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A Wong-Rosay type theorem for proper holomorphic self-maps

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Abstract. — In this short paper, we show that the only proper holomorphic self-maps of bounded domains in $\mathbb{C}^k$ whose iterates approach a strictly pseudoconvex point of the boundary are automorphisms of the euclidean ball. This is a Wong-Rosay type theorem for a sequence of maps whose degrees are *a priori* unbounded.

Résumé. — Dans cette note, nous prouvons que les seules auto-applications holomorphes propres des domaines bornés de $\mathbb{C}^k$ dont les itérées accumulent un point de stricte-pseudoconvexité du bord sont des automorphismes de la boule. Il s'agit d'un résultat de type Wong-Rosay pour une suite d'applications dont les degrés sont *à priori* non bornés.

Introduction

In 1977, Wong proved that the only strictly pseudoconvex domain with non-compact automorphism group is the ball [16]. This result was generalized by Rosay [12] (see also [11]).

Theorem [Wong-Rosay]. — *Let Ω be a bounded domain in $\mathbb{C}^k$ and (f_n) a sequence of its automorphisms. Assume that the orbit of a point of Ω under (f_n) accumulates a smooth strictly pseudoconvex point of $b\Omega$. Then Ω is biholomorphic to the euclidean ball.*

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This theorem remains valid for a sequence of correspondences provided that their degrees remain bounded [10]. In this paper, we prove that the theorem above also holds true in presence of unbounded degree, when the sequence of automorphisms is replaced by the iterates of a proper holomorphic self-map.

Theorem 1. — *Let Ω be a bounded domain in $\mathbb{C}^k$ with a proper holomorphic self-map f. If there is a point y of Ω whose orbit under the iterates of f accumulates a smooth strictly pseudoconvex point a of $b\Omega$ (that is $f^{n_k}(y) \longrightarrow a$), then Ω is biholomorphic to the euclidean ball and f is an automorphism.*

In [9], the question of wether a proper holomorphic self-map of a smoothly bounded domain in $\mathbb{C}^k$ has to be an automorphism of the domain was considered. In $\mathbb{C}^2$ for instance, it was proved that non-injective proper self-maps of such domains has a non-compact dynamics (all the limit maps of the dynamics have value on the boundary of Ω). Theorem 1 goes one step further in this direction : the limit maps even take values in the weakly pseudoconvex part of the boundary.

The main ingRedient for this result is a local version of Wong-Rosay's theorem concerning sequences of CR-maps. It was first obtained by Webster in the wake of Chern-Moser's theory of strictly pseudoconvex hypersurfaces [15].

Theorem 2 (Webster). — *Let (Σ, a) and S be two germs of strictly pseudoconvex hypersurfaces. Assume there is a sequence of CR-embeddings of S into Σ whose images converge to a. Then S is spherical, i.e. locally CR-diffeomorphic to the euclidean sphere.*

The idea behind the proof of theorem 1 is to consider the CR-maps induced by f^n on the boundary rather than the maps f^n themselves. Using techniques developped in [9], we study the way these CR-maps degenerate and check that theorem 2 applies : around a, the boundary of Ω is spherical. The local biholomorphism between our domain and the ball then extends to the whole of Ω thanks to the dynamical situation.

The paper is organised as follows. We first collect some trivial dynamical facts about f and the automorphisms of the ball which will allow us to propagate the local sphericity to the whole domain. In section 2, we prove theorem 1 *modulo* the central question of the local sphericity around a. In section 3, we finally turn back to this problem.

1. Preliminary remarks

Surprisingly enough, the convergence hypothesis on $f^{n_k}(y_0)$ in theorem 1 has very strong (though very classical) implications in the holomorphic context. The aim of this section is to clarify some of them, as well as pointing out the well-known properties of the dynamics of the automorphisms of the ball which will be usefull to us. Henceforth, Ω, f and a are as in theorem 1.

First of all, this hypothesis may seem weaker than it actually is. Indeed, the contracting property of holomorphic maps for the Kobayashi distance (which is a genuine distance on bounded domains) leads to the following classical fact :

LEMMA 1.1. — *Any sequence of holomorphic maps between bounded domains Ω and Ω', which takes a point y in Ω to a sequence converging to a strictly pseudoconvex point of the boundary of Ω', converges locally uniformly to this point on Ω. For instance, the sequence f^{n_k} converges locally uniformly to a on Ω.*

COROLLARY 1.2. — *The map f extends smoothly to a neighbourhood of a in $b\Omega$ and $f(a) = a$. Moreover, f is a local biholomorphism (resp. CR-automorphism) in a neighbourhood of a (resp. in $b\Omega$).*

Proof. — Call $z_k := f^{n_k}(y)$ and $w_k := f(z_k)$. Since $w_k = f(f^{n_k}(y)) = f^{n_k}(f(y))$, both z_k and w_k tend to a because of the previous lemma. Since a is a strictly pseudoconvex point, an observation of Berteloot ensures that f extends continuously to a neighbourhood of a in $b\Omega$ [3], with $f(a) = \lim f(z_k) = \lim w_k = a$. Such an extension is automatically smooth because a is a strictly pseudoconvex smooth point of $b\Omega$ [2]. Since we are close to a strictly pseudoconvex point of the boundary, branching is prohibited and f must be one-to-one (see [5]). □

Let us now discuss the dynamical type of the fixed point a. Although it attracts part of the dynamics, it is not obvious at first glance that a is not a repulsive fixed point. The orbit of y_0 could *in principle* jump close to a from time to time, then get expelled away from a. The following lemma shows that such a behaviour does not occur in our holomorphic context.

LEMMA 1.3. — *The point a is a non-repulsive fixed point of f.*

Proof. — Assume by contradiction that f is repulsive at a. By definition, there is an open neighbourhood U of a on which the inverse f^{-1} of f is well defined, takes values in U, and is even contracting : $d(f^{-1}_{|U}(z), a) < d(z, a)$

for any $z \in U$. By assumption, there is a point $y_0 \in \Omega$ such that $f^{n_k}(y_0) \in U$ as soon as k is large enough. Define then

$$n'_k := \min\{n \mid f^i(y_0) \in U, \qquad \forall i \in [n, n_k]\},$$

so that $f^{n'_k-1}(y_0) \notin U$. Since $f^{-1}_{|U}$ is contracting, the point $f^{n'_k}(y_0)$ is closer to a than $f^{n_k}(y_0)$, so it tends to a (in particular (n'_k) is an extraction). Equivalently $f^{n'_k-1}(f(y_0))$ tends to a, so $f^{n'_k-1}$ converges locally uniformly to a by lemma 1.1. This is in contradiction with $f^{n'_k-1}(y_0) \notin U$. □

Let us finally discuss the dynamics of the automorphisms of the ball. Since there are very few of them (they form a finite dimensional group), their dynamics is rather poor and any small piece of information on it may give rise to strong restrictions. Recall the following well-known classification (see [13], section 2.4).

Proposition 1.4. — *Let g be an automorphism of the unit ball in $\mathbb{C}^n$. Then the dynamics of g is*

- *either hyperbolic (North-South) : there exist exactly two fixed points $N, S \in bB$ of g and g^n converges locally uniformly to S on $\overline{B}\backslash\{N\}$.*
- *or parabolic (South-South) : there exists a unique fixed point $S \in bB$ of g and g^n converges locally uniformly to S on $\overline{B}\backslash\{S\}$.*
- *or recurrent (compact) : the g-orbits remain at fixed distance from bB. If g has a fixed point on bB then it has a whole complex pointwise fixed line through this point (see also [6]).*

What will be of interest for us in this classification is contained in the following lemma, whose proof is straightforward from the classification.

Lemma 1.5. — *Let g be a ball automorphism which has a non-repulsive fixed point p on bB, and no interior fixed point near p. Then the dynamics of g is either hyperbolic or parabolic, with south pole p (meaning that S is p in the previous classification). Moreover, given any neighbourhood U of p, there is a point z in U whose orbit remains in U and converges to p.*

2. Proof of theorem 1

In this section, we prove theorem 1 leaving aside the central question of the sphericity of $b\Omega$ around a, which will be dealt with in the next section. Let us first fix the notation. Let (Ω, f, a) be a triple as in theorem 1. By a global change of coordinates in $\mathbb{C}^k$, we can take a to the origin, the tangent plane of $b\Omega$ at a to $\{\operatorname{Re} z_1 = 0\}$, and make Ω strictly convex locally near a. For α small enough, define U_α and Ω_α as being the connected components of a in $b\Omega \cap \{\operatorname{Re} z_1 < \alpha\}$ and $\Omega \cap \{\operatorname{Re} z_1 < \alpha\}$.

The first step of the proof, postponed to the following section, consists in showing that $b\Omega$ is spherical around a.

Lemma 2.1. — *A neighbourhood of a in $b\Omega$ is spherical.*

This means that there exists a CR-diffeomorphism $\Phi : U_\varepsilon \longrightarrow V \subset bB$. A classical extension theorem even shows that Φ extends to a biholomorphism $\Phi : \Omega_\varepsilon \longrightarrow D$ where D is an open set of B whose boundary contains V (see [4]). This biholomorphism allows to transport f to a local automorphism of B, defined by

$$\begin{array}{cccc} g & : & \Phi(\Omega_\varepsilon \cap f^{-1}(\Omega_\varepsilon)) & \longrightarrow & \Phi(\Omega_\varepsilon) \\ & & x & \longmapsto & \Phi \circ f \circ \Phi^{-1}(x). \end{array}$$

The key point of the whole proof is the following extension phenomena discovered by Alexander [1] (see also [11, 14] for the form of the result we use here). The local biholomorphism g uniquely extends to a global automorphism of the ball, again denoted by g.

The second step consists in using the dynamics of f and the injectivity of g (which we got for free thanks to Alexander's theorem) to propagate the local sphericity, and produce a biholomorphism between Ω and B. Let us first discuss the possible dynamics of g. By lemma 1.3, a is not a repulsive fixed point of f so $\Phi(a)$ is neither one for g. Moreover, since f has no fixed point inside Ω (because of lemma 1.1), g has also no fixed point in V. By lemma 1.5, g is either hyperbolic with attractive fixed point $\Phi(a)$ or parabolic with only fixed point $\Phi(a)$. From now on, we will denote $S := \Phi(a)$. The same lemma also guarantees that there are points in D whose (positive) orbits under g remain in D and tend to S. Since their whole orbits remain in D, the conjugacy thus allows to get the following informations on f in return.

LEMMA 2.2. — *The whole sequence of iterates* (f^n) *(rather than only a subsequence) converges to* a *on* Ω*. Moreover, the set*

$$\Omega'_\varepsilon := \{z \in \Omega_\varepsilon\ ,\ f^n(z) \in \Omega_\varepsilon\ \ \forall n \in \mathbb{N}\}$$

is a non-empty open invariant set of f*.*

Proof. — From the discussion above, we conclude that there is a point y in Ω_ε such that $f^n(y)$ remains in Ω_ε (thus Ω'_ε is not empty). Its orbit also converges to a. By lemma 1.1, f^n must therefore converge to a locally uniformly on Ω. The set Ω'_ε is obviously invariant by f. Finally, it is open because the Kobayashi metric decreases under f. Indeed, if z is in Ω'_ε, so is a Kobayashi δ-neighbourhood of this point (take $\delta := d_K\big(\Omega \cap \{\mathrm{Re}\, z_1 = \varepsilon\}\,, \mathrm{Orbit}(z)\big)$). $\square$

COROLLARY 2.3. — *The map* Φ *extends to a holomorphic map from* Ω *to* B*.*

Proof. — Let O_i denote $f^{-i}(\Omega'_\varepsilon)$. Because of the invariance of Ω'_ε by f and since f^n converges to a on Ω, we conclude that $(O_i)_i$ is a growing sequence of open sets which exhausts Ω. Define therefore

$$\begin{array}{cccll} \Phi & : & \Omega = \cup O_i & \longrightarrow & B \\ & & z \in O_i & \longmapsto & g^{-i} \circ \Phi_{|\Omega'_\varepsilon} \circ f^i(z). \end{array}$$

This map is obviously holomorphic (because Ω'_ε is open), and coincides with Φ on Ω'_ε. It is therefore an extension of Φ itself. $\square$

The remaining point to prove is that Φ is a biholomorphism. Let us first prove that it is proper.

LEMMA 2.4. — *The map* Φ *is a proper map from* Ω *to* B*.*

Proof. — Recall that the dynamics of g is either hyperbolic or parabolic. Moreover, $\Phi^{-1} \circ g(w) = f \circ \Phi^{-1}(w)$ for any $w \in D$ such that $g(w)$ belongs to D (recall that $\Phi : \Omega_\varepsilon \longrightarrow D$ is a biholomorphism). A basic consequence of these two facts is that $\Phi(O_n \backslash O_{n-1})$ goes to bB with n. Indeed, the f-orbit of a preimage by Φ of a point w in this set reaches $O_0 = \Omega'_\varepsilon$ only at time n, so the g-orbit of w cannot remain in D before the same time (if $g^k(w) \in D$ for $k \geqslant N$, then $f^k(\Phi^{-1}(w)) = \Phi^{-1}(g^k(w))$ is in Ω_ε for $k \geqslant N$ also). If n is large, $\Phi(z)$ has to be very close to some pole of the dynamics which is either S if g is parabolic or another point of bB if g is hyperbolic. Anyway $\Phi(z)$ is close to the boundary of B.

For an arbitrary sequence $(z_i)_{i\in\mathbb{N}} \in \Omega$ converging to $b\Omega$, we must show that $\Phi(z_i)$ tends to the boundary of B. For this, fix a positive real number δ and an integer n_0 such that $d(\Phi(O_n\backslash O_{n-1}), bB) \leqslant \delta$ for all $n \geqslant n_0$, meaning that $\Phi(\Omega\backslash O_{n_0})$ is δ-close from bB. Split then (z_i) into two subsequences, one containing all the elements which belong to O_{n_0}, the other one those which escape from O_{n_0} :

$$\begin{aligned}(z_i^1) &:= \{z_{n_i} \in \{z_n\} \mid z_{n_i} \notin O_{n_0}\},\\ (z_i^2) &:= \{z_{n_i} \in \{z_n\} \mid z_{n_i} \in O_{n_0}\}.\end{aligned}$$

By construction, $d(\Phi(z_i^1), bB) \leqslant \delta$. Since $f^{n_0}(z_i^2) \subset O_0$ and since f^{n_0}, $\Phi_{|O_0}$ and g are proper maps, $\Phi(z_i^2) = g^{-n_0} \circ \Phi_{|O_0} \circ f^{n_0}(z_i^2)$ is also δ-close to bB for i large enough. $\square$

Finally, we need to show that Φ is a biholomorphism. It is not yet clear since there exist holomorphic coverings of the ball. Anyway we know that any proper map to a bounded domain has a finite degree (see [13], chap. 15). In particular, there is an integer d which bounds the numbers of preimages of Φ :

$$\#\Phi^{-1}(z) \leqslant d, \qquad\qquad \forall z \in B.$$

Notice now that the degree of Φ bounds this of f^n for all n because $\Phi = g^{-n} \circ \Phi \circ f^n$. The degree of f^n is thus bounded on one hand and equal to $(\deg f)^n$ on the other. So f is an automorphism of Ω. The injectivity of Φ is now immediate since $\Phi_{|O_i} = g^{-i} \circ \Phi_{|O_0} \circ f^i$ is a composition of injective maps for all fixed i. $\square$

3. Local sphericity near the attractive point

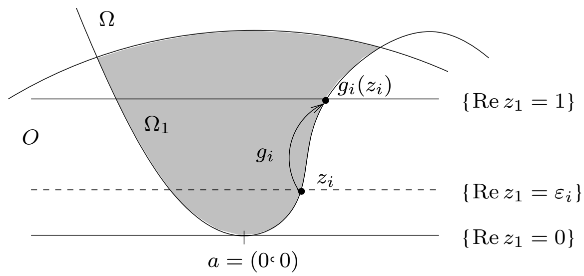
In this last section, we prove lemma 2.1, namely that a neighbourhood of a in $b\Omega$ is spherical. We recall that all the results proved in the previous section used this fact, so we have to go back to the general situation of theorem 1. Nevertheless, remind that we can speak of the action of f on $b\Omega$, at least close to a, thanks to lemma 1.2. The idea behind this technical part of the proof is based on previous results concerning behaviours of sequences of CR-maps (see [9, 8]). Unformally speaking, they explain that non-equicontinuous sequences of CR-maps on strictly pseudoconvex hypersurfaces dilate a certain (anisotropic) distance. The proof of the sphericity then goes as follows. Either f^{n_k} converges to a on $\mathcal{SPC}(b\Omega)$ and theorem 2 gives the sphericity. Or f^{n_k} is not equicontinuous on $\mathcal{SPC}(b\Omega)$ and it is dilating. Then the inverse branches of f^{n_k} are contracting CR-diffeomorphisms and theorem 2 gives the sphericity. Let us first fix the easy situation where f^{n_k} converges to a on $\mathcal{SPC}(b\Omega)$.

Proposition 3.1. — *Assume f^{n_k} converges locally uniformly to a on a neighbourhood of a in $b\Omega$. Then $b\Omega$ is spherical near a.*

Proof. — Theorem 2 explains that it is enough to find a contracting sequence of CR-automorphisms on a neighbourhood of a. We are assuming here that (f^{n_k}) is a sequence of contracting CR-maps on a piece of $\mathcal{SPC}(b\Omega)$. Also, corollary 1.2 shows that f is a local diffeomorphism at a. We thus only need to prove that there is a fixed neighbourhood of a on which all f^{n_k} are injective. To see this, first assume that f^n, and not only f^{n_k}, converges to a. Fix then a neighbourhood U of a on which f is injective. Since f^n converges to a on U, $f^n(U) \subset U$ for all large enough integers $n \geqslant n_0$. Consider now a neighbourhood U' of a in U whose images $U', f(U'), \dots, f^{n_0}(U')$ are all contained in U. Such a set exists because f is continuous and a is a fixed point of f. By construction $f^n(U') \subset U$ for all $n \in \mathbb{N}$, and the restriction of f^n to U' is injective as a composition of injective maps.

In the general setting, let us first check that in fact, the convergence of the subsequence f^{n_k} to a implies the convergence of the whole dynamics of an iterate $h = f^p$ to a. Pick again a small neighbourhood U of a in $\mathcal{SPC}(b\Omega)$ and an integer $p = n_{k_0}$ such that $f^p(U) \subset U$. The map $h := f^p$ restricts to U to a local diffeomorphism from U to itself, whose sequence of images $h^n(U)$ is obviously decreasing (*i.e.* $h^i(U) \supset h^{i+1}(U)$). Observe then that the subsequence $(h^{n'_k})$ defined by $n'_k := E(n_k/p) + 1$ converges uniformly to a on U. Indeed, $h^{n'_k} = f^{pn'_k} = f^{n_k+i}$ with $i < p$, so $h^{n'_k}(U) \subset \cup_{i \leqslant p} f^i(f^{n_k}(U))$. Since $f^{n_k}(U)$ is close to a by hypothesis (for k large enough) and a is a fixed point of f, the continuity of f implies that $h^{n'_k}(U)$ is also close to a. Since the sequence $h^n(U)$ decreases, it thus converges to a. Replacing f by h, we can therefore apply the above argument, so a neighbourhood of a is indeed spherical. □

Consider now the situation when f^{n_k} does not converge to a on a neighbourhood of a. Let us first describe the figure and notation. As in the previous section, we assume that Ω is strongly convex in a neighbourhood O of a, that a is the origin and that $\Omega \cap O$ is contained in $\{\operatorname{Re} z_1 \geqslant 0\}$. We put $\Omega_\varepsilon := \Omega \cap O \cap \{\operatorname{Re} z_1 \leqslant \varepsilon\}$, $U_\varepsilon := b\Omega \cap O \cap \{\operatorname{Re} z_1 \leqslant \varepsilon\}$ and we assume without loss of generality that $\Omega_1 \Subset O$. Also since all the arguments to come are purely local and occur in O, we will consider in the sequel that f extends smoothly to the boundary (lemma 1.2), without explicitly mentionning any further the necessary restriction of f to O. The non-convergence of f^{n_k} means the existence of a sequence of points $z_i \in b\Omega$ tending to a, and integers k_i such that the points $f^{n_{k_i}}(z_i)$ lay out of a fixed neighbourhood of a, say U_1. Since a is fixed by $f^{n_{k_i}}$, we can even assume that $f^{n_{k_i}}(z_i) \in bU_1 = b\Omega \cap O \cap \{\operatorname{Re} z_1 = 1\}$ by moving z_i closer to a. Finally, put $f_i := f^{n_{k_i}}$ and define ε_i by $z_i \in \{\operatorname{Re} z_1 = \varepsilon_i\}$.



The main point of this section is that f^{n_k} has a strong expanding behaviour.

PROPOSITION 3.2 *(see also [8]). — For all ε there exists an integer $k = k(\varepsilon)$ such that $f_k(U_\varepsilon) \supset U_1 \backslash U_\varepsilon$.*

The sphericity near a is a direct consequence of this proposition :

COROLLARY 3.3. — *If (f^{n_k}) does not converge to a in a neighbourhood of a then $b\Omega$ is spherical near a.*

Proof. — Fix an open contractible set V compact in $U_1 \backslash \{a\}$. For ε small enough, $V \subset U_1 \backslash U_\varepsilon$ and there is an integer k_ε such that $f_{k_\varepsilon}(U_\varepsilon) \supset V$. Moreover, there are no critical value of $f_{k_\varepsilon |U_\varepsilon}$ inside V because both U_ε and V are strictly pseudoconvex (see [5]). Since V is simply connected, there exists an inverse branch of $f_{k_\varepsilon |U_\varepsilon}$ on V, which means a CR-diffeomorphism $h_\varepsilon : V \longrightarrow U_\varepsilon$ with $f_{k_\varepsilon} \circ h_\varepsilon =$Id. The sequence h_ε is therefore contracting on V, and theorem 2 implies that V is spherical. We have thus proved the local sphericity of $U_1 \backslash \{a\}$, which even proves the sphericity of U_1 because a is a strictly pseudoconvex point. Indeed, Chern-Moser's theory expresses the sphericity of an open strictly pseudoconvex hypersurface by the vanishing of a continuous invariant tensor. Since this tensor vanishes on $U_1 \backslash \{a\}$, it also vanishes on the whole of U_1 so U_1 itself is spherical. In the spirit of [8], It would be pleasant to get a more down-to-earth argument for this last point. □

The proof of proposition 3.2 relies on the following lemma.

LEMMA 3.4. — *For all ε there exists a diverging sequence $c_i \longrightarrow +\infty$ such that for all $p \in U_1$ with $f_i(p) \notin U_\varepsilon$ we have :*

$$\|f_i'(p)u\| \geqslant c_i \|u\| \qquad \forall u \in T_p^{\mathbb{C}} b\Omega.$$

Proof. — The idea is that Hopf's lemma gives estimates on the normal derivative of f_i, which transfer automatically to complex tangential estimates in strictly pseudoconvex geometry. For $p \in U_1$, let $\vec{N}(p)$ be the unit vector normal to $b\Omega$ pointing inside Ω and

$$B^+_\delta(p) := B(p + \delta\vec{N}(p), \delta) \cap \{\langle \vec{N}(p), \cdot\rangle \geqslant \delta\}.$$

When δ is small enough but fixed, $B^+_\delta(p)$ is in Ω and its image by f_i for i large is in $\Omega_{\frac{\varepsilon}{2}}$ (f_i converges to a inside Ω). Thus if $f_i(p) \notin U_\varepsilon$, the non-positive *p.s.h* function $\varphi := -\langle \vec{N}(f_i(p)), f_i(\cdot) - f_i(p)\rangle$ vanishes at p while it is less than $-c\varepsilon^2$ on $B^+_\varepsilon(p)$ (c is a constant depending only on the curvature of $b\Omega$ at a). Hopf's lemma then asserts that

$$n_i(p) := \langle f_i'(p)\vec{N}(p), \vec{N}(f_i(p))\rangle = \|\vec{\nabla}\varphi(p)\| \geqslant \frac{c'\varepsilon^2}{\delta}.$$

Since δ was arbitrary, we could take it much smaller than ε^2, so that the radial escape rate $n_i(p)$ is large. To transfer this radial estimate on the derivatives of f_i to complex tangential ones, consider the Levi form $\mathcal{L}$ of $b\Omega$ defined by

$$\mathcal{L}(p, u) := \langle [u, iu], i\vec{N}(p)\rangle, \qquad u \in T^{\mathbb{C}}_p b\Omega,$$

where u stands for the vector in $T^{\mathbb{C}}_p b\Omega$ as well as any extension of it to a vector field of $T^{\mathbb{C}} b\Omega$. The smoothness and strict pseudoconvexity of U_1 implies the existence of geometric constants c_1, c_2 such that

$$c_1\|u\|^2 \leqslant \mathcal{L}(p, u) \leqslant c_2\|u\|^2 \qquad \forall p \in U_1,\ \forall u \in T^{\mathbb{C}}_p b\Omega.$$

Easy computations show that :

$$c_2\|f_i'(p)u\|^2 \geqslant \mathcal{L}(f_i(p), f_i'(p)u) = n_i(p)\mathcal{L}(p, u) \geqslant c_1 n_i(p)\|u\|^2.$$

Since $n_i(p)$ is large when i is, this serie of inequalities implies lemma 3.4. □

The previous lemma asserts that f_i dilates the complex tangential directions of $b\Omega$ if $f_i(p)$ is not close to a. The last observation we need to make in order to prove proposition 3.2 is that this "complex tangential dilation" property implies a genuine dilation.

A path γ in $b\Omega$ will be called a complex path if $\dot\gamma(t) \in T^{\mathbb{C}}_{\gamma(t)} b\Omega$ for all t. Its euclidean length will be denoted by $\ell(\gamma)$. For $x, y \in U_1$, define the CR-distance $d^{\mathrm{CR}}(x, y)$ between x and y as the infimum of the lengths of complex paths joining x to y. The point is that the strict pseudoconvexity condition means that the complex tangential distribution is contact so complex paths can join any two points. Even more, the open set $U_1\backslash U_\varepsilon$ is d^{CR}-bounded (see theorem 4 of [7], or [9]).

Proof of proposition 3.2. — Fix $\tau > 0$ such that $B^{\mathrm{CR}}(z_i,\tau) \subset U_\varepsilon$ for all i large enough. Since $U_1\backslash U_\varepsilon$ is d^{CR}-bounded, it is enough to prove that

$$bf_i(B^{\mathrm{CR}}(z_i,\tau)) \cap B^{\mathrm{CR}}(f_i(z_i), c_i\tau) \cap \big(U_1\backslash U_\varepsilon\big) = \emptyset$$

because $c_i\tau$ can be made greater than the CR-diameter of $U_1\backslash U_\varepsilon$. Take a point $x \in bf_i(B^{\mathrm{CR}}(z_i,\tau)) \cap U_1\backslash U_\varepsilon$ and let us prove that

$$d^{\mathrm{CR}}(f_i(z_i), x) \geqslant c_i\tau. \tag{3.1}$$

Consider an arc-length parameterized complex path γ in $U_1\backslash U_\varepsilon$ joining $f_i(z_i)$ to x. Since f_i is a local CR-diffeomorphism at each point of $B^{\mathrm{CR}}(z_i,\tau)$ whose image lies in the strictly pseudoconvex part of $b\Omega$, the connected component of $f_i(z_i)$ in $\gamma \cap f_i(B^{\mathrm{CR}}(z_i,\tau))$ can be lifted to a complex path $\widetilde{\gamma}$ through f_i. Thus there exists $l \leqslant \ell(\gamma)$ and $\widetilde{\gamma} : [0,l] \longrightarrow B^{\mathrm{CR}}(z_i,\tau)$ joining z_i to $bB^{\mathrm{CR}}(z_i,\tau)$ such that $f_i \circ \widetilde{\gamma}(t) = \gamma(t)$ for all $t \in [0,l]$. Since $\widetilde{\gamma}(t) \in U_1$ and $f_i(\widetilde{\gamma}(t)) \in U_1\backslash U_\varepsilon$ for all t, the estimates obtained in lemma 3.4 yield :

$$\ell(\gamma) \geqslant l = \int_0^l \|\dot{\gamma}(t)\|dt = \int_0^l \|f_i'(\widetilde{\gamma}(t))\dot{\widetilde{\gamma}}(t)\|dt \geqslant c_i \int_0^l \|\dot{\widetilde{\gamma}}(t)\|dt \geqslant c_i\ell(\widetilde{\gamma}).$$

This proves (3.1) since $\widetilde{\gamma}$ joins z_i to $bB^{\mathrm{CR}}(z_i,\tau)$ (so $\ell(\widetilde{\gamma}) \geqslant \tau$) and γ is any complex path joining $f_i(z_i)$ to x. $\square$

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