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On the stability by convolution product of a resurgent algebra

YAFEI OU⁽¹⁾

ABSTRACT. — We consider the space of holomorphic functions at the origin which extend analytically on the universal covering of $\mathbb{C} \setminus \omega\mathbb{Z}$, $\omega \in \mathbb{C}^*$. We show that this space is stable by convolution product, thus is a resurgent algebra.

RÉSUMÉ. — On considère l'espace des fonctions holomorphes à l'origine qui se prolongent analytiquement sur le recouvrement universel de $\mathbb{C} \setminus \omega\mathbb{Z}$, $\omega \in \mathbb{C}^*$. Nous démontrons que cet espace est stable par produit de convolution, donc elle définit une algèbre de résurgence.

1. Introduction

For $\zeta \in \mathbb{C}$ we denote by \mathcal{O}_ζ the set of all germs of holomorphic functions at ζ .

For $\varphi, \psi \in \mathcal{O}_0$ two germs of holomorphic functions at the origin, we define their convolution product by

$$\varphi * \psi(\zeta) = \int_0^\zeta \varphi(\eta)\psi(\zeta - \eta) d\eta \quad (1.1)$$

where the integral is taken along the segment $[0, \zeta]$ for ζ close enough to the origin. So

$$\varphi * \psi(\zeta) = \int_0^1 \varphi(\zeta t)\psi(\zeta(1 - t)) \zeta dt$$

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and the classical Lebesgue type theorems ensure that the convolution product $\varphi * \psi$ defines a germ of holomorphic functions at the origin.

In the paper we assume that $\omega \in \mathbb{C}^*$ is a fixed nonzero complex number.

For any $\omega \in \mathbb{C}^*$ we now introduce the \mathbb{C} -vector space $\mathcal{R}_\omega \subset \mathcal{O}_0$ made of all germs of holomorphic functions at the origin which extend analytically on the universal covering of $\mathbb{C} \setminus \omega\mathbb{Z}$, $\omega \in \mathbb{C}^*$. Then:

THEOREM 1.1. — *Let $\omega \in \mathbb{C}^*$. The space \mathcal{R}_ω is stable under convolution product. Therefore \mathcal{R}_ω is a resurgent algebra.*

Our aim in this paper is to show this theorem which plays an important role in resurgent theory (cf. [1, 2, 3, 4, 5, 6, 7, 8]) and for which surprisingly, to the best of our knowledge, there does not exist a complete written proof.

2. Some recalls and the statement of the problem

2.1. Analytic continuation along a path

For the very classical results hereafter, see e.g.,[9]

2.1.1. Analytic continuation of the germ of holomorphic functions

We denote by $\mathcal{O} = \bigsqcup_{\zeta} \mathcal{O}_{\zeta}$ the set of all germs of holomorphic functions.

We recall that \mathcal{O} is a topologically separated and locally compact space: if $\varphi_{\zeta} \in \mathcal{O}_{\zeta}$ is the germ of $\varphi \in \mathcal{O}(U)$ at ζ , where $U \subset \mathbb{C}$ is a neighborhood of ζ , then a neighborhood of φ_{ζ} is defined as the set of all germs $\varphi_{\xi} \in \mathcal{O}_{\xi}$ of φ for ξ in a neighborhood of ζ in U .

With the projection $p : \begin{array}{l} \mathcal{O} \rightarrow \mathbb{C} \\ \varphi \in \mathcal{O}_{\zeta} \mapsto \zeta \in \mathbb{C} \end{array}$ which associates to a germ its support, the space \mathcal{O} becomes a so-called “espace étalé” (étalé space), that is p is a local homeomorphism.

An analytic continuation of a germ φ is a connected subset of \mathcal{O} containing φ

2.1.2. Analytic continuation along a path

Hereafter a path λ is any continuous map $\lambda \in \mathcal{C}^0([0, 1], \mathbb{C})$.

Let $\lambda : [0, 1] \rightarrow \mathbb{C}$ be a path starting from $\zeta_0 = \lambda(0)$. The analytic continuation of the germ $\varphi \in \mathcal{O}_{\zeta_0}$ along λ is the image of a path $\lambda.\varphi : [0, 1] \rightarrow \mathcal{O}$ such that $\lambda.\varphi(0) = \varphi$ and whose projection by p is λ :

$$\begin{array}{ccc} & \mathcal{O} & \\ \lambda.\varphi \nearrow & & \searrow p \\ [0, 1] & \longrightarrow & \mathbb{C} \\ & \lambda & \end{array}$$

In other words one can define a subdivision $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ of the interval $[0, 1]$, open sets $U_j \subset \mathbb{C}$ with $\lambda([t_{j-1}, t_j]) \subset U_j$, and holomorphic functions $\varphi_j \in \mathcal{O}(U_j)$ whose germ at $\lambda(t_{j-1})$ is given by $\lambda.\varphi(t_{j-1}) \in \mathcal{O}_{\lambda(t_{j-1})}$, $j = 1, \dots, n$, such that (see Fig. 1) :

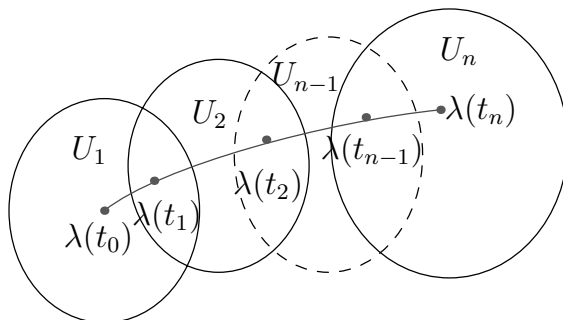


Figure 1

- (i) $\lambda.\varphi(t_0) = \varphi$
- (ii) $\varphi_j = \varphi_{j+1}$ on the connected component V_j of $\lambda(t_j)$ in $U_j \cap U_{j+1}$, $j = 1, \dots, n - 1$.

In what follows :

- we note $\lambda.\varphi(t) \in \mathcal{O}_{\lambda(t)}$ the analytic continuation along λ of the germ $\varphi \in \mathcal{O}_{\lambda(0)}$ at $\lambda(t)$,
- we write $\varphi(\lambda(t)) = \lambda.\varphi(t)(\lambda(t))$.

Obviously :

LEMMA 2.1. — *We assume that $\varphi \in \mathcal{O}_{\zeta_0}$ extends analytically along a path λ starting from $\zeta_0 = \lambda(0)$. Let λ' be a path deduced from λ by reparametri-*

sation¹, that is there exists an increasing homeomorphism h from $[0, 1]$ onto itself such that $\lambda' = \lambda \circ h$. Then φ extends analytically along λ' .

DEFINITION 2.2. — We denote by \mathfrak{R} the set of paths $\lambda : [0, 1] \rightarrow \mathbb{C}$ starting from 0.

We equip the space \mathfrak{R} with the uniform norm topology :

$$\forall \lambda_1, \lambda_2 \in \mathfrak{R}, \|\lambda_1 - \lambda_2\| = \max_{t \in [0,1]} |\lambda_1(t) - \lambda_2(t)|.$$

LEMMA 2.3. — Assume that $\varphi \in \mathcal{O}_0$ can be analytically continued along the path $\lambda \in \mathfrak{R}$. Then there exists a neighbourhood $V \subset \mathfrak{R}$ of λ such that for every path $\lambda' \in V$ the germ φ continues analytically along λ' , and for any $t \in [0, 1]$ the germs $\lambda.\varphi(t)$ and $\lambda'.\varphi(t)$ at $\lambda(t)$ and $\lambda'(t)$ respectively are the germs of the same analytic function.

Proof. — Consider the mapping $f : t \in [0, 1] \mapsto \rho(\lambda.\varphi(t)) \in \mathbb{R}^{+*}$ where $\rho(\lambda.\varphi(t))$ stands for the radius of the maximal disc of analyticity of the germ $\lambda.\varphi(t) \in \mathcal{O}_{\lambda(t)}$. (See Fig. 2). This map f is clearly continuous. Set $r = \min_{[0,1]} f > 0$. Then it is sufficient to assume that V is the open ball $\{\lambda' \in \mathfrak{R}, \|\lambda - \lambda'\| < r\}$. \square

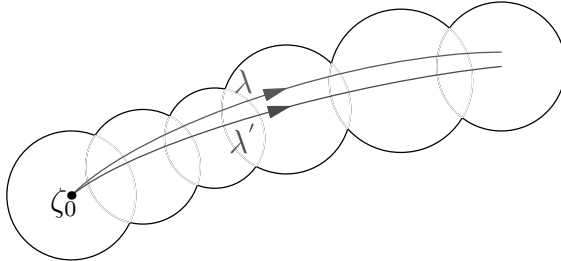


Figure 2

2.2. Analytic continuations of a convolution product

We consider $\varphi, \psi \in \mathcal{O}_0$ and $\lambda_{\zeta_0} \in \mathfrak{R}$ a path ending at $\zeta_0 = \lambda_{\zeta_0}(1)$. We assume that

- the germ φ continues analytically along the path λ_{ζ_0} ,
- the germ ψ continues analytically along the path $\lambda_{\zeta_0}^* : t \in [0, 1] \mapsto \lambda_{\zeta_0}^*(t) = \lambda_{\zeta_0}(1) - \lambda_{\zeta_0}(1 - t)$,

⁽¹⁾ Note that a reparametrisation is a homotopy.

According to lemma 2.2, these properties are still satisfied by small variations of the path. This allows to assume that λ_{ζ_0} is \mathcal{C}^1 by part and furthermore we can consider instead the nearby family of paths

$$\lambda_\zeta : t \in [0, 1] \mapsto \lambda_{\zeta_0}(t) + t(\zeta - \zeta_0)$$

for $|\zeta - \zeta_0|$ small enough. Under these conditions, the integral

$$I(\zeta) = \int_{\lambda_\zeta} \varphi(\eta)\psi(\zeta - \eta) d\eta = \int_0^1 \varphi(\lambda_\zeta(t))\psi(\lambda_\zeta^*(1-t)) \frac{d\lambda_\zeta}{dt}(t) dt$$

is well defined and $I(\zeta)$ defines a germ of analytic function at ζ_0 (by holomorphic dependance of λ_ζ with ζ and applying classical results in integration theory).

However, it should be noted that $I(\zeta)$ has no reason to be the analytic continuation along λ_{ζ_0} of the germ of analytic function at the origin defined by the convolution product $\varphi * \psi(\zeta)$. For that other conditions are necessary. The following lemma gives sufficient conditions.

LEMMA 2.4. — *One considers $\varphi, \psi \in \mathcal{O}_0$ and $\lambda \in \mathfrak{R}$. We assume that there is a continuous map $\Gamma : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma(s, t) = \Gamma_t(s) \in \mathbb{C}$ such that :*

- $\forall s \in [0, 1], \Gamma_0(s) = 0,$
- $\forall t \in [0, 1], \Gamma_t(0) = 0, \Gamma_t(1) = \lambda(t),$
- $\forall t \in [0, 1],$ the germ φ continues analytically along the path $\Gamma_t : s \in [0, 1] \mapsto \Gamma_t(s),$
- $\forall t \in [0, 1],$ the germ ψ continues analytically along the path $\Gamma_t^* : s \in [0, 1] \mapsto \Gamma_t^*(s) = \Gamma_t(1) - \Gamma_t(1 - s).$

*Then the germ of analytic functions $\varphi * \psi \in \mathcal{O}_0$ continues analytically along the path λ .*

Proof. — The map $\Gamma : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma(s, t) \in \mathbb{C}$ is continuous on a compact thus, by regularization, it can be uniformly approached by means of \mathcal{C}^∞ functions :

for every $\varepsilon > 0,$ there exists $\Gamma^\varepsilon : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma^\varepsilon(s, t) = \Gamma_t^\varepsilon(s) \in \mathbb{C},$ $\Gamma^\varepsilon \in \mathcal{C}^\infty$ such that

1. $\forall s \in [0, 1], \Gamma_0^\varepsilon(s) = 0,$

2. $\forall t \in [0, 1], \Gamma_t^\varepsilon(0) = 0,$
3. $\max_{(s,t) \in [0,1]^2} |\Gamma^\varepsilon(s, t) - \Gamma(s, t)| < \varepsilon.$

Also, using the same notations used in the proof of Lemma 2.3, we consider the mappings $f : t \in [0, 1] \mapsto \min_{s \in [0,1]} \rho(\Gamma_t \cdot \varphi(s)) \in \mathbb{R}^{+*}$ and $g : t \in [0, 1] \mapsto \min_{s \in [0,1]} \rho(\Gamma_t^* \cdot \psi(s)) \in \mathbb{R}^{+*}$. Since Γ_t depends continuously on t , both f and g are continuous. We note $r = \min\{\min_{[0,1]} f, \min_{[0,1]} g\} > 0$. Assuming $0 < \varepsilon < r$ we deduce that

4. $\forall t \in [0, 1],$ the germ φ continues analytically along the path $\Gamma_t^\varepsilon,$
5. $\forall t \in [0, 1],$ the germ ψ continues analytically along the path $\Gamma_t^{\varepsilon*} : s \in [0, 1] \mapsto \Gamma_t^{\varepsilon*}(s) = \Gamma_t^\varepsilon(1) - \Gamma_t^\varepsilon(1 - s).$

We note

$$\lambda^\varepsilon : t \in [0, 1] \mapsto \lambda^\varepsilon(t) = \Gamma_t^\varepsilon(1)$$

so that

$$\|\lambda^\varepsilon - \lambda\| < \varepsilon. \tag{2.2}$$

Now for every $t \in [0, 1]$ the integral

$$\begin{aligned} I(\lambda^\varepsilon(t)) &= \int_{\Gamma_t^\varepsilon} \varphi(\eta) \psi(\lambda^\varepsilon(t) - \eta) d\eta \\ &= \int_0^1 \varphi(\Gamma_t^\varepsilon(s)) \psi(\Gamma_t^{\varepsilon*}(1 - s)) \frac{\partial \Gamma_t^\varepsilon}{\partial s}(s) ds \end{aligned}$$

is well defined by virtue of conditions 4. and 5.. Furthermore, by conditions 1., 2., $I(\lambda^\varepsilon(t))$ coincides with $\varphi * \psi(\lambda^\varepsilon(t))$ for t close to 0. By the arguments discussed above, $\varphi * \psi$ is therefore analytically continuable along λ^ε . Taking $\varepsilon > 0$ small enough, we conclude by (2.2) and Lemma 2.3 that $\varphi * \psi$ extends analytically along λ . \square

In practice in order to apply lemma 2.4 one has to construct the homotopy map Γ for a given path λ . To solve this problem, we are going to introduce a class of symmetric paths which are convenient for the germs of analytic functions we consider in this paper.

3. Some spaces of paths

3.1. The spaces \mathcal{R}_ω and \mathfrak{R}_ω

DEFINITION 3.1. — We denote by $\tilde{\mathfrak{R}}_\omega \subset \mathfrak{R}$ the set of paths λ avoiding the one-dimensional lattice $\omega\mathbb{Z}$ except for the origin:

$$\tilde{\mathfrak{R}}_\omega = \{\lambda \in \mathfrak{R} \text{ with } \lambda([0, 1]) \subset \mathbb{C} \setminus \omega\mathbb{Z}\}$$

DEFINITION 3.2. — We denote by $\mathcal{R}_\omega \subset \mathcal{O}_0$ the \mathbb{C} -vector space of germs of holomorphic functions at the origin extending analytically along every path belonging to $\tilde{\mathfrak{R}}_\omega$.

DEFINITION 3.3. — We denote by $\mathfrak{R}_\omega \subset \mathfrak{R}$ the subset of paths belonging to \mathfrak{R} along which every germ of analytic functions $\varphi \in \mathcal{R}_\omega$ can be analytically continued.

The space \mathfrak{R}_ω is endowed with the topology induced by that of \mathfrak{R} .

We mention that $\tilde{\mathfrak{R}}_\omega \subset \mathfrak{R}_\omega$ strictly. For instance, every path $\lambda \in \mathfrak{R}$ whose image is contained in the star-shaped domain $\mathbb{C} \setminus ([\omega, \omega\infty[\cup]-\omega, -\omega\infty[)$ belongs to \mathfrak{R}_ω . This is due to the specific role played by the origin.

For completeness we describe \mathfrak{R}_ω . We first introduce new definitions.

DEFINITION 3.4. — We define :

- the set of loops \mathfrak{l}_ω :

$$\mathfrak{l}_\omega = \{\sigma \in \mathfrak{R} \text{ with } \sigma(1) = 0, \sigma([0, 1]) \subset \mathbb{C} \setminus \omega\mathbb{Z}\} \cup \{\sigma \equiv 0\},$$

- its subset \mathfrak{L}_ω made of paths homotopic to the constant path:

$$\mathfrak{L}_\omega = \{\sigma \in \mathfrak{l}_\omega / \exists f : t \in [0, 1] \mapsto f_t \in \mathfrak{l}_\omega, f_0 \equiv 0, f_1 \equiv \sigma, f \in \mathcal{C}^0\},$$

- the set $\tilde{\mathfrak{L}}_\omega \subset \mathfrak{R}$ as the group generated from \mathfrak{L}_ω by multiplication up to reparametrisation²

It is not hard to show that

$$\forall \sigma \in \mathfrak{L}_\omega, \forall \varphi \in \mathcal{R}_\omega, \sigma \cdot \varphi \text{ is well defined and } \sigma \cdot \varphi(1) = \varphi.$$

This means that $\mathfrak{L}_\omega \subset \mathfrak{R}_\omega$. As a consequence one has $\tilde{\mathfrak{L}}_\omega \subset \mathfrak{R}_\omega$. Then:

⁽²⁾ For $\sigma_1, \sigma_2 \in \tilde{\mathfrak{L}}_\omega$ we define their multiplication as usual by $\sigma_1 \sigma_2(t) = \begin{cases} \sigma_1(2t), & t \in [0, 1/2] \\ \sigma_2(2t - 1), & t \in [1/2, 1] \end{cases}$ and we consider its class modulo reparametrisations.

LEMMA 3.5. — \mathfrak{R}_ω coincides with the set of paths $\Lambda_\omega = \{\sigma\lambda \text{ mod reparametrisation } / \sigma \in \tilde{\mathfrak{L}}_\omega, \lambda \in \tilde{\mathfrak{R}}_\omega\}$.

Proof. — From what precedes one has $\Lambda_\omega \subset \mathfrak{R}_\omega$. We now show that $\mathfrak{R}_\omega \subset \Lambda_\omega$. Since for instance the meromorphic function $\sum_{n \in \mathbb{Z}^*} \frac{1}{(\zeta - n\omega)^2}$ belongs to \mathcal{R}_ω (when considered as defining a germ of holomorphic functions at the origin) one just has to show that: if $\lambda \in \mathfrak{R}$ is a loop which avoids $\omega\mathbb{Z}^*$ then

$$\left(\forall \varphi \in \mathcal{R}_\omega, \lambda.\varphi \text{ is well defined} \right) \text{ implies } \lambda \in \tilde{\mathfrak{L}}_\omega.$$

To prove that, just consider some finite linear combinations of germs $\varphi_k(\zeta) = \frac{1}{\zeta} \ln \left(1 - \frac{\zeta}{k\omega} \right), \varphi_k(0) = -\frac{1}{k\omega}, k \in \mathbb{Z}^*$. (For instance if the index of the loop λ with respect to $k\omega$ is nonzero, then consider φ_k to deduce that $\lambda \notin \mathfrak{R}_\omega$). \square

By the very definitions of \mathcal{R}_ω and \mathfrak{R}_ω , for a given $\lambda \in \mathfrak{R}_\omega$ and for $s \in [0, 1]$:

- either for every $\varphi \in \mathcal{R}_\omega$ the germ $\lambda.\varphi(s)$ extends holomorphically in a disc containing 0 and in this case

$$\forall \varphi \in \mathcal{R}_\omega, \rho(\lambda.\varphi(s)) \geq \min_{n \in \mathbb{Z}^*} |\lambda(s) - n\omega| > 0,$$

(we remind that where $\rho(\lambda.\varphi(s))$ stands for the radius of the maximal disc of analyticity of the germ $\lambda.\varphi(s) \in \mathcal{O}_{\lambda(s)}$)

- or

$$\forall \varphi \in \mathcal{R}_\omega, \rho(\lambda.\varphi(s)) \geq \min_{n \in \mathbb{Z}} |\lambda(s) - n\omega| > 0.$$

Therefore

$$\forall \lambda \in \mathfrak{R}_\omega, \forall s \in [0, 1], \inf_{\varphi \in \mathcal{R}_\omega} \rho(\lambda.\varphi(s)) > 0.$$

Now for $\lambda \in \mathfrak{R}_\omega$ and for every $\varphi \in \mathcal{R}_\omega$, since λ is continuous, the mapping

$$t \in [0, 1] \mapsto \rho(\lambda.\varphi(t)) \in \mathbb{R}^{+\ast}$$

is continuous. Thus there exists $t^\star \in [0, 1]$ such that $\rho(\lambda.\varphi(t^\star)) = \inf_{t \in [0, 1]} \rho(\lambda.\varphi(t))$.

Using the previous property one deduces that for a given $\lambda \in \mathfrak{R}_\omega$,

$$\inf_{\varphi \in \mathcal{R}_\omega} \inf_{t \in [0, 1]} \rho(\lambda.\varphi(t)) > 0.$$

This justifies the following definition:

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DEFINITION 3.6. — For $\lambda \in \mathfrak{R}_\omega$ we call

$$d_\omega(\lambda) = \inf_{\varphi \in \mathcal{R}_\omega} \inf_{t \in [0,1]} \rho(\lambda, \varphi(t)) > 0$$

the distance of λ to the lattice $\omega\mathbb{Z}$.

PROPOSITION 3.7. — \mathfrak{R}_ω is an open subspace of \mathfrak{R} and \mathfrak{R}_ω is connected.

Proof. — To show that \mathfrak{R}_ω is an open subspace of \mathfrak{R} we introduce $\lambda \in \mathfrak{R}_\omega$ and consider its distance $d_\omega(\lambda)$ to the lattice $\omega\mathbb{Z}$. Then the disc $\{\lambda' \in \mathfrak{R}, \|\lambda' - \lambda\| < d_\omega(\lambda)\} \subset \mathfrak{R}$ belongs to \mathfrak{R}_ω .

To show that \mathfrak{R}_ω is connected, we show instead that \mathfrak{R}_ω is connected by arc : if $\lambda_0, \lambda_1 \in \mathfrak{R}_\omega$, just consider the continuous map

$$\lambda : s \in [0, 1] \mapsto \lambda_s \in \mathfrak{R}_\omega, \quad \lambda_s : t \in [0, 1] \mapsto \begin{cases} \lambda_0((1-2s)t) & \text{si } s \in [0, 1/2] \\ \lambda_1((2s-1)t) & \text{si } s \in [1/2, 1] \end{cases}$$

□

Notice that the germ obtained by analytic continuation of a given germ $\varphi \in \mathcal{R}_\omega$ along a path $\lambda \in \mathfrak{R}_\omega$ depends only on the homotopy class of this path in the space \mathfrak{R}_ω .

DEFINITION 3.8. — We note $\underline{\mathfrak{R}}_\omega$ the set of the homotopy classes of paths with fixed extremities in \mathfrak{R}_ω .

The set $\underline{\mathfrak{R}}_\omega$ is endowed with the quotient topology. Then it is easy to show that:

LEMMA 3.9. — With the projection $\underline{\pi} : [\lambda] \in \underline{\mathfrak{R}}_\omega \mapsto \lambda(1) \in \mathbb{C}$, the set $\underline{\mathfrak{R}}_\omega$ is an étalé space on $\mathbb{C} \setminus \omega\mathbb{Z}^*$.

Note that $\underline{\pi}^{-1}(0)$ is reduced to a single point, the homotopy class of the constant path $t \in [0, 1] \mapsto 0$.

When pulling back by $\underline{\pi}$ the complex structure of $\mathbb{C} \setminus \omega\mathbb{Z}^*$, the étalé space $(\underline{\mathfrak{R}}_\omega, \underline{\pi})$ becomes a Riemann surface and we deduce from the above definitions and properties that:(see also [10])

PROPOSITION 3.10. — \mathcal{R}_ω can be identify with the space of analytic functions on the Riemann surface $(\underline{\mathfrak{R}}_\omega, \underline{\pi})$.

3.2. The space of \mathfrak{R}_ω -symmetric paths

3.2.1. Definition

DEFINITION 3.11. — A path $\gamma \in \mathfrak{R}$ is called a symmetric path if it is symmetric with respect to its mid-point $\frac{\gamma(1)}{2}$. (See Fig. 3). In other words,

$$\forall t \in [0, 1], \quad \text{we have} \quad \gamma^*(t) = \gamma(t) \quad \text{where} \quad \gamma^*(t) = \gamma(1) - \gamma(1 - t). \quad (3.3)$$

We denote by \mathfrak{R}^{sym} the subset of symmetric paths of \mathfrak{R} .

obviously:

LEMMA 3.12. — \mathfrak{R}^{sym} is a vectorial \mathbb{C} -space. Also,

$$\forall \lambda \in \mathfrak{R}, \quad \gamma = \lambda + \lambda^* \in \mathfrak{R}^{sym}.$$

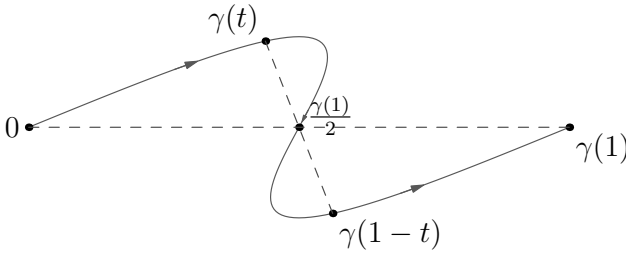


Figure 3

DEFINITION 3.13. — We say that a path γ is \mathfrak{R}_ω -symmetric if $\gamma \in \mathfrak{R}_\omega \cap \mathfrak{R}^{sym}$.

We denote by $\mathfrak{R}_\omega^{sym}$ the set of \mathfrak{R}_ω -symmetric paths.

3.2.2. Deformations of \mathfrak{R}_ω -symmetric paths

In a moment we shall have to construct continuous deformations in the space of \mathfrak{R}_ω -symmetric paths. This will be based on the following arguments.

PROPOSITION 3.14. — We consider $\gamma_0 \in \mathfrak{R}_\omega^{sym}$, $\zeta_0 \in \mathbb{C}^*$. We assume that

$$\inf_{v \in [0, 2], k \in \mathbb{Z}} |\gamma_0(1) + \zeta_0 v - k\omega| \geq 2R. \quad (3.4)$$

where $0 < 2R < |\omega|$. We also fix a constant r such that $0 < r < R$. Then there exists a continuous mapping

$$\gamma : (s, u) \in [0, 1]^2 \mapsto \gamma(s, u) \in \mathbb{C}$$

such that:

- $\forall s \in [0, 1], \gamma(s, 0) = \gamma_0(s)$.
- $\forall u \in [0, 1], \gamma(\cdot, u) \in \mathfrak{R}_\omega^{sym}$.
- $\forall u \in [0, 1], \gamma(1, u) = \gamma_0(1) + \zeta_0 u$.

Moreover, if $d_\omega(\gamma_0) \geq r$ then

$$\forall u \in [0, 1], d_\omega(\gamma(\cdot, u)) \geq r.$$

Proof. — We consider $\gamma_0 \in \mathfrak{R}_\omega^{sym}$ and $\zeta_0 \in \mathbb{C}^*$.

1. We introduce the constant vector field $\zeta \in \mathbb{C} \mapsto \zeta_0 \in \mathbb{C}^*$ and its flow $a : (\zeta, v) \in \mathbb{C} \times \mathbb{R} \mapsto a(\zeta, v) = \zeta + v\zeta_0$. Now defining for $u \geq 0$ the continuous mapping,

$$A(s, u) = a(\gamma_0(s), us) = \gamma_0(s) + \zeta_0 su, \quad \frac{dA}{du}(s, u) = \zeta_0 s, \quad A(s, 0) = \gamma_0(s),$$

one obtains:

- for $u \geq 0$ small enough, $A(\cdot, u) \in \mathfrak{R}_\omega$ (since \mathfrak{R}_ω is open, Proposition 3.7),
 - $\forall u \geq 0$, $A(\cdot, u) \in \mathfrak{R}_\omega^{sym}$, that is $A^*(s, u) = A(s, u)$ where $A^*(s, u) = A(1, u) - A(1 - s, u)$, $s \in [0, 1]$.
2. The previous first property fails in general when $u \geq 0$ is large enough. We now introduce R satisfying $0 < 2R < |\omega|$ and r such that $0 < r < R$. These conditions and a partition of unity allow to define a C^∞ function $f_{\omega, R, r} : \mathbb{C} \mapsto [0, 1]$ such that

$$\forall k \in \mathbb{Z}, \begin{cases} f_{\omega, R, r}(\zeta) = 0 & \text{if } |\zeta - k\omega| \leq r \\ f_{\omega, R, r}(\zeta) = g(|\zeta - k\omega|) & \text{if } r \leq |\zeta - k\omega| \leq R \end{cases},$$

$$\text{else } f_{\omega, R, r}(\zeta) = 1.$$

We consider the vector field $\zeta \in \mathbb{C} \mapsto \zeta_0 f_{\omega, R, r}(\zeta) \in \mathbb{C}$ and its flow $b : (\zeta, v) \in \mathbb{C} \times \mathbb{R} \mapsto b(\zeta, v)$,

$$\frac{db}{dv}(\zeta, v) = \zeta_0 f_{\omega, R, r}(b(\zeta, v)), \quad b(\zeta, 0) = \zeta.$$

(Note that the flow is indeed defined globally). Then for $u \geq 0$ we consider the continuous mapping $B(s, u) = b(\gamma_0(s), us)$,

$$B(s, u) = b(\gamma_0(s), us), \quad \frac{dB}{du}(s, u) = \zeta_0 s f_{\omega, R, r}(B(s, u)), \quad B(s, 0) = \gamma_0(s), \quad (3.5)$$

that is

$$B(s, u) = \gamma_0(s) + \zeta_0 s \int_0^u f_{\omega, R, r}(B(s, u')) du' = \gamma_0(s) + \zeta_0 s v(s, u)$$

or also

$$B(s, u) = A(s, u) + \zeta_0 s [v(s, u) - u],$$

where $v(s, \cdot) : u \geq 0 \mapsto v(s, u)$ is monotonous increasing with $v(s, u) \in [0, u]$.

One has

- for $u \geq 0$, $B(\cdot, u) \in \mathfrak{R}_\omega$, since the vector field $\zeta \mapsto \zeta_0 f_{\omega, R, r}(\zeta)$ vanishes when $|\zeta - k\omega| \leq r$, $k \in \mathbb{Z}$. This property has the following consequence: if $d_\omega(\gamma_0) \geq r$ then

$$\forall u \in [0, 1], d_\omega(B(\cdot, u)) \geq r.$$

- However as a rule $B(\cdot, u) \notin \mathfrak{R}^{sym}$.

3. For this reason we finally modify (3.5), defining the following continuous mapping $\gamma : (s, u) \mapsto \gamma(s, u)$ for $s \in [0, 1]$ and $u \geq 0$:

$$\begin{cases} \frac{d\gamma}{du}(s, u) = \zeta_0 s f_{\omega, R, r}(\gamma(s, u)) + \zeta_0(1-s) [1 - f_{\omega, R, r}(\gamma(1-s, u))] \\ \gamma(s, 0) = \gamma_0(s) \end{cases} \quad (3.6)$$

that is also

$$\begin{cases} \gamma(s, u) = \gamma_0(s) + \zeta_0 V(s, u), & 0 \leq V(s, u) \leq u \\ V(s, u) = s \int_0^u f_{\omega, R, r}(\gamma(s, u')) du' \\ \quad \quad \quad + (1-s) \int_0^u [1 - f_{\omega, R, r}(\gamma(1-s, u'))] du'. \end{cases} \quad (3.7)$$

It is easy to see that $\gamma(s, u)$ is well defined for $s \in [0, 1]$ and $u \geq 0$. Since $\gamma_0 \in \mathfrak{R}_\omega^{sym}$ one easily gets from (7) that

$$\forall s \in [0, 1], \forall u \geq 0, \gamma(s, u) + \gamma(1-s, u) = \gamma_0(1) + \zeta_0 u.$$

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According to (3.6) one has $\frac{d\gamma}{du}(1, u) = \zeta_0 f_{\omega, R, r}(\gamma(1, u))$. This property associated with condition (3.4) which we assume from now on implies that $\gamma(1, u) = \gamma_0(1) + \zeta_0 u$ for $u \in [0, 2]$. Therefore

$$\forall s \in [0, 1], \forall u \in [0, 2], \gamma(s, u) + \gamma(1 - s, u) = \gamma(1, u)$$

which means that $\forall u \in [0, 2], \gamma(\cdot, u) \in \mathfrak{R}^{sym}$ (see Fig 4)

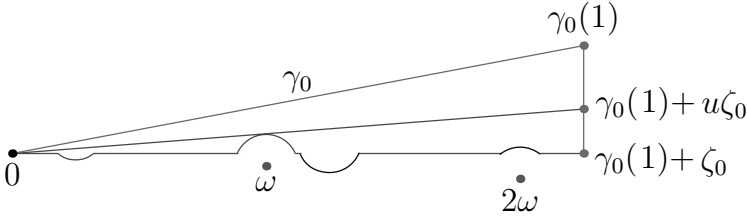


Figure 4

LEMMA 3.15. — For a given $s \in [0, 1]$, if

$$\exists k \in \mathbb{Z}, \exists u \in [0, 1], |\gamma(s, u) - k\omega| < R$$

then

$$\forall k \in \mathbb{Z}, \forall u \in [0, 1], |\gamma(1 - s, u) - k\omega| \geq R.$$

Proof. — For a given $s \in [0, 1]$ we assume that

$$\exists k_1 \in \mathbb{Z}, \exists u_1 \in [0, 1], |\gamma(s, u_1) - k_1\omega| < R$$

and

$$\exists k_2 \in \mathbb{Z}, \exists u_2 \in [0, 1], |\gamma(1 - s, u_2) - k_2\omega| < R.$$

We deduce that

$$|\gamma_0(1) + \zeta_0[V(s, u_1) + V(1 - s, u_2)] - (k_1 + k_2)\omega| < 2R.$$

This contradicts condition (3.4) because $0 \leq V(s, u) \leq u \leq 1$ for $(s, u) \in [0, 1]^2$. \square

This lemma has the following consequences: for a given $s \in [0, 1]$,

- either $\exists k \in \mathbb{Z}, \exists u \in [0, 1], |\gamma(s, u) - k\omega| < R$, and in this case

$$\forall u \in [0, 1], 1 - f_{\omega, R, r}(\gamma(1 - s, u)) = 0.$$

This means that

$$\forall u \in [0, 1], \gamma(s, u) = B(s, u).$$

- or $\forall k \in \mathbb{Z}, \forall u \in [0, 1], |\gamma(s, u) - k\omega| \geq R$.

We conclude that $\forall u \in [0, 1], \gamma(\cdot, u) \in \mathfrak{R}_\omega$. Moreover, if $d_\omega(\gamma_0) \geq r$ then

$$\forall u \in [0, 1], d_\omega(\gamma(\cdot, u)) \geq r.$$

This shows the Proposition 3.14. \square

3.3. \mathfrak{R}_ω -symmetric allowed paths

DEFINITION 3.16. — *The path $\lambda \in \mathfrak{R}_\omega$ is said to be \mathfrak{R}_ω -symmetric allowed if there exists a continuous map $\Gamma : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma(s, t) = \Gamma_t(s)$ such that:*

1. $\Gamma_0 \equiv 0$,
2. $\forall t \in [0, 1], \Gamma_t \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t(1) = \lambda(t)$.

It is worth mentioning that in this definition, the path λ is not supposed to be symmetric³. This is where our definition differs from that of a so-called “symmetric contractile path” ([5],[1] p.56).

PROPOSITION 3.17. — *We introduce the set*

$$\mathcal{H} = \{\lambda \in \mathfrak{R}_\omega, \lambda \text{ is } \mathfrak{R}_\omega\text{-symmetric allowed}\}.$$

Then \mathcal{H} is open in \mathfrak{R}_ω .

Proof. — One considers $\lambda_0 \in \mathcal{H}$. By definition 3.16, there is a homotopy map $\Gamma : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma(s, t) = \Gamma_t(s)$ such that:

1. $\Gamma_0 \equiv 0$,
2. $\forall t \in [0, 1], \Gamma_t \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t(1) = \lambda_0(t)$.

We introduce

$$d_\omega(\Gamma) = \inf_{t \in [0, 1]} d_\omega(\Gamma_t)$$

where $d_\omega(\Gamma_t)$ is the distance of $\Gamma_t \in \mathfrak{R}_\omega^{sym}$ to the lattice $\omega\mathbb{Z}$. Since Γ is continuous one gets that

$$d_\omega(\Gamma) > 0.$$

⁽³⁾ Note however that λ is homotopic to the symmetric path Γ_1 . Just consider the homotopy

$$H : (t, s) \in [0, 1]^2 \mapsto H(t, s), \begin{cases} H(t, 0) = \Gamma_t(1) = \lambda(t), \\ \text{for } s \in]0, 1], H(t, s) = \begin{cases} \Gamma_s(t/s), & t \in [0, s] \\ \Gamma_t(1) = \lambda(t), & t \in [s, 1] \end{cases} \end{cases}$$

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We now consider the open ball $V(\lambda_0, d_\omega(\Gamma)) = \{\lambda \in \mathfrak{R}_\omega, \|\lambda - \lambda_0\| < d_\omega(\Gamma)\} \subset \mathfrak{R}_\omega$ centred on λ_0 . For $\lambda \in V(\lambda_0, d_\omega(\Gamma))$ and $t \in [0, 1]$ we consider the path

$$\tilde{\Gamma}_t : s \in [0, 1] \mapsto \tilde{\Gamma}_t(s) = \Gamma_t(s) + (\lambda(t) - \lambda_0(t))s.$$

One has:

- $\tilde{\Gamma}_0 = \Gamma_0 + (\lambda(0) - \lambda_0(0))Id \equiv 0$,
- for $s \in [0, 1]$ we have, since $\Gamma_t \in \mathfrak{R}_\omega^{sym}$,

$$\tilde{\Gamma}_t(s) + \tilde{\Gamma}_t(1-s) = \Gamma_t(s) + \Gamma_t(1-s) + (\lambda(t) - \lambda_0(t)) = \Gamma_t(1) + (\lambda(t) - \lambda_0(t)).$$

Note that $\Gamma_t(1) = \lambda_0(t)$, therefore

$$\tilde{\Gamma}_t(s) + \tilde{\Gamma}_t(1-s) = \lambda(t) = \tilde{\Gamma}_t(1).$$

Thus $\tilde{\Gamma}_t \in \mathfrak{R}_\omega^{sym}$ and $\tilde{\Gamma}_t(1) = \lambda(t)$, $\forall t \in [0, 1]$.

- The mapping $\tilde{\Gamma} : (s, t) \in [0, 1]^2 \mapsto \tilde{\Gamma}_t(s)$ is obviously continuous and one has

$$\max_{(s,t) \in [0,1]^2} |\tilde{\Gamma}(s, t) - \Gamma(s, t)| < d_\omega(\Gamma).$$

One deduces that

$$\forall t \in [0, 1], \tilde{\Gamma}_t \in \mathfrak{R}_\omega$$

and therefore $\lambda \in \mathcal{H}$.

Thus \mathcal{H} is an open subset of \mathfrak{R}_ω . \square

We suspect that \mathcal{H} is also closed in \mathfrak{R}_ω but we have been unable to demonstrate this property. If this is true, then $\mathcal{H} = \mathfrak{R}_\omega$ since \mathfrak{R}_ω is connected (Proposition 3.7).

However the following theorem will be enough for our purpose. Furthermore its proof has the advantage to provide an explicit way of constructing the homotopy map associated with a \mathfrak{R}_ω -symmetric allowed path.

THEOREM 3.18. — *Every path $\lambda \in \tilde{\mathfrak{R}}_\omega$ is \mathfrak{R}_ω -symmetric allowed.*

Proof. — We consider $\lambda_0 \in \tilde{\mathfrak{R}}_\omega$ and we note $d_0 = d_\omega(\lambda_0) \leq |\omega|$. We introduce a sequence $t_0 = 0 < t_1 < t_2 < \dots < t_N = 1$ with the following properties.

- We choose t_1 so that

$$\forall t \in [t_0, t_1], |\lambda_0(t)| \leq \frac{d_0}{2}.$$

- We define

$$d = \inf_{t \in [t_1, 1], k \in \mathbb{Z}} |\lambda_0(t) - k\omega| \leq d_0.$$

Since $\lambda_0 \in \tilde{\mathfrak{X}}_\omega$ one has $d > 0$. We choose now $(t_n)_{n=2, \dots, N-1}$ so that

$$\forall n = 1, \dots, N-1, \forall t \in [t_n, t_{n+1}], |\lambda_0(t) - \lambda_0(t_n)| < \frac{d}{8}.$$

(this is possible because λ_0 is continuous).

We introduce the path λ defined by:

$$\left\{ \begin{array}{l} \forall t \in [0, t_1], \lambda(t) = \lambda_0(t) \\ \text{for } n = 1, \dots, N-1, \forall t \in [t_n, t_{n+1}], \lambda(t) \\ \quad = \lambda_0(t_n) + \frac{t - t_n}{t_{n+1} - t_n} (\lambda_0(t_{n+1}) - \lambda_0(t_n)) \end{array} \right.$$

In other words, λ coincide with λ_0 for $t \in [0, t_1]$ while for $t \geq t_1$ the path λ is a piecewise linear path which interpolates λ_0 at the points $\lambda_0(t_n)$, $n = 1, \dots, N$.

We easily obtains that

$$\|\lambda - \lambda_0\| < \frac{d}{4} \tag{3.8}$$

and that

$$\left\{ \begin{array}{l} \forall n = 1, \dots, N-1, \inf_{u \in [0, 2], k \in \mathbb{Z}} \left| \lambda(t_n) + (\lambda(t_{n+1}) - \lambda(t_n))u - k\omega \right| \geq 2R. \\ \text{where } R = \frac{7d}{16} \end{array} \right. \tag{3.9}$$

(hint: consider the case $u \in [0, 1]$ then $u \in [1, 2]$)

- We define the following continuous mapping Γ^1 ,

$$\Gamma^1 : (s, t) \in [0, 1] \times [t_0, t_1] \mapsto \Gamma^1(s, t) = \Gamma_t^1(s) = s\lambda(t).$$

From the condition on t_1 one has

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- $\Gamma_0^1 \equiv 0$,
- $\forall t \in [t_0, t_1], \Gamma_t^1 \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t^1(1) = \lambda(t)$,
- Also, obviously

$$\forall t \in [t_0, t_1], d_\omega(\Gamma_t^1) \geq \frac{d_0}{2} \geq \frac{d}{4}.$$

- For a given $n \in \{1, \dots, N-1\}$ assume that there exists a continuous mapping Γ^n ,

$$\Gamma^n : (s, t) \in [0, 1] \times [t_0, t_n] \mapsto \Gamma^n(s, t) = \Gamma_t^n(s)$$

such that

- $\Gamma_0^n \equiv 0$,
- $\forall t \in [t_0, t_n], \Gamma_t^n \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t^n(1) = \lambda(t)$,
- $\forall t \in [t_0, t_n], d_\omega(\Gamma_t^n) \geq \frac{d}{4}$.

Following (3.9), one can apply Proposition 3.14 with $\gamma_0 = \Gamma_{t_n}^n$, $\zeta_0 = \lambda(t_{n+1}) - \lambda(t_n)$, $R = \frac{7d}{16} < \frac{|\omega|}{2}$ and $r = \frac{d}{4} < R$: there exists a continuous mapping

$$\gamma : (s, u) \in [0, 1]^2 \mapsto \gamma(s, u) \in \mathbb{C}$$

such that:

- $\forall s \in [0, 1], \gamma(s, 0) = \Gamma_{t_n}^n(s)$.
- $\forall u \in [0, 1], \gamma(\cdot, u) \in \mathfrak{R}_\omega^{sym}$.
- $\forall u \in [0, 1], \gamma(1, u) = \lambda(t_n) + (\lambda(t_{n+1}) - \lambda(t_n))u$.
- $\forall u \in [0, 1], d_\omega(\gamma(\cdot, u)) \geq r$, since $d_\omega(\Gamma_{t_n}^n) \geq r$.

This allows to extend continuously Γ^n into the following continuous mapping

$$\left\{ \begin{array}{l} \Gamma^{n+1} : (s, t) \in [0, 1] \times [t_0, t_{n+1}] \mapsto \Gamma^{n+1}(s, t) = \Gamma_t^{n+1}(s) \\ \forall (s, t) \in [0, 1] \times [t_0, t_n], \Gamma^{n+1}(s, t) = \Gamma^n(s, t), \\ \forall (s, t) \in [0, 1] \times [t_n, t_{n+1}], \Gamma^{n+1}(s, t) = \gamma\left(s, \frac{t - t_n}{t_{n+1} - t_n}\right) \end{array} \right.$$

which satisfies:

- $\Gamma_0^{n+1} \equiv 0$,
- $\forall t \in [t_0, t_{n+1}], \Gamma_t^{n+1} \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t^{n+1}(1) = \lambda(t)$,
- $\forall t \in [t_0, t_{n+1}], d_\omega(\Gamma_t^{n+1}) \geq r$.

One thus deduces by a finite induction that the path $\lambda \in \mathfrak{R}_\omega$ is \mathfrak{R}_ω -symmetric allowed. More precisely, there is a homotopy map $\Gamma : (s, t) \in [0, 1] \times [0, 1] \mapsto \Gamma(s, t) = \Gamma_t(s)$ such that:

1. $\Gamma_0 \equiv 0$,
2. $\forall t \in [0, 1], \Gamma_t \in \mathfrak{R}_\omega^{sym}$ and $\Gamma_t(1) = \lambda(t)$.
3. moreover $d_\omega(\Gamma) = \inf_{t \in [0, 1]} d_\omega(\Gamma_t) \geq \frac{d}{4}$.

Since by (8) one has $\|\lambda - \lambda_0\| < \frac{d}{4}$, we deduce that λ_0 is itself \mathfrak{R}_ω -symmetric allowed (see the proof of Proposition 3.17). This ends the proof of Theorem 3.18. \square

4. Proof of theorem 1.1

We consider $\varphi, \psi \in \mathcal{R}_\omega$. Using Lemma 2.2 it is straightforward to show that their convolution product $\varphi * \psi$ is analytically continuable along every \mathfrak{R}_ω -symmetric allowed paths. Theorem 3.18 thus implies that $\varphi * \psi$ is indeed analytically continuable along every path $\lambda \in \mathfrak{R}_\omega$ and the proof of Theorem 1.1 is thus complete.

5. Conclusion

We have shown that the space of holomorphic functions at the origin which extend analytically on the universal covering of $\mathbb{C} \setminus \omega\mathbb{Z}$, $\omega \in \mathbb{C}^*$, is stable by convolution product. This provides the simplest resurgent algebra.

More generally, resurgent algebras are spaces of holomorphic functions which are endlessly continuable [6] or even continuable without cuts [7]. As far as we know, there are no complete convincing written proofs showing the stability by convolution products of these spaces. More than the results by themselves, which are considered to be true by the specialists, these written proofs could be interesting in their possible generalization to other weighted products with a wide spectrum of applications in the field of parametric resurgence [8]. We hope to be able to make some advances in that direction in other articles.

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