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On the number of zeros of Melnikov functions

SERGEY BENDITKIS⁽¹⁾, DMITRY NOVIKOV⁽¹⁾

ABSTRACT. — We provide an effective uniform upper bound for the number of zeros of the first non-vanishing Melnikov function of a polynomial perturbations of a planar polynomial Hamiltonian vector field. The bound depends on degrees of the field and of the perturbation, and on the order k of the Melnikov function. The generic case $k = 1$ was considered by Binyamini, Novikov and Yakovenko [BNY10]. The bound follows from an effective construction of the Gauss-Manin connection for iterated integrals.

RÉSUMÉ. — Nous donnons une borne supérieure effective et uniforme pour le nombre de zéros de la première fonction de Melnikov d'une perturbation polynomiale d'un champ de vecteurs hamiltonien polynomial sur le plan. La borne dépend des degrés du champ et de la perturbation, et de l'ordre k de la fonction de Melnikov. Le cas générique $k = 1$ a été considéré par Binyamini, Novikov et Yakovenko [BNY10]. La borne est obtenue à l'aide d'une construction effective de la connection de Gauss-Manin pour les intégrales itérées.

1. Introduction

1.1. Infinitesimal Hilbert 16th problem

The second part of 16th Hilbert problem asks: How many limit cycles may have a planar polynomial vector field? The question has a long history,

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and was at the origin of several theories, see [I02]). The most remarkable achievement, Ecalle-Ilyashenko theorem, claims that the number of limit cycles is finite for any individual vector field, see [E92, I]. However, existence of a uniform upper bound for this number even for quadratic vector fields is an open problem.

A weaker form of the same question concerns perturbations of Hamiltonian vector fields. Let $H(x, y)$ be a bivariate polynomial (further called Hamiltonian). The corresponding Hamiltonian system can be written in Pfaffian form as

$$dH = 0. \quad (1.1)$$

Consider its polynomial perturbation

$$dH + \varepsilon\omega = 0, \quad \text{where } \omega = P(x, y)dx + Q(x, y)dy, \quad P, Q \in \mathbb{R}[x, y], \quad (1.2)$$

and $\varepsilon \in (\mathbb{R}^1, 0)$.

Consider a nest of cycles $\{\delta_t \subset \{H = t\}, t \in [a, b] \subset \mathbb{R}\}$ of (1.1). We ask how many limit cycles of (1.2) converge to this nest as $\varepsilon \rightarrow 0$.

It is easy to see that closed trajectories δ_t that survive after the perturbation should produce zero value of the Poincaré integral (aka first Melnikov function)

$$I = I(\delta_t, \omega) = \oint_{\delta_t} \omega,$$

the so-called Poincaré-Andronov-Pontryagin criterion, see [IY, §26A]. Therefore estimates on the number of zeros of this so-called Abelian integral have direct relation to the Hilbert 16th problem. Binyamini, Novikov, Yakovenko studied the case of *non-conservative* perturbations, namely, when the Poincaré integral does not vanish identically.

THEOREM 1.1 [BNY10]. — *Assume that $I \not\equiv 0$ for the nest of cycles of (1.1). Assume that $\deg \omega < \deg H$. Then the number of cycles δ_t providing the zero value of Poincaré integral is at most $2^{P(\deg H)}$, where $P(n)$ is some explicit polynomial of degree at most 61.*

This upper bound serves also as an upper bound for the cyclicity of an open nest of the limit cycles (which is defined as a supremum of cyclicities of all closed subnests of the open nest, see e.g. [GN10]).

For generic Hamiltonians identical vanishing of I implies exactness of ω (again, assuming $\deg \omega < \deg H$), so the perturbation remains integrable, see [I69]. However, for degenerate Hamiltonians one has to consider Melnikov functions of higher order.

1.2. Melnikov functions and the main theorem

DEFINITION 1.2. — For a cycle δ of (1.1) choose a transversal σ with coordinate z chosen in such a way that δ intersect σ at $z = 0$.

Denote by $\Delta : \sigma \rightarrow \sigma$ the holonomy map of cycle γ considered as a function of the parameters h, ε . Being analytic function of its arguments, Δ can be expanded in the converging series

$$\Delta(z, \varepsilon) = z + \varepsilon M_1(z) + \dots + \varepsilon^k M_k(z) + \dots, \quad (1.3)$$

where $M_k(z)$ are real analytic functions defined in some common neighborhood of the origin $z = 0$. The function M_k is called k -th Melnikov function.

Assume that the first nonzero function $M_k(z)$ has N isolated zeros (counted with their multiplicities) in the closed interval $\{|z| \leq \rho\}$.

PROPOSITION 1.3 [IY, Proposition 26.1]. — There exists a small positive value $r > 0$ such that for all $|\varepsilon| < r$ the foliation (1.2) has no more than N limit cycles intersecting σ at $\{|h| \leq \rho\}$.

Our main result provides an upper bound for the number of isolated zeros of the first non-zero Melnikov function.

THEOREM 1.4. — The number of isolated zeros of the first non-zero Melnikov function M_K is bounded by $\exp(\exp(d^{O(1)} n^{O(K)}))$, where $n + 1 = \deg H$, $d = \deg \omega$, and the absolute constants in $O(1), O(K)$ can be explicitly computed.

This bound is certainly not exact, and construction of lower bounds is a difficult problem, unsolved even for the Abelian integrals.

Note that the order K of the first non-zero Melnikov function cannot be easily bounded in terms of degree of H : this problem includes, as a particular case, the center-focus problem.

1.3. Iterated integrals and algebraic motivation

It is well-known, see [G05, IY], that M_K can be represented as a linear combination of so-called *iterated integrals* of length at most K .

DEFINITION 1.5. — Let $\gamma(s) : [0, 1] \rightarrow \mathbb{C}^2$ be parameterization of a curve $\gamma \subset \mathbb{C}^2$. For a k -tuple of forms $\omega_1, \dots, \omega_k \in \Lambda^1(\mathbb{C}^2)$ we define the iterated integral as

$$\int_{\gamma} \omega_1 \dots \omega_k = \int_0^1 \left(\int_0^{s_1} \left(\dots \left(\int_0^{s_k} \gamma^* \omega_k \right) \gamma^* \omega_{k-1} \right) \dots \right) \gamma^* \omega_1.$$

We will always assume that γ lies on a Riemann surface $\{H = t\}$. The fundamental question then is the dependence of the iterated integral on the path of integration γ . Unlike Abelian integrals, the iterated integrals are *not* additive function of γ . However, iterated integrals are preserved by homotopy with fixed endpoints of γ , in particular by reparametrization. If γ is closed, this means that iterated integrals depend on $[\gamma] \in \pi_1(\{H = t\}, \gamma(0))$ only.

Iterated integrals were extensively studied from various points of view, see e.g. [Ch, H, MN08]. Our goal is to investigate their oscillation properties. Let us choose a straight line as a transversal to the nest of cycles. Iterated integrals define functions on this transversal: to any point p of the transversal corresponds the value of the iterated integral over the cycle of the foliation passing through it, with p being the initial point of the path of integration (note that, unlike the Melnikov function, the iterated integrals do depend on the choice of the initial point of the cycle).

The main step of the proof of Theorem 1.4 is an explicit construction of a meromorphic flat connection whose horizontal sections are given by basic iterated integrals (see (3.7) for definition), a higher order analogue of the Gauss-Manin connection for Abelian integrals. We prove in Section 4 that this connection belongs to the class of connections considered in the paper of Binyamini, Novikov and Yakovenko [BNY10], see the next section for formulation of the result. Estimates on the complexity of the connection, proved in Section 3, allow to apply their main result not only to linear combinations of basic iterated integrals, but also to their combinations with coefficients polynomially dependent on z from (1.3). In Section 5 we represent M_K in this form.

Note that the main result of the paper holds for Melnikov functions of any order, not only for the first non-vanishing one. However, the zeros of these higher order Melnikov functions do not seem to have any meaning.

2. Non-oscillation of horizontal sections of meromorphic connections

In this section we briefly recall the main result of [BNY10]. Let Ω be a rational $l \times l$ -matrix of rational differential 1-forms on a complex manifold M , with a singular locus Σ . It defines a connection

$$dX = \Omega \cdot X \tag{2.1}$$

on trivial vector bundle $M \times \mathbb{C}^l$. We denote by Σ the singular locus of the connection.

2.1. Regular integrable connections

DEFINITION 2.1. — *The form Ω is integrable or locally flat if $d\Omega - \Omega \wedge \Omega = 0$.*

This condition is equivalent to local existence of a basis of horizontal sections of (2.1) near each nonsingular point $a \notin \Sigma$.

DEFINITION 2.2. — *The Picard-Fuchs system (2.1) (and the corresponding matrix 1-form Ω) is called regular at $a \in M$, if for any germ of a holomorphic curve $\gamma : (\mathbb{C}, 0) \rightarrow (M, a)$ the pull-back of the connection to $(\mathbb{C}, 0)$ has a regular singularity at the origin:*

$$\forall C > 0 \quad \exists p = p(C) \in \mathbb{R} \quad \|X(\gamma(s))\|^{\pm 1} = O(|s|^{-p}) \quad (2.2)$$

as $s \rightarrow 0$ in the sector $\{\arg s \leq C\}$.

Connection is called regular on M if it is regular at each point $a \in M$.

Regular connections remain regular after pull-backs, push-forwards, (semi)direct products etc., see [D].

2.2. Quasiunipotent connections

DEFINITION 2.3. — *For a point $a \in M$ a small loop around a is a closed path γ , such that exists a holomorphic mapping $\{|z| \leq 1\} \rightarrow M$ which maps 0 to a , $\{|z| = 1\}$ to γ and such that the image of $\{|z| \leq 1\} \setminus \{0\}$ is disjoint from Σ .*

Recall that an operator is called *quasiunipotent* if all its eigenvalues are roots of unity, i.e. belong to $\exp(2\pi i\mathbb{Q})$.

DEFINITION 2.4. — *The integrable form Ω is called quasiunipotent at a point $a \in M$, if the monodromy operator associated with any small loop around a is quasiunipotent. The system is (globally) quasiunipotent, if it is quasiunipotent at every point of $\mathbb{C}P^n$.*

Similarly to regularity condition, quasiunipotency at generic points implies quasiunipotency at all points:

THEOREM 2.5. — *(Kashiwara theorem [K81]). A regular integrable system that is quasiunipotent at each point outside an algebraic subset of codimension 2, is globally quasiunipotent.*

In general, quasiunipotency does not imply that every monodromy operator associated to Ω is quasiunipotent.

2.3. Degree of rational function

We define degree of a rational function to be the minimum of sums of degrees of numerator and denominator over all its representations as a ratio of two polynomials. Degree of the form is defined in such a way that the operator d has degree 0.

2.4. Notion of size

In this work, similarly to [BNY10], we are studying various objects, like matrices, functions, differential forms, defined over \mathbb{Q} , the field of rational numbers. To obtain quantitative characteristics of these objects, we need to use the notion of *size*, or *complexity* of the objects.

DEFINITION 2.6. — *The norm of a multivariate polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$, $P(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$ (in the standard multi-index notation) is the sum of absolute values of its coefficients, $\|P\| = \sum_{\alpha} |c_{\alpha}|$. Clearly, this norm is multiplicative,*

$$\|PQ\| \leq \|P\| \cdot \|Q\|$$

DEFINITION 2.7. — *The size $\mathbf{S}(P)$ of an integer polynomial $P \in \mathbb{Z}[z_1, \dots, z_n]$ is set to be equal to its norm, $\mathbf{S}(P) = \|P\|$.*

The size of a rational fraction $R \in \mathbb{Q}(z_1, \dots, z_n)$ is

$$\mathbf{S}(R) = \min_{P, Q} \{ \|P\| + \|Q\| : R = P/Q; P, Q \in \mathbb{Z}[z_1, \dots, z_n] \}$$

The size of a (polynomial or rational) 1-form on \mathbb{P}^m or on $\mathbb{P}^m \times \mathbb{P}^1$ defined over \mathbb{Q} , is the sum of sizes of its coefficients in the standard affine chart \mathbb{C}^m .

The size of a vector or matrix rational function (resp., 1-form) defined over \mathbb{Q} , is the sum of the sizes of its components.

Note that, unlike polynomials, for rational functions we have only

$$\begin{aligned} \deg \left(\sum R_i \right) &\leq 2 \sum_i \deg R_i \\ \mathbf{S} \left(\sum_{i=1}^n R_i \right) &\leq (n+1) \prod_{i=1}^n \mathbf{S}(R_i). \end{aligned} \tag{2.3}$$

2.4.1. Counting number of zeros of the solution

Let Ω be a rational $l \times l$ -matrix 1-form of degree d on the product $\mathbb{C}P^m$, and consider the restriction of the corresponding Picard-Fuchs system (2.1) to some line $\ell \cong \mathbb{C}P^1 \subset \mathbb{C}P^m$. We are interested in the number of zeros of a linear combinations of entries of the fundamental matrix of (2.1). In general, restriction of the fundamental matrix to this line produces a multivalued matrix function on $\ell \setminus \Sigma$, so to count zeros one should choose a simply connected domain in $\ell \setminus \Sigma$. One can easily see that the geometric complexity of the domain should be taken into account.

DEFINITION 2.8. — *We denote by $\mathcal{N}(\ell) = \mathcal{N}(\Omega|_\ell)$ the supremum over all constant matrices B and all closed triangles T lying in $\ell \setminus \Sigma$ of the number of isolated zeros of the function $\text{Tr}BX$ in T .*

Without extra assumption, this supremum could be easily infinite. However,

THEOREM 2.9 [BNY10, Theorems 7,8]. — *Let Ω be a rational $l \times l$ -matrix 1-form of degree d on the product $\mathbb{C}P^m \times \mathbb{C}P^1$. Assume that Ω is integrable, regular and quasiunipotent, is defined over \mathbb{Q} and its size is $s = \mathbf{S}(\Omega)$. Then*

$$\forall \ell \cong \mathbb{C}P^1 \subset \mathbb{C}P^m \times \mathbb{C}P^1 \quad \mathcal{N}(\ell) \leq s^{2^{C(at^4 m)^5}}$$

for some universal constant C .

3. Construction of Gauss-Manin connection for iterated integrals

3.1. Base spaces: notations

Let $\mathbb{C}_{n+1}[x, y]$ denote the space of all bivariate polynomials of degree at most $n + 1$. We will denote the points of its projectivization $P\mathbb{C}_{n+1}[x, y]$ by λ . In standard coordinates $H = \sum_{0 \leq i+j \leq n+1} \lambda_{ij} x^i y^j$. By $\tilde{\lambda}$ we denote the tuple of all elements of λ except the last one, λ_{00} .

An important role plays the space $\tilde{\mathbb{C}}_{n+1}[x, y]$ of polynomials vanishing at the origin, of dimension smaller by 1. The tuples $\tilde{\lambda}$ parametrize the points of its projectivization $P\tilde{\mathbb{C}}_{n+1}[x, y]$.

3.2. Gauss-Manin connection for Abelian integrals

DEFINITION 3.1. — Let $H \in \mathbb{C}[x, y]$ be a polynomial of degree $n+1$. The Petrov module P_H is the $\mathbb{C}[t]$ -module defined as the quotient space

$$P_H = \frac{\Lambda^1}{dH \cdot \Lambda^0 + d\Lambda^0}$$

of polynomial 1-forms over the space of relatively exact forms $f \cdot dH + dg$, where f, g are polynomials.

Recall that a bivariate polynomial is called *ultra-Morse* if it has Morse critical points with distinct critical values and its highest homogeneous part has no multiple factors.

PROPOSITION 3.2 [IY, Theorem 26.21]. — The set of all ultra-Morse polynomials H for which the forms $\omega_{ij} = x^{i-1}y^j dx$, $1 \leq i, j \leq n$ form a basis of P_H over $\mathbb{C}[h]$ is a Zarisky open subset $\mathcal{UM} \subset P\mathbb{C}n_{+1}[x, y]$.

To simplify the notations, we will sometimes reenumerate the set of basic forms $\mathcal{B} = \{\omega_{ij} = x^{i-1}y^j dx, i, j = 1, \dots, n\}$ as $\mathcal{B} = \{\omega_l\}$, in such a way that $\deg \omega_l$ is non-decreasing with l . \mathcal{B} provides a convenient global trivialization of homological Milnor bundle over \mathcal{UM} : a cycle $\delta \in H_1(\{H=0\}, \mathbb{C})$ corresponds to the vector $\{\int_{\delta} \omega_l\} \in \mathbb{C}^{n^2}$. The Gauss-Manin connection in this trivialization can be written explicitly and this fact relates the main result of [BNY10] to Infinitesimal Hilbert 16th problem. Let us formulate this result.

Let H be a polynomial satisfying the conditions of Proposition 3.2, such that the affine curve $\Gamma_H = \{H=0\} \subset \mathbb{C}^2$ is smooth. Choose a point $p_0 \in \Gamma_H$. Γ_H is a Riemann surface of genus $\frac{n(n-1)}{2}$ with $n+1$ removed points. Therefore its fundamental group $\pi_1(\Gamma_H, p_0)$ is a free group in $N = n^2$ generators.

Choose $\delta_1, \dots, \delta_N \in \pi_1(\Gamma_H, p_0)$ in such a way that their homology classes form a basis in $H_1(\Gamma_H, \mathbb{Z})$ and consider the matrix

$$S_1 = \left\{ \oint_{\delta_k} \omega_l \right\}_{k,l=1}^N. \quad (3.1)$$

In [G98] it was proved that S_1 is non-degenerate for H . Moreover, S_1 is a fundamental system of solutions of a Picard-Fuchs equation, see [AGV].

THEOREM 3.3 ([BNY10]). — $S_1 = S_1(H)$ is the matrix of fundamental solutions of the Picard-Fuchs equation

$$dS_1 = \Omega_1 S_1, \quad (3.2)$$

which is defined over \mathbb{Q} and has the size $s = \mathbf{S}(\Omega)$, dimension ℓ and the degree $d = \deg \Omega$ explicitly bounded from above as

$$s \leq 2^{\text{Poly}(n)}, \quad d \leq O(n^2), \quad \ell = n^2. \quad (3.3)$$

Using these estimates and Theorem 2.9, one gets the main result of [BNY10].

Our goal is to generalize this construction for iterated integrals of length $K > 1$. To this end we will need more detailed results. We will consider forms $\theta \in \Lambda^1[\mathbb{C}^2] \otimes \mathbb{C}(\lambda)$, i.e. the one-forms on \mathbb{C}^2 whose coefficients are polynomials in x, y and the coefficients of these polynomials are rational functions of λ . If the coefficients of these rational functions in λ are rational numbers, i.e. $\theta \in \Lambda^1[\mathbb{C}^2] \otimes \mathbb{Q}(\lambda)$, then we will say that θ is a *polynomial 1-form on \mathbb{C}^2 defined over $\mathbb{Q}(\lambda)$* .

PROPOSITION 3.4. — Let θ be a polynomial one-form of degree d on $\mathbb{C}_{x,y}^2$ defined over $\mathbb{Q}(\lambda)$. Let θ be of degree ν in λ and of size s . Denote by $\tilde{\lambda} = \lambda \setminus \{\lambda_{00}\}$ the tuple of all coefficients of H except the first one. Then one can write a decomposition

$$\theta = \sum_{i=1}^N (f_i \circ H) \omega_i + f dH + dg, \quad f, g \in \mathbb{Q}(\tilde{\lambda})[x, y], \quad f_i \in \mathbb{Q}(\tilde{\lambda})[h], \quad (3.4)$$

with $\deg_{x,y} f, \deg_{x,y} g \leq d$ and $\deg_h f_i \leq \frac{d}{n+1}$. Moreover, coefficients of f, g and f_i can be chosen to be of degree at most $\nu + O(d^3)$ in $\tilde{\lambda}$ and of sizes bounded by $sd^{O(d^3)}$.

Proof. — It is well known that for any fixed sufficiently generic $\tilde{\lambda}$ one can write decomposition (3.4) with this bounds on degrees in x, y and h and some numerical coefficients, see e.g. [G98]. The functions f, g in this decomposition are not uniquely defined, but the functions f_i are defined uniquely. To understand dependence of coefficients of f_i, f, g on $\tilde{\lambda}$, consider (3.4) as a system of linear equations on the coefficients of f, g and f_i (these are rational functions of $\tilde{\lambda}$)

$$A\hat{F} = \hat{\theta}, \quad (3.5)$$

where $\hat{F} \in \mathbb{Q}(\tilde{\lambda})^{D_U}$ denotes the vector of coefficients of f, g and f_i , and $\hat{\theta} \in \mathbb{Q}(\tilde{\lambda})^{D_E}$ denotes the vector of coefficients of θ .

Note that the matrix A in (3.5) does not depend on the form θ , only on its degree. Let us denote determinant of its biggest non-degenerate minor by Δ_d .

Assume that $d > n$ (otherwise there is nothing to prove). The number of equations is equal to the number of coefficients of θ , i.e. is equal to $D_E = (d + 1)(d + 2) = O(d^2)$. The number of unknowns (i.e. of coefficients of f_i, f and g) is $D_U = \frac{dn^2}{n+1} + O(d^2) = O(d^2)$. Coefficients of the matrix A in (3.5) are polynomials in λ , of degrees and sizes being $O(d)$ (coming from H^j) and $n^{O(d)}$ correspondingly. By assumption, on the right hand side of (3.5) are polynomials of degree at most ν_1 , divided by some common polynomial R of degree ν_2 , $\nu_1 + \nu_2 = \nu$. Their sizes are at most s . Multiplying by R , applying Cramer rule and dividing back by R , we conclude that the coefficients of f, g and f_i can be chosen to be polynomials of degree $\nu_1 + O(d^3)$ and of size $sn^{O(d^3)}$ divided by a product of $R\Delta_d$. One can see that Δ_d is of degree $O(d^3)$ and of size $n^{O(d^3)}$, so the entries of \hat{F} are of degree $\nu + O(d^3)$ and of sizes $sd^{O(d^3)}$. Since the denominators of all entries are equal, the same bounds hold for the degrees and sizes of f, g and f_i . \square

3.3. Chen homomorphism

Here we prove an analogue of the first claim of Theorem 3.3 for iterated integrals.

Let U be the space of all formal infinite series in non-commuting variables $X_1, \dots, X_N, N = n^2$. Let \mathfrak{m} denote the maximal ideal $\mathfrak{m} = \langle X_1, \dots, X_N \rangle \subset U$. We denote the units of U and of $\pi_1(\Gamma_H, p_0)$ by e .

DEFINITION 3.5. — *Define Chen homomorphism $\varphi : \pi_1(\Gamma_H, p_0) \rightarrow U$ as*

$$\varphi(\delta) = e + \sum_{(\omega_{i_1} \dots \omega_{i_k})} \oint_{\delta} \omega_{i_1} \dots \omega_{i_k} X_{i_1} \dots X_{i_k}, \quad (3.6)$$

where summation is over the set of all non-empty words in alphabet ω_l .

Let $j^K : U \rightarrow U/\mathfrak{m}^{K+1}U$ be the natural homomorphism. Define φ_K to be the composition $\varphi_K = j^K \varphi : \pi_1(\Gamma_H, p_0) \rightarrow U/\mathfrak{m}^{K+1}U$.

One can easily show that φ (and therefore φ_K) is a group homomorphism to the set of invertible elements of U (of U/\mathfrak{m}^{K+1} resp.), see [H].

Note that the space $U/\mathfrak{m}^{K+1}U$ is finite-dimensional, and has standard basis of monomials $\{X_{i_1}X_{i_2}\cdots X_{i_k}, 0 \leq k \leq K, 1 \leq i_j \leq N\}$. We claim that the image of φ_K spans $U/\mathfrak{m}^{K+1}U$.

LEMMA 3.6. — *Let $\Delta^{\leq K}$ be the set of products of length at most K of the generators δ_j of $\pi_1(\Gamma_H, p_0)$. The set $\{j^K\varphi(\delta), \delta \in \Delta^{\leq k}\}$ is a basis of $U/\mathfrak{m}^{K+1}U$.*

Proof. — The statement of the Lemma holds simultaneously for all base in $H^1(\Gamma_H, \mathbb{Z})$, so, by replacing ω_i by their linear combinations, we can assume that $\{[\omega_i]\}$ form a basis of $H^1(\Gamma_H, \mathbb{Z})$ dual to $\{[\delta_i]\} \subset H_1(\Gamma_H, \mathbb{Z})$.

We have $\varphi(e) = e$. This implies the statement for $K = 0$.

If $K = 1$, then $\varphi(\delta_i) - \varphi(e) = X_i + \mathfrak{m}^2U$, so X_i is in the span of $j^1\varphi(\Delta^{\leq 1})$.

For $K > 1$, we see from the previous equality that

$$X_{i_1}X_{i_2}\cdots X_{i_k} = (\varphi(\delta_{i_1}) - \varphi(e)) \cdot \dots \cdot (\varphi(\delta_{i_k}) - \varphi(e)) + \mathfrak{m}^{K+1}U,$$

and, since φ is homomorphism, the right hand side is a linear combination of elements of $\{\varphi(\delta), \delta \in \Delta^{\leq k}\} \pmod{\mathfrak{m}^{K+1}U}$.

So $\{\varphi_K(\delta), \delta \in \Delta^{\leq k}\}$ spans $U/\mathfrak{m}^{K+1}U$, and, by cardinality reason (here we use that $\pi_1(\Gamma_H, p_0)$ is a free group), is a basis. \square

3.4. Construction of the horizontal section

Abelian integrals are iterated integrals of length 1. The direct analogue of the matrix S_1 of (3.1) for iterated integrals of length at most K is the matrix

$$S_K(H) = \left\{ \oint_{\delta_{j_1}\dots\delta_{j_k}} \omega_{i_1}\dots\omega_{i_l}, \quad j_r, i_s = 1, \dots, N \right\}_{k,l=0}^K \quad (3.7)$$

of iterated integrals of length at most K of basic forms $\omega_j \in \mathcal{B}$ over the cycles $\delta = \delta_{j_1}\dots\delta_{j_k} \in \Delta^{\leq K}$ (we adopt convention $\int_{\delta} \emptyset = 1$). We call these integrals *the basic iterated integrals*.

The iterated integrals depend on the choice of the base point of $\pi_1(\Gamma_H, p_0)$, so we choose p_0 as one of the points of intersection of $\{H = 0\}$ with the line $\sigma = \{x = 0\}$ (generically, there are $n + 1$ such points). Columns of S_K are just the coordinates of $\varphi_K(\delta), \delta \in \Delta^{\leq K}$ written in the standard basis of $U/\mathfrak{m}^{K+1}U$.

For a generic H for all \tilde{H} sufficiently close to H the pairs $(\Gamma_{\tilde{H}}, p_0(\tilde{H}))$ are diffeomorphic to $(\Gamma_H, p_0(H))$ by a diffeomorphism close to identity. This diffeomorphism is unique up to isotopy, so we can identify the fundamental groups $\pi_1(\Gamma_{\tilde{H}}, p_0(\tilde{H}))$. This means that any path $\delta \in \pi_1(\Gamma_H, p_0(H))$ can be continuously extended to a family $\delta(\tilde{H})$ defined in some neighborhood of H . Therefore S_K can be extended holomorphically to some neighborhood of H , and, by analytic continuation, to a multivalued matrix function holomorphic on some Zarisky open subset of $PC_{n+1}[x, y]$.

Lemma 3.6 claims that S_K is non-degenerate for a generic choice of H . Therefore near a generic H the matrix S_K describes a basis of space of sections of the trivial vector bundle $PC_{n+1}[x, y] \times U/\mathfrak{m}^{K+1}$.

Our goal is to explicitly write coefficients of the connection for which the matrix S_K is a basis of horizontal sections. We construct this connection locally in a neighborhood of some generic point $H \in PC_{n+1}[x, y]$. The coefficients of the connection matrix $\Omega_K = dS_K S_K^{-1}$ turn out to depend rationally on H and the point of intersection $p \in \{H = 0\} \cap \sigma$, which is an algebraic function of H .

To eliminate the algebraic multivaluedness of Ω_K we lift the bundle and the connection to the corresponding algebraic cover. Namely, for any $n > 0$ we define B_n to be the product

$$B_n = P\tilde{C}_{n+1}[x, y] \times \mathbb{C}P^1 \tag{3.8}$$

of the space of all polynomials of degree at most $n + 1$ vanishing at $(0, 0)$, and of the line $\sigma = \{x = 0\}$. Define the mapping $\text{ev} : B_n \rightarrow PC_{n+1}[x, y]$ by $\text{ev}(H, y) = H - H(y)$. Lifting $\tilde{\Omega}_K = \text{ev}^* \Omega_K$ defines a meromorphic connection on $B_n \times U/\mathfrak{m}^{k+1}$. We prove that the resulting connection matrix is rational on B_n and satisfies the conditions of Theorem 2.9.

3.5. Differentiation of iterated integrals

Our main tool in construction of the connection is a formula of differentiating of integrals, a version of the Gelfand-Leray formula for non-closed paths. We follow closely [G05].

Let R be a function holomorphic in some open set $W \subset \mathbb{C}^2$, and assume that its non-critical level $\{R = 0\}$ is smooth and intersects transversely the line $\sigma = \{x = 0\}$ at a point $p_0(0)$. Choose a path δ lying on $\{R = 0\}$ and starting from $p_0(0)$ with endpoint $p_1(0)$. For any point p in a neighborhood of $p_1(0)$ we can define a path $\delta(p)$ close to δ , lying on $\{R = R(p)\}$ and joining p and the point $p_0(p)$ of transversal intersection of $\{R = R(p)\} \cap \sigma$.

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PROPOSITION 3.7 ([G05, Lemma 2.2]). — Let ω be a differential 1-form holomorphic in W . Then for the integral $\int_{\delta(p)} \omega$, the following equation holds

$$d \int_{\delta(p)} \omega = \left(\int_{\delta(p)} \frac{d\omega}{dR} \right) dR + \omega - (\sigma \circ R)^* \omega, \quad (3.9)$$

where $\sigma : (\mathbb{C}, 0) \rightarrow \{x = 0\}$ is the parameterization of $\{x = 0\}$ by values of R : $\sigma(t) = \{R = t\} \cap \{x = 0\}$.

Here, as usual, the Gelfand-Leray derivative $\frac{d\omega}{dR}$ is defined by $\omega = dR \wedge \frac{d\omega}{dR}$, see [IY]. Restricting to $p \in \sigma$ and closed path δ we get the standard Gelfand-Leray formula.

Assume now that the initial path $\delta = \delta(0)$ is closed, and the endpoint p of the path $\delta(p)$ varies on σ . As in Definition 1.2, denote the resulting nest of cycles by $\delta(t)$, where $t = R(p)$. Let $\omega_1, \dots, \omega_n$ be differential 1-forms holomorphic near δ_0 . Assume in addition that the pullbacks

$$(\sigma \circ R)^* \omega_i = 0 \quad (3.10)$$

for the transversal line σ .

PROPOSITION 3.8. — The following equation holds:

$$\begin{aligned} \frac{d}{dt} \oint \omega_1 \dots \omega_r &= \sum_{i=1}^r \oint \omega_1 \dots \omega_{i-1} \frac{d\omega_i}{dR} \omega_{i+1} \dots \omega_r \\ &\quad - \sum_{i=1}^{r-1} \oint \omega_1 \dots \omega_{i-1} \frac{\omega_i \wedge \omega_{i+1}}{dR} \omega_{i+2} \dots \omega_r \end{aligned} \quad (3.11)$$

Proof. — Let us denote $\eta_i = \omega_1 \dots \omega_i$ and $\theta_j = \omega_j \dots \omega_r$. Denote

$$\begin{aligned} \varphi_i(p) &= \int_{p_0}^p \omega_i \dots \omega_r = \int_{p_0}^p \theta_i \\ \varphi_{r+1} &\equiv 1, \quad \theta_{r+1} \equiv 1 \end{aligned}$$

Also let us define $\psi_i(q) = \int_{p_0}^q \rho_i$, where

$$\rho_i = \frac{d(\omega_i \varphi_{i+1})}{dR}$$

Then we have

$$\begin{aligned} \int_{p_0}^p \eta_{i-1} \rho_i &= \int_{p_0}^p \eta_{i-1} \frac{d(\omega_i \varphi_{i+1})}{dR} = \int_{p_0}^p \eta_{i-1} \frac{d\varphi_{i+1} \wedge \omega_i + \varphi_{i+1} d\omega_i}{dR} \quad 1 \leq i < r \\ \int_{p_0}^p \eta_{r-1} \rho_r &= \int_{p_0}^p \eta_{r-1} \frac{d\omega_r}{dR} \end{aligned}$$

and, by (3.9) and (3.10)

$$d\varphi_{i+1} = \psi_{i+1}dR + \omega_{i+1}\varphi_{i+2} \quad (3.12)$$

hence

$$\frac{d\varphi_{i+1} \wedge \omega_i}{dR} = \omega_i\psi_{i+1} - \frac{\omega_i \wedge \omega_{i+1}}{dR}\varphi_{i+2}$$

and then

$$\begin{aligned} \int_{p_0}^p \eta_{i-1}\rho_i &= \int_{p_0}^p \eta_i\psi_{i+1} - \int_{p_0}^p \eta_{i-1} \frac{\omega_i \wedge \omega_{i+1}}{dR} \theta_{i+2} + \int_{p_0}^p \eta_{i-1} \frac{d\omega_i}{dR} \theta_{i+1}, \quad 1 \leq i < r \\ \int_{p_0}^p \eta_{r-1}\rho_r &= \int_{p_0}^p \eta_{r-1} \frac{d\omega_r}{dR} \end{aligned}$$

Observe that

$$\int \eta_i\psi_{i+1} = \int \eta_i\rho_{i+1}$$

So we obtain

$$\int_{p_0}^p \rho_1 = \sum_{i=1}^r \int_{p_0}^p \eta_{i-1} \frac{d\omega_i}{dR} \theta_{i+1} - \sum_{i=1}^{r-1} \int_{p_0}^p \eta_{i-1} \frac{\omega_i \wedge \omega_{i+1}}{dR} \theta_{i+2}$$

Now assume that $p = p_0$, so $\delta(t)$ are cycles. We will use Gelfand-Leray formula to obtain

$$\frac{d}{dt} \oint \theta_1 = \frac{d}{dt} \oint \omega_1 \varphi_2 = \oint \frac{d(\omega_1 \varphi_2)}{dR} = \oint \rho_1$$

Hence

$$\frac{d}{dt} \oint \theta_1 = \sum_{i=1}^r \oint \eta_{i-1} \frac{d\omega_i}{dR} \theta_{i+1} - \sum_{i=1}^{r-1} \oint \eta_{i-1} \frac{\omega_i \wedge \omega_{i+1}}{dR} \theta_{i+2}$$

□

COROLLARY 3.9. — *Let us assume that $\omega_i = x^{\beta_i} y^{\gamma_i} dx$, then*

$$\frac{d}{dt} \oint \omega_1 \dots \omega_r = \sum_{i=1}^r \oint \omega_1 \dots \omega_{i-1} \frac{d\omega_i}{dR} \omega_{i+1} \dots \omega_r \quad (3.13)$$

Proof. — True since $(\tau \circ R)^*\omega_i = 0$ and $\omega_i \wedge \omega_j = 0$. □

3.6. Exact forms in iterated integrals

Let $\omega_1, \dots, \omega_n$ be holomorphic differential 1-forms, g be a holomorphic function in a domain V and $\delta \subset V$ be a path connecting points p_0 and p_1 . We assume that $R|_\delta \equiv \text{const}$ for some analytic function R in a neighborhood of δ . Our goal is to express iterated integrals involving the exact form dg in terms of iterated integrals of smaller length.

Clearly

$$\int_{p_0}^{p_1} dg = g(p_1) - g(p_0)$$

For iterated integrals of length greater than 1, integrating by parts and using (3.12) gives

$$\int_{p_0}^{p_1} (dg)\omega_1 \dots \omega_n = \int_{p_0}^{p_1} (dg)\varphi_1 = g\varphi_1 \Big|_{p_0}^{p_1} - \int_{p_0}^{p_1} g(\psi_1 dR + \omega_1 \varphi_2).$$

But $dR = 0$ on level curves, so we have

$$\int_{p_0}^{p_1} (dg)\omega_1 \dots \omega_n = g(p_1) \int_{p_0}^{p_1} \omega_1 \dots \omega_n - \int_{p_0}^{p_1} (g\omega_1)\omega_2 \dots \omega_n$$

Next,

$$\int_{p_0}^{p_1} \eta_i (dg)\theta_{i+1} = \int_{p_0}^{p_1} \eta_i \left(g(q) \int_{p_0}^q \theta_{i+1} - \int_{p_0}^q (g\omega_{i+1})\theta_{i+2} \right)$$

Hence

$$\begin{aligned} \int_{p_0}^{p_1} \omega_1 \dots \omega_i (dg)\omega_{i+1} \dots \omega_n &= \int_{p_0}^{p_1} \omega_1 \dots (\omega_i g)\omega_{i+1} \dots \omega_n \\ &\quad - \int_{p_0}^{p_1} \omega_1 \dots \omega_i (g\omega_{i+1}) \dots \omega_n \end{aligned} \quad (3.14)$$

And the third formula:

$$\begin{aligned} \int_{p_0}^{p_1} \omega_1 \dots \omega_n (dg) &= \int_{p_0}^{p_1} \omega_1 \dots \omega_n (g(q) - g(p_0)) \\ &= \int_{p_0}^{p_1} \omega_1 \dots (\omega_n g) - g(p_0) \int_{p_0}^{p_1} \omega_1 \dots \omega_n \end{aligned}$$

3.7. Construction of Picard-Fuchs system

Let H be a Hamiltonian of degree $n + 1$, which we can write in multi-index form

$$H = \sum_{0 \leq i+j \leq n+1} \lambda_{ij} x^i y^j, \quad \lambda = (\lambda_{ij}) \in PC_{n+1}[x, y].$$

We assume that $\mathcal{B} = \{x^{i-1}y^j dx\} = \{\omega_l, l = 1, \dots, n^2\}$ form a basis of the Petrov module P_H , and that the curve $\{H = 0\}$ is smooth and intersects the line $\sigma = \{x = 0\}$ transversely. We will compute the connection matrix Ω_K locally near H , and then, by analytic continuation, this expression will be valid everywhere.

Let $\delta \subset \{H = 0\}$ be a cycle with an initial point at $p(H) \in \delta \cap \sigma$, and consider the vector of coefficients of $\varphi(\delta)$ from (3.6) in the standard basis $\{X_{i_1} \cdots X_{i_k}, k = 1, \dots, K\}$ of U/\mathfrak{m}^{K+1} :

$$I(\lambda) = \begin{pmatrix} 1 \\ I_1 \\ I_2 \\ \dots \\ I_{N_K} \end{pmatrix} \quad (3.15)$$

where $\lambda = \{\lambda_\alpha\}_{|\alpha| \leq n+1}$. We assume that the integrals I_1, \dots, I_{N_K} are ordered by length, i.e. $I_j = \int \eta_1 \dots \eta_k, \eta_j = \omega_{i_j} \in \mathcal{B}$, if and only if $N_{k-1} < j \leq N_k$, where $N_k = \dim U/\mathfrak{m}^{k+1}$.

Our first goal is to provide an analogue of Proposition 3.4 for iterated integrals.

PROPOSITION 3.10. — *If $\eta_1 \in \mathcal{B}, \dots, \eta_K \in \mathcal{B}$, and θ is a polynomial 1-form of degree d , then, for any $1 \leq i \leq K + 1$,*

$$\oint_{\delta} \eta_1 \dots \eta_{i-1} \theta \eta_i \dots \eta_K = \sum_{j=1}^{N_{K+1}} h_j I_j, \quad h_j \in \mathbb{C}(\lambda)[p], \quad (3.16)$$

where $p = \delta(0)$ is the initial point of the cycle δ . Degrees of h_j in p do not exceed $d + O(nK)$.

Moreover, if θ is defined over $\mathbb{Q}(\lambda)$, its degree and size do not exceed ν and s correspondingly and $d \geq n$, then the polynomials h_j are also defined over $\mathbb{Q}(\lambda)$ and their degrees and sizes do not exceed $\nu + O(K^5 d^4)$ and $s(Kd)^{O(K^5 d^3)}$ respectively.

Proof. — Using Proposition 3.4, we can write

$$\begin{aligned} \oint \eta_1 \dots \eta_{i-1} \theta \eta_i \dots \eta_K &= \oint \eta_1 \dots \eta_{i-1} \left(\sum_{j=1}^m (f_j \circ H) \omega_j + dg \right) \eta_i \dots \eta_K \\ &= \sum_{j=1}^m f_j(0) \oint \eta_1 \dots \eta_{i-1} \omega_j \eta_i \dots \eta_K + \oint \eta_1 \dots \eta_{i-1} (dg) \eta_i \dots \eta_K, \end{aligned} \quad (3.17)$$

since $\delta \subset \{H = 0\}$. The latter term can be rewritten as a sum of two iterated integrals of length K , using the equations of §3.6. These iterated integrals of length K are similar to the iterated integral on the left-hand side of (3.16), only of smaller length and with θ replaced by $g\eta_{i-1}$, i.e. by the form of degree $d + \deg \eta_{i-1}$ (recall that $\deg g \leq d$ by Proposition 3.4). Repeating the same steps, we reduce everything to a combination of iterated integrals of length $K-1$ and so on. During the steps of induction process the terms on the right-hand side of (3.16) will appear if the non-basic form $\tilde{\theta}$ appearing on this step becomes the last or the first in the tuple of forms under the integral sign. In this cases equations of §3.6 will produce an iterated integral of smaller length plus a basic integral multiplied by a polynomial of degree not exceeding $\deg \tilde{\theta}$. Therefore $\deg_p h_j \leq d + \sum \deg \eta_i \leq d + 2nK$. Evidently, h_j are defined over $\mathbb{Q}(\lambda)$ as Proposition 3.4), the only tool we use, preserves rationality of coefficients.

We estimate the degrees and sizes of h_j in λ using bounds of Proposition 3.4) in this inductive process.

One can show that all denominators of coefficients of the polynomials f_i, g appearing during induction are factors of $\Delta = \left(\prod_{j=1}^{d+2nK} \Delta_j\right)^K$, where Δ_j were defined after (3.5). We get $\deg \Delta = O(K^5 d^4)$ by Proposition 3.4. Therefore, multiplying θ by Δ , we can assume that all f_i, g on all steps of induction are polynomial in λ . This implies that the coefficients of h_j in (3.16) are also polynomial in λ . Denote by $b(K+1, d, \nu)$ the maximum of degrees in λ of the coefficients of h_j . Then

$$b(K+1, d, \nu) \leq \max\{\nu + O(d^3), b(K, d+2n, \nu + O(d^3))\}$$

Indeed, the first step of induction reduces (3.16) to a sum of similar equations for iterated integrals of length K and with a polynomial form $g\eta_i$ of degree at most $d+2n$ and with coefficients of degree $\nu + O(d^3)$ (by Proposition 3.4). Since, in addition, we know that everything is in fact polynomial in λ , this allows to replace sums of degrees in (2.3) by maximum of the degrees.

This implies $b(K+1, d, \nu) \leq \nu + O(K^4 d^3)$. Adding the degree of Δ , we get the required estimate.

Similarly, assume that the polynomials f_i, g obtained on all inductive steps are polynomials, and denote by $\mathfrak{s}(K+1, d, s)$ the size of the coefficients of the polynomials h_j . We have

$$\mathfrak{s}(K+1, d, s) \leq \max\left\{sd^{O(d^3)}, 2\mathfrak{s}(K, d+2n, sd^{O(d^3)})\right\},$$

where factor 2 appears due to (3.14). Note that on i -th step of induction we get iterated integrals of length $K + 2 - i$, so the iterated integrals from different steps do not add up. This implies that

$$\mathfrak{s}(K + 1, d, s) \leq s(Kd)^{O(K^4 d^3)}.$$

Now, size of Δ is $(Kn)^{O(K^5 n^3)}$, so applying the previous estimate to $\Delta\theta$ (and remembering $d > n$), we get the result. \square

We can prove more general statement:

PROPOSITION 3.11. — *Let $\theta_1, \dots, \theta_K \in \Lambda^1(\mathbb{C}^2) \otimes \mathbb{C}(\lambda)$ be 1-forms of degree at most d , and of degree in λ at most ν . Then*

$$\oint_{\delta} \theta_1 \dots \theta_K = \sum_{j=1}^{N_K} h_j I_j, \quad h_j \in \mathbb{C}(\lambda)[p], \quad (3.18)$$

with degrees of h_j in λ, p bounded by $\nu d^{O(K^2)}$. This decomposition is

Proof. — Indeed, decomposing $\theta_i = \sum_{\omega \in \mathcal{B}} (f_{i\omega} \circ H)\omega + f_i dH + dg_i$ as in Proposition 3.4, we see that

$$\oint_{\delta} \theta_1 \dots \theta_K = \sum_{\phi} \left(\prod_{i=1}^K f_{i\phi(i)}(0) \right) \oint_{\delta} \phi(1) \dots \phi(K) + \dots, \quad (3.19)$$

where summation is over all mappings $\phi : \{1, \dots, K\} \rightarrow \mathcal{B}$ and the dots denote $(n^2 + 1)^K - n^{2K}$ iterated integrals of length K with at least one exact form dg_i . These can be represented as iterated integrals of lesser length, and the result follow by induction.

To estimate the degrees and sizes of the coefficients in (3.16), note that the degrees in λ of $\prod_{i=1}^K f_{i\phi(i)}$ are $K\nu + O(Kd^3)$ by Proposition 3.4. The remaining $(n^2 + 1)^K - n^{2K}$ terms can be rewritten by formulae of §3.6, as at most $2((n^2 + 1)^K - n^{2K})$ iterated integrals of length $K - 1$ with coefficients being rational in λ and polynomial in p , of degree at most $K\nu + O(Kd^3)$ in λ and at most d in p . Under the integral sign stand some tuple of elements of \mathcal{B} , dg_i -s and product of some g_i with elements of \mathcal{B} . So these are forms of degrees at most $3d$ in x, y . By (2.3), we get

$$b(K, d, \nu) \leq 4(K\nu + O(Kd^3) + b(K - 1, 3d, 2\nu + O(d^3)))((n^2 + 1)^K - n^{2K}),$$

where $b(K, d, \nu)$ denote an upper bound for the degrees in λ, p of the coefficients of the h_j . This implies that $b(K, d, \nu) \leq \nu d^{O(K^2)}$. \square

However, in order to prove Theorem 2.9, we will need only dependence on $\lambda_{00} = -t$ and on p , which are much better:

PROPOSITION 3.12. — *Assume that the forms θ_i are independent on λ_{00} . Then the coefficients h_j in (3.18) are polynomial in p, λ_{00} , and for their degrees in p, λ_{00} we get*

$$\deg H \deg_{\lambda_{00}} h_j + \deg_p h_j + \sum \deg \omega_{j_k} \leq \sum \deg \theta_i. \quad (3.20)$$

Proof. — Polynomiality follows from Proposition 3.4. Therefore the relation between the degrees can be computed by asymptotics at infinity, as $\lambda_{00} \rightarrow \infty$, similar to [G98]. For homogeneous forms and generic homogeneous Hamiltonians counting homogeneity degrees we get

$$\deg H \deg_{\lambda_{00}} h_j + \deg_p h_j + \sum_{k=1}^m \deg \omega_{j_k} = \sum_{i=1}^K \deg \theta_i,$$

where $I_j = \int \omega_{j_1} \dots \omega_{j_m}$. For non-homogeneous in x, y forms and Hamiltonians, this becomes an inequality. \square

PROPOSITION 3.13. — *Let, as before, H be a Morse-plus polynomial such that the curve $\Gamma_H = \{H = 0\}$ is smooth and intersects transversely the line $\sigma = \{x = 0\}$. Let $S_K(\lambda)$ be the $N_K \times N_K$ -matrix valued function defined in a neighborhood of H as in (3.7). Then*

$$dS_K(\lambda) = \Omega_K(\lambda, p)S_K(\lambda), \quad (3.21)$$

with $N_K \times N_K$ matrix $\Omega_K(\lambda, p(\lambda))$ of one-forms on $PC_{n+1}[x, y]$ with coefficients being rational functions of λ and polynomials in $p(\lambda)$. Here $p = p_0(\lambda)$ is a starting point of integration, $p \in \Gamma_H \cap \sigma$.

The coefficients of $\Omega_K(\lambda, p)$ have degree in p at most $O(nK)$, and their degrees in λ and sizes are at most $O(K^6 n^7)$ and $(Kn)^{O(K^6 n^5)}$.

Proof. — According to Corollary 3.9 for any tuple $\{\eta_i\}_{i=1}^K$, with $\eta_i \in \mathcal{B}$,

$$\frac{\partial}{\partial \lambda_\alpha} \oint \eta_1 \dots \eta_K = - \sum_{i=1}^K \oint \eta_1 \dots \eta_{i-1} \left(\frac{d\eta_i}{d\lambda_\alpha} \right) \eta_{i+1} \dots \eta_K. \quad (3.22)$$

Our goal is to express the Gelfand-Leray derivatives $\frac{d\omega_i}{d\lambda_\alpha}$ as rational combinations of forms ω_j . Consider the form $Hd\omega_i$. Differentials of the elements

of \mathcal{B} form a basis of $\Lambda^2(\mathbb{C}^2)/dH \wedge \Lambda^1(\mathbb{C}^2)$, see [G98], so one can write

$$Hd\omega_i = \sum_{j=1}^{n^2} a_{ij}d\omega_j + \theta_i \wedge dH, \quad i = 1, \dots, n^2, \quad (3.23)$$

where θ_i is a polynomial one-form of degree equal to the degree of ω_i (we follow closely [BNY10, A.3]). Multiplying by the monomial $\frac{\partial H}{\partial \lambda_\alpha}$, dividing by dH and decomposing $\frac{\partial H}{\partial \lambda_\alpha}\theta_i$ as in Proposition 3.4 we get modulo some multiple of dH ,

$$H \frac{d\omega_i}{d\lambda_\alpha} = \sum_j a_{ij} \frac{d\omega_j}{d\lambda_\alpha} + \left(\sum_j b_{ij}^\alpha(H)\omega_j + f_i^\alpha dH + dg_i^\alpha \right), \quad i, j = 1, \dots, n^2. \quad (3.24)$$

Multiplying both sides of this system of equations by inverse \check{A} of the matrix $A = \{a_{ij}\}$ we get

$$H \sum \check{a}_{ij} \frac{d\omega_j}{d\lambda_\alpha} = \frac{d\omega_i}{d\lambda_\alpha} + \sum_{j=1}^{n^2} q_{ij}^\alpha(H)\omega_j + dg_i^\alpha \quad (3.25)$$

where coefficients $q_{ij}^\alpha(H)$ are polynomial in H with coefficients being rational functions of λ .

Integrating formula (3.22) over $\delta \subset \{H = 0\}$ and substituting (3.25) (cancelling its zero left-hand side), we get

$$\begin{aligned} \frac{\partial}{\partial \lambda_\alpha} \oint \eta_1 \dots \eta_K &= \sum_{i=1}^K \sum_{j=1}^{n^2} q_{ij}^\alpha(0) \oint \eta_1 \dots \eta_{i-1}(\omega_j)\eta_{i+1} \dots \eta_K \\ &+ \sum_{i=1}^K \oint \eta_1 \dots \eta_{i-1}(dg_i^\alpha)\eta_{i+1} \dots \eta_K \end{aligned} \quad (3.26)$$

Using Proposition 3.10, we can express the integrals from the right hand side of the latter equation as a combination of basic ones.

Now, let us estimate the degrees and sizes of the coefficients of $\Omega_K(\lambda, p)$. The degree in x, y of the form θ_i is equal to the degree of $\omega_i \in \mathcal{B}$, i.e. is at most $2n$. For each ω_i the equation (3.23), after dividing by $dx \wedge dy$, produces an equality between two polynomials in x, y of degree $O(n)$. By equating coefficients of these polynomials we get a system of $O(n^2)$ linear equations on $a_{ij}, j = 1, \dots, \#\mathcal{B}$, and coefficients of η_i . This system is defined over $\mathbb{Q}(\lambda)$ and its coefficients are of degree at most 1 in λ and of size $O(n)$. Therefore

a_{ij} and η_i can be chosen to be defined over $\mathbb{Q}(\lambda)$, of degree $O(n^2)$ in λ and of size $n^{O(n^2)}$.

Bounds of Proposition 3.4 imply that the degrees $\deg_H b_{ij}^\alpha \leq 3 = O(1)$, degree in x, y of \tilde{g}_i^α is bounded by $O(n)$, and degrees in λ and sizes of b_{ij}^α , \tilde{g}_i^α are bounded from above by $O(n^3)$ and $n^{O(n^3)}$ respectively. This implies that degrees and sizes of \tilde{a}_{ij} are bounded by $O(n^4)$ and $n^{O(n^2)}$ respectively, so q_{ij}^α are polynomials in H of degree $O(1)$, of degree $O(n^6)$ in λ and of size $n^{O(n^2)}$. The functions g_i^α are polynomial in x, y of degree $O(n)$, and their degrees in λ and sizes are $O(n^7)$ and $n^{O(n^5)}$ correspondingly. Bounds of Proposition 3.10 now imply that the coefficients of $\Omega_K(\lambda, p)$ are rational functions of p, λ , polynomial in p of degree at most $O(nK)$, and of degrees in λ and sizes at most $O(K^6 n^7)$ and $(Kn)^{O(K^6 n^5)}$. \square

3.8. Changing the variables (lifting)

The coefficients of the connection (3.21) depend algebraically on $\lambda \in P\mathbb{C}_{n+1}[x, y]$, since the base point $p = p(\lambda)$ of the fundamental group is not defined uniquely. Let $\text{ev} : B_n \rightarrow P\mathbb{C}_{n+1}[x, y]$ be the map $\text{ev}(H, y) = H - H(y)$, where B_n was defined in (3.8). Let $\tilde{S}_K = S_K \circ \text{ev}$ be the lifting of the matrix S_K to B_n . The coefficients of the pulled-back connection

$$d\tilde{S}_K = \tilde{\Omega}_K \tilde{S}_K, \quad \tilde{\Omega}_K = \text{ev}^* \Omega_K, \quad (3.27)$$

on $B_n \times (U/\mathfrak{m}^{K+1})$ are rational one-forms on B_n .

PROPOSITION 3.14. — *The degree and the size of the matrix $\tilde{\Omega}_K$ are bounded by $O(K^6 n^8)$ and $(Kn)^{O(K^6 n^7)}$ correspondingly.*

Proof. — Degree of the mapping ev is equal to $n + 1$, and it has coefficients equal to 0 or 1. Therefore degree of $\tilde{\Omega}_K$ is at most $O(K^6 n^8)$. Sizes of coefficients of $\tilde{\Omega}_K$ will not exceed sizes of coefficients of $\Omega_K(\lambda, p)$ multiplied by the size of ev raised to $\deg_\lambda \Omega_K(\lambda, p)$, i.e. $(Kn)^{O(K^6 n^5)} n^{O(K^6 n^7)} = (Kn)^{O(K^6 n^7)}$. \square

4. Properties of the system

4.1. Quasi-Unipotency

PROPOSITION 4.1. — *Connection (3.27) is quasiunipotent and regular.*

Proof. — From (3.26) we see that derivatives of an iterated integral are linear combination of iterated integrals of smaller or equal length. This means that Ω_K is a lower-block-triangular matrix:

$$\Omega_K = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ * & \Theta_{11} & 0 & & 0 \\ & * & \Theta_{22} & & 0 \\ & * & & \dots & 0 \\ & * & & & \Theta_{nn} \end{pmatrix},$$

where each block Θ_{ii} is $k_i \times k_i$ matrix corresponding to the integrals of length exactly i . Note that Θ_{11} is just the pull-back by ev^* of the matrix of the Gauss-Manin connection for Abelian integrals, so it has quasiunipotent monodromy by [K81].

Since $\tilde{\Omega}_K$ is lower-block-triangular, it preserves the flag $\mathcal{F} = \{0\} = \mathcal{F}_{K+1} \subset \mathcal{F}_K \subset \dots \subset \mathcal{F}_0 = U/\mathfrak{m}^{K+1}$, where $\mathcal{F}_i = \mathfrak{m}^i/\mathfrak{m}^{K+1}$, so any monodromy operator corresponding to Ω_K preserves this flag as well, i.e. is lower-triangular with blocks M_{ii} on diagonal. Therefore its eigenvalues are just the eigenvalues of the monodromy operators M_{ii} corresponding to Θ_{ii} , the connection induced by $\tilde{\Omega}_K$ on the factor-bundle with fiber $\mathcal{F}_i/\mathcal{F}_{i+1} = \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

Recall definition of Kronecker product of two matrices:

$$\{a_{ij}\}_{i,j=1}^{m,n} \otimes \{b_{kl}\}_{k,l=1}^{p,q} = \{c_{rs}\}_{r,s=1}^{mp,nq}, \quad \text{where } c_{(i-1)p+k,(j-1)q+l} = a_{ij}b_{kl}.$$

In a more invariant language, if A and B represent linear transformations $V_1 \rightarrow W_1$ and $V_2 \rightarrow W_2$, respectively, then $A \otimes B$ represents the tensor product of the two maps, $V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$. The Kronecker sum of two matrices is the operation induced by the Kronecker product on the corresponding Lie algebras:

$$\begin{aligned} A_1 \oplus \dots \oplus A_n &= \left. \frac{d}{dt} \right|_{t=0} [\exp(tA_1) \otimes \dots \otimes \exp(tA_n)] = \\ &= A_1 \otimes I \dots \otimes I + I \otimes A_2 \otimes I \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes A_n. \end{aligned} \quad (4.1)$$

LEMMA 4.2. — For all $1 \leq k \leq n$, $\Theta_{kk} = P(\Theta_{11}^{\oplus k})P^{-1}$ for some permutation matrix P . Correspondingly, $M_{kk} = PM_{11}^{\otimes k}P^{-1}$.

The first claim is just a way to say that the differentiation of iterated integrals satisfies Leibniz rule, up to iterated integrals of smaller length. This can be seen from (3.22). Now, derivation of the products of Abelian integrals satisfy the same rule, so horizontal sections of Θ_{kk} are described by these

products, and monodromy operators of are just tensor powers of monodromy M_{11} of Abelian integrals (the permutation matrix appears since we did not prescribe exactly the ordering of I_k in (3.15)). This proves quasiunipotency of $\tilde{\Omega}_K$ since tensor powers of quasiunipotent operators are quasiunipotent.

Regularity of $\tilde{\Omega}_K$ follows from regularity of Θ_{kk} and the fact that semidirect product of regular connections is regular, see [D]. \square

5. Proof of Theorem 1.4

Let $\delta \subset \{H = 0\}$ be a cycle and U its small neighborhood, and let M_K be the first non-zero Melnikov function defined as in (1.3). Recall the construction, going back to at least [F96, Y95], expressing M_K as a polynomial in iterated integrals of the perturbation form ω and its Gelfand-Leray derivatives up to order K .

DEFINITION 5.1. — *A real analytic 1-form $\alpha \in \Lambda^1(U)$ is relatively exact with respect to the integrable foliation $\mathcal{F} = \{dH = 0\}$ in a domain U , if*

$$\alpha = h \cdot dH + dg, \quad h, g \in \mathcal{O}(U) \tag{5.1}$$

Clearly, the integral of a relatively exact form α along any closed oval on any level curve $\{H = z\} \subset U$, vanishes:

$$\forall \text{ oval } \delta \subseteq \{f = z\} \quad \oint_{\delta} \alpha = 0 \tag{5.2}$$

Define the sequence of real analytic 1-forms $\omega_1, \omega_2, \dots, \omega_k$ as follows:

1. (Base of induction). $\omega_1 = \omega$ is the perturbation form from (1.2)
2. (Induction step). If the forms $\omega_1, \dots, \omega_j$ are already constructed and turned out to be relatively exact, then $\omega_j = h_j \cdot dH + dg_j$. In this case we define

$$\omega_{j+1} = -h_j \omega \tag{5.3}$$

THEOREM 5.2 [IY, Theorem 26.7]. — *If $\omega_k, k \geq 2$, is the first not relatively exact 1-form in the sequence $\omega_1, \dots, \omega_{k-1}, \omega_k$, constructed inductively by (5.3), then*

$$M_k(z) = - \oint_{\{H=z\}} \omega_k \tag{5.4}$$

Evidently, the functions h_j can be restored as $h_j = -\int \frac{d\omega_j}{dH}$, so

$$\omega_{j+1} = \omega \int \frac{d}{dH} \left(-\omega \int \frac{d}{dH} \left(\dots \left(-\omega \int \frac{d\omega}{dH} \right) \dots \right) \right).$$

Denote by ϕ the algebraic function $H|_{\sigma}^{-1} \circ H$ of x, y which maps the point (x, y) to the (one of $d + 1$) point of the transversal $\sigma = \{x = 0\}$ lying on the same level curve of H as (x, y) . In other words, $p = \phi(x, y, \tilde{\lambda})$.

LEMMA 5.3. — *The function h_j in (5.3) is a linear combination of iterated integrals of differential one-forms with coefficients polynomial in x, y . The coefficients of this combination are rational in $\tilde{\lambda}, p$.*

Proof. — We have $h_0 = 1$, which is of the required form trivially. We proceed by induction on j . Assume that h_j is a finite sum of terms of the type $R(\lambda, p) \int \theta_1 \dots \theta_k$, where θ_i are polynomial in x, y one-forms, and R is a rational function. Now, $h_{j+1} = \int \frac{d(h_j \omega)}{dH}$, so it is enough to consider the case of $h_j = R(\lambda, p) \int \theta_1 \dots \theta_k$. Applying repeatedly Proposition 3.7 and Proposition 3.8, gives

$$\begin{aligned} h_{j+1} &= \int \frac{d}{dH} \left(R(\lambda, p) \omega \int \theta_1 \dots \theta_k \right) = \frac{\partial R}{\partial p} (H|'_{\sigma}(p))^{-1} \int \omega \theta_1 \dots \theta_k + \\ &R \int \frac{d\omega}{dH} \theta_1 \dots \theta_k + R \sum_{i=1}^k \int \omega \theta_1 \dots \frac{d\theta_i}{dH} \dots \theta_k - R \int \frac{\omega \wedge \theta_1}{dH} \theta_2 \dots \theta_k \quad (5.5) \\ &R \int \omega \sum_{i=1}^{k-1} \theta_1 \dots \frac{\theta_i \wedge \theta_{i+1}}{dH} \dots \theta_k - R \int \omega \theta_1 \dots \frac{\theta_{k-1} \wedge (\sigma \circ H)^* \theta_k}{dH}. \end{aligned}$$

The first term is clearly of the required type.

Denote by $m(H)$ the product $\prod_{i=1}^{n^2} (H - c_i)$, where c_i are critical values of H (repeated if multiple). It is well known, see [G05][Prop.2.4], that $m(H)$ lies in the Jacobian ideal of H , so the operator $\frac{m(H)}{dH}$ preserves polynomial one-forms. Therefore

$$\begin{aligned} R \int \omega \theta_1 \dots \frac{\theta_i \wedge \theta_{i+1}}{dH} \dots \theta_k &= \frac{R}{m(H(p))} \int \omega \theta_1 \dots \frac{m(H) \theta_i \wedge \theta_{i+1}}{dH} \dots \theta_k \\ R \sum_{i=1}^k \int \omega \theta_1 \dots \frac{d\theta_i}{dH} \dots \theta_k &= \frac{R}{m(H(p))} \sum_{i=1}^k \int \omega \theta_1 \dots \frac{m(H) d\theta_i}{dH} \dots \theta_k \quad (5.6) \end{aligned}$$

all terms in (5.5) except the last one are of the required type.

On the number of zeros of Melnikov functions

Finally, $(\sigma \circ H)^*\theta = \phi_k(p)dH$, where $\phi_k(p) = \frac{\theta(p)(\partial_y)}{dH(p)(\partial_y)}$ is a rational function of p , so the last term of (5.5) is also of the required type. \square

LEMMA 5.4. — *For a form ω of degree $d > n$ in x, y , we have*

$$M_K = \sum_{i=1}^{N_K} h_i I_i,$$

where h_j depend rationally on p and has degree at most $2^{O(K)} dn^6$ in p .

Proof. — In the inductive step (5.5) one iterated integral of forms of degrees at most d_k with coefficient R of degree ν_k generated $O(k)$ iterated integrals of forms of degrees at most $2d_k + O(n^3)$. The coefficients of these new integrals are obtained from R by a combination of a differentiation and division either by $H|'_\sigma(p)$ or by $m(H(p))$ or just by division by one of these polynomials. Applying these operation K times we can increase degree of R by at most $O(Kn^3)$. Summing together, we get a representation of M_K as a sum of $2^{O(K^2)}$ iterated integrals of forms of degree at most $2^{O(K)}(d + O(n^3))$, with rational in p coefficients of degree at most $O(Kn^3)$, with common denominator of degree $O(Kn^3)$.

Applying Proposition 3.12, we represent each of these iterated integrals as in (3.18), and the coefficients of these representations have degrees in p at most $2^{O(K)}(d + n^3)$. Summing these representations together, we arrive to the statement of the Proposition. \square

Multiplying by the common denominator, we see that the estimate of Theorem 1.4 is implied by the following

LEMMA 5.5. — *Linear combination $\sum_{i=1}^{N_K} h_i I_i$ of iterated integrals of length at most K , with coefficients h_i being polynomial in p of degree at most μ , has at most $\exp(\exp(\mu^{O(1)} n^{O(K)}))$ zeros on each line $\lambda \times \mathbb{C}_p$.*

Proof. — We can construct in a standard way a connection whose horizontal sections are described by these functions. More exact, let J_μ be the Jordan block of size $(\mu + 1) \times (\mu + 1)$. Then the Kronecker product $\tilde{S}_K \otimes \exp(pT_\mu)$ describes horizontal sections of a connection with the connection matrix being the Kronecker sum $\tilde{\Omega}_K^\mu = \tilde{\Omega}_K \oplus (J_\mu dp)$. Using estimates on the size and degree of $\tilde{\Omega}_K$, we conclude that $\tilde{\Omega}_K^\mu$ is defined over $\mathbb{Q}(\lambda)$, and is of dimension $\mu N_K = \mu n^{O(K)}$, of degree $O(K^6 n^8)$ and of size $\mu(Kn)^{O(K^6 n^7)}$. Any linear combination $\sum_{i=1}^{N_K} h_i I_i$ is a linear combination of

components of some horizontal section of this connection, so Theorem 2.9 is applicable and gives the required upper bound. \square

Substitution of $\mu = 2^{O(K)} dn^6$ from Lemma 5.4 provides the estimate of the Theorem 1.4.

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