ANNALES
DE LA FACULTÉ DES SCIENCES

DE TOULOUSE

Mathématiques

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Codimension one foliations on complex tori

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Abstract. — We prove a structure theorem for codimension one singular foliations on complex tori, from which we deduce some dynamical consequences.

Résumé. — On démontre un théorème de structure pour les feuilletages singuliers de codimension 1 sur les tores complexes, et on en déduit des conséquences dynamiques.

1. Introduction

Following a suggestion of Ohsawa [Ohs], we will study in this paper some dynamical properties of holomorphic (and possibly singular) foliations of codimension one on complex tori.

First of all, we shall prove a structure theorem for such foliations, in respect of their normal bundle:

Theorem 1.1. — *Let $X = \mathbb{C}^n/\Gamma$ be a complex torus and let $\mathcal{F}$ be a codimension one foliation on X. Then one and only one of the following possibilities occurs:*

(1) $\mathcal{F}$ is a linear foliation;

(2) $\mathcal{F}$ is a turbulent foliation;

(3) there exists a complex torus $Y = \mathbb{C}^m/\Gamma'$ ($m \leqslant n$), a codimension one foliation $\mathcal{G}$ on Y, a linear projection $\pi : X \to Y$, such that:

(3.1) $\mathcal{F} = \pi^*(\mathcal{G})$;

(3.2) the normal bundle $N_\mathcal{G}$ of $\mathcal{G}$ is ample.

A foliation on a torus is **linear** if its lifting to the universal covering $\mathbb{C}^n$ is a foliation by parallel hyperplanes. The dynamics of a linear foliation is well understood; in particular, either every leaf is dense in the ambient torus, or every leaf is dense in a real codimension one subtorus, or every leaf is compact (a complex codimension one subtorus). **Turbulent** foliations will be defined and studied below. They are possibly singular generalizations of a class of nonsingular foliations introduced by Ghys in [Ghy]. The dynamics of turbulent foliations is also rather well understood. Thus, the meaning of Theorem 1.1 is that, up to some well understood classes, the study of codimension one foliations on complex tori is reduced to the case of foliations with *ample* normal bundle.

Concerning this last case, we will prove:

THEOREM 1.2. — *Let X be a complex torus of dimension $n \geqslant 3$ and let $\mathcal{F}$ be a codimension one foliation on X with ample normal bundle. Then every leaf of $\mathcal{F}$ accumulates to (part of) the singular set $Sing(\mathcal{F})$.*

This answers to a conjecture of [Bru], in the special case of complex tori. In the case of projective spaces ($X = \mathbb{C}P^n$, $n \geqslant 3$) the analogous result has been proved by Lins Neto in [Lin] (for a foliation on a projective space the normal bundle is always ample). A related result, for foliations on tori satisfying some topological assumptions, can also be found in [Ohs].

The main step in the proof of Theorem 1.2 is the construction of a *strictly* plurisubharmonic exhaustion of the complement of $\overline{L}$, where L is a leaf of $\mathcal{F}$ which, by contradiction, does not accumulate to $Sing(\mathcal{F})$. As in [Bru], such a function will be constructed by exploiting a positive curvature metric on $N_\mathcal{F}$. This approach should be compared with [Ohs], where the author constructs a (non strictly) plurisubharmonic exhaustion by exploiting flat metrics on the torus.

Along the proof of Theorem 1.2, we shall obtain some "new" examples of Stein domains inside some compact complex surfaces. This will be explained in the last part of the paper.

2. General structure of foliations on complex tori

2.1. Normal reduction

Let $X = \mathbb{C}^n/\Gamma$ be a complex torus and let $\mathcal{F}$ be a (possibly singular) codimension one foliation on X.

The main basic fact that we shall use below is that the normal bundle $N_{\mathcal{F}}$ of $\mathcal{F}$ is *effective*. Indeed, we may take a global holomorphic vector field v on X, not everywhere tangent to $\mathcal{F}$, and project it to $N_{\mathcal{F}}$ under the natural map $TX \to N_{\mathcal{F}}$. The result is a global holomorphic section $s_v \in H^0(X, N_{\mathcal{F}})$, vanishing exactly on the hypersurface $\Sigma_v \subset X$ along which v is not transverse to $\mathcal{F}$.

More explicitly, if the foliation $\mathcal{F}$ is locally defined by holomorphic 1-forms $\omega_j \in \Omega^1(U_j)$ (with zero set of codimension at least two, as usual), then the section s_v is locally defined by the functions $f_j = i_v\omega_j \in \mathcal{O}(U_j)$: we have $\omega_j = g_{jk}\omega_k$ on $U_j \cap U_k$, where $\{g_{jk}\}$ is (by definition) the multiplicative cocycle defining $N_{\mathcal{F}}$, hence $f_j = g_{jk}f_k$ and therefore $\{f_j\}$ defines a global section of $N_{\mathcal{F}}$.

According to standard results about line bundles on complex tori [Deb, Théorème VI.5.1], from the effectivity of $N_{\mathcal{F}}$ we deduce that there exists a complex torus $Y = \mathbb{C}^m/\Gamma'$, a linear projection $\pi : X \to Y$ and an *ample* line bundle L on Y such that

$$N_{\mathcal{F}} = \pi^*(L).$$

This map π, which is uniquely and canonically defined by $\mathcal{F}$, will be called **normal reduction** of $\mathcal{F}$.

It may happen that $m = 0$, i.e. Y is a point. This occurs if and only if $N_{\mathcal{F}}$ is trivial, that is $\mathcal{F}$ is globally defined by a holomorphic 1-form on X and, therefore, $\mathcal{F}$ is a linear foliation. It may also happen that $m = n$, which means that $N_{\mathcal{F}}$ itself is ample. From now on we shall suppose $1 \leqslant m \leqslant n-1$.

Denote by F_y, $y \in Y$, the fibers of π, which are complex subtori of X of dimension $\ell = n - m$. The fibration π is locally trivial: all the fibers F_y are isomorphic to the same complex torus F, and given $y \in Y$ there exists a neighbourhood $U \subset Y$ of y such that

$$V = \pi^{-1}(U) \simeq U \times F.$$

Let us study the foliation on such a neighbourhood V.

If U is sufficiently small, then $L|_U$ is trivial, and so $N_{\mathcal{F}}|_V$ is trivial too. This means that $\mathcal{F}|_V$ is defined by a holomorphic 1-form $\omega \in \Omega^1(V)$, with zero set of codimension two or more. Denote by $\{dw_1, \ldots, dw_\ell\}$ a basis of $\Omega^1(F)$, and by $\{z_1, \ldots, z_m\}$ coordinates on U. Then the 1-forms $\{dz_j, dw_k\}$, lifted to $V = U \times F$, form a basis of the $\mathcal{O}(V)$-module $\Omega^1(V)$, and therefore ω can be written as a linear combination of those forms with coefficients in $\mathcal{O}(V)$. Taking into account that $\mathcal{O}(V) = \pi^*(\mathcal{O}(U))$, we obtain

$$\omega = \pi^*(\omega_0) + \sum_{j=1}^{\ell} a_j(z)dw_j$$

where $\omega_0 \in \Omega^1(U)$ and $a_j \in \mathcal{O}(U)$ for every j.

From this local (over Y) expression, we deduce:

LEMMA 2.1. — *If the fibers of the normal reduction $\pi : X \to Y$ are all $\mathcal{F}$-invariant, then there exists a foliation $\mathcal{G}$ on Y such that $\mathcal{F} = \pi^*(\mathcal{G})$.*

Indeed, the hypothesis of this lemma means that the above functions $\{a_j\}$ are all identically zero, hence $\omega = \pi^*(\omega_0)$ and $\mathcal{G}$ is the foliation on Y defined by ω_0 on $U \subset Y$. We also have $N_{\mathcal{G}} = L$, hence $N_{\mathcal{G}}$ is ample and we are in case (3) of Theorem 1.1.

2.2. Turbulent foliations

We now analyse the case in which the fibers of the normal reduction are not all $\mathcal{F}$-invariant. We start with a definition, close to [Ghy].

DEFINITION 2.2. — *Let X be a complex torus and let $\mathcal{F}$ be a codimension one foliation on X with normal reduction $\pi : X \to Y$, $0 < \dim Y < \dim X$. Then $\mathcal{F}$ is said to be a **turbulent foliation** if it can be defined by a meromorphic 1-form η of the type*

$$\eta = \pi^*(\eta_0) + \eta_1$$

where:

    (i) η_0 is a closed meromorphic 1-form on Y;

    (ii) η_1 is a holomorphic 1-form on X, not vanishing on the fibers of π.

Remark that such a 1-form η is not only integrable, but even *closed*. The decomposition $\eta = \pi^*(\eta_0) + \eta_1$ is not unique: we may replace η_0 with

$\eta_0 + \gamma$, where γ is any holomorphic 1-form on Y, and η_1 with $\eta_1 - \pi^*(\gamma)$. The nonvanishing condition in (ii) is required in order to exclude the case $\mathcal{F} = \pi^*(\mathcal{G})$, already met before. Note that $N_{\mathcal{F}}$ is represented by the divisor $(\eta)_\infty - (\eta)_0$, which is equal to $\pi^*((\eta_0)_\infty)$, and so L is represented by the polar divisor of η_0. In particular, this polar divisor is nontrivial, because L is ample (and $\dim Y > 0$).

The dynamics of turbulent foliations will be described below, let us firstly conclude the proof of Theorem 1.1 by proving that, if the fibers of the normal reduction are not all $\mathcal{F}$-invariant, then $\mathcal{F}$ is a turbulent foliation.

We can firstly choose a global holomorphic vector field v on X, tangent to the fibers of π but not everywhere tangent to $\mathcal{F}$. Then we take the meromorphic 1-form η on X uniquely defined by the conditions

$$\eta|_{\mathcal{F}} \equiv 0 \qquad \text{and} \qquad i_v\eta \equiv 1.$$

This is a 1-form which defines $\mathcal{F}$, and we claim that it admits a decomposition as in Definition 2.2.

To see this, let us look again at the neighbourhood $V = U \times F$, where $\mathcal{F}$ is defined also by the holomorphic 1-form $\omega = \pi^*(\omega_0) + \sum_{j=1}^{\ell} a_j dw_j$. The 1-forms $\{dw_j\}$ appearing here are, in fact, restriction to V of global holomorphic 1-forms on X, still denoted by $\{dw_j\}$. The functions $i_v dw_j$ (on X) are therefore constant, and up to a change of basis we may assume that

$$i_v dw_1 = 1 \qquad \text{and} \qquad i_v dw_j = 0 \text{ for } j \geqslant 2.$$

The meromorphic 1-form η, on V, is necessarily proportional to ω: $\eta = f\omega$ for some meromorphic function $f \in \mathcal{M}(V)$. From $i_v\eta = 1$ we then obtain $f = 1/a_1$. Thus, setting $\beta_0 = \frac{1}{a_1}\omega_0$ and $b_j = \frac{a_j}{a_1}$, we get

$$\eta|_V = \pi^*(\beta_0) + \sum_{j=1}^{\ell} b_j(z) dw_j$$

with b_1 identically equal to 1.

LEMMA 2.3. — *The 1-form β_0 is closed and the functions $\{b_j\}$ are constant.*

Proof. — This is a consequence of the integrability condition $\eta \wedge d\eta \equiv 0$, which develops into three sets of conditions:

(1) $\beta_0 \wedge d\beta_0 = 0$

(2) $\pi^*(\beta_0) \wedge db_j + b_j \pi^*(d\beta_0) = 0$ for every j

(3) $b_j db_k - b_k db_j = 0$ for every j, k.

Because $b_1 \equiv 1$, condition (3) with $k = 1$ gives the constancy of b_j for every j, and condition (2) gives the closedness of β_0.   □

   Set

$$\eta_1 = \sum_{j=1}^{\ell} b_j dw_j,$$

which is now, thanks to the previous lemma, a holomorphic 1-form globally defined on X. The difference $\eta - \eta_1$, restricted to V, is equal to $\pi^*(\beta_0)$. By connectedness, the same holds on the full X: there exists a meromorphic 1-form η_0 on Y, which extends β_0, such that $\eta - \eta_1 = \pi^*(\eta_0)$. The 1-form η_0 is closed, because β_0 is. Hence $\mathcal{F}$ is a turbulent foliation.

2.3. Dynamics of turbulent foliations

   Let $\mathcal{F}$ be a turbulent foliation, with normal reduction $\pi : X \to Y$ ($\dim X = n$, $\dim Y = m > 0$, $\ell = n - m > 0$), defined by $\eta = \pi^*(\eta_0) + \eta_1$ as in Definition 2.2. As already observed, this decomposition of η (and η itself) is not uniquely determined by $\mathcal{F}$. However, the polar divisor of η_0,

$$D = (\eta_0)_\infty,$$

is an object intrinsically associated to $\mathcal{F}$: a point $y \in Y$ belongs to D if and only if the fiber F_y is not everywhere transverse to $\mathcal{F}$. Moreover, if $D = \sum_{j=1}^{r} k_j D_j$ is the decomposition into irreducible components, and

$$E_j = \pi^{-1}(D_j),$$

then E_j is an hypersurface invariant by $\mathcal{F}$, along which $\mathcal{F}$ and π are tangent at order k_j.

   The case $m = 1$ is somewhat special [Ghy]: the foliation $\mathcal{F}$ has no singularity, each D_j is a point, each E_j is a compact leaf of $\mathcal{F}$, whereas all the other leaves are noncompact and accumulate to every E_j.

   On the other hand, if $m \geqslant 2$ then $\mathcal{F}$ is certainly singular, and more precisely every E_j must contain some singularity of $\mathcal{F}$ (recall that $D \neq \emptyset$, for $L = \mathcal{O}(D)$ is ample). Indeed, in the opposite case where $E_j \cap Sing(\mathcal{F}) = \emptyset$, the normal bundle $N_{\mathcal{F}}$ would be flat on E_j (by Bott's vanishing principle, see e.g. [Suw, Theorem VI.6.4]), hence $L = \pi_*(N_{\mathcal{F}})$ would be flat on D_j, contradicting its ampleness (if $m \geqslant 2$). Also when $m \geqslant 2$ all the leaves

outside $E = \cup_j E_j$ accumulate to every E_j. In particular, every leaf of $\mathcal{F}$ accumulates to some singular point of the foliation.

On $X \setminus E$ the dynamics of $\mathcal{F}$ can be visualized as follows: there is a "vertical" dynamics, which is given by the periods of η_1 along the fibers of π, and a "horizontal" dynamics, which depends also on the periods of η_0. More precisely, we have on X a natural real fibration by real subtori $h : X \to M$: a fiber of h is a real subtorus obtained by taking the *closure* of a leaf of $(\ker \eta_1) \cap (F_y)$, $y \in Y$. The normal reduction factorizes through h, i.e. $\pi = g \circ h$ with $g : M \to Y$ a real fibration. The space M is a real torus, and the fibers of g are either points (i.e. $M = Y$, i.e. η_1 defines a linear foliation with dense leaves on every fiber of π), or circles (i.e. η_1 defines on every fiber a foliation whose leaves are dense in real codimension one tori), or real 2-tori (i.e. η_1 defines on every fiber a foliation with compact leaves).

Set $M_0 = g^{-1}(Y \setminus D) = h(X \setminus E)$. The foliation $\mathcal{F}|_{X \setminus E}$ "projects" to a real foliation $\mathcal{L}$ on M_0, whose leaves are transverse to the fibers of g and of complementary dimension. The dynamics of $\mathcal{L}$ is described by its monodromy representation $\pi_1(Y \setminus D) \to \mathcal{T}rsl$, where $\mathcal{T}rsl$ is the group of translations of a fiber of M_0. This monodromy representation, which is the horizontal dynamics of $\mathcal{F}$, can be computed from the periods of η_0 and those of η_1 in the directions transverse to the fibers of π.

3. Foliations with ample normal bundle

3.1. A general convexity result

In order to prove Theorem 1.2, we start with a general result, motivated by [Bru] and valid on arbitrary complex manifolds.

PROPOSITION 3.1. — *Let X be a complex manifold and let $\mathcal{F}$ be a codimension one foliation on X. Let $\mathcal{M} \subset X$ be a compact $\mathcal{F}$-invariant subset, disjoint from $Sing(\mathcal{F})$, and suppose that the normal bundle $N_\mathcal{F}$ admits an hermitian metric with positive curvature on a neighbourhood of $\mathcal{M}$. Then there exists a smooth function $\Phi : X \setminus \mathcal{M} \to \mathbb{R}$ such that:*

(1) $\Phi(p) \to +\infty$ as $p \to \mathcal{M}$;

(2) Φ is strictly plurisubharmonic on a neighbourhood of $\mathcal{M}$.

We can resume the conclusion of this proposition by saying that the end (or the ends) of $X \setminus \mathcal{M}$ converging to $\mathcal{M}$ is (or are) strongly pseudoconvex [Pet].

Let us fix some notation. We cover a neighbourhood U of $\mathcal{M}$ with foliated charts $\{U_j\}_{j=1}^{\ell}$. On each U_j, the foliation $\mathcal{F}$ is defined by a holomorphic submersion

$$f_j : U_j \to V_j \subset \mathbb{C}.$$

The differential $df_j \in \Omega^1(U_j)$ is therefore a nowhere vanishing section of the conormal bundle $N_{\mathcal{F}}^*$ over U_j. We denote by $\|df_j\|$ its norm with respect to the metric on $N_{\mathcal{F}}^*$ which is dual to the metric on $N_{\mathcal{F}}$ appearing in the statement of Proposition 3.1. Thus, the (1,1)-form on U

$$\Theta = i\partial\bar{\partial}\log\|df_j\|$$

is a *positive* form, up to shrinking U (modulo a positive constant factor, it is just the curvature of $N_{\mathcal{F}}$).

Set $\mathcal{M}_j = \mathcal{M} \cap U_j$ and $K_j = f_j(\mathcal{M}_j) \subset V_j$, and denote by

$$\delta_j : V_j \setminus K_j \to \mathbb{R}$$

the euclidean distance from K_j :

$$\delta_j(z) = \inf_{w \in K_j} |z - w| \ ,$$

where $|\cdot|$ denotes the standard norm on $V_j \subset \mathbb{C}$. Remark that $-\log\delta_j$ is a continuous subharmonic function on $V_j \setminus K_j$, being the supremum of harmonic functions, and that $-\log\delta_j(z)$ tends to $+\infty$ as $z \to K_j$.

Define, for every j,

$$h_j : U_j \setminus \mathcal{M}_j \to \mathbb{R}$$

$$h_j(p) = \log\frac{\|df_j(p)\|}{\delta_j(f_j(p))} \ .$$

By the above properties of f_j and δ_j, we immediately see that:

(i) h_j is continuous, and $h_j(p) \to +\infty$ as $p \to \mathcal{M}_j$;

(ii) $i\partial\bar{\partial}h_j \geqslant \Theta$ (in the sense of currents).

Consider now the difference $h_j - h_k$ on two overlapping charts U_j and U_k.

LEMMA 3.2. — $h_j(p) - h_k(p) \to 0$ *as* $p \to \mathcal{M} \cap U_j \cap U_k$.

Proof. — We have $f_j = \varphi \circ f_k$ on $U_j \cap U_k$, for some holomorphic diffeomorphism

$$\varphi : f_k(U_j \cap U_k) \to f_j(U_j \cap U_k)$$

(we omit the jk-index), and therefore

$$\|df_j(p)\| = |\varphi'(f_k(p))| \cdot \|df_k(p)\|.$$

Remark, in particular, that the quotient $\|df_j\|/\|df_k\|$ is constant along the leaves, as well as the difference $h_j - h_k$, and so our problem is actually one dimensional.

Set $V = f_k(U_j \cap U_k)$ and $K = f_k(\mathcal{M} \cap U_j \cap U_k)$ (a closed subset of V).

For every $p \in U_j \cap U_k$ we have, by definition, $\delta_j(f_j(p)) = \inf_{w' \in K_j} |f_j(p) - w'| = \inf_{w' \in K_j} |\varphi(f_k(p)) - w'|$. But, on some neighbourhood of $\mathcal{M} \cap U_j \cap U_k$, this expression can be rewritten as

$$\delta_j(f_j(p)) = \inf_{w \in K} |\varphi(f_k(p)) - \varphi(w)|$$

(as soon as the infimum over K_j is realized by a point in $\varphi(K) \subset K_j$). Similarly, and still on some neighbourhood of $\mathcal{M} \cap U_j \cap U_k$, we have

$$\delta_k(f_k(p)) = \inf_{w \in K} |f_k(p) - w|$$

(as soon as the infimum over K_k is realized by a point in $K \subset K_k$).

Thus, the conclusion of the lemma is equivalent to the following one: the function $\lambda : V \setminus K \to \mathbb{R}$ defined by

$$\lambda(z) = |\varphi'(z)| \cdot \frac{\inf_{w \in K} |z - w|}{\inf_{w \in K} |\varphi(z) - \varphi(w)|}$$

tends to 1 as $z \to K$.

This claim can be checked as follows, by elementary calculus. We factorize $\varphi(z) - \varphi(w)$ as $\psi(z, w) \cdot (z - w)$, where ψ is a holomorphic function on $V \times V$ and $\psi(z, z) = \varphi'(z)$. If $z_n \to z_\infty \in K$, then $\inf_{w \in K} |z_n - w|$ is realized by some point $w_n \in K$, for n sufficiently large. Hence

$$\lambda(z_n) = \frac{|\varphi'(z_n)| \cdot |z_n - w_n|}{\inf_{w \in K} |\varphi(z_n) - \varphi(w)|} \geqslant \frac{|\varphi'(z_n)| \cdot |z_n - w_n|}{|\varphi(z_n) - \varphi(w_n)|} = \frac{|\varphi'(z_n)|}{|\psi(z_n, w_n)|} .$$

This last quantity tends to 1, because $w_n \to z_\infty$ (even if w_n is possibly not uniquely determined). In particular, we obtain that $\liminf_{n \to +\infty} \lambda(z_n) \geqslant 1$. Similarly, $\inf_{w \in K} |\varphi(z_n) - \varphi(w)|$ is realized by some $\tilde{w}_n \in K$, with $\tilde{w}_n \to z_\infty$, and we get $\limsup_{n \to +\infty} \lambda(z_n) \leqslant 1$. Hence $\lambda(z_n) \to 1$, as desired. $\square$

Using Lemma 3.2 and properties (i) and (ii) above, we can now construct, by a glueing procedure, the asymptotically strictly plurisubharmonic exhaustion $\Phi : X \setminus \mathcal{M} \to \mathbb{R}$. [1]

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[1] I learned the following glueing technique from J.-P. Rosay

Set $U_0 = X \setminus \mathcal{M}$ and let $\{g_j\}_{j=0}^{\ell}$ be a smooth partition of unity associated to $\{U_j\}_{j=0}^{\ell}$. Set $I = \{1, \ldots, \ell\}$. For every $j \in I$, we can choose a constant $\varepsilon_j > 0$ so that $\varepsilon_j i \partial \bar{\partial} g_j \geqslant -\frac{1}{2}\Theta$, hence the function

$$\tilde{h}_j = h_j + \varepsilon_j g_j$$

on $U_j \setminus \mathcal{M}_j$ still satisfies (i) and (ii), except that Θ is replaced with $\frac{1}{2}\Theta$. Take now

$$h : U \setminus \mathcal{M} \to \mathbb{R}$$

$$h(p) = \sup_{j \in I(p)} \tilde{h}_j(p)$$

where $I(p) = \{j \in I \mid p \in U_j \}$.

LEMMA 3.3. — *On a sufficiently small neighbourhood of $\mathcal{M}$, the function h is continuous and $i \partial \bar{\partial} h \geqslant \frac{1}{2}\Theta$.*

Proof. — Take $q \in \mathcal{M}$ and let us distinguish two cases (even if, formally, this is not indispensable).

(1). The set valued function $I(p)$ is locally constant around q. This means that there exists a neighbourhood U_q of q such that, for every $j \in I$, either $U_q \subset U_j$ or $U_q \cap U_j = \emptyset$. Denote by J the set of those $j \in I$ for which the former possibility occurs. Then the function $h|_{U_q \setminus \mathcal{M}}$ can be expressed as $\sup_{j \in J} \tilde{h}_j$, and now the functions appearing there are all defined on the *full* $U_q \setminus \mathcal{M}$. The continuity and the strict plurisubharmonicity of $h|_{U_q \setminus \mathcal{M}}$ are then a consequence of standard and easy facts.

(2). The set valued function $I(p)$ is not locally constant around q. This means that q belongs to the boundary of one or more charts. Fix $j_0 \in I$ such that $g_{j_0}(q) > 0$. Then we can find a neighbourhood $U_q \subset U_{j_0}$ of q such that, for every $j \in I$, either $U_q \subset U_j$ or $\varepsilon_j g_j < \varepsilon_{j_0} g_{j_0}$ on $U_q \cap U_j$. By Lemma 3.2, and up to restricting U_q, we may replace this inequality with $\tilde{h}_j < \tilde{h}_{j_0}$. This means that, as in case (1), the function $h|_{U_q \setminus \mathcal{M}}$ can be expressed as a supremum of functions *fully* defined on $U_q \setminus \mathcal{M}$, continuous and strictly plurisubharmonic, and we conclude as before. $\square$

Standard regularisation results (Richberg) allow now to approximate h with a smooth function $\Phi : U \setminus \mathcal{M} \to \mathbb{R}$, which is still exhaustive towards $\mathcal{M}$ and still strictly plurisubharmonic close to $\mathcal{M}$. This completes the proof of Proposition 3.1.

3.2. Ample foliations on tori

We return now to the case of a foliation on a torus $X = \mathbb{C}^n/\Gamma$, and we prove Theorem 1.2.

Suppose, by contradiction, that some leaf of $\mathcal{F}$ does not accumulate to $Sing(\mathcal{F})$. Taking closure, we get a compact subset $\mathcal{M} \subset X$ as in Proposition 3.1. Therefore, the open subset $X \setminus \mathcal{M}$ is strongly pseudoconvex.

According to classical results of Grauert and Remmert [Pet], $X \setminus \mathcal{M}$ is a point modification of a Stein space. However, a complex torus cannot contain an exceptional set, i.e. an analytic subset of positive dimension collapsible to a point (for instance, any analytic subset of a torus can be shifted away by using a holomorphic flow, and this clearly prevents its collapsibility). Thus, $X \setminus \mathcal{M}$ is itself a Stein space.

Now we can use the singular argument of [Lin]. By Baum-Bott formula [Suw, Theorem VI.3.7], the cohomology class

$$c_1^2(N_{\mathcal{F}}) \in H^4(X, \mathbb{R})$$

is represented by a cycle $\sum_{j=1}^{\alpha} \lambda_j S_j$, where $\{S_j\}$ are the codimension 2 irreducible components of $Sing(\mathcal{F})$ and $\{\lambda_j\}$ are complex numbers (Baum-Bott residues). Because $N_{\mathcal{F}}$ is ample, this class is not zero, and because $n = \dim X \geqslant 3$, we infer that $Sing(\mathcal{F})$ has positive dimension. Moreover, $Sing(\mathcal{F}) \cap \mathcal{M} = \emptyset$, and so $Sing(\mathcal{F}) \subset X \setminus \mathcal{M}$. But this contradicts the Steinness of $X \setminus \mathcal{M}$.

Remark 3.4. — Our construction of the plurisubharmonic exhaustion of $X \setminus \mathcal{M}$ can be compared with the construction of [Ohs] in the following way. Take, as in [Ohs], a flat metric on X. Because $N_{\mathcal{F}}$ is (outside the singularities) a quotient of TX, we get an induced metric on $N_{\mathcal{F}}$, and by a standard comparison principle such a metric as *semipositive* curvature. If, by chance, this curvature is *positive*, then we can apply our construction (Proposition 3.1), and we get an exhaustion which is not so far from Ohsawa's one (our $\exp(-h_j)$ is a sort of "transverse distance from $\mathcal{M}$", and if the metric on $N_{\mathcal{F}}$ arises from an ambient metric then this transverse distance is not very different from the ambient distance...). However, even if $N_{\mathcal{F}}$ is ample, it can happen that the above quotient metric has some flat points, corresponding to points where the leaf has a higher order tangency with an hyperplane. These flat points are source of difficulties also in [Ohs]. Whereas Ohsawa's strategy consists, in some sense, in getting rid of these flat points by working simultaneously with several flat metrics on X, our strategy is rather to perturb the metric on $N_{\mathcal{F}}$ to a positive curvature one. Of course, this needs the ampleness hypothesis, which is however furnished by Theorem 1.1.

Remark 3.5. — Theorem 1.2 holds, more generally, when X is a homogeneous manifold. The proof is the same as in the case of tori.

3.3. An example

Let us return to the general Proposition 3.1. When $n = \dim X \geqslant 3$, then we actually do not know any example satisfying the hypothesis of Proposition 3.1. They exist, instead, when $n = 2$, as we will now show (see also the end of [Bru]). It is also worth observing that if $n \geqslant 3$ then, by Rossi filling theorem, a neighbourhood of $\mathcal{M}$ as in the conclusion of Proposition 3.1 can be embedded into a compact complex manifold (of the same dimension), and then the foliation can be extended to such a compact manifold. Thus, possible examples in higher dimension should be investigated in a compact setting.

Let $C = \mathbb{D}/\Gamma$ be a compact, connected, complex curve, of genus $g \geqslant 2$, and let $\rho : \pi_1(C) \to Aut(\mathbb{P})$ be a representation of the fundamental group of C into the group of Möbius transformations of the projective line. Then $\pi_1(C)$ acts on $\mathbb{D} \times \mathbb{P}$ via the diagonal representation (Γ, ρ), and the quotient

$$X = (\mathbb{D} \times \mathbb{P})/(\Gamma, \rho)$$

is a compact complex surface, ruled over C. On X we have a (nonsingular) foliation $\mathcal{F}$, arising from the horizontal foliation on $\mathbb{D} \times \mathbb{P}$, and transverse to the ruling. If $K \subset \mathbb{P}$ is a compact subset invariant by ρ, then $\mathbb{D} \times K$ projects on X to a compact subset $\mathcal{M}$, invariant by $\mathcal{F}$. Because we are interested in the strong pseudoconvexity of $X \setminus \mathcal{M}$, we may assume, without loss of generality, that $\mathcal{M}$ is a proper subset of X with empty interior (otherwise we replace $\mathcal{M}$ with $\partial\mathcal{M}$).

We shall assume also that the following property holds:

(•) X contains an irreducible curve D with $D \cdot D < 0$.

This is related to the instability of the rank 2 vector bundle over C whose projectivization is X [Fri].

LEMMA 3.6. — *The normal bundle $N_{\mathcal{F}}$ can be decomposed, as a $\mathbb{Q}$-bundle, as $L \otimes \mathcal{O}(\ell D)$, where L is an ample $\mathbb{Q}$-bundle and ℓ is a positive rational number.*

Proof. — Fix a fiber F of the ruling $X \to C$. Then $N_{\mathcal{F}}$ is represented, as a $\mathbb{Q}$-bundle, by a divisor of the form $aF + bD$, for some rational numbers a, b,

because F and D generate $Pic(X) \otimes \mathbb{Q}$. From $c_1(N_{\mathcal{F}}) \cdot F = 2$ (for $N_{\mathcal{F}}|_F \simeq TF$ and $F \simeq \mathbb{P}$) and $c_1^2(N_{\mathcal{F}}) = 0$ (Bott's vanishing), we get $a = -(D \cdot D)/(F \cdot D)^2$ and $b = 2/(F \cdot D)$. Note that a and b are both positive. But any $\mathbb{Q}$-divisor on X of the type $aF + \varepsilon D$, $\varepsilon > 0$ sufficiently small, is ample (use, e.g., Nakai's criterion [Fri]). Hence $N_{\mathcal{F}}$ can be decomposed as $L \otimes \mathcal{O}(\ell D)$, with $L = \mathcal{O}(aF + \varepsilon D)$ ample and $\ell = b - \varepsilon \in \mathbb{Q}^+$. $\qquad \square$

Observe now that the curve D cannot be totally contained in $\mathcal{M}$: otherwise, by $\text{int}(\mathcal{M}) = \emptyset$, D would be a leaf of $\mathcal{F}$, but this is forbidden by $D \cdot D < 0$. Hence we may take a point $p \in D \cap (X \setminus \mathcal{M})$ and a neighbourhood $U \subset\subset X \setminus \mathcal{M}$ of p.

LEMMA 3.7. — *The normal bundle $N_{\mathcal{F}}$ admits an hermitian metric with positive curvature on $X \setminus \overline{U}$ (which is a neighbourhood of $\mathcal{M}$).*

Proof. — By the previous lemma, it is sufficient to prove that the line bundle $\mathcal{O}(D)$ admits an hermitian metric with semipositive curvature on $X_0 = X \setminus \overline{U}$.

We firstly put a *singular* metric on $\mathcal{O}(D)$ with curvature equal to $\delta_D =$ integration current on D (Poincaré-Lelong equation). By standard results, there exists a Stein neighbourhood $V \subset X$ of $D \setminus (D \cap U)$, and an equation $f \in \mathcal{O}(V)$ for $D' = D \cap V$; we may assume also that $|f| > 1$ on some neighbourhood of $\partial V \cap X_0$. Over V, the line bundle $\mathcal{O}(D)$ is trivial, and so the above singular metric is represented by a weight function ψ, which satisfies $i \partial \bar{\partial} \psi = \delta_{D'}$. Hence we can write $\psi = \psi_0 + \frac{1}{2\pi} \log |f|^2$, where ψ_0 is a pluriharmonic function on V.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi' \geqslant 0$, $\chi'' \geqslant 0$, $\chi(t) = t$ for $t \geqslant 0$, $\chi(t) = -1$ for $t \leqslant -2$. Then $\psi_\chi = \psi_0 + \frac{1}{2\pi} \chi(\log |f|^2)$ is smooth and plurisubharmonic on the full V, and equal to ψ around $\partial V \cap X_0$ (where $|f| > 1$). Thus, we can replace the weight ψ with ψ_χ, and obtain a well defined smooth metric on $\mathcal{O}(D)$ over X_0, coinciding with the previous one outside V. The curvature of this metric is semipositive. $\qquad \square$

By Lemma 3.7 and Proposition 3.1, we get:

PROPOSITION 3.8. — *Under assumption ($\bullet$), $X \setminus \mathcal{M}$ is strongly pseudoconvex.*

Examples which satisfy ($\bullet$) can be constructed as follows. We take a Kleinian group $\Lambda \subset Aut(\mathbb{P})$, i.e. a discrete subgroup which acts in a proper and freely discontinuous way on some nonempty open subset $\Omega \subset \mathbb{P}$ [Mas]. The quotient $C = \Omega/\Lambda$ is a complex curve, and there are many examples

in which C is even compact, connected, of genus $g \geqslant 2$ [Mas]. Then $X = (\Omega \times \mathbb{P})/(\Lambda, \Lambda)$ is a ruled surface as before, with a foliation $\mathcal{F}$ transverse to the ruling as before. Here we can take as $\mathcal{M}$ the quotient of $\Omega \times \partial\Omega$. Then X contains an irreducible curve D with $D \cdot D < 0$: the quotient of the diagonal $\Delta \subset \Omega \times \Omega \subset \Omega \times \mathbb{P}$. The negativity of the selfintersection comes from the fact that the tangent and the normal bundle of D are naturally isomorphic, and its genus is $\geqslant 2$. Remark that in this case we have not only $D \not\subset \mathcal{M}$, but even $D \cap \mathcal{M} = \emptyset$.

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