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Voiculescu's Entropy and Potential Theory

Thomas Bloom[1]

Abstract. — We give a new proof, relying on polynomial inequalities and some aspects of potential theory, of large deviation results for ensembles of random hermitian matrices.

Résumé. — Nous donnons une démonstration nouvelle, s'appuyant sur des inégalités polynomiales et certains aspects de la théorie du potentiel, des résultats de grande déviation pour des ensembles de matrices hermitiennes aléatoires.

Introduction

That the (negative of) the logarithmic energy of a planar measure can be obtained as a limit of volumes originated with work of D. Voiculescu ([Vo1], [Vo2]). His motivation came from operator theory and free probability theory. Ben Arous and A. Guionnet [Be-Gu] put that result in the framework of large deviations. Other results in that direction are due to Ben Arous and Zeitouni [Be-Ze] and Hiai and Petz [Hi-Pe]. These authors use potential theory and retain the basic form of Voiculescu's original proof.

Informally, these results express the asymptotic value (as $d \to \infty$) of the average of a "weighted" VanDerMonde determinant of a point $(\lambda_1, \cdots, \lambda_d) \in E^d$, as the discrete measures $\kappa_d(\lambda) := \frac{1}{d}\sum_{j=1}^{d} \delta(\lambda_j)$ approach a fixed probability measure μ with compact support E in $\mathbb{C}$. Such weighted VanDerMonde

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(VDM) determinants arise, for example, as the joint probability distribution of the eigenvalues of certain ensembles of random Hermitian matrices and also in the study of certain determinental point processes. Specifically, we prove, for measures μ with support in a rectangle H:

Theorem 3.1. —

$$\inf_{G\ni\mu}\lim_{d\to\infty}\frac{1}{d^2}\log\int_{\tilde{G}_d(\mu)}|\mathrm{VDM}_d^w(\lambda)^2|d\tau(\lambda)=\Sigma(\mu)-2\int Qd\mu$$

where the infimum is over all neighborhoods of μ in the weak topology of measures on H, $\tilde{G}_d(\mu) := \{\lambda \in H^d \mid \kappa_d(\lambda) \in G\}$, $Q = -\log w$, $\Sigma(\mu) = \iint \log|z-t|d\mu(z)d\mu(t)$, τ is a measure satisfying the density condition on H: there are constants r_0, $T > 0$ such that*

$$\tau(D(z_0,r)) \geqslant r^T \quad \text{for all} \quad z_0 \in H \quad \text{and} \quad r \leqslant r_0.$$

Here $D(z_0,r)$ denotes the disc center z_0 radius r. $\mathrm{VDM}_d^w(\lambda)$ is a weighted VanDerMonde determinant (see (2.7)) with w continuous and >0 on H.

Note that the right side of the statement of theorem 3.1 is independent of the measure τ,as long as the density condition is satisfied. It is also independent of the rectangle H containing the support of μ. It depends only on μ and the values of Q on the support of μ. It is, in fact,the (negative of)the weighted energy of μ (see section 3).

This result is not essentially new , however the proof is new. The lower bound in theorem 3.1 is obtained by using Markov's polynomial inequality on the weighted VanDerMondes when the weight is a real polynomial, the general case being obtained by approximation.

Voiculescu's method (and those of the authors cited above) uses a "discretization" argument on the measure μ (the method has been used in other situations ([Ze]-[Ze])). This method relies on the factorization of the VDM determinant into linear factors. The method of this paper does not use such factorization-the interest in doing so, being in higher dimensional versions of these results (The methods of this paper were the basis for the announcement of some higher dimensional results [Bl, talk],see also,the paper of the author and N.Levenberg, Pluripotential energy,arxiv: 1007.2391).

R. Berman ([Be1], [Be2]) has recently proven large deviation results and a version of the above result in general higher dimensional situations. Reduced to the one-dimensional case of compact subsets of $\mathbb{C}$, his proof is different than that of Voiculescu or this paper.

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1. Topology on $\mathcal{M}(E)$

Let E be a closed subset of $\mathbb{C}$ (which we identify with $\mathbb{R}^2$). We let $\mathcal{M}(E)$ denote the set of positive Borel probability measures on E with the weak* topology.

The weak* topology on $\mathcal{M}(E)$ is given as follows (see [E], appendix A8). A neighborhood basis of any $\mu \in \mathcal{M}(E)$ is given by sets of the form

$$\{\nu \in \mathcal{M}(E) \Big| |\int_E f_i(d\mu - d\nu)| \leqslant \epsilon \quad \text{for} \quad i = 1, \cdots, k\} \tag{1.1}$$

where $\epsilon > 0$ and $f_1, \cdots, f_k$ are bounded continuous functions on E.

$\mathcal{M}(E)$ is a complete metrizable space and for E compact a neighborhood basis of $\mu \in \mathcal{M}(E)$ is given by sets of the form

$$G(\mu, k, \epsilon) := \{\nu \in \mathcal{M}(E) \Big| |\int_E x^{n_1} y^{n_2} (d\mu - d\nu)| < \epsilon\}$$

$$\text{for} \quad k, n_1, n_2 \in \mathbb{N}, \; n_1 + n_2 \leqslant k \quad \text{and} \quad \epsilon > 0. \tag{1.2}$$

That is $G(\mu, k, \epsilon)$ consists of all probability measures on E whose (real) moments, up to order k, are within ϵ of the corresponding moment for μ.

If $k_1 \geqslant k$ and $\epsilon_1 \leqslant \epsilon$ then

$$G(\mu, k_1, \epsilon_1) \subset G(\mu, k, \epsilon). \tag{1.3}$$

Now for $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{C}^d$, we let

$$\kappa_d(\lambda) := \frac{1}{d} \sum_{j=1}^{d} \delta(\lambda_j) \tag{1.4}$$

where δ is the Dirac δ-measure at the indicated point.

We let

$$\tilde{G}_d(\mu, k, \epsilon) := \{\lambda \in E^d \Big| \kappa_d(\lambda) \in G(\mu, k, \epsilon)\}. \tag{1.5}$$

It follows from (1.3) that

$$\tilde{G}_d(\mu, k_1, \epsilon_1) \subset \tilde{G}_d(\mu, k, \epsilon) \quad \text{for} \quad k_1 \geqslant k \quad \text{and} \quad \epsilon_1 \leqslant \epsilon. \tag{1.6}$$

For $\lambda \in \mathbb{C}^d$ we let

$$\Delta_d(\lambda) = \{\lambda' \in \mathbb{C}^d \Big| |\lambda'_j - \lambda_j| \leqslant e^{-\sqrt{d}} \quad \text{for} \quad j = i, \cdots, d\}. \tag{1.7}$$

Proposition 1.1 and 1.2 follow immediately from the definition of the weak* topology in $\mathcal{M}(E)$ (for E compact).

Proposition 1.1. — *Let f be continuous on E and $\mu \in \mathcal{M}(E)$. Given $\epsilon_1 > 0$ there exist k, ϵ such that*

$$\Big| \int_E f(d\mu - \kappa_d(\lambda)) \Big| \leqslant \epsilon_1 \quad \text{for} \quad \lambda \in \tilde{G}_d(\mu, k, \epsilon).$$

Proposition 1.2. — *Let $\nu \in G(\mu, k, \epsilon)$. Then there exists k_1, ϵ_1, such that $G(\nu, k_1, \epsilon_1) \subset G(\mu, k, \epsilon)$.*

Proposition 1.3. — *Let $\lambda \in \tilde{G}_d(\mu, k, \epsilon)$. Then $\Delta_d(\lambda) \in \tilde{G}_d(\mu, k, 2\epsilon)$ for all d sufficiently large.*

Proof. — The proof follows from the fact that monomials satisfy a Lipshitz condition on E. □

2. Markov's Polynomial Inequality

The classical Markov polynomial inequality for real polynomials on an interval $I \subset \mathbb{R}$ is an estimate for the derivative of the polynomial in terms of its degree and sup norm on I. Specifically ([Be-Er], theorem 5.1.8)

$$|p'(x)| \leqslant Ak^2 \|p\|_I \quad \text{for} \quad x \in I \tag{2.1}$$

where $k = \deg(p)$ and A is a constant > 0. For $I = [-1, 1]$ on may take $A = 1$.

Numerous extensions of (2.1) to multivariable settings have been established (see e.g. [Ba], [Pl]).

We will however use a version of (2.1) for rectangles $H \subset \mathbb{R}^2$ which is an immediate consequence of (2.1). (We will always assume that rectangles have sides parallel to the axes). Let $p(x, y)$ be a polynomial of degree $\leqslant k$ in each varaible, then

$$|\text{grad } (p)(x)| \leqslant Ak^2 \|p\|_H \tag{2.2}$$

where $A > 0$ is a constant.

Integrating (2.2) over the straight line joining z_1 to z_2 in H we have

$$|p(z_1) - p(z_2)| \leqslant Ak^2 \|p\|_H |z_1 - z_2|. \tag{2.3}$$

We will now use (2.3) to show in quantitative terms that the value of polynomials at points near a point where it assumes its maximum is close to the maximum value.

Let $\{\Lambda_d\}_{d=1,2\cdots}$ be a sequence of polynomials on $(\mathbb{R}^2)^d$, non negative on H^d, such that for some constants $c_1 > 0,\ \gamma_1 > 0$ each polynomial Λ_d is of degree $\leqslant c_1 d^{\gamma_1}$ in each of its $2d$ real variables. Let $z^M := (z_1^M, \cdots, z_d^M)$ be a point in $H^d \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$ where Λ_d assumes its maximum i.e. $\Lambda_d(z^M) = \|\Lambda_d\|_{H^d}$.

THEOREM 2.1. — *For $z \in \Delta_d(z^M) \cap H^d$. Then*

$$\Lambda_d(z) \geqslant \Lambda_d(z^M)\psi(d)$$

where $\psi(d) = 1 - cd^\gamma e^{-\sqrt{d}}$ for some constants $c, \gamma > 0$ (independent of d).

Proof. — We write $\Lambda_d(z^M) - \Lambda_d(z)$ in the form

$$\Lambda_d(z^M) - \Lambda_d(z) = \sum_{j=1}^{d} \Lambda_d(z_1, \cdots, z_{j-1}, z_j^M, \cdots, z_d^M) - \Lambda_d(z_1, \cdots, z_j, z_{j+1}^M, \cdots, z_d^M). \tag{2.4}$$

But for $z_1, \cdots, z_{j-1}, z_{j+1}^M, \cdots, z_d^M$ fixed, $t \to \Lambda_d(z_1, \cdots, z_{j-1}, t, z_{j+1}^M, \cdots, z_d^M)$ is a polynomial in $\mathbb{R}^2$ of deg $\leqslant c_1 d^{\gamma_1}$ in each real variable. Applying (2.3) and the fact that $z \in \Delta_d(z^M)$ to each term on the right side of (2.4) we have an estimate of the form

$$\Lambda_d(z^M) - \Lambda_d(z) \leqslant dA(c_1 d^{\gamma_1})^2 \Lambda_d(z^M) e^{-\sqrt{d}}. \tag{2.5}$$

The result follows. $\square$

We will apply this result to sequences of polynomials constructed as follows: Let

$$\mathrm{VDM}_d(\lambda) = \mathrm{VDM}_d(\lambda_1, \cdots, \lambda_d) = \prod_{1 \leqslant i < j \leqslant d} (\lambda_i - \lambda_j) \tag{2.6}$$

be the VanDerMonde determinant of the points $(\lambda_1, \cdots, \lambda_d)$ in $\mathbb{C}$. Also, for w a function on $\mathbb{C}$ we let

$$\mathrm{VDM}_d^w(\lambda) := \mathrm{VDM}_d(\lambda) \prod_{i=1}^{d} w(\lambda_i)^d. \tag{2.7}$$

Thus if w is a real polynomial the sequence of polynomials $\Lambda_d := |\mathrm{VDM}^w_d(\lambda)|^2$ for $d = 1, 2, \cdots$ satisfies the hypothesis of theorem 2.1. In this situation, points $z^M \in H^d$ at which $|\mathrm{VDM}^w_d(\lambda)|^2$ assumes its maximum are known as a w-Fekete set.

3. Energy as a limit of volumes

Let E be a compact subset of $\mathbb{C}$ and w an admissible weight function on E (i.e. w is uppersemicontinuous, $w \geqslant 0$, $w > 0$ on a non-polar subset of E. In particular, E is non-polar.

The weighted equilibrium measure (see [Sa-To], theorem I 1.3), denoted $\mu_{eq}(E, w)$ is the unique probability measure which minimizes the functional $I_w(\nu)$ over all $\nu \in \mathcal{M}(E)$ where

$$\begin{aligned} I_w(\nu) : &= \iint \log\Big(\frac{1}{|z-t|w(z)w(t)}\Big) d\nu(z)d\nu(t) \\ &= -\iint \log|z-t| d\nu(z)d\nu(t) + 2\int Q(z)d\nu(z) \end{aligned} \tag{3.1}$$

where

$$Q(z) := -\log w(z). \tag{3.2}$$

$I_w(\nu)$ is termed the weighted energy of the measure ν. We also use the notation

$$\Sigma(\nu) := \iint \log|z-t| d\nu(z)d\nu(t). \tag{3.3}$$

$\Sigma(\nu)$ is termed the free entropy of ν (it may assume the value $-\infty$). We let

$$\delta^w_d := \mathrm{Max}_{\lambda \in E^d} |\mathrm{VDM}^w_d(\lambda)|^{\frac{2}{d(d-1)}}. \tag{3.4}$$

Then (see [Sa-To], chapter III, theorem 1.1)

$$\delta^w := \lim_{d\to\infty} \delta^w_d \tag{3.5}$$

exists and

$$\log \delta^w = -I_w(\mu_{eq}(E,w)) = \Sigma(\mu_{eq}(E,w)) - 2\int Q(z)d\mu_{eq}(E,w). \tag{3.6}$$

We now consider conditions on the measure τ in the left side of the statement of theorem3.1.

We say that the triple (E, w, τ) satisfies the weighted Bernstein-Markov (B-M) inequality if, for all $\epsilon > 0$, these exists a constant $c > 0$ such that, for all analytic polynomials p of degree $\leqslant k$ we have

$$\|w^k p\|_E \leqslant c(1+\epsilon)^k \|w^k p\|_{L^2(\tau)}. \tag{3.7}$$

We set

$$Z_d := \int_{E^d} |\mathrm{VDM}_d^w(\lambda)|^2 d\tau(\lambda). \tag{3.8}$$

where $d\tau(\lambda) = d\tau(\lambda_1)\cdots d\tau(\lambda_d)$ is the product measure on E^d. Then if (E, w, τ) satisfies the weighted B-M inequality ([Bl-Le2]).

$$\lim_{d\to\infty} Z_d^{d^{-2}} = \delta^w. \tag{3.9}$$

In particular,the above limit is independent of the measure τ as long as the B-M inequality is satisfied.

We consider measures τ which satisfy the following condition on the rectangle H(satisfied by any measure that is a positive continuous function times Lebesgue measure):

There are constants r_0, $T > 0$ such that

$$\tau(D(z_0, r)) \geqslant r^T \quad \text{for all} \quad z_0 \in H \quad \text{and} \quad r \leqslant r_0. \tag{3.10}$$

Here $D(z_0, r)$ denotes the disc center z_0 radius r

Proposition 3.1. — *Let H be a rectangle in $\mathbb{C}$ and let τ satisfy (3.10). Then for all continuous functions $w > 0$ on H, (H, w, τ) satisfies the weighted B-M inequality.*

Proof. — First we can consider $H \subset \mathbb{C} \simeq \mathbb{R}^2$ as a subset of $\mathbb{C}^2$. Then, using [Bl-Le1], theorem 2.2 and [B1], theorem 3.2. (H, w, τ) satisfies the weighted B-M inequality as a subset of $\mathbb{C}^2$ (the definition of which is an obvious adaptation of (3.7) to the several variable case-see [Bl]). But every analytic polynomial $p(z)$ on $\mathbb{C}$ is the restriction to $\mathbb{C} \simeq \mathbb{R}^2 \subset \mathbb{C}^2$ of the analytic polynomial $p(z_1 + iz_2)$. Hence the result $\square$

Hence, if τ satisfies (3.10) then (3.9) is satisfied for any continuous $w > 0$ on H. This fact is used only in the statement of the large deviation result given in section 5,although (3.10) is used in the proof of theorem 3.1.

Let H be a rectangle in $\mathbb{C}$, τ, $\mu \in \mathcal{M}(H)$, with τ satisfing (3.10), and let $\phi > 0$ be a continuous function on H. Let $S = -\log\phi$. We will consider integrals of the form

$$J_d^\phi(\mu, k, \epsilon) := \int_{\tilde{G}_d(\mu,k,\epsilon)} |\mathrm{VDM}_d^\phi(\lambda)|^2 d\tau(\lambda). \tag{3.11}$$

The integral in (3.11) is of the same form as that in (3.8) used to define Z_d however here we only integrate over a subset of the product sets H^d. Theorem 3.1 below establishes asymptotic properties of such integrals. The leading term depends only on μ and S. (and as mentioned in the introduction, the result is not essentially new but goes back to results of Voiculescu ([Vo1], [Vo2]).

THEOREM 3.1. —

$$\inf_{k,\epsilon}\Big\{\lim_{d\to\infty}\frac{1}{d^2}\log J_d^{\phi}(\mu,k,\epsilon)\Big\} = \Sigma(\mu) - 2\int S d\mu$$

Proof. — To prove this result we will show

(a) $\inf_{k,\epsilon}\Big\{\overline{\lim_{d\to\infty}}\frac{1}{d^2}\log J_d^{\phi}(\mu,k,\epsilon)\Big\} \leqslant \Sigma(\mu) - 2\int S d\mu$ and

(b) $\inf_{k,\epsilon}\Big\{\underline{\lim_{d\to\infty}}\frac{1}{d^2}\log J_d^{\phi}(\mu,k,\epsilon)\Big\} \geqslant \Sigma(\mu) - 2\int S d\mu.$

To prove the upper bound (a) we will follow ([Be1], proposition 3.4). The proof does not use (3.10). Let w be continuous > 0 on H. Then

$$\prod_{i=1}^{d} w(\lambda_i)^{2d}|\mathrm{VDM}_d^{\phi}(\lambda)|^2 = |\mathrm{VDM}_d^{w}(\lambda)|^2\prod_{i=1}^{d}\phi(\lambda_i)^{2d}. \tag{3.12}$$

Hence,

$$|\mathrm{VDM}_d^{\phi}(\lambda)|^2 \leqslant (\delta_d^w)^{d(d-1)}\exp\big(2d^2\int_H (Q-S)\kappa_d(\lambda)\big). \tag{3.13}$$

Let $\lambda^d \in \overline{\tilde{G}_d(\mu,k,\epsilon)}$ be a point at which the maximum of $|\mathrm{VDM}_d^{\phi}(\lambda)|$ over $\overline{\tilde{G}_d(\mu,k,\epsilon)}$ is attained. (3.13) implies that

$$J_d^{\phi}(\mu,k,\epsilon)\tau(H)^d \leqslant (\delta_d^w)^{d(d-1)}\exp(2d\int_H (Q-S)\kappa_d(\lambda^d)). \tag{3.14}$$

For any sequence of d's we may pass to a subsequence and assume that the sequence of measures $\kappa_d(\lambda^d)$ converges to a measure $\sigma \in \overline{G(\mu,k,\epsilon)}$.

We deduce that

$$\overline{\lim_{d\to\infty}}\frac{1}{d^2}\log J_d^{\phi}(\mu,k,\epsilon) \leqslant \log\delta^w + 2\int_H (Q-S)d\sigma. \tag{3.15}$$

Taking the inf over k, ϵ, the σ's converge to μ so

$$\inf_{k,\epsilon} \overline{\lim} \frac{1}{d^2} \log J_d^{\phi}(\mu, k, \epsilon) \leqslant \log \delta^w + 2 \int_H (Q - S) d\mu.$$

This is valid for any w continuous and > 0 on H.

Now take a sequence of continuous weights w such that $\mu_{eq}(H, w)$ converges to μ in $\mathcal{M}(H)$ and $\Sigma(\mu_{eq}(H, w))$ converges to $\Sigma(\mu)$ (see proof of (b) (iii)).

Then using (3.6) we obtain (a).

For the lower bound (b) we proceed as follows.

We prove (b) when

(i) $\mu = \mu_{eq}(H, w)$, w is a polynomial > 0 and $\phi = w$.

(ii) μ as in (i) but the restriction on ϕ is dropped.

(iii) general μ.

(i) We consider points $z^M \in H^d$ at which $|\mathrm{VDM}_d^w(\lambda)|^2$ assumes its maximum (i.e. w-Fekete points). It is known that $\kappa_d(z^M)$ converges to μ in $\mathcal{M}(H)$ so, for d large, $\kappa_d(z^M) \in \tilde{G}_d(\mu, k, \epsilon)$. Then for d sufficiently large, using proposition 1.3

$$J_d^w(\mu, k, 2\epsilon) \geqslant \tau(\Delta_d(z^M)) \mathrm{Min}_{\lambda \in \Delta_d(z^M) \cap H} |\mathrm{VDM}_d^w(\lambda)|^2. \tag{3.16}$$

By (3.10) $\tau(\Delta_d(z^M)) \geqslant e^{-Td\sqrt{d}}$ and using theorem 2.1 on the sequence of polynomials $|\mathrm{VDM}_d^w(\lambda)|^2$ we have

$$\underline{\lim}_{d \to \infty} \frac{1}{d^2} \log J_d^w(\mu, k, 2\epsilon) \geqslant \log \delta^w = \Sigma(\mu) - 2 \int Q d\mu. \tag{3.17}$$

(ii) Recall that $S = -\log \phi, \mathrm{and} Q = -\log w$. Given $\epsilon_1 > 0$, by proposition 1.1, choose k, ϵ so that

$$\int (Q - S)(d\mu - \kappa_d(\lambda)) \leqslant \epsilon, \quad \text{for all } \lambda \in \tilde{G}_d(\mu, k, \epsilon). \tag{3.18}$$

This yields

$$\prod_{i=1}^{d} w(\lambda_i)^{2d} \leqslant \prod_{i=1}^{d} \phi(\lambda_i)^{2d} \exp(2d^2[\epsilon_1 - \int (Q - S) d\mu]). \tag{3.19}$$

Multiplying by $|\mathrm{VDM}_d(\lambda)|^2$ and integrating over $\tilde{G}_d(\mu,k,\epsilon)$ gives

$$\exp(2d^2[-\epsilon_1 + \int (Q-S)d\mu])J_d^w(\mu,k,\epsilon) \leqslant J_d^\phi(\mu,k,\epsilon). \tag{3.20}$$

Then using (i) and the fact that $\epsilon_1 > 0$ is arbitrary gives

$$\inf_{k,\epsilon} \varliminf_{d\to\infty} \left\{\frac{1}{d^2}\log J_d^\phi(\mu,k,\epsilon)\right\} \geqslant \log\delta^w + 2\int (Q-S)d\mu \tag{3.21}$$

and using (3.6) completes (ii).

For (iii) we will use an approximation argument. First we note that it is an immediate consequence of proposition 1.2 that

$$\mu \to \inf_{k,\epsilon} \varliminf_{d\to\infty} \frac{1}{d^2}\log J_d^\phi(\mu,k,\epsilon) \quad \text{is}$$

uppersemicontinuous on $\mathcal{M}(H)$. So, it suffices to show that any $\mu \in \mathcal{M}(H)$ may be approximated by measures $\{\mu_s\} \in \mathcal{M}(H)$ where each μ_s satisfies (i) above and $\Sigma(\mu_s)$ converges to $\Sigma(\mu)$. ($\mu \to \Sigma(\mu)$ is uppersemicontinuous on $\mathcal{M}(H)$ but not, in general, continuous). First, we may assume $\mathrm{supp}(\mu) \subset \mathrm{int}(H)$ since, taking H centered at 0 the measures $\mu_s = \pi_s^*(\mu)$, the push forward of μ under the scaling $z \to sz (s < 1)$ satisfy $\Sigma(\mu_s)$ converges to $\Sigma(\mu)$. Next for μ with compact support in $\mathrm{int}(H)$ we approximate μ by $\mu_s = \mu * \rho_s$ where $\rho = \rho(|z|)$ is a standard smoothing kernel for subhamonic functions on $\mathbb{C}$ and $\rho_s = s^{-2}\rho\left(\frac{|z|}{s}\right)$. Then $\rho(|z|)\rho(|t|)$ is a standard smoothing kernel for plurisubhamonic functions on $\mathbb{C}^2$ so $\log|z-t| * \Big(\rho_z(|z|)\rho_s(|t|)\Big)$ decreases pointwise to $\log|z-t|$.

Now, for ν a positive measure with compact support in $\mathbb{R}^n$, ψ a smooth function with compact support such that $\psi(x) = \psi(-x)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} \psi(\nu * h)dm \;=\int_{\mathbb{R}^n}(\psi * h)d\nu$$

where dm denotes Lebesque measure and $*$ convolution.

Applying this formula to $\mathbb{R}^4 \simeq \mathbb{C}^2$ with (z,t) as coordinates, $\nu = \mu \otimes \nu$, $\psi = \rho_s(|z|)\rho_z(|t|)$ and $h = \log|z-t|$, then using the Lebesgue monotone convergence theorem yields $\Sigma(\mu_s) \to \Sigma(\mu)$ (as $s \to 0$). Finally, for μ a smooth function with compact support times Lebesgue measure let Q be a smooth potential for μ which, adding a constant, we may assume is < 0 on H. Then $\mu = \mu_{eq}(H,w)$ where $w = e^{-Q}$ and one may approximate

μ by $\mu_s = \mu_{eq}(H, w_s)$ where w_s are real polynomial weights converging uniformly to w on H. To see that $\Sigma(\mu_s)$ converges to $\Sigma(\mu)$ we may use ([Sa-To], theorem 6.2 (c), chapter I) - which is stated for monotonically decreasing sequences of weights but the conclusion also holds for uniformly convergent sequences of weights.

4. Entropy

Let $\mu \in \mathcal{M}(H)$. The free entropy of μ (see (3.3)) defined as an integral may be obtained via discrete measures as follows:

Let $W(\mu)$ be defined via

$$W_d(\mu, k, \epsilon) := \sup\{|\mathrm{VDM}_d(\lambda)|^{\frac{2}{d(d-1)}} \Big| \kappa_d(\lambda) \in \tilde{G}_d(\mu, k, \epsilon)\} \tag{4.1}$$

and let

$$W(\mu, k, \epsilon) = \varlimsup_{d\to\infty} W_d(\mu, k, \epsilon) \tag{4.2}$$

and

$$W(\mu) = \inf_{k,\epsilon} W(\mu, k, \epsilon). \tag{4.3}$$

Then

THEOREM 4.1. — *$\log W(\mu) = \Sigma(\mu)$.*

Proof. — The proof consists of establishing the two inequalities

(a) $\log W(\mu) \leqslant \Sigma(\mu)$ and

(b) $\Sigma(\mu) \leqslant \log W(\mu)$.

For (a) let $\kappa_d(\lambda^d) = \frac{1}{d}\sum_{j=1}^{d} \delta(\lambda_j^d)$ be, for $d = 1, 2, \cdots$, a sequence of discrete measures converging to μ weak* such that

$$\log W(\mu) = \lim_{d\to\infty} \frac{1}{d^2} \sum_{j\neq k} \log|\lambda_j^d - \lambda_k^d|. \tag{4.4}$$

Now,

$$\lim_{d\to\infty} \frac{1}{d^2} \sum_{j\neq k} \delta(\lambda_j^d, \lambda_k^d) = \mu \otimes \mu \text{ weak}^* \tag{4.5}$$

and so, since $\log|z - t|$ is u.s.c.

(a) follows from ([Sa-To], theorem 1.4, chapter O).

For (b) we note that by definition of the quatities involved

$$J_d(\mu,k,\epsilon) \leqslant W_d(\mu,k,\epsilon)^{d(d-1)}\tau(H)^d.$$

so that

$$\overline{\lim_{d\to\infty}} \frac{1}{d^2}\log J_d(\mu,k,\epsilon) \leqslant \log W(\mu,k,\epsilon)$$

Taking the inf over k,ϵ and using theorem 3.1, (b) follows. □

Corollary 4.1. — *Define $W_d^\phi(\mu,k,\epsilon)$ analogously to the definition of $W_d(\mu,k,\epsilon)$ in (4.1). That is $W_d^\phi(\mu,k,\epsilon) = \sup\{|\mathrm{VDM}_d^\phi(\lambda)|^{\frac{2}{d(d-1)}}|\kappa_d(\lambda) \in \tilde{G}_d(\mu,k,\epsilon)\}$ and define $W^\phi(\mu)$ analogously to the definition of $W(\mu)$ (see (4.3)). Then*

$$\Sigma(\mu) - 2\int S d\mu = \log W^\phi(\mu).$$

5. Large Deviation

Consider the sequence of probability measures on H^d (for $d = 1, 2, \cdots$) given by

$$\frac{|\mathrm{VDM}_d^\phi(\lambda)|^2 d\tau(\lambda)}{Z_d^\phi} := \mathrm{Prob}_d. \tag{5.1}$$

Then

$$\frac{1}{d^2}\log \mathrm{Prob}_d(\tilde{G}_d(\mu,k,\epsilon)) = \frac{1}{d^2}\log \mathrm{Prob}_d(\lambda|\kappa_d(\lambda) \in G(\mu,k,\epsilon)). \tag{5.2}$$

Using theorem 3.1 and (3.9) gives

$$\inf_{k,\epsilon}\lim_{d\to\infty}\frac{1}{d^2}\log \mathrm{Prob}_d(\tilde{G}_d(\mu,k,\epsilon)) = I_\phi(\mu_{eq}(H,\phi)) - I_\phi(\mu).$$

The functional $\mu \to I_\phi(\mu) - I_\phi(\mu_{eq}(H,\phi)) =: I(\mu)$ attains its minimum value of zero at the unique measure $\mu = \mu_{eq}(H,\phi)$.

Then $I(\mu)$ is a good rate functional and the sequence of discrete random measures $\kappa_d(\lambda)$ satisfy a large devation principle in the scale d^{-2} (see discussion [Hi-Pe], page 211).

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