact that $|d_2 - b_2| \leqslant \tau$. We will construct a finite number of functions to go from f_0 to f_1. First, let $K = E_Y^1 \cap \overline{B}(0, 3 \cdot 2^8)$. Then for each $z \in K$, there is a radius r_z such that E_Y^1 is Bi-Hölder equivalent to a 2-plane P_z, with Hölder exponent $1 + \tau$. Since K is compact, we can cover K by a finite number of balls $B(z_i, r_{z_i}), 1 \leqslant i \leqslant N$. Finally, we choose $\eta > 0$ which is smaller than $\frac{1}{10} \min\{r_{z_i}\}, 1 \leqslant i \leqslant N$.

Next, let $\{x_i\}, 1 \leqslant i \leqslant l$ be a maximal collection of points in K such that $|x_i - x_j| \geqslant \eta$ for $i \neq j$. Set $\tilde{\varphi}_j$ a bump function with support in $B(x_j, 2\eta)$ and such that $\tilde{\varphi}_j(x) = 1$ for $x \in \overline{B}(x_j, \eta)$ and $0 \leqslant \tilde{\varphi}_j(x) \leqslant 1$ everywhere. We note that $\sum_j \tilde{\varphi}_j(x) \geqslant 1$ for $x \in E_Y^1 \cap B(0, 3 \cdot 2^8)$ since x must lie in one of the ball $B(x_j, \eta)$ by the maximality of the family $\{x_i\}$. Set $\tilde{\varphi}_0$ a C^∞ function in $\mathbb{R}^4$ such that $\tilde{\varphi}_0(x) = 0$ for $|x| \leqslant 3 \cdot 2^8 - \eta$ and $\tilde{\varphi}_0(x) = 1$ for $|x| \geqslant 3 \cdot 2^8$ and $0 \leqslant \tilde{\varphi}_0(x) \leqslant 1$ everywhere. We have then $\sum_{j=0}^{l} \tilde{\varphi}_j(x) \geqslant 1$ on E_Y^1 and we set

$$\varphi_j(x) = \tilde{\varphi}_j(x) \{\sum_{j=0}^{l} \tilde{\varphi}_j(x)\}^{-1} \text{ for } x \in E_Y^1 \text{ and } 0 \leqslant j \leqslant l. \tag{3.2.4}$$

The functions $\varphi_j, 0 \leqslant j \leqslant l$ have the following properties.

$$\varphi_j \text{ has support in } B(x_j, 2\eta) \text{ for } j \geqslant 1, \tag{3.2.5}$$

$$\begin{aligned} \sum_{j=0}^{l} \varphi_j(x) &= 1 \text{ for } x \in E_Y^1, \\ \sum_{j=1}^{l} \varphi_j(x) &= 1 \text{ for } x \in E_Y^1 \cap B(0, 3 \cdot 2^8 - \eta), \end{aligned} \tag{3.2.6}$$

since $\varphi_0(x) = 0$ on $B(0, 3 \cdot 2^8 - \eta)$. Our first approximation is a sequence of functions given by

$$g_k = f_0 + \sum_{0<j<k} \varphi_j (f_1 - f_0), \tag{3.2.7}$$

with $0 \leqslant k \leqslant l$. Then $g_0 = f_0$ and

$$g_l(x) = f_1(x) \text{ for } x \in E \cap B(0, 3 \cdot 2^8 - \eta). \tag{3.2.8}$$

We note that for $k \geqslant 1$

$$g_k(x) - g_{k-1}(x) = \varphi_k(x)(f_1(x) - f_0(x)) \text{ is supported in } B(x_k, 2\eta). \tag{3.2.9}$$

We compute the number of solutions in E_Y^1 of the equations $g_k(x) = 0$. We will modify f_0 and the g_k such that they have only a finite number of zeroes. We modify first f_0.

SUB-LEMMA 3.2.1.— *There exists a continuous function h_0 on E_Y^1 such that*

$$|h_0(x) - f_0(x)| \leqslant 10^{-6} \textit{for } x \in E_Y^1, \tag{3.2.9}$$

h_0 has exactly one zero b_1 in E_Y^1, and b_1 is a simple, non-degenerate zero of h_0.

Here, we say that $\xi \in E_Y^1$ is a non-degenerate, simple zero of a continuous function h on E_Y^1 if $h(\xi) = 0$ and there is a ball $B(\xi, \rho)$ and a Bi-Hölder function γ with Hölder exponent $1+\tau$ which maps $E_Y^1 \cap B(\xi, \rho)$ to an open set V of a 2-plane, such that $h \circ \gamma^{-1}$ is of class C^1 on V and the differential $D(h \circ \gamma^{-1})$ at the point $\gamma(\xi)$ is of rank 2.

Proof.— We modify f_0 in a neighborhood of d_1. We have already our Bi-Hölder homeomorphism ψ_1 which satisfies (6),(7) and (8). Next, since E_Y^1 is the connected component of E_Y which contains b_1, we have

$$E_Y \cap B(d_1, 1) = E_Y^1 \cap B(d_1, 1),$$

thus

$$E_Y^1 \cap B(d_1, 1/3) \subset \psi_1(B(L_1 \cap B(d_1, 1/2))) \subset E_Y^1 \cap B(d_1, 1), \tag{3.2.10}$$

here L_1 is the 2-face of T that contains d_1, which is Bi-Höder equivalent to E_Y^1 in the ball $B(d_1, 1)$.

Set $h_0 = f_0$ outside the ball $B(d_1, 1/2)$. In $B(d_1, 1/4)$, we set $h_0 = f_0 \circ \psi^{-1}$. In the region between the two balls $R = \overline{B}(d_1, 1/2) \setminus B(d_1, 1/4)$, we set

$$h_0(x) = \alpha(x) f_0(x) + (1 - \alpha(x)) f_0 \circ \psi^{-1}(x), \tag{3.2.11}$$

where $\alpha(x) = 4|x-d_1|-1$. We have then $|h_0(x) - f_0(x)| \leqslant |f_0(x) - f_0 \circ \psi_1^{-1}(x)| \leqslant C\tau$ for $x \in B(d_1, 1/2)$ since $|\psi_1(x) - x| \leqslant \tau$ and the differential of f_0 is bounded in this ball. We have then (3.2.9).

Since $f_0(x) = (x_4, |x|^2 - 4^9)$, so $|f_0(x)| \geqslant 1/500$ for $x \in E_Y^1 \setminus B(d_1, 10^{-2})$. By consequence, all the zeroes of h_0 must lie in the ball $B(d_1, 1/4)$.

We verify next that h_0 has exactly one zero in $B(d_1, 1/4)$, which is simple and non-degenerate. Set $\gamma_1(x) = \psi_1^{-1}(x)$ for $x \in E_Y^1 \cap B(d_1, 1/4)$. Then γ_1 is a homeomorphism from $E_Y^1 \cap B(d_1, 1/4)$ onto its image, which is an open set in L_1.

Since $h_0 = f_0 \circ \psi_1^{-1} = f_0 \circ \gamma_1$ on $E_Y^1 \cap B(d_1, 1/4)$, we have that $h_0(\xi) = 0$ for $\xi \in E_Y^1 \cap B(d_1, 1/4)$ if and only if $\gamma_1(\xi)$ is a zero of $f_0(x) = (x_4, |x|^2 - 4^9)$ in $L_1 \cap B(d_1, 1/2)$, which can only be d_1. The verification that Df_0 is of maximal rank at d_1 is clear. The sub-lemma follows.

We need another sub-lemma which allows us to go from h_{k-1} to h_k.

Sub-lemma 3.2.2. — *We can find continuous functions $\theta_k, 1 \leqslant k \leqslant l$, such that*

$$\theta_k \text{ is supported in } B(x_k, 3\eta), \tag{3.2.12}$$

and

$$||\theta_k||_\infty \leqslant 2^{-k}10^{-6}, \tag{3.2.13}$$

and if we set

$$h_k = h_{k-1} + \varphi_k(f_1 - f_0) + \theta_k, \tag{3.2.14}$$

for $1 \leqslant k \leqslant l$, then
(3.2.15)

each h_k has a finite number of zeroes in E_Y^1, which are all simple and non-degenerate.

Proof. — We will construct h_k by induction. For $k = 0$, the function h_0 satisfy clearly (3.2.15). Let $k \geqslant 1$, and we suppose that we have already constructed h_{k-1} such that (3.2.15) holds.

We note that $h_{k-1} + \varphi_k(f_1 - f_0)$ coincide with h_{k-1} outside the ball $B(x_k, 2\eta)$, by (3.2.5). We take a thin annulus

$$A = \overline{B}(x_k, \rho_2) \setminus B(x_k, \rho_1), 2\eta < \rho_1 < \rho_2 < 3\eta, \tag{3.2.16}$$

which doesn't meet the finite set of zeroes of h_{k-1}. Recall that there is a Bi-Hölder function $\psi_k : B(x_k, 20\eta) \to \mathbb{R}^4$ and a 2-plane P_k passing through

x_k such that $|\psi_k(x) - x| \leqslant 10\eta\tau$ for $x \in B(x_k, 20\eta)$ and

$$E_Y^1 \cap B(x_k, 19\eta) \subset \psi_k(P_k \cap B(x_k, 20\eta)) \subset E_Y^1. \tag{3.2.17}$$

We choose θ_k such that θ_k is supported in $B(x_k, \rho_2)$ and $||\theta_k||_\infty < \min\{2^k 10^{-6}, \inf_{x\in A} |h_{k-1}(x)|\}$, of course $\inf_{x\in A} |h_{k-1}(x)| > 0$ since A doesn't meet the set of zeroes of h_{k-1}. Then $h_k = h_{k-1}$ outside the ball $B(x_k, \rho_2)$.

We will control h_k in the ball $B(x_k, \rho_1)$. Set $\gamma(x) = \psi_k^{-1}(x)$ for $x \in E_Y^1 \cap B(x_k, \rho_1)$. By (3.2.17) and since ψ_k is Bi-Hölder on $B(x_k, 20\eta)$, γ is a Bi-Hölder homeomorphism from $E_Y^1 \cap B(x_k, \rho_1)$ onto an open set V of the 2-plane P_k.

By the density of C^1 function in the space of bounded continuous functions on V with the sup norm, we can choose θ_k with the above properties and such that

$$h_k \circ \theta_k \text{ is of class } C^1 \text{ on } V. \tag{3.2.18}$$

We can also add a very small constant $w \in \mathbb{R}^2$ to θ_k on $E_Y^1 \cap B(x_k, \rho_1)$, and then interpolate continuously on A. We verify that for almost every choice of w,

$$h_k \text{ has a finite number of zeroes in } E_Y^1 \cap B(x_k, \rho_1). \tag{3.2.19}$$

For this, we set $Z_y = \{z \in V; h_k \circ \psi_k(z) = y\}$. By (3.2.18), we can apply the co-area formula ([9, 3.2.22]) for $h_k \circ \psi_k$ on V, and we obtain

$$\int_V J(z) dH^2(z) = \int_{y\in\mathbb{R}^2} H^0(Z_y) dH^2(y), \tag{3.2.20}$$

here, $J(z)$ denote the Jacobian of $h_k \circ \psi_k$ at z, which is clearly bounded. We deduce that Z_y is finite for almost-every $y \in \mathbb{R}^2$. If we choose w such that Z_w is finite and then add $-w$ to θ_k in $E_Y^1 \cap B(x_k, \rho_1)$, then the new Z_0 will be finite, and we have (3.2.19).

We consider now the rank of the differential. By Sard's theorem, the set of critical values of $h_k \circ \psi_k$ has measure 0 in $\mathbb{R}^2$. So if we choose $w \in \mathbb{R}^2$ which is not a critical value, and add $-w$ to θ_k in $E_Y^1 \cap B(x_k, \rho_1)$, then the differential of the new function $h_k \circ \psi_k$ at each zero of $h_k \circ \psi_k$ is of rank 2.

So we take w very small with the above properties, and add $-w$ to θ_k in $B(x_k, \rho_1)$; next, we interpolate in the region A, we obtain a function h_k having a finite number of zeroes in $E_Y^1 \cap B(x_k, \rho_1)$ which are all simple and non-degenerate. The sub-lemma follows.

Now let $N(k)$ be the number of zeroes of h_k in E_Y^1. Then $N(0) = 1$ since the only zero of h_0 in E_Y^1 is b_1. Let us check that for the last index l,

$N(l) = 0$. First we have

$$h_l - h_0 = \sum_{1\leqslant k\leqslant l} (h_k - h_{k-1}) = \sum_{1\leqslant k\leqslant l} \varphi_k(f_1 - f_0) + \sum_{1\leqslant k\leqslant l} \theta_k.$$

If $x \in E_Y^1 \cap B(0, 3\cdot 2^8 - \eta)$, then $\sum_{1\leqslant k\leqslant l} \varphi_k(x) = 1$, thus

$$h_l(x) = h_0(x) + f_1(x) - f_0(x) + \sum_{1\leqslant k\leqslant l} \theta_k(x)$$

so that

$$\begin{aligned} |h_l(x)| &\geqslant |f_1(x)| - |h_0(x) - f_0(x)| - \sum_{1\leqslant k\leqslant l} |\theta_k(x)| \\ &\geqslant 1/4 - 10^{-6} - \sum_{1\leqslant k\leqslant l} 2^{-k}10^{-6} > 0 \end{aligned}$$

by (3.2.3), (3.2.6) and (3.2.13).

If $x \in E_Y^1 \cap B(0, 2^{10}) \setminus B(0, 3\cdot 2^8 - \eta)$, then $\sum_{1\leqslant k\leqslant l} \varphi_k(x) = 1 - \varphi_0(x)$, so

$$h_l(x) = h_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x)) + \sum_{1\leqslant k\leqslant l} \theta_k(x)$$

which implies

$$\begin{aligned} &|h_l(x) - f_0(x) - (1 - \varphi_0(x))(f_1(x) - f_0(x))| \\ &\qquad\leqslant |h_0(x) - f_0(x)| + \sum_{1\leqslant k\leqslant l} |\theta_k(x)| \leqslant 2.10^{-6}. \end{aligned}$$

But the second coordinate of $f_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x))$ is

$$\begin{aligned} &|x|^2 - 4^9 + (1 - \varphi_0)(x)(|x - d_2|^2 - |x|^2 + 4^9) \\ &\qquad = \varphi_0(x)(|x|^2 - 4^9) + (1 - \varphi_0(x))|x - d_2|^2 \geqslant 1/4, \end{aligned}$$

by (3.2.2) and because $|x| \geqslant 3\cdot 2^8 - \eta$. Thus $h_l(x) \neq 0$ in this case also. We deduce that h_l has no zero in E_Y^1, and $N(l) = 0$.

Sub-lemma 3.2.3. — *$N(k) - N(k-1)$ is even for $1 \leqslant k \leqslant l$.*

Proof. — We observe that h_{k-1} don't vanish on A, where A is the annulus defined in (3.2.16), and we took $||\theta_k||_\infty$ very small so that h_k does not vanish on A as well. Next, by definition of φ_k, $\varphi_k = 0$ on A. Setting

$$m_t(x) = h_{k-1}(x) + t[h_k(x) - h_{k-1}(x)] = h_{k-1}(x) + \theta_k(x), \qquad (3.2.21)$$

for $x \in E_Y^1 \cap \overline{B}(x_k, \rho_2)$ and $0 \leqslant t \leqslant 1$. Then $m_0 = h_{k-1}$ and $m_1 = h_k$ on $E_Y^1 \cap \overline{B}(x_k, \rho_2)$. Since $m_t(x) = h_{k-1}(x) + t\theta(x)$ for $x \in E_Y^1 \cap A$ and $0 \leqslant t \leqslant 1$, so $m_t(x) \neq 0$ if we take θ small enough. Let $\beta_k > 0$ such that $|m_t(x)| \geqslant \beta_k$ for $x \in E_Y^1 \cap A$. Set $S_\infty = \mathbb{R}^2 \cup \{\infty\}$, so that S_∞ can be stereographically identified with a sphere of dimension 2, we define $\pi : \mathbb{R}^2 \to S_\infty$ by

$$\pi(x) = \infty \text{ if } |x| \geqslant \beta_k \text{ and } \pi(x) = \frac{x}{\beta_k - |x|} \text{ otherwise.} \tag{3.2.22}$$

Next, we set

$$p_t(x) = \pi(m_t(x)) \text{ for } x \in E_Y^1 \cap \overline{B}(x_k, \rho_2) \text{ and } 0 \leqslant t \leqslant 1. \tag{3.2.23}$$

Then $p_t(x)$ is a continuous function of x and t, which takes values in S_∞. By the definition of β_k,

$$p_t(x) = \infty \text{ for } x \in E_Y^1 \cap A \text{ and } 0 \leqslant t \leqslant 1. \tag{3.2.24}$$

We want to replace the domain $E_Y^1 \cap \overline{B}(x_k, \rho_2)$ by an open set in a 2-plane P_k. We keep our Bi-Hölder function ψ_k as above, which maps an open set V of a 2-plane P_k onto $E_Y^1 \cap B(x_k, \rho_2)$ and its inverse γ which is also Bi-Hölder and maps $E_Y^1 \cap B(x_k, \rho_2)$ onto V. For $0 \leqslant t \leqslant 1$, we set

$$q_t(x) = p_t(\psi_k(x)) \text{ for } x \in V \text{ and } q_t(x) = \infty \text{ for } x \in P_k \setminus V. \tag{3.2.25}$$

We check that q_t is continuous in $P_k \times [0,1]$. It is continuous in $V \times [0,1]$, since p_t is continuous in $[E_Y^1 \cap B(x_k, \rho_2)] \times [0,1]$. It is also continuous in $[P_k \setminus \overline{V}] \times [0,1]$, because it is ∞ here. Now if $x \in \partial V$, then $\psi_k(x) \in E_Y^1 \cap \partial B(x_k, \rho_2)$, so there is a neighborhood of $\psi_k(x)$ in $\overline{B}(x_k, \rho_2)$ which is contained in A, and we have $p_t(\psi_k) = \infty$ on this neighborhood, so $q_t = \infty$ near x.

We set $q_t(\infty) = \infty$, so q_t is well defined on $S' = P_k \cup \{\infty\}$ and it is clear that each q_t is continuous for $0 \leqslant t \leqslant 1$.

Now since q_0 and q_1 are two continuous functions from the 2-sphere S' to the 2-sphere S_∞, we can compute their degrees. First, as q_0 and q_1 are homotopic, they have the same degrees. We compute the degree of q_0, for example. Let

$$q_0^{-1}(\{0\}) = \{y_1, y_2, ..., y_m\}, \tag{3.2.26}$$

the set of zeroes of q_0. This is a finite set since q_t has only finite number of zeroes for $t \leqslant 1$. Since each zero of q_0 is simple and non-degenerate, for each $1 \leqslant k \leqslant m$, there exists a neighborhood W_k of y_k such that

$$q_0 \text{ is a homeomorphism from } W_k \text{ to } q_0(W_k), \tag{3.2.27}$$

and

$$W_k \cap W_l = \varnothing \text{ if } k \neq l. \tag{3.2.28}$$

So the degree of q_0 is computed as follows. We begin by 0, next, for $1 \leqslant k \leqslant m$, if q_0 preserve the orientation of W_k, we add 1, if q_0 doesn't preserve the orientation of W_k, we add -1. Then it is clear that

$$d(q_0) \text{ is of the same parity as } m. \tag{3.2.29}$$

Here $d(q)$ denote the degree of the function q. By the same arguments, we have

$$d(q_1) \text{ is of the same parity as the number of zeroes of } q_1. \tag{3.2.30}$$

But $d(q_0) = d(q_1)$ as above, we obtain

the number of zeroes of q_0 is of the same parity as the number of zeroes of q_1. (3.2.31)

We want to prove next that the number of zeroes of h_{k-1} is of the same parity as the number of zeroes of h_k. Since $h_{k-1} = h_k$ outside the ball $B(x_k, \rho_2)$ and they both don't vanish on $E_Y^1 \cap A$, we need only to consider their number of zeroes in $E_Y^1 \cap B(x_k, \rho_1)$. We verify that

the number of zeroes of h_{k-1+s} in $E_Y^1 \cap B(x_k, \rho_1)$ is equal to the number of zeroes of q_s in S' for $s = 0, 1$. (3.2.32)

We verify for $s = 0$. If $q_0(x) = 0$, then $x \in V$ (otherwise $q_0(x) = \infty$), so $q_0(x) = p_0(\psi_k(x))$ and then $p_0(\psi_k(x)) = 0$. Since $m_0(\psi_k(x)) = 0$, we have $h_{k-1}(\psi_k(x)) = 0$. Because $x \in V$, we have $\psi_k(x) \in B(x_k, \rho_1)$. So if $q_0(x) = 0$, then $\psi_k(x) \in B(x_k, \rho_1)$ and is a zero of h_{k-1}.

Conversely, if $y \in B(x_k, \rho_1)$ is such that $h_{k-1}(y) = 0$, then $p_0(y) = 0$ and then there exists $y' \in V$ such that $\psi_k(y') = y$ because ψ_k is a homeomorphism from V to $B(x_k, \rho_1)$. Now $q_0(y') = p_0(\psi_k(y')) = 0$ and thus y' is a zero of q_0.

So we have (3.2.32) for $s = 0$. The case $s = 1$ is the same, and we have then (3.2.32). By (3.2.31), we obtain that the number of zeroes of h_{k-1} is of the same parity as the number of zeroes of h_k, which means that $N(k) - N(k-1)$ is even. The sub-lemma follows.

Now by sub-lemma 3.2.3, we know that $N(0) - N(1)$ is even, but it is 1, so we obtain a contradiction, and we finish the proof of Lemma 3.2. □

3.3. Proof of Theorem 2

Let $U(y), y \in E_Y \cap B(0, 3 \cdot 2^8)$ be the set of connected components V of $B(0, 2^{10}) \backslash E$ such that $y \in \overline{V}$. Since for each $y \in E_Y$, there is a neighborhood W of y on which E is Bi-Hölder equivalent to a $\mathbb{Y}$, we see that $U(y)$ is locally constant. By Lemma 3.2, we can connect b_1 to another point b_i, $i \neq 1$, by a curve in E_Y^1, and we can suppose that $i = 2$. Because $b_1, b_2 \in E_Y$ and $U(y)$ is locally constant on E_Y, we have $U(b_1) = U(b_2)$. By Lemma 3.1, and the fact that E is Bi-Hölder equivalent to a $\mathbb{Y}$ near each point of type $\mathbb{Y}$, we have

$$\{V_2, V_3, V_4\} = U(b_1)$$

and

$$\{V_1, V_3, V_4\} = U(b_2),$$

where $V_i, 1 \leqslant i \leqslant 4$ is as in Lemma 3.1. So we see that $U(b_1) \neq U(b_2)$, which is a contradiction. We finish the proof of Theorem 2. $\square$

Bibliography

[1] Almgren (F. J.). — Some interior regularity theorems for minimal surfaces and an extension of Bernstein's Theorem, Ann. of Math (2), Vol. 84, p. 277-292 (1966).

[2] Allard (W. K.). — On the First Variation of a Varifold, Ann. of Math (2), Vol. 95, p. 417-491 (1972).

[3] David (G.). — Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$, Ann. Fac. Sci. Toulouse Math, (6), 18(1) p. 65-246 (2009).

[4] David (G.). — $C^{1+\alpha}$-regularity for two dimensional almost-minimal sets in $\mathbb{R}^n$, Journal of Geometric Analysis, Vol 20, Number 4, p. 837-954.

[5] David (G.). — Singular sets of minimizers for the Mumford-Shah functional, Progress in Mathematics 233 (581p.), Birkhauser (2005).

[6] David (G.), De Pauw (T.), and Toro (T.). — A generalization of Reifenberg's theorem in R^3, Geom. Funct. Anal. Vol. 18, p. 1168-1235 (2008).

[7] David (G.) and Semmes (S.). — Uniform rectifiability and quasiminimizing sets of arbitrary codimension, Memoirs of the A.M.S. Number 687, Vol 144 (2000).

[8] Dugundji (J.). — Topology, Allyn and Bacon, Boston (1966).

[9] Federer (H.). — Geometric measure theory, Grundlehren der Mathematishen Wissenschaften 153, Springer Verlag (1969).

[10] Simons (J.). — Minimal varieties in riemannian manifolds, Ann. of Math, (2), Vol. 88, p. 62-105 (1968).

[11] Taylor (J.). — The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces, Ann. of Math. (2) 103, no. 3, p. 489-539 (1976).