ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XX, nº S2 (2011), p. 211-239.

<http://afst.cedram.org/item?id=AFST_2011_6_20_S2_211_0>

Annales de la Faculté des Sciences de Toulouse Vol. XX, n° Spécial, 2011
pp. 211–239

Extendible bases and Kolmogorov problem on asymptotics of entropy and widths of some class of analytic functions

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Dedicated to Professor Nguyen Than Van

Abstract. — Let K be a compact set in an open set D on a Stein manifold Ω of dimension n. We denote by $H^{\infty}(D)$ the Banach space of all bounded and analytic in D functions endowed with the uniform norm and by A_K^D a compact subset of the space $C(K)$ consisted of all restrictions of functions from the unit ball $\mathbb{B}_{H^{\infty}(D)}$. In 1950ies Kolmogorov posed a problem: does

$$\mathcal{H}_{\varepsilon}\left(A_K^D\right) \sim \tau \left(\ln \frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0,$$

where $\mathcal{H}_{\varepsilon}\left(A_K^D\right)$ is the ε-entropy of the compact A_K^D. We give here a survey of results concerned with this problem and a related problem on the strict asymptotics of Kolmogorov diameters of the set A_K^D wih respect to the unit ball in the space $C(K)$. We decribe a progress in studying of these problems, beginning with initial results of 1950ies, in the closed connection with the problem on existence of a common basis for the spaces $A(K)$ and $A(D)$ with good estimates on sublevel sets of extremal plurisubharmonic function for the pair (condenser) (K, D). The survey is concluded by a discussion of some open problems.

Résumé. — Soit K un sous-ensemble compact d'un ouvert D d'une variété de Stein Ω de dimension n. On désigne par $H^{\infty}(D)$ l'espace de Banach des fonctions analytiques et bornées sur D functions muni de la norme de la convergence uniforme et par A_K^D une partie compacte de l'espace $C(K)$ consituée de toutes les restrictions des fonctions de la boule unité $\mathbb{B}_{H^{\infty}(D)}$. Dans les année 1950, Kolmogorov posa le problème suivant: Existe-t-il une constante τ telle qu'on ait l'asymptotique

$$\mathcal{H}_{\varepsilon}\left(A_K^D\right) \sim \tau \left(\ln \frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0,$$

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où $\mathcal{H}_\varepsilon\left(A_K^D\right)$ est la ε-entropie du compact A_K^D? On donne ici une revue des résultats concernant ce problème et un problème qui lui est relié concernant l'asymptotoque stricte des diamètres de Kolmogorovde l'ensemble A_K^D par rapport à la boule unité de l'espace $C\left(K\right)$. On décrit les progrès réalisés dans l'étude de ces problèmes, commençant par les résultats initiaux des années 1950, en relation étroite avec le problème de l'existence des bases communespour les espaces $A\left(K\right)$ and $A\left(D\right)$ avec de bonnes estimations sur les ensembles de sous-niveau de la fonction plurisouharmonique extrémalede la paire (condensateur) (condenser) (K, D). On conclut cette présentation avec une discussion des problèmes ouverts.

1. Introduction

Kolmogorov [K1] introduced (developing an idea of Pontryagin and Schnirelman in their supplement to the Russian translation of the book [GW]) the following important characteristics of massiveness of a precompact set A in a metric space $X = (X, d)$: the *ε-entropy* $\mathcal{H}_\varepsilon\left(A\right) = \mathcal{H}_\varepsilon\left(A, X\right) := \ln N_\varepsilon\left(A, X\right)$, where $N_\varepsilon\left(A, X\right)$ is the smallest integer N such that A can be covered by N sets of diameter not greater than 2ε and the *ε-capacity* $\mathcal{C}_\varepsilon\left(A\right) = \mathcal{C}_\varepsilon\left(A, X\right) := \ln M_\varepsilon\left(A, X\right)$, where $M_\varepsilon\left(A, X\right)$ is the largest integer M such that there is a finite set $\{x_j : j = 1, \dots M\} \subset A$ with the property: $d\left(x_i, x_j\right) \geqslant \varepsilon$, $i \neq j$ (hereafter we prefer to use the natural logarithm instead of more traditional $\log_2$). We refer to the survey [KT] for properties of these characteristics: one can found there also a discussion of important investigations (Vitushkin, Kolmogorov, Tikhomirov, Babenko, Erokhin, Arnold), in particular, those related to the 13th Hilbert problem and to the probability information theory.

Here we restrict our consideration to the Kolmogorov's problem of 1950ies on *the strict asymptotics of ε-entropy of the special important class of analytic functions A_K^D* (which will be described just now) and the problem on *existence of an extendible bases in spaces of analytic functions*, closely connected with the first one. Let K be a compact set in an open set D on a Stein manifold Ω of dimension n. We denote by $H^\infty\left(D\right)$ the Banach space of all bounded and analytic in D functions endowed with the uniform norm and by A_K^D a compact subset of the space $C\left(K\right)$ consisted of all restrictions of functions from the unit ball $\mathbb{B}_{H^\infty(D)}$. **We always assume that the restriction operator $R : H^\infty\left(D\right) \to C\left(K\right)$ is injective**, so one can think that $A_K^D = \mathbb{B}_{H^\infty(D)}$). Everywhere in this article those pairs (K, D) will be called shortly "*condensers*".

Problem 1.1. — *For which condensers* (K, D) *there exists a constant* τ *such that the strict asymptotics*

$$\mathcal{H}_\varepsilon\left(A_K^D\right) \sim \tau\left(\ln\frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0, \tag{1.1}$$

holds? What is a nature of the constant?

For a set A in a Banach space X the *Kolmogorov diameters (*or *widths) of* A *with respect to the unit ball* $\mathbb{B}_X$ of the space X are the numbers

$$d_k(A) = d_k(A, \mathbb{B}_X) := \inf_{L\in\mathcal{L}_k}\ \sup_{x\in A}\ \inf_{y\in L}\ \|x-y\|_X\,, \quad k = 0, 1, \dots, \tag{1.2}$$

here $\mathcal{L}_k$ is the set of all k-dimensional subspaces of X.

Problem 1.1 is connected closely (see Section 2) with the following problem which is usually attributed to Kolmogorov too.

Problem 1.2. — *Describe the condensers* (K, D) *such that the strict asymptotics*

$$-\ln d_k\left(A_K^D\right) \sim \sigma k^{1/n}, \ k \longrightarrow \infty \tag{1.3}$$

hold with some constant σ.

Notice that it was known that the weak asymptotics

$$\mathcal{H}_\varepsilon\left(A_K^D\right) \asymp \left(\ln\frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0, \ -\ln d_k\left(A_K^D\right) \asymp k^{1/n}, \quad k \longrightarrow \infty.$$

hold for good enough pairs (K, D) [K1].

Notation. — Hereafter we use the notation:

- $|f|_E := \sup\{|f(x)| : x \in E\}$ for a given function $f : E \longrightarrow \mathbb{C}$
- given a positive sequence $a = (a_k)$ we set $m_a(t) := \sharp\{k : a_k \leqslant t\}$
- $X \hookrightarrow Y$ stays always for a linear continuous imbedding with dense image, where X and Y are linear topological spaces
- $\mathbb{B}_X$ denotes a closed unit ball in a Banach space X
- Ω always stays for a Stein manifold, $\dim\Omega = n$, D for an open set in Ω and K for a compact set in D;

- $A(D)$ is the *space of all functions analytic in* D with the usual *locally convex topology of locally uniform convergence in* D;
- $A(K)$ is the space of all analytic germs on K with the usual inductive limit topology;
- $\mathcal{B}^2_\rho(D)$ is a weighted Bergman space, i.e. the space of all functions analytic in D with a finite norm $\|x\|_{\mathcal{B}^2_\rho(D)} = \left(\int |x(z)|^2\, dm\right)^{1/2}$, where m is a Hermitian measure on D; in the case $D \subset \mathbb{C}^n$ one can take $m = \frac{\lambda}{|z|^2+1}$, where λ is Lebesgue measure on $\mathbb{C}^n$;
- given K and a Borel positive measure μ on K, $AL^2(K,\mu)$ is the space obtained by the completion of $A(K)$ in $L^2(K,\mu)$;
- for a couple of Hilbert spaces $H_1 \hookrightarrow H_0$ we denote by $(H_0)^{1-\alpha}(H_1)^\alpha$, $\alpha \in \mathbb{R}$, the Hilbert scale connecting these spaces (see, e.g., [Kr, Li, KPS])

On the early stage the strict asymptotics (1.1) was known only for some special condensers (Vitushkin [V1, V2], see also [KT]): two concentric disks, interval and ellipse with foci in its ends or, more generally, a continuum and its Faber domain; in several complex variables, the strict asymptotics (1.1) was done for condensers which are Cartesian products of above ones. What was important that in all these examples *some classical extendible bases were applied* (Taylor, Chebyshev or Faber polynomial bases) in evaluation of entropy; in [KT], Section 7, some general approach, which modelled results from [V1, V2], was developed and applied for various concrete examples.

Kolmogorov, analysing Vitushkin's results, predicted that for a good enough condenser, *in one-dimensional case*, the strict asymptotics (1.1) should hold with the constant τ which coincides with the Green capacity $\tau(K,D)$ of the condenser (K,D). Remind that the Green capacity of a condenser (K,D) in $\mathbb{C}$ is the number

$$\tau(K,D) := \frac{1}{2\pi}\int_K \triangle\omega,$$

where $\omega(z) = \omega(D,K;z)$ is the generalized Green potential, which we define here in a Perron style:

$$\omega(D,K;z) := \limsup_{\zeta\longrightarrow z} \sup\left\{u(\zeta) : u \in Sh(D),\ u|_K \leqslant 0,\ u(\zeta) < 1 \text{ in } D\right\};$$

this function is subharmonic in D, harmonic in $D \setminus K$, therewith $\triangle\omega$ is a non-negative Borel measure supported by K. It provides the generalized solution of the Dirichlet problem for the Laplace equation in the domain $D \setminus K$ with the boundary value $u(z)$ equal to 1 on ∂D and 0 on $\partial(\mathbb{C} \setminus K)$. We say that the *condenser* (K, D) in $\mathbb{C}$ *is regular* if (a) $\omega(z) \equiv 0$ on K and $\lim_{z \longrightarrow \partial D} \omega(z) = 1$, (b) $A(D)$ is dense in $A(K)$ (i.e. that for each component G of D the set $G \setminus K$ is connected) and D has no component disjoint with K.

2. Relationship between the entropy and widths

Mityagin [M] investigated how the entropy of an absolutely convex compact set in a Banach space can be estimated through some special counting functions related to Kolmogorov diameters. Here we consider a refined version of those estimates for an arbitrary absolutely convex compact set in a Banach space; therewith the left estimate is an easy adaptation of the right inequality of Theorem 4, [M] to the complex case, while the right asymptotic inequality is new; it is obtained in [Z10] by a modification of the estimate (20) from [LT].

Theorem 2.1. — *Let A be an absolutely convex compact set in a complex Banach space X. Then there exists a constant M such that*

$$2\int_0^{\frac{1}{2\varepsilon}} \frac{m_c(t)}{t}\, dt \leqslant \mathcal{H}_\varepsilon(A, X) \lesssim 2\int_0^{\frac{M}{\varepsilon}} \frac{m_a(t)}{t}\, dt, \ \ \varepsilon = \searrow 0, \tag{2.1}$$

where $a = (1/d_{k-1}(A, \mathbb{B}_X))$ and $c = (k/d_{k-1}(A, \mathbb{B}_X))$.

Notice that for special A and X these estimates may be considerably better, like in Theorem 3 of [M], where $X = l_p$ and A is an l_p-ellipsoid.

The estimates (2.1) make it possible to show that the problem on the asymptotics (1.1) is equivalent to the related problem on asymptotics for diameters of the set A_K^D.

Lemma 2.2. — *Let K be a compact set in an open set D on a Stein manifold Ω of dimension n. The asymptotics (1.1) is true if and only if the asymptotics (1.3) holds with the constant $\sigma = \left(\frac{2}{(n+1)\tau}\right)^{1/n}$.*

For $\Omega = \mathbb{C}$ the part "if" was proved in [LT] for $n = 1$. In general case this lemma was stated without proof in [Z9]; its complete proof, developing methods of [M] and [LT], is done in [Z10].

In view of Lemma 2.2 one may prove only one of the asymptotics (1.3) or (1.1).

3. One-dimensional case

An essential step toward the problem on the asymptotics (1.1), (1.3) was done by Babenko [B] and Erokhin [E1, E2] in 1958. They proved the following result, which confirmed the Kolmogorov's conjecture in a quite general case.

THEOREM 3.1. — *Let K be a continuum in $\mathbb{C}$ with connected complement, consisted of more than one point, D a simply connected open neighborhood of K, $D \neq \mathbb{C}$. Then the asymptotics (1.1) holds with the constant $\tau = \tau(K, D) = \frac{1}{\ln R}$, where R is a conformal modulus of a domain $D \setminus K$.*

The proofs in [B] and [E1, E2] (see also the posthumous publication of the Erokhin's thesis [E3]) are completely different. The first one is based on rational interpolation methods in a spirit of Walsh's book [Wl] and, since it is out of the main line in further discussion, we do not go into details about it. The Erokhin's proof, quite the contrary, will be considered here in detail, because his main idea, to construct and apply (similarly with [V1, V2] and [KT]) a common basis for the spaces $A(K)$ and $A(D)$, turned to be crucial for further investigations, especially in the case of several variables. Notice that the technics of both proofs was *exclusively one-dimensional.* In particular, Erokhin's construction of common bases [E1], is based on his brilliant result related to geometric function theory, which unfortunately has no analogue for several complex variables (and even for merely general plane condensers).

LEMMA 3.2. — *Suppose (K, D) is a condenser in $\mathbb{C}$ such that K and $\mathbb{C} \setminus D$ are continuums with connected complement consisted of more than one point and $F : D \setminus K \to \{1 < |z < R|\}$ is an analytic bijection. Then there exist analytic bijections $\varphi : D \to G := \varphi(D)$ and $\psi : \mathbb{C} \setminus L \to \{|z| > 1\}$, $L := \varphi(K)$, such that $F(z) = \psi(\varphi(z))$, $z \in D \setminus K$ and $\psi(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots$ in a neighborhood of ∞.*

In the notation of this lemma, consider closed paths

$$\Gamma_\alpha = \Gamma_\alpha(t) := F^{-1}\left(R^\alpha\, e^{it}\right),\ \gamma_\alpha = \gamma_\alpha(t) = \psi^{-1}\left(R^\alpha\, e^{it}\right),$$

$$t \in [0, 2\pi],\ 0 < \alpha < 1$$

and two one-parameter families of bounded domains

$$D_\alpha,\ G_\alpha \text{with} \partial D_\alpha = \Gamma_\alpha,\ \partial G_\alpha = \gamma_\alpha,\ 0 < \alpha < 1. \qquad (3.1)$$

Then the mapping

$$g(w) \to f(z) = g(\varphi(z))$$

preserves uniform norms: $|g|_{G_\alpha} = |f|_{D_\alpha}$, $0 < \alpha < 1$, hence it represents an isomorphism of each space $A(G)$, $A(L)$, $A(G_\alpha)$, $A\left(\overline{G_\alpha}\right)$ onto the corresponding space $A(D)$, $A(K)$, $A(D_\alpha)$, $A\left(\overline{D_\alpha}\right)$, $0 < \alpha < 1$. Thus the system

$$f_n(z) = \frac{1}{2\pi} \int_{\Gamma_\alpha} \frac{F(\zeta)^n \ \varphi'(\zeta) \ d\zeta}{\varphi(\zeta) - \varphi(z)} = \frac{1}{2\pi} \int_{\gamma_\alpha} \frac{\psi(w)^n \ dw}{w - \varphi(z)} = \Phi_n(\varphi(z)), \tag{3.2}$$

where $\{\Phi_n(w)\}_{n=0}^{\infty}$ is the classical Faber polynomial basis for the compact set $L = \varphi(K)$, *forms a common basis in all the spaces* $A(D)$, $A(K)$, $A(D_\alpha)$, $A\left(\overline{D_\alpha}\right)$. This basis inherits very nice properties of the Faber basis $\Phi_n(w)$ (see, e.g., [Mk]), which are important for estimating of entropy $\mathcal{H}_\varepsilon\left(A_K^D\right)$ in the Erokhin's proof of Theorem 3.1.

In his thesis [E3] Erokhin gave also a separate proof of the strict asymptotics $-\ln d_k\left(A_K^D\right) \sim k \ \ln R$, $\ k \to \infty$ in the conditions of Theorem 3.1. As it was mentioned above, due to [LT, Z10], it turns to be clear that it is sufficient to study only one of the asymptotics (1.1) or (1.3).

Levin and Tikhomirov [LT] proved the asymptotics

$$\ln d_k\left(A_K^D\right) \sim -\frac{1}{\tau(K,D)} \ k, \ k \longrightarrow \infty, \tag{3.3}$$

for condensers (K, D) in $\mathbb{C}$ such that both K and D are bounded by finitely many analytic Jordan curves; they derived also the strict asymptotics

$$\mathcal{H}_\varepsilon\left(A_K^D\right) \sim \tau(K,D) \ \left(\ln\frac{1}{\varepsilon}\right)^2, \quad \varepsilon \longrightarrow 0. \tag{3.4}$$

from the asymptotics (3.3) by their general result mentioned in the previous section.

In [Z1] Zakharyuta, aiming to find a method suitable also for several variables, suggested another approach to the construction of Erokhin-type common basis based on the following classical fact of the functional analysis which is a corollary of the theorem on eigenvectors of a compact self-adjoint operator.

Lemma 3.3 *(see, e.g., [Be, Li, Kr, M]). — Let* (H_0, H_1) *be a couple of Hilbert spaces with compact dense imbedding:* $H_1 \hookrightarrow H_0$. *Then there is a system* $\{e_k\}_{k=1}^{\infty} \subset H_1$ *which is a common orthogonal basis for the spaces* H_1 *and* H_0 *such that*

$$\|e_k\|_{H_0} = 1, \quad \mu_k = \mu_k(H_0, H_1) := \|e_k\|_{H_1} \nearrow \infty.$$

This basis is unique up to the permutations do not changing the sequence μ_k. *The system* $\{e_k\}$ *forms also a basis in all the spaces of the Hilbert scale* $H_\alpha = H_0^{1-\alpha} H_1^\alpha$, *therewith* $\|e_k\|_{H_\alpha} = \mu_k^\alpha$, $\alpha \in \mathbb{R}$.

DEFINITION 3.4. — *The basis from Lemma 3.3 will be called the (canonical) doubly orthogonal basis for the couple of Hilbert spaces* (H_0, H_1).

In the conditions of Theorem 3.1 consider conformal mappings $\varphi : D \longrightarrow \{|w| < 1\}$ and $\psi : \overline{\mathbb{C}} \setminus K \longrightarrow \{|w| > 1\}$ such that $\psi(\infty) = \infty$ and $\psi(z) \sim \gamma z$ as $z \longrightarrow \infty$. Then the system $\left\{\varphi(z)^k\right\}_{k=0}^\infty$ is a basis in $A(D)$, while the system of Faber polynomials $\{\Phi_k(z)\}_{k=0}^\infty$ forms a basis in the space $A(K)$. Introduce two Hilbert spaces, generated by this bases: the space H_1 of all functions $f(z) = \sum_{k=0}^\infty \xi_k \varphi(z)^k$ with $\|f\|_{H_1} := \left(\sum_{k=0}^\infty |\xi_k|^2\right)^{1/2} < \infty$ and the space H_0 of all formal expansions into the series by Faber polynomials $g = \sum_{k=0}^\infty \xi_k \Phi_k$ with $\|g\|_{H_0} := \left(\sum_{k=0}^\infty |\xi_k|^2 \gamma^{-2k}\right)^{1/2} < \infty$. Then

$$H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0.$$

THEOREM 3.5 *([Z1])([Z1]). — The canonical doubly orthogonal basis* $\{e_k\}_{k=1}^\infty$ *for the couple* (H_0, H_1), *chosen above, forms also a basis in the spaces* $A(D)$, $A(K)$, $A(D_\alpha)$, $A\left(\overline{D_\alpha}\right)$, *where* D_α *is defined in (3.1),* $0 < \alpha < 1$. *This basis satisfies the estimates*

$$\frac{1}{C}\ R^{\alpha(1-\varepsilon)k} \leqslant |e_k|_{D_\alpha} \leqslant C\ R^{\alpha(1+\varepsilon)k}, \quad k \in \mathbb{N},\ 0 < \alpha < 1,\ \varepsilon > 0$$

with some constant $C = C(\alpha, \varepsilon)$. *Therewith the Hilbert scale* $H_\alpha := H_0^{1-\alpha} H_1^\alpha$ *satisfies the conditions:*

$$A\left(\overline{D_\alpha}\right) \hookrightarrow H_\alpha \hookrightarrow A(D_\alpha), \quad 0 < \alpha < 1 \tag{3.5}$$

and

$$\ln \mu_k(H_0, H_1) \sim k\ \ln R,\ k \longrightarrow \infty. \tag{3.6}$$

The proof of the asymptotics (3.6) in Theorem 3.5, which is important for gaining the asymptotics (1.1), was quite artificial: some linear topological invariants under common isomorphisms of pairs of linear topological spaces were applied (see [Z1], Section 3); the evaluation of this invariants was specifically one-dimensional.

In 1972 Nguyen Thanh Van [Ng1], applying the methods from [Z1], extended Theorem 3.5 to the case of *regular condenser* (K, D) in $\mathbb{C}$ (it is supposed additionally that the open sets D and $\mathbb{C} \setminus K$ are connected). Namely, he proved that, for a special couple of Hilbert spaces $H_0,\ H_1$ determined by the condenser, the Hilbert scale $H_\alpha := (H_0)^{1-\alpha}\ (H_1)^\alpha$ satisfies the conditions (3.5) and the doubly orthogonal basis for $H_0,\ H_1$ turns to be a common basis in the spaces $A(D),\ A(K),\ A(D_\alpha),\ A\left(\overline{D_\alpha}\right)$ with

$$D_\alpha := \{z \in D : \omega(D, K; z) < \alpha\}, \quad 0 < \alpha < 1.$$

This basis satisfies the estimates

$$\frac{1}{C}\ \exp\frac{\alpha(1-\varepsilon)k}{\tau(K,D)} \leqslant |e_k|_{D_\alpha} \leqslant C\ \exp\frac{\alpha(1+\varepsilon)k}{\tau(K,D)},\quad k \in \mathbb{N},\ 0<\alpha<1,\ \varepsilon>0$$

with some constant $C = C(\alpha, \varepsilon)$.

Let us describe the Hilbert spaces H_0 and H_1, utilized in [Ng1]: $H_0 = AL^2(K,\mu)$ is the subspace obtained by completion of the subspace of polynomials in the space $L^2(K,\mu)$ with Borel measure μ admissible in the sense of Widom [Wd1] (an additional restriction that K has no holes occurs just here); the space H_1 is constructed (similarly to [Z1]), using an extendible basis $\{\varphi_j\}$ in the space $A(D)$, which is biorthogonal (by Grothendieck-Köthe-Silva duality) to the basis $g_j(\zeta) = \Lambda_j\left(a + \frac{1}{\zeta}\right)$ in $A(D^*)$, where Λ_j is the Leja polynomial basis [Le] in the space $A(F),\ F = \left\{w = \frac{1}{z-a} : z \in D^*\right\}$ with some $a \in D$.

Notice that, in order to get the asymptotics

$$\ln \mu_k(H_0, H_1) \sim \frac{k}{\tau(K,D)},\ k \longrightarrow \infty,$$

Nguyen applies linear topological invariants for pairs of linear topological spaces from [Z1] and evaluates them still in a specifically one-dimensional way.

He considered several applications of such bases, but, in the context of the present survey, the most interesting is the following result

Theorem 3.6 *([Ng1]). — Suppose that (K, D) is a regular condenser in $\mathbb{C}$, such that D is a domain with the boundary consisted of countable set of Jordan curves and K is polynomially convex. Then the asymptotics (3.4) holds.*

H. Widom [Wd2] investigated the asymptotics (1.3) in one-dimensional case in a close connection with studying the asymptotics of the best approximation by rational functions. We cite here only his result about the asymptotics (1.3).

Theorem 3.7. — *Let (K,D) be a condenser in $\overline{\mathbb{C}}$. Then*

$$\limsup_{k\longrightarrow\infty}\left(d_k\left(A_K^D\right)\right)^{1/k}\leqslant\exp\left(-1/\tau\left(K,D\right)\right). \tag{3.7}$$

If, additionally, the complement of D has countably many connected components, then also

$$\liminf_{k\longrightarrow\infty}\left(d_k\left(A_K^D\right)\right)^{1/k}\geqslant\exp\left(-1/\tau\left(K,D\right)\right).. \tag{3.8}$$

It is worth to be noted that the Riemann boundary value problem is used in the proof of the inequality (3.8).

The asymptotics (1.1) and (1.3) for condensers on a one-dimensional open Riemann surface Ω were studied in [ZS, S]; Hilbert scales methods were central in these considerations.

Theorem 3.8 *[S]. — Suppose that (K,D) is a condenser on an open Riemann surface Ω, $n=\dim\Omega=1$; K is a perfect compact set such that $D\smallsetminus K$ is connected and has no polar portion; $D\Subset\Omega$ is a Caratheodory domain (that is, $\partial D=\partial\overline{D}$) such that the set of all points $\zeta\in\partial D$, irregular for the open set $\Omega\smallsetminus\overline{D}$, has zero harmonic measure (with respect to the domain D). Then both asymptotics (3.4) and (1.3) with $\sigma=\frac{1}{\tau(K,D)}, n=1$, are true.*

This result strengthens Theorem 2 from [ZS], where it was assumed that K is a regular compact set such that $D\smallsetminus K$ is connected and D is a finitely connected Caratheodory domain.

N. Skiba [S] showed that the upper estimate could be considerably better than in (3.7) even for regular condensers. Namely, the following inequality holds

$$\limsup_{k\longrightarrow\infty}\left(d_k\left(A_K^D\right)\right)^{1/k}\leqslant\exp\left(-1/\zeta\left(K,D\right)\right)=\exp\left(-1/\tau\left(K,\Delta\right)\right) \tag{3.9}$$

with $\zeta=\zeta\left(K,D\right):=\inf\left\{\tau\left(K,G\right)\right\}$, where infimum is taken over all open sets $G\supset D$, such that $\partial G\subset\partial D$ and $\partial D\smallsetminus\partial G$ is a portion of zero analytic capacity, and Δ is a union of those sets G. In particular, if the open set Δ satisfies the conditions of Theorem 3.8, then the asymptotics (1.3) holds with the constant $\sigma=\frac{1}{\tau(K,\Delta)}$. It is clear from the following example that the estimate (3.9) can be better than (3.7). Suppose that $D=\Delta\smallsetminus E$, where E is the standard Cantor set, Δ is any simply connected open neighborhood of E, and K is any regular compact subset of D. Then, since E has positive capacity but zero analytic capacity, we have that $\zeta(K,D) = \tau(K,\Delta) < \tau(K,D)$.

Fisher and Miccelli [FM] studied the asymptotics of widths (1.2), where A is the restriction of the unit ball of the Hardy space $H^p(D)$ and X is either $C(K)$ or $L^p(\nu)$ with some probability measure on K.

Notice that the following well-known fact of potential theory was essentially used almost in all one-dimensional investigations (see, e.g., [Wd2, ZS, S]).

Proposition 3.9. — *For any regular condenser (K,D) on an open Riemann surface there exists a sequence of finite sets $F_s = \left\{\zeta_j^{(s)} : j = 1,\dots,m_s\right\}$ and $\alpha^{(s)} = \left(\alpha_j^{(s)}\right) \in \mathbb{R}_+^{m_s}$ such that the sequence $\sum_{j=1}^{m_s} \alpha_j^{(s)}\ g_D\left(z,\zeta_j^{(s)}\right)$ converges to the function $\omega(D,K;z) - 1$ locally uniformly in $D \setminus K$, where $g_D(z,\zeta)$ is the Green function with the singularity $\ln|z-\zeta|$ at ζ; therewith $\sum_{j=1}^{m_s} \alpha_j^{(s)} \longrightarrow \tau(K,D)$.*

4. Multidimensional case (extendible bases)

It was demonstrated in [Z1] that the methods based on Lemma 3.3 are applicable to several complex variables as well. Namely, it was proved that a common Erokhin-type basis in $A(D)$ and $A(K)$ exists for a condenser (K,D) in $\mathbb{C}^n$ if D is a $(p_1,\dots,p_n)$-circular domain and $K = \overline{G}$, where G is a $(q_1,\dots,q_n)$-circular domain. In particular, they allowed to give a positive solution to the problem posed by L. Aizenberg on existence of a common bases, consisted of polynomials homogeneous with respect to $z_1^{1/p_1},\dots,z_n^{1/p_n}$, where $(p_1,\dots,p_n) = (q_1,\dots,q_n)$.

Another application of the Hilbert scales method for constructing of a common basis for the spaces $A(D)$ and $A(K)$ was displayed in [Z2], where D is a bounded convex domain in $\mathbb{C}^n$ and a compact set $K \subset D$ is a closure of some convex domain. The main goal in [Z2] was establishing of the isomorphism of the spaces $A(D)$ and $A(K)$ to the canonical spaces of analytic functions in polydiscs; for this purpose only the weak asymptotics $\ln\mu_k(H_0,H_1) \asymp k^{1/n}$ is sufficient.

Notice that there was no idea how to get a strict asymptotics for $\ln\mu_k(H_0,H_1)$ in the both above cases as well as in more general results about common bases ([Z3, Z4, Z6, Z7, Z8, A1, A2, Ng2, NZ1, Ze2, Ze3, Ze4] etc.) considered in this section below (see Theorem 4.16 and its discussion below).

Before dealing with them several definitions need to be introduced.

Definition 4.1. — *The* ***Green pluripotential*** *(pluricomplex Green function, relative extremal plurisubharmonic function)* ***of a condenser*** *(K, D) on a Stein manifold Ω is the function*

$$\omega(z) = \omega(D, K; z) := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \in \mathcal{P}(K, D)\}, \tag{4.1}$$

where $\mathcal{P}(K, D)$ is the class of all functions u plurisubharmonic in D and such that $u|_K \leqslant 0$ and $u(\zeta) \leqslant 1$ in D.

Definition 4.2. — ***An open set*** *$D \subset \Omega$ is* ***pluriregular*** *if there is a negative plurisubharmonic function $u(z)$ in D such that $u(z_j) \to 0$ for each sequence $\{z_j\} \subset D$ without limit points in D, shortly, $\lim\limits_{z \longrightarrow \partial D} u(z) = 0$; a* ***compact set*** *$K \subset \Omega$ is said to be* ***pluriregular*** *if $\omega\left(\widetilde{G}, K; z\right) = 0,\ z \in K$ for every open neighborhood G of K, where $\widetilde{G}$ is a holomorphic envelope of G.*

Definition 4.3. — ***An open set*** *$D \subset \Omega$ is* ***strictly pluriregular*** *if there exists a continuous plurisubharmonic function u in an open set $G \Supset D$ such that $D = \{z \in G : u(z) < 0\}$.*

Definition 4.4. — *A* ***condenser*** *(K, D) on a Stein manifold is said to be* ***pluriregular*** *if (i) K and D are pluriregular in the above sense and (ii) D has no component disjoint with K and $\widehat{K}_D = K$ (it is known that for a pluriregular condenser (K, D) the function (4.1) is continuous in D [Z4]).*

The notion of pluriregularity of an open set in Ω appeared (under the name *strong pseudoconvexity*) [Z3, Z4, Z6] in connection with the characterization of the isomorphism $A(D) \simeq A(\mathbb{U}^n)$ (do not confuse with the *strict pseudoconvexity*) and (under the name *hyperconvexity*) [St] in connection with Serre conjecture on holomorphic fiber bundles (see also [KR, A1, A2]). We find the name "pluriregular" for an open set in Ω more natural because in the classical potential theory the analogous condition characterizes the regularity of an open set.

The following result is a generalization of the one-dimensional result about Hilbert scales [Z1, Ng1, ZS, S]; it proved to be an important tool in solving of several problems in complex analysis (isomorphism of spaces of analytic functions, separate analyticity, orthogonal polynomials etc.; see, e.g., [Z8]).

THEOREM 4.5. — *Suppose* (K, D) *is a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$, *and* D *is strictly pluriregular;* H_0, H_1 *are Hilbert spaces satisfying the condition:*

$$A\left(\overline{D}\right) \hookrightarrow H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0 \hookrightarrow AC(K). \tag{4.2}$$

Then the Hilbert scale $H_\alpha := (H_0)^{1-\alpha} \, (H_1)^\alpha$ *satisfies the condition:*

$$A(K_\alpha) \hookrightarrow H_\alpha \hookrightarrow A(D_\alpha), \quad 0 < \alpha < 1, \tag{4.3}$$

where

$$D_\alpha := \{z \in D : \omega(D, K; z) < \alpha\}, \quad K_\alpha := \{z \in D : \omega(D, K; z) \leqslant \alpha\}. \tag{4.4}$$

Notice that the existence of couples H_0, H_1 satisfying the conditions (4.2) follows from nuclearity of the spaces $A(D)$ and $A(K)$, due to Pietsch ([Pt], section 4.4).

The left imbeddings in (4.3) are equivalent to some interpolation inequality for norms of analytic functionals, which is an analogue of the two constant theorem for analytic functions (see, e.g., [Si1, Si2, Z3, Z4]).

THEOREM 4.6 *([Z7, Z8]). — Let* (K, D) *be as in Theorem 4.5. Suppose that* X_α, $0 \leqslant \alpha \leqslant 1$, *are Banach spaces satisfying the conditions*

$$A\left(\overline{D}\right) \hookrightarrow X_1 \hookrightarrow A(D) \hookrightarrow A(K_\alpha) \hookrightarrow X_\alpha \hookrightarrow A(D_\alpha) \hookrightarrow A(K) \hookrightarrow X_0 \hookrightarrow AC(K),$$

where $0 < \alpha < 1$. *Set*

$$\|x'\|_\alpha^* := \sup\left\{|x'(x)| : x \in X_\alpha, \ \|x\|_{X_\alpha} \leqslant 1\right\}, \ x' \in X'_\alpha.$$

Then for every $\varepsilon > 0$ *and* $0 < \alpha < 1$ *there exists a constant* $C = C(\alpha, \varepsilon)$ *such that*

$$\|x'\|_\alpha^* \leqslant C \, \left(\|x'\|_0^*\right)^{1-\alpha+\varepsilon} \, \left(\|x'\|_1^*\right)^{\alpha-\varepsilon}, \ x' \in X'_0. \tag{4.5}$$

An alternative proof of this interpolational property for analytic functionals, based on Aytuna's technique from [A1, A2], can be found in [NZ2].

DEFINITION 4.7. — *Let* X *be a Fréchet space,* X' *its dual space,* $\left\{|\cdot|_p, \ p \in \mathbb{N}\right\}$ *non-decreasing sequence of seminorms defining its topology and*

$$|x'|_p^* := \sup\left\{|x'(x)| : |x|_p \leqslant 1\right\}, \ p \in \mathbb{N}$$

the system of polar (non-bounded) norms. The space X belongs to the class $\mathcal{D}_2$ [Z5] (see also [Vg, MV] under the notation $\overline{\Omega}$) if for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that fore each $r \in \mathbb{N}$ there is a constant $C > 0$ such that

$$\left(|x'|_q^*\right)^2 \leqslant C \quad |x'|_p \cdot |x'|_r \ , \quad x' \in X'.$$

Definition 4.8. — *We say that a Banach space $E \hookrightarrow X$ is **adherent** to the Fréchet space X if for every $p \in \mathbb{N}$ and $0 < \alpha < 1$ there exist $q \in \mathbb{N}$ and $C > 0$ such that*

$$|x'|_q^* \leqslant C \quad (|x'|_{E^*})^\alpha \left(|x'|_p\right)^{1-\alpha}, \quad x' \in X'. \tag{4.6}$$

The name "dead end spaces" is also used for those spaces.

Lemma 4.9. — *(Zakharyuta [Z4, Z6], see also [Z7, Z8]) Let D be an open set on a Stein manifold Ω. Then $A(D) \in \mathcal{D}_2$ if and only if D is pluriregular.*

Notice that the part "if" is an easy consequence of Theorem 4.5.

Definition 4.10. — *A Banach space $X \hookleftarrow A(K)$ is **adherent** to $A(K)$ if the dual space $X' \subset A(K)'$ is adherent to the space $A(K)'$. A couple of Banach spaces (X_0, X_1) satisfying the condition:*

$$X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0 \tag{4.7}$$

*is **adherent to the couple** $(A(K), A(D))$ if X_1 is adherent to $A(D)$ and X_0 is adherent to $A(K)$.*

It follows from Theorem 4.5 that any Hilbert space H_0 satisfying (4.2) is adherent to $A(K)$ if K is pluriregular and has a Runge neighborhood in Ω. In fact there are adherent spaces with norms weaker than the uniform norm on K, for example the space $AL_2(K, \mu_0)$ is adherent to $A(K)$ if $\mu_0 = (dd^c\omega)^n$ is the equilibrium measure generated by the pluripotential (4.1) (see [NZ1, Z7, Ze1, NS, Z8, NZ2, Lv] for this and more general conditions providing that $AL_2(K, \mu)$ is adherent to $A(K)$ for a measure μ on K; notice that for the one-dimensional case conditions of such kind were considered in [Wd1, Ul]).

In the conditions of Theorem 4.5, any couple (H_0, H_1) satisfying (4.2) is adherent to the couple $(A(K), A(D))$. But the existence of a space adherent to $A(D)$ for an arbitrary pluriregular open set turned to be a hard nut to crack - it became clear, for the first time, due to the following important general result (combined with Lemma 4.9).

LEMMA 4.11 *(Vogt [Vg], Lemma 1.4, [MV], Lemma 29.16). — A Fréchet space X belongs to $\mathcal{D}_2$ if and only if there exists an adherent Banach space $E \hookrightarrow X$.*

Later A. Aytuna [A1, A2], using Hörmander's theory on L^2-estimates for solutions of the $\overline{\partial}$-equation, showed that an adherent space can be realized as a concrete functional space.

LEMMA 4.12 *([A1, A2]). — Suppose D is a pluriregular open set on a Stein manifold having finite set of components, $\rho = e^{\varphi}$, $\varphi \in Psh(D)$. Then the weighted Bergman space $B^2_\rho(D)$ (see Introduction for the definition) is adherent to the space $A(D)$; therewith the weight $\rho(z) \equiv 1$ is available if $D \Subset \Omega$.*

Applying Lemma 4.11 (or 4.12) and considerations after Definition 4.10, we obtain

COROLLARY 4.13. — *For any pluriregular condenser (K, D) there exists a wide class of Banach (Hilbert) couples $(X_0.X_1)$ adherent to the couple $(A(K), A(D))$; in particular, if $(X_0.X_1)$ to $(A(K), A(D))$ is adherent, then each couple (Y_0, Y_1), satisfying the conditions*

$$X_0 \hookrightarrow Y_0 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow Y_1 \hookrightarrow X_1 \tag{4.8}$$

is also adherent to $(A(K), A(D))$.

Theorem 4.5 can be generalized to any normal regular Banach scale connecting a given couple of Banach spaces (X_0, X_1) adherent to $(A(K), A(D))$; we refer to [KPS], chapter 3, for definitions and facts of the Banach scales theory which are not explained here.

THEOREM 4.14 *(cf., [Z9]). — Let (X_0, X_1) be a couple of Banach spaces, satisfying the imbeddings (4.7) and such that $\mathbb{B}_{X_1}$ is closed in the topology induced on X_1 by the topology of X_0. Then the couple (X_0, X_1) is adherent to the couple of spaces $(A(K), A(D))$ if and only if any (some) regular scale X_α, $\alpha \in [0,1]$, connecting X_0 and X_1 (for their existence see [KPS], chapter 3), satisfies the imbeddings*

$$A(D_\alpha) \hookrightarrow X_\alpha \hookrightarrow A(K_\alpha), \quad \alpha \in (0,1),$$

where $D_\alpha = \{z \in D : \omega(D, K; z) < \alpha\}$, $K_\alpha = \{z \in D : \omega(D, K; z) \leqslant \alpha\}$, $0 < \alpha < 1$.

In [Z9] this result was proved under the stronger assumption that $\mathbb{B}_{X_1}$ is closed in X_0, but the proof suggested there remains valid in the above formulation. We emphasize that this statement became having content for any pluriregular pair, due to Vogt's Lemma 4.11 (or Aytuna's Lemma 4.12), providing existence of the left adherent space.

Corollary 4.15. — *The conclusion in Theorem 4.6 remains correct for a pluriregular condenser (K, D) if X_0, X_1 is a couple of Banach spaces such that X_1 is adherent to $A(D)$ and X_0 is any Banach space such that $A(K) \hookrightarrow X_0$ (remember that density of $A(K)$ in X_0 is included in this condition, see the notation in Introduction).*

The next theorem is a generalization of one-dimensional results about existence of Erokhin type bases ([Z1, Ng1, ZS, S]), though the problem about the strict asymptotics for the numbers $\ln \mu_k(H_0, H_1)$ involved in the estimates of the bases had been proved in such generality much later; we shall discuss this problem in the next chapter.

Theorem 4.16 *([Z4, Z7, Z8]). — Let (K, D) be a pluriregular condenser on a Stein manifold. Then for any couple of Hilbert spaces (H_0, H_1), adherent to the couple $(A(K), A(D))$ (such couples exist due to the Vogt's Lemma 4.11), the canonical doubly orthogonal basis $\{e_k\}$ forms a common basis in the spaces $A(K)$, $A(D)$, $A(D_\alpha)$, $A(K_\alpha)$, where D_α and K_α are as in Theorem 4.14. This basis satisfies the estimates*

$$\frac{1}{C(\alpha, \varepsilon)}\ \mu_k^{\alpha-\varepsilon} \leqslant |e_k|_{D_\alpha} \leqslant C(\alpha, \varepsilon)\ \mu_k^{\alpha+\varepsilon},\ 0 < \alpha < 1,\ \varepsilon > 0, \tag{4.9}$$

where

$$\ln \mu_k = \ln \mu_k(H_0, H_1) \asymp k^{1/n},\ k \longrightarrow \infty. \tag{4.10}$$

Conversely, if $\{f_k\}$ is a common basis for the spaces $A(K)$, $A(D)$ then there is a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ and a positive sequence $\{t_k\}$ such that the system $\left\{t_k\ f_{\sigma(k)}\right\}$ satisfies the estimates (4.9) and coincides with the canonical doubly orthogonal basis $\{e_k\}$ for some couple of Hilbert spaces (H_0, H_1) adherent to the couple $(A(K), A(D))$. The pluripotential (4.1) can be expressed by the following formula:

$$\omega(D, K; z) = \limsup_{\zeta} \limsup_{k \longrightarrow \infty} \frac{\ln |e_k(\zeta)|}{\ln \mu_k}, \quad z \in D \smallsetminus K. \tag{4.11}$$

This result was proved first in [Z4] under an additional assumption that the open set D is *strictly pluriregular*; then in [Z7] (see also [Z8]) it was derived from the results of [Z4] in the above form, due to the Vogt's result of Lemma 4.11. An alternative proofs, based on Aytuna's approach [A1, A2] and Vogt's results on the spaces of the class D_2, were done in [NS, NZ2].

THEOREM 4.17. — *Let $D \subset \Omega$ be an open set, consisted of finite set of components, $\dim \Omega = n$. Then the space $A(D)$ is isomorphic to $A(\mathbb{U}^n)$ if and only if D is pluriregular.*

The first proof of this result [Z6] was based on Theorem 4.5 and Corollary 4.13 (hence on Lemma 4.11 of Vogt). A. Aytuna [A1, A2] obtained an alternative proof of the part "if" in Theorem 4.17, using a weighted Bergman space $\mathcal{B}^2_\rho(D)$ as an adherent space to $A(D)$ (see Lemma 4.12).

DEFINITION 4.18. — *([Z7, Ll]) (Multipole Green Pluripotential) Let D be a pluriregular open set on a Stein manifold Ω, $F = \{\zeta_j : j = 1, \ldots, m\} \subset D$ a finite set and $\alpha = (\alpha_j) \in \mathbb{R}^m_+$. The* ***multipole Green Pluripotential*** *(pluricomplex Green function in [Kl]) in D with logarithmic singularities of given masses α at the points of F is the function*

$$g_D(F, \alpha; z) := \limsup_{\zeta \longrightarrow z} \sup \{u(\zeta) : u \in \mathcal{P}(\Omega, F, \alpha)\},$$

where $\mathcal{P}(\Omega, F, \alpha)$ is the class of all non-negative functions $u \in Psh(D)$ such that for each $j = 1, \ldots, m$ the estimate $u(\zeta) \leqslant \alpha_j \ \ln|t_j(\zeta)| + C$, $\zeta \in V_j$ holds with some constant $C = C_j(u)$, where V_j is some open neighborhood of ζ_j, $t_j : V_j \longrightarrow \mathbb{C}^n$ is a coordinate mapping such that $t_j(\zeta_j) = 0$.

THEOREM 4.19 *([Z7, Z8, Z9]). — Let $F = \{\zeta_\mu : \mu = 1, \ldots, m\}$ be a finite subset in a pluriregular open set $D \subset \Omega$, such that each component of D has non-empty intersection with F; $\alpha = (\alpha_\mu)_{\mu=1}^m$, $\alpha_\mu > 0$, $\mu = 1, \ldots, m$;*

$$D_\lambda := \{z \in D : g_D(F, \alpha; z) < -\lambda\}, \ F_\lambda := \{z \in D : g_D(F, \alpha; z) \leqslant -\lambda\},$$

with $0 < \lambda < \infty$, and

$$\sigma_n = \left(\frac{n!}{\sum_{\mu=1}^m (\alpha_\mu)^n} \right)^{1/n}.$$

Let $H \hookrightarrow A(D)$ be a Hilbert space adherent to the space $A(D)$. Then there exists a system $\{f_i(z)\}_{i \in \mathbb{N}}$, orthonormal in H, which forms a common basis in all the spaces

$$A(\Omega), \ A(F), \ A(\Omega_\lambda), \ A(F_\lambda), \ 0 < \lambda < \infty,$$

such that for each $\lambda : 0 < \lambda < \infty$ and $\varepsilon > 0$ the estimates

$$\frac{1}{C} \ \exp \sigma_n(-\lambda - \varepsilon) i^{1/n} \leqslant |f_i(z)|_{F_\lambda} \leqslant C \ \exp \sigma_n(-\lambda + \varepsilon) i^{1/n}, \ i \in \mathbb{N}, \tag{4.12}$$

hold with some constant $C = C(\lambda, \varepsilon)$. The Green multipole function can be expressed via the basis by the formula:

$$\sigma_n \ g_D(F, \alpha; z) = \limsup_{\zeta \to z} \ \limsup_{i \to \infty} \frac{\ln |f_i(\zeta)|}{i^{1/n}}, \ z \in \Omega \setminus F. \tag{4.13}$$

These bases are of a mixed nature: on the one hand, they are orthonormal in a Hilbert space connected with D and on the other hand, they have an interpolation nature with respect to given points of K. It is possible to take in this theorem the Bergman space $H = \mathcal{B}^2_\rho(\Omega)$, described in Aytuna's Lemma 4.12. Such bases are an important tool in establishing of the asymptotics (1.1) and (1.3) which will be discussed in the next section.

Zeriahi ([Ze4]) considered bases extendible into sublevel domains of a Green pluripotential with *countable set of weighted logarithmic singularities*; this generalizes the Kadampatta-Zakharyuta's result ([ZK]) on existence of common bases for a pair $K \subset D \subset \mathbb{C}$, where D is a regular domain and K is an arbitrary polar compact set in D. Notice that it remains open the problem about a complete multidimensional analogue of the Kadampatta-Zakharyuta's result.

PROBLEM 4.20. — *Suppose D be a pluriregular open set on Ω and $K \subset D$ is a compact set having non-empty intersection with every component of D and complete pluripolar, that is there exists $u \in Psh(D)$ such that $K = \{z \in D : u(z) = -\infty\}$. Does there exist a common (orthonormal-interpolation) basis for the spaces $A(D)$ and $A(K)$ (extendible onto sublevel sets of some plurisubharmonic function)?*

5. Multidimensional case (entropy and width asymptotics)

For a long time the asymptotics (1.1) or (1.3) for several complex variables were known in quite special cases. Notice that usually in particular cases one can obtain asymptotic formulas much finer than (1.1) and (1.3) which would not be expected in general. For example, A. Vitushkin ([V1, V2]) proved that for the condenser, formed by two concentric polydiscs

$$\overline{\mathbb{U}_r}, \ \mathbb{U}_R, \ \ r = (r_\nu), \ R = (R_\nu), \ 0 < r_\nu < R_\nu, \ \nu = 1, \ldots, n, \tag{5.1}$$

the following asymptotics have place

$$\mathcal{H}_\varepsilon\left(A^{\mathbb{U}_R}_{\mathbb{U}_r}\right) = \tau \ \left(\ln \frac{1}{\varepsilon}\right)^{n+1} + O\left(\left(\ln \frac{1}{\varepsilon}\right)^n \ \ln \ln \frac{1}{\varepsilon}\right), \ \varepsilon \longrightarrow 0 \tag{5.2}$$

with $\tau = \frac{2}{(n+1)!\ \sqcap_{\nu=1}^{n} \ln \frac{R_\nu}{r_\nu}}$. It is obvious that this asymptotics implies the asymptotics (1.1) with the same constant τ. Besides this classical result next two results about the asymptotics (1.1) and (1.3) are worth to be mentioned here .

The following result for special analytic polyhedra was proposed in [Z4], Proposition 10.6 and Corollary 10.1 without proof as an easy consequence from Theorem 4.5.

PROPOSITION 5.1. — *Suppose that (K, D) is a condenser on a Stein manifold Ω, $\dim \Omega = n$ and there is an open set $G \Supset D$ and an analytic mapping $g = (g_\nu) : G \longrightarrow \mathbb{C}^n$ so that $K = g^{-1}\left(\overline{\mathbb{U}_r}\right),\ D = g^{-1}\left(\mathbb{U}_R\right)$, where $\overline{\mathbb{U}_r},\ \ \mathbb{U}_R$ are defined in (5.1). Then the asymptotics (1.1) holds with the constant*

$$\tau = \frac{2\ \kappa}{(n+1)!\ \sqcap_{\nu=1}^{n} \ln \frac{R_\nu}{r_\nu}} \tag{5.3}$$

where κ is the multiplicity of the mapping g over D.

A set $E \subset \mathbb{C}^n$ is said to be *complete n-circular* if $z = (z_\nu) \in E$ implies that each $w = (w_\nu)$ with $|w_\nu| \leqslant |z_\nu|$ also belongs to E; to each complete n-circular set E we correspond its characteristic function:

$$h_E(\theta) := \sup\left\{\sum_{\nu=1}^{n} \theta_\nu\ \ln|z_\nu| : z = (z_\nu) \in E\right\},\ \theta = (\theta_\nu) \in \mathbb{R}^n_+.$$

In 1970th Ronkin and Zakharyuta considered independently (not published) the case of a complete n-circular condenser (K, D). For the proof of the following result see [ARZ].

PROPOSITION 5.2. — *Suppose (K, D) is a condenser in $\mathbb{C}^n$ such that both sets are complete n-circular and*

$$\Theta = \left\{\theta = (\theta_\nu) \in \mathbb{R}^n_+ : h_D(\theta) - h_K(\theta) \leqslant 1\right\}.$$

Then the asymptotics (1.3) holds with the constant $\sigma = \left(\frac{1}{Vol\Theta}\right)^{1/n}$ and the asymptotics (1.1) with the constant $\tau = \frac{2\ Vol\Theta}{n+1}$.

Remark 5.3. — In the conditions of Propositions 5.1 and 5.2, the asymptotics can be essentially refined to the additive form similar to the classical Vitushkin's asymptotics (5.2).

The question about an eventual constants for the asymptotics (1.1), (1.3) in more or less general case had been open until Bedford and Taylor [BT2] (see also Sadullaev [Sd]) introduced a capacity, which, for a pluriregular condenser (K, D), has a form

$$C(K, D) := \int_K (dd^c \omega(z))^n, \tag{5.4}$$

here the complex Monge-Ampére operator $u \longrightarrow (dd^c u)^n$ associates to any function $u \in Psh(D) \cap L^\infty_{loc}(D)$ some non-negative Borel measure. In particular, the so-called *equilibrium measure* $(dd^c \omega(z))^n$ is supported by K (for details see [BT1, BT2, Sd, Kl]). It is convenient to consider also the *pluricapacity* $\tau(K, D) = \frac{1}{(2\pi)^n} C(K, D)$, which differs from the capacity (5.4) by a natural factor so that it coincides with the Green capacity in the case $n = 1$. Soon afterwards it was conjectured in [Z7, Z8] that, for a good enough condenser (K, D) on a Stein manifold of dimension n, the asymptotics (1.3) ought to hold with the constant $\sigma = \left(\frac{n!}{\tau(K,D)}\right)^{1/n}$ (respectively, (1.1) with the constant $\tau = \frac{2\ \tau(K,D)}{(n+1)!}$). This conjecture is confirmed by the cases considered above, because in the conditions of Proposition 5.1

$$\tau(K, D) = \frac{\kappa}{\sqcap_{\nu=1}^n \ln \frac{R_\nu}{r_\nu}},$$

and in Proposition 5.2 we have (see [ARZ])

$$\tau(K, D) = n!\ Vol\Theta.$$

Moreover, some approach was suggested in [Z7, Z8] how to reduce the question about those asymptotics to the problem of the pluripotential theory on the *approximation of the function* $\omega(D, K; z) - 1$ *by multipole Green pluripotentials* (see Definition 4.18 above). This problem has been solved recently in positive by S. Nivoche [N1, N2] and E. Poletsky [P]. Namely they proved the following

THEOREM 5.4 *([N1, N2, P]). — For a pluriregular condenser* (K, D) *on a n-dimensional Stein manifold* Ω *there exists a sequence of finite sets* $F_s = \left\{\zeta_j^{(s)} : j = 1, \ldots, m\right\}$ *and masses* $\alpha^{(s)} = \left(\alpha_j^{(s)}\right) \in \mathbb{R}^n_+$ *such that the sequence of multipole Green pluripotentials* $g_D\left(F_s, \alpha^{(s)}; z\right)$ *converges locally uniformly to the function* $\omega(D, K; z) - 1$ *on* $D \setminus K$.

This statement, generalizing Proposition 3.9, first appeared as a conjecture in [Z7] (see also [Z8]). Together with the approach suggested in [Z7, Z8], it plays an important role in a final proof of the asymptotics (1.3) and (1.1) under quite general assumptions about condensers ([Z9], see below Theorems 5.6, 5.8, etc.).

DEFINITION 5.5. — *A couple of Banach spaces* (X_0, X_1), *satisfying the condition (4.7), is called admissible for a condenser* (K, D) *if for each couple of Banach spaces* (Y_0, Y_1) *satisfying the condition (4.8) we have* $\ln d_k(\mathbb{B}_{X_1}, \mathbb{B}_{X_0}) \sim \ln d_k(\mathbb{B}_{Y_1}, \mathbb{B}_{Y_0})$ *as* $k \to \infty$.

If a couple (X_0, X_1) is adherent for the couple $(A(K), A(D))$, then it is admissible for a condenser (K, D) ([Z9]). Hence, by Corollary 4.13 there exist a lot of admissible couples of Banach spaces (hence, Hilbert spaces) for a pluriregular condenser (K, D).

THEOREM 5.6. — *Let* (K, D) *be a pluriregular condenser on Stein manifold* Ω, $\dim \Omega = n$. *Then for any couple of Banach spaces* (X_0, X_1) *admissible for this condenser (they exist due to Corollary 4.13) the asymptotics have place*

$$-\ln d_k(\mathbb{B}_{X_1}, \mathbb{B}_{X_0}) \sim \left(\frac{n!\ k}{\tau(K, D)}\right)^{1/n}, \ k \longrightarrow \infty \tag{5.5}$$

$$\mathcal{H}_\varepsilon(\mathbb{B}_{X_1}, X_0) \sim \frac{2\ \tau(K, D)}{(n+1)!}\left(\ln \frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0 \tag{5.6}$$

The main idea of the proof is, using the basis from Theorem 4.19, first to get estimates of widths for condensers formed by level sets of the multipole Green pluripotential and then, applying Theorem 5.4, to approximate the condenser (K, D) by those level sets condensers (K_j, D_j) and then apply Bedford-Taylor result [BT2] providing that $\tau(K_j, D_j) \longrightarrow \tau(K, D)$.

Theorem 5.6 allows us to improve essentially the formulas (4.10) and (4.11), so that the bases constructed in Theorem 4.16 becomes a complete analogue of one dimensional Erokhin-type bases, considered in Theorem 3.5.

PROPOSITION 5.7. — *Under the conditions of Theorem 4.16 we have*

$$\ln \mu_k(H_0, H_1) \sim \left(\frac{n!\ k}{\tau(K, D)}\right)^{1/n}$$

and hence

$$\left(\frac{\tau(K, D)}{n!}\right)^{1/n} \omega(D, K; z) = \limsup_{\zeta \to z} \limsup_{k \to \infty} \frac{\ln |e_k(\zeta)|}{k^{1/n}}, \quad z \in D \setminus K.$$

It is clear that for any admissible couple of Banach spaces (Y_0, Y_1) the asymptotics (1.3) could occur with *the smallest possible constant* σ among all Banach couples satisfying the conditions (4.7), while, due to Lemma 2.2, the asymptotics (1.1) for admissible couples could happen with *the largest possible constant* τ.

Theorem 5.8. — *Let* (K, D) *be a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$. *Then the following statements are equivalent:*

(a) *the couple* $(H_\infty(D), AC(K))$ *is admissible for the condenser* (K, D);

(b) *the asymptotics (1.3) holds with the constant* $\sigma = \left(\frac{n!}{\tau(K,D)}\right)^{1/n}$;

(c) *the asymptotics (1.1) holds with the constant* $\tau = \frac{2\tau(K,D)}{(n+1)!}$.

The equivalence of (a) and (b) has been proved in [Z9], Theorem 1.5 and Corollary 1.7. The equivalence of (b) and (c) is by Lemma 2.2.

Theorem 5.9 *([Z9]).* — *Suppose* (K, D) *is a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$, *and* D *is strictly pluriregular. Then the asymptotics (1.3) holds with* $\sigma = \left(\frac{n!}{\tau(K,D)}\right)^{1/n}$.

Applying Aytuna's Lemma 4.12 and considerations after Definition 4.10, we derive from Theorem 5.6 the following

Proposition 5.10. — *Suppose* (K, D) *is a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$; $H_1 = \mathcal{B}^2_\rho(\Omega)$, *where* $\rho = e^{-\varphi}$, $\varphi \in Psh\,(D)$; $H_0 = AL_2(K, \mu_0)$ *with the equilibrium measure* $\mu_0 = (dd^c\omega)^n$ *for the pluripotential (4.1). Then*

$$-\ln d_k(\mathbb{B}_{H_1}, \mathbb{B}_{H_0}) = \ln \mu_k(H_0, H_1) \sim \left(\frac{n!}{\tau(K,D)}\right)^{1/n} k^{1/n}, \quad k \longrightarrow \infty. \tag{5.7}$$

6. Conclusion and some open questions

It is worth noting that there are numerous interesting results on asymptotics, estimates or even precise computations of diameters for particular classes of condensers and concrete related couples of Banach spaces (Yu. A. Farkov, S. D. Fisher, K. Yu. Osipenko, O.G. Parfenov, M. I. Stessin, V. M. Tikhomirov, et al). We do not discuss them here, because our main goal was to give a survey of results just concerning the classical Kolmogorov problem in their development from initial steps of 1950ies till quite general nowadays results.

We conclude this survey by some comments and problems remained open (see also [Z9], Section 9; some of remarks from there we repeat here).

The Widom's one-dimensional inequality (3.7) can be generalized for an arbitrary condenser on a Stein manifold, due to Theorem 5.6.

Proposition 6.1. — *Let $(K, D) \subset \Omega$, $\dim \Omega = n$ be a condenser and (X_0, X_1) a Banach couple satisfying (4.7). Then*

$$\limsup_{k \longrightarrow \infty} \frac{\ln d_k\left(\mathbb{B}_{X_1}, \mathbb{B}_{X_0}\right)}{k^{1/n}} \leqslant -\left(\frac{n!}{\tau(K, D)}\right)^{1/n}. \tag{6.8}$$

In particular,

$$\limsup_{k \longrightarrow \infty} \frac{\ln d_k\left(A_K^D\right)}{k^{1/n}} \leqslant -\left(\frac{n!}{\tau(K, D)}\right)^{1/n}. \tag{6.9}$$

Proof. — Consider a sequence of pluriregular condensers (K_j, D_j), $j \in \mathbb{N}$, such that $K_j \Supset K_{j+1}$, $\cap K_j = K$ and $D_j \Subset D_{j+1}$, $\cup D_j = D$. Then $\tau(K_j, D_j) \searrow \tau(K, D)$ (see, e.g., [BT2]) and (6.8) follows from Theorem 5.6. $\square$

For the sake of simplicity, we considered above mainly pluriregular condensers. In fact, the restrictions on the compact set K in Theorem 5.6 can be seriously eased.

Proposition 6.2. — *Suppose that (K, D) is a condenser such that D is a pluriregular open set in Ω, $\dim \Omega = n$, K is a non-pluripolar perfect compact set and there exists a sequence of pluriregular compact sets K_j such that $K_j \subset K_{j+1}$ and $\tau(K_j, D) \longrightarrow \tau(K, D)$. If $H^\infty(D)$ is adherent to $A(D)$, then the asymptotics (1.3) holds.*

Indeed, since $d_k\left(A_K^D\right) \geqslant d_k\left(A_{K_j}^D\right)$, we obtain from Theorem 5.6 that the inequality converse to (6.9) holds. Thus, combining this with (6.9), we obtain (1.3).

The next statement can be considered as a multidimensional generalization of the Skiba's estimate (3.9).

Proposition 6.3. — *Suppose that (K, D) be a pluriregular condenser and there exists an open set $G \supset D$ forming with K a pluriregular condenser*

(G,K) such that every function $f \in H^\infty(D)$ is extended to a function $F \in H^\infty(G)$ preserving its uniform norm. Then

$$\limsup_{k\longrightarrow\infty} \frac{\ln d_k\left(A_K^D\right)}{k^{1/n}} \leqslant -\left(\frac{n!}{\tau(K,G)}\right)^{1/n}.$$

If, additionally, $H^\infty(G)$ is adherent to $A(G)$, then $\leqslant$ is to be changed to $=$.

In connection with this fact, Example 9.9 in [Z9] (suggested by Sibony in [Sib]) is of interest.

Consider an extremal plurisubharmonic function ([Z4, Sib], see also [Z9]):

$$\gamma(D,K;z) := \limsup_{\zeta\longrightarrow z} \{u(\zeta) : u \in \mathcal{A}(D,K)\},$$

where $\mathcal{A}(D,K)$ is the class of all functions $u(z) = \alpha \ \ln|f(z)|$ with $\alpha > 0$ and $f \in H^\infty(D)$ such that $u|_K \leqslant 0$ and $u(z) < 1$ in D. It was proved in [Z9] that the condition

$$\gamma(D,K;z) = \omega(D,K;z), \ z \in D \tag{6.10}$$

is necessary for $H^\infty(D)$ to be adherent to the space $A(D)$.

Problem 6.4. — *Let (K,D) be a pluriregular condenser on Ω. Is the condition (6.10) sufficient for the adherence of $H^\infty(D)$ to $A(D)$?*

Although Theorems 5.6, 5.8, 5.9 give general necessary and sufficient conditions for the couples of Banach spaces related to a pluriregular condenser (K,D) for which the asymptotics (5.5) and (5.6) hold, the problem to check these conditions for concrete classes of condensers and couples of Banach spaces remains open even in one-dimensional case.

In connection with Aytuna's Lemma 4.12 arises

Problem 6.5. — *Given a pluriregular open set $D \subset \Omega$, characterize all weights ρ providing that the weighted Bergman space $\mathcal{B}_\rho^2(D) =: H_1$ is adherent to $A(D)$.*

Some sufficient conditions for $AL_2(K,\mu)$ to be adherent to $A(K)$ were mentioned above (see the paragraph next to Definition 4.10). In this connection arises

Problem 6.6. — *Given a pluriregular compact set $K \subset \Omega$, characterize all measures μ supported by K providing that the space $H_0 := AL_2(K,\mu)$ is adherent to $A(K)$.*

The question how to characterize open sets $D \subset \Omega$ with the property "$H^\infty(D)$ is adherent to $A(D)$" remains open even in the one-dimensional case as it follows from Example 9.8 from [Z9]; for the space $H^\infty(D)$, considered in that example, it has been proved only recently ([GM]) that it is dense in the space $A(D)$, while the condition of *adherence yields quite fast bounded approximation on every compact set* $K \subset D$. More precisely, see Corollary 6.9 below.

DEFINITION 6.7. — *A pluriregular condenser* (K, D) *satisfies the bounded power approximation property (BPAP) if*

$$\inf\{|x-y|_K : y \in H^\infty(D),\ |y|_D \leqslant M\} \leqslant C\ M^{-\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}} \tag{6.11}$$

for every $x \in A(D_\alpha)$ *with* $|x|_{D_\alpha} \leqslant 1$ *and* $\varepsilon > 0$, $0 < \alpha < 1$ *with some constant* $C = C(\alpha, \varepsilon)$ *(the sublevel domains* D_α *are defined in (4.4)).*

PROPOSITION 6.8 *(cf. Theorem 3.2.9 and Proposition 3.2.10 in [Z8]). Let* (K, D) *be a pluriregular condenser on* Ω*. Then* $H^\infty(D)$ *is adherent to* $A(D)$ *if and only if* (K, D) *satisfies BPAP.*

Proof. — Suppose that $X_1 := H^\infty(D)$ is adherent to $A(D)$. Applying Corollary 4.15 with $X_0 := AC(K)$, $X_\alpha = AC\left(\overline{D_\alpha}\right)$, $0 < \alpha < 1$ and using Lemma 29.13 from [MV] we can rewrite the estimates (4.5) in the form

$$\mathbb{B}_{X_\alpha} \subset \frac{C}{M^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}}}\ \mathbb{B}_{X_0} + M\ \mathbb{B}_{X_1}$$

with some constant $C = C(\alpha, \varepsilon) > 0$ for any $M > 0$, which is equivalent to (6.11). On the other hand, the estimates (6.11) imply the inequalities (4.5), which provide that $H^\infty(D)$ is adherent to $A(D)$. $\square$

COROLLARY 6.9. — *Let* D *be a pluriregular open set with finite set of components and* $H^\infty(D)$ *is adherent to* $A(D)$*. Let* $L \subset D$ *be a compact set. Then for each* $x \in A(D)$ *there exists a sequence* $y_m \in H^\infty(D)$, $|y_m|_D \leqslant m$, $m \in \mathbb{N}$*, such that*

$$\lim_{m \longrightarrow \infty} \frac{\ln|x-y_m|_L}{\ln m} = -\infty.$$

Indeed, one can take a pluriregular compact $K \supset L$ forming a pluriregular condenser with D and then apply Proposition 6.8, taking into account that each $x \in A(D)$ belongs to $AC(D_\alpha)$, $0 < \alpha < 1$.

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