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Renormalized solution for nonlinear degenerate problems in the whole space

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Abstract. — We consider the general degenerate parabolic equation :

$$u_t - \Delta b(u) + div\ \tilde{F}(u) = f \qquad \text{in} \ \ Q \ =]0,T[\times \mathbb{R}^N, \ \ T > 0.$$

We suppose that the flux $\tilde{F}$ is continuous, b is nondecreasing continuous and both functions are not necessarily Lipschitz. We prove the existence of the renormalized solution of the associated Cauchy problem for L^1 initial data and source term. We establish the uniqueness of this type of solution under a structure condition $\tilde{F}(r) = F(b(r))$ and an assumption on the modulus of continuity of b. The novelty of this work is that $\Omega = \mathbb{R}^N$, u_0, $f \in L^1$, b, $\tilde{F}$ are not Lipschitz functions and the techniques are different from those developed in the previous works.

Résumé. — Nous considérons l'équation parabolique dégénérée général :

$$u_t - \Delta b(u) + div\ \tilde{F}(u) = f \qquad \text{dans} \ \ Q \ =]0,T[\times \mathbb{R}^N, \ \ T > 0.$$

Nous supposons que le flux $\tilde{F}$ est continu, b est continue et croissante au sens large et les deux fonctions ne sont pas nécessairement lipschitziennes. Nous prouvons l'existence de solution renormalisée du problème de Cauchy associé à cette équation avec des données (terme source et condition initiale) dans L^1. Nous établissons l'unicité de cette solution sous une condition dite de structure du type $\tilde{F}(r) = F(b(r))$ et sous une hypothèse sur le module de continuité de b. La nouveauté dans le travail vient du fait que $\Omega = \mathbb{R}^N$, u_0, $f \in L^1$, b, $\tilde{F}$ ne sont pas des fonctions nécessairement lipschitziennes et les techniques sont différentes de celles développées dans les travaux antérieurs.

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1. Introduction

Let $Q =]0, T[\times \mathbb{R}^N$ with $T > 0$.

We consider the Cauchy problem $(CP) = (CP)(b, F, f, u_0)$:

$$(CP) \qquad \begin{cases} u_t - \Delta b(u) + div\ F(b(u)) = f & \text{in } Q \\ u(0,.) = u_0 & \text{on } \mathbb{R}^N, \end{cases}$$

where F is a Lipschitz continuous function on $\mathbb{R}$ and b is a nondecreasing continuous function. For normalization, we set $F(0) = 0$ and $\mathrm{b}(0) = 0$.

$(CP)(b, F, f, u_0)$ is a model of degenerate second order diffusion-convection motions of fluid ; it has important applications in two phase flows in porous media (*cf.* $[CJ]$) and sedimentation-consolidation processes (*cf.* $[BCBT]$). It is well known that there is no classical solution and the weak solution is not unique. There exists a vast literature on the degenerate parabolic equation we consider. In this literature (cf. [AL], [BG], [BT1], [BW], [BT2], [DT], [YJ], [BM], [BR], [O], [BBGPV]), many results are proved about the existence of a weak solution and the uniqueness under various conditions.

In [MT3], we prove the existence and the uniqueness of the entropy solution of $(CP)(b, F, f, u_0)$ when u_0, f are bounded and $\Omega = \mathbb{R}^N$.

In [IW], authors prove that $(CP)(b, F, f, u_0)$ is well posed in the sense of a renormalized solution when Ω is a bounded domain and u_0, $f \in L^1$.

In [MK], the existence and uniqueness of the renormalized solution of (CP) are given when $\Omega = \mathbb{R}^N$, and $\tilde{F}$, b are Lipschitz functions.

Here we extend these previous works to the case of u_0, f in L^1, $\Omega = \mathbb{R}^N$ and b, $\tilde{F}$ not necessarily Lipschitz functions. More precisely, under the structure condition $\tilde{F}(r) = F(b(r))$ and the assumption (H_2) on the modulus of continuity of b, we prove the existence and uniqueness of the renormalized solution of $(CP)(b, F, f, u_0)$ in the whole space $\mathbb{R}^N$ with u_0, f in L^1. This condition on the modulus of continuity of b appears at the first time in [BK] for conservation laws where the authors give a counterexample to set the optimality of the condition.

Note that if $b(r) = r^m$, this condition is $m > \dfrac{N-1}{N}$.

The techniques, ideas and estimations developed in the present paper are based essentially on [C1], [C2], [IU1], [IU2], [IW], [MT3] and [M].

The present work falls into three sections. Section 1 is the introduction.

Section 2 is intended to prove the existence and uniqueness of the weak (entropy) solution when the data u_0 and f are bounded. Section 3, on the other hand, using the results of the previous section, aims at proving the existence and uniqueness of the renormalized solution of (CP) in the case of L^1 initial data and source term.

2. Existence and uniqueness of the weak solution

In this section we suppose that the initial data and source term satisfy the following hypothesis :

$$(H1)\qquad \begin{cases} 1)\ \ u_0 \in L^\infty(\mathbb{R}^N); \\ 2)\ \ f \in L^1_{Loc}(Q) \ \text{ and for } \ a.e \ \ t \in]0,T[, \quad f(t) \in L^\infty(\mathbb{R}^N); \\ 3)\ \ \displaystyle\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^N)} dt < \infty. \end{cases}$$

We define the operators H_ϵ, H, H_0 and the truncation function at the level k by :

$$H_\epsilon(s) = \min(\frac{s^+}{\epsilon}, 1), \quad H(s) = \begin{cases} 1 & \text{if} \quad s > 0 \\ [0,1] & \text{if} \quad s = 0 \\ 0 & \text{if} \quad s < 0, \end{cases} \quad H_0(s) = \begin{cases} 1 & \text{if} \ \ s > 0 \\ 0 & \text{if} \ \ s \leqslant 0, \end{cases}$$

$$T_k(s) = \begin{cases} k & \text{if} \quad s > k \\ s & \text{if} \quad |s| \leqslant k \\ -k & \text{if} \quad s < -k. \end{cases}$$

Definition 2.1. — *(Weak solution of $(CP)(b, F, f, u_0)$).*

Let u_0 and f be such that $(H1)$ is fulfilled. A weak solution of (CP) is a function $u \in L^\infty(Q)$, such that :

$$u_t \in L^2\left((0,T), H^{-1}_{loc}(\mathbb{R}^N)\right) + L^1\left((0,T), L^\infty(\mathbb{R}^N)\right), \tag{2.1}$$

$$b(u) \in L^2\left((0,T), H^1_{loc}(\mathbb{R}^N)\right), \tag{2.2}$$

$$u_t - \Delta b(u) + div F(b(u)) = f \qquad \text{in} \qquad \mathcal{D}'(Q)$$
$$\text{and} \ \ u(0,x) = u_0 \ \text{on} \quad \mathbb{R}^N.$$

The last condition must be understood in the sense that

$$\int_0^T < u_t, \xi > \ dt = -\int_Q u \ \xi_t \ dx \ dt - \int_{\mathbb{R}^N} u_0 \ \xi(0) \ dx \tag{2.3}$$

for any $\xi \in L^2\left((0,T); \mathcal{D}(\mathbb{R}^N)\right) \cap W^{1,1}\left((0,T); L^\infty(\mathbb{R}^N)\right)$ so that $\xi(T) = 0$ and $<,>$ represents the duality product between $H^{-1}(\mathbb{R}^N)$ and $\mathrm{H}^1(\mathbb{R}^N)$.

DEFINITION 2.2 *(Entropy solution of (CP) (b, F, f, u_0)). — Let u_0 and f verify $(H1)$. An entropy solution u of (CP) (b, F, f, u_0) is a weak solution of the same problem such that :*

$$\begin{cases} \displaystyle\int_Q H_0(u-s)\{\nabla b(u)\nabla\xi - (F(b(u)) - F(b(s)))\nabla\xi - (u-s)\xi_t - f\xi\}\ dx\ dt \\ \qquad\qquad \displaystyle - \int_{I\!R^N} (u_0 - s)^+\xi(0)\ dx \leqslant 0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} \displaystyle\int_Q H_0(s-u)\{\nabla b(u)\nabla\xi - (F(b(u)) - F(b(s)))\nabla\xi - (u-s)\xi_t - f\xi\}\ dx\ dt \\ \qquad\qquad \displaystyle + \int_{I\!R^N} (s - u_0)^+\xi(0)\ dx \geqslant 0 \end{cases} \tag{2.5}$$

for any $s \in I\!R$ and $\xi \in \mathcal{D}(Q)$, $\xi \geqslant 0$.

THEOREM 2.3. — *We suppose that u_0, f verify $(H1)$ and that b is not a Lipschitz function.*

Let ω be the modulus of continuity of b ; we suppose that ω satisfies

$$\begin{cases} \displaystyle\liminf_{\varepsilon \to 0} \frac{\omega(\varepsilon)^N}{\varepsilon^{N-1}} < +\infty \quad \textit{if} \quad N > 2 \quad \textit{and} \quad \liminf_{\varepsilon \to 0} \frac{\omega(\varepsilon)^2}{\varepsilon} = 0 \quad \textit{if} \quad N = 2\,; \\ \text{for} \quad N = 1 \quad \textit{there is no condition on } \omega. \end{cases} \tag{$H2$}$$

Then the Cauchy problem $(CP)(b, F, u_0, f)$ has a unique weak solution.

Remark 2.4. — *If $b = r^m$, we can see by an elementary calculus that (H_2) means that $m > \dfrac{N-1}{N}$.*

Proof of Theorem 2.3. —

1. Given that an entropy solution of (CP) is a weak solution of the same problem, the existence of the weak solution is a consequence of Theorem 3.7 of [MT3].

2. For the uniqueness we use Theorem 3.11 in [MT3]. However, since we are working in the whole space, we cannot take a test function identically equal to one in Kato's inequality (2.8) below to get the comparison principle. An additional condition on b is required (hypothesis $(H2)$).

Let u be a weak solution of $(CP)(b, F, u_0, f)$, $s \in \mathbb{R}$ such that $b(s) \notin E$ where

$$E = \{r \in \ \mathrm{Im}(b)/; (b^{-1})_0 \quad \text{is discontinuous at r}\}.$$

Since $(b^{-1})_0$ is a monotone function, E is a countable subset of $\mathbb{R}$.

Let $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geqslant 0$. Consider that R is big enough such that $supp\xi \subset B(0,R) =: \Omega$; in particular we have $\xi \equiv 0$ on $\partial\Omega$. It is known that u is a weak solution of (CP) on $\Omega \times (0,T)$; by Lemma 4 in [IU1], Lemma 5 and Theorem 6 in [C2] we have :

$$\begin{cases} \displaystyle\int_Q H_0(u-s)\{(\nabla b(u) + F(b(s)) - F(b(u)))\nabla\xi + (s-u)\xi_t - f\xi\}\ dx\ dt \\ \qquad\qquad \displaystyle - \int_{\mathbb{R}^N} (u_0 - s)^+ \xi(0)\ dx = \lim_{\epsilon \longrightarrow 0} \int_Q |\nabla b(u)|^2 H'_\epsilon(b(s) - b(u))\xi\ dx\ dt \end{cases} \tag{2.6}$$

and

$$\begin{cases} \displaystyle\int_Q H_0(s-u)\{(\nabla b(u) + F(b(s)) - F(b(u)))\nabla\xi + (s-u)\xi_t - f\xi\}\ dx\ dt \\ \qquad\qquad \displaystyle - \int_{\mathbb{R}^N} (s - u_0)^+ \xi(0)\ dx = \lim_{\epsilon \longrightarrow 0} \int_Q |\nabla b(u)|^2 H'_\epsilon(b(s) - b(u))\xi\ dx\ dt \end{cases} \tag{2.7}$$

for all $s \in \mathbb{R}$ such that $b(s) \notin E$ and $\xi \in \mathcal{D}(Q)$, $\xi \geqslant 0$.

By the method of doubling variables introduced by S.N. Kružkhov for the conservation laws and adapted by J. Carrillo for second order parabolic equations, we obtain the so called Kato's inequality :

For (u_{01}, f_1) and (u_{02}, f_2) satisfying $(H1)$, let u_1, u_2 be weak solutions of $(CP)(b, F, f_1, u_{01})$, $(CP)(b, F, f_2, u_{02})$ respectively. Then

$$\begin{cases} \displaystyle\int_Q \{(\nabla(b(u_1) - b(u_2))^+ \\ +H_0(u_1 - u_2)(F(b(u_2)) - F(b(u_1))))\nabla\xi - (u_1 - u_2)^+\xi_t\}\ dx\ dt \\ \displaystyle - \int_{\mathbb{R}^N} (u_{01} - u_{02})^+\xi(0)\ dx \leqslant \int_Q \nu(f_1 - f_2)\xi\ dx\ dt \end{cases} \tag{2.8}$$

for any $\nu \in H(u_1 - u_2)$ a.e. and $\xi \in \mathcal{D}([0, T[\times\mathbb{R}^N)$, $\xi \geqslant 0$.

Let now $W = (u_1 - u_2)^+$, $W_0 = (u_{01} - u_{02})^+$ and $h = (f_1 - f_2)^+$. By using the Theorem 3.11 in [MT3], (H2) and the fact that

$$|b(u_1) - b(u_2)|\chi_{\{u_1 > u_2\}} \leqslant \omega(W), \quad |F(b(u_1)) - F(b(u_2))|\chi_{\{u_1 > u_2\}} \leqslant C\omega(W),$$

one has

$$\int_{\mathbb{R}^N}(u_1(t)-u_2(t))^+\,dx \leqslant \int_{\mathbb{R}^N}(u_{01}-u_{02})^+\,dx+\int_{Q_t}\nu(f_1-f_2)\,dx\,ds \quad (2.9)$$

for $\nu \in H(u_1-u_2)$ a.e..

The uniqueness and comparison principle of weak solutions are the consequences of this last inequality (2.9). □

3. Existence and uniqueness of the renormalized solution

In this section we suppose that the initial data and source term satisfy the following hypothesis :

$$(H3) \qquad u_0 \in L^1(\mathbb{R}^N), \qquad f \in L^1(Q),$$

b satisfies the condition $(H2)$ and F is a Lipschitz continuous function.
It is important to mention that b and $\tilde{F}$ are not necessarily Lipschitz functions as in $[MK]$.

Definition 3.1 *(Renormalized solution of (CP) (b,F,f,u_0). — A renormalized solution of (CP) (b,F,f,u_0)) is a function u such that :*

1. *$u \in L^1(Q)$;*
2. *$T_k b(u) \in L^2(0,T,H^1_{loc}(\mathbb{R}^N))$ for any $k>0$;*
3. *For all $\xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$ and $h \in \mathcal{C}(\mathbb{R})$,*

$$\begin{cases} -\int\int_Q \xi_t(\int_{u_0}^u h(b(r))\,dr)\,dx\,dt + \int\int_Q (\nabla b(u) \\ -F(b(u))).\nabla(h(b(u))\xi)\,dx\,dt = \int\int_Q fh(b(u))\xi\,dx\,dt. \end{cases} \quad (3.10)$$

In addition

$$\int\int_{Q\cap[n\leqslant|b(u)|\leqslant n+1]}\{|\nabla b(u)|^2-F(b(u)).\nabla b(u)\}dx\,dt \to 0 \ \text{ as } n\to+\infty. \quad (3.11)$$

Proposition 3.2. — *Let $f_1,\ f_2 \in L^1(Q)$ and $u_{01},\ u_{02} \in L^1(\mathbb{R}^N)$. If u_i is a renormalized solution of (CP) (b,F,f_i,u_{0i}) for $i=1,2$ then*

$$\int_{\mathbb{R}^N}(u_1(t)-u_2(t))^+dx \leqslant \int_{\mathbb{R}^N}(u_{01}-u_{02})^+dx + \int_0^t\int_{\mathbb{R}^N}\nu(f_1-f_2)dx\,ds \quad (3.12)$$

where $\nu \in Sign^+(u_1-u_2)$.

Proof of Proposition 3.2. — The main idea of the proof is to state that a renormalized solution of (CP) (b, F, f, u_0) satisfies an entropy inequality for an auxiliary problem and use Theorem 2.3 to get the uniqueness.

To be more precise, let $h \in \mathcal{C}_c(\mathbb{R})$ with $h \geqslant 0$, $h(0) > 0$ and

$$j_h(r) = \int_0^r h(b(s))ds, \quad b_h(r) = \int_0^{b(r)} h(s)ds,$$

$$F_h(r) = h(b(r))F(b(r)), \quad f_h(r) = h(b(r))f(r),$$

$$G_h(r) = (-\nabla b(r) + F(b(r))).\nabla h(b(r)).$$

We consider the Cauchy problem below :

$$(CP)_h \quad \begin{cases} \dfrac{\partial}{\partial t} j_h(u) - \nabla(\nabla b_h(u) - F_h(u)) = f_h(u) + G_h(u) & \text{in } Q \\ j_h(u(0)) = j_h(u_0) & \text{on } \mathbb{R}^N. \end{cases}$$

The first step is to ensure that if u is a renormalized solution of (CP) (b, F, f, u_0) then u satisfies the entropy inequality below of the auxiliary problem $(CP)_h$:

$$\begin{cases} \displaystyle\int\int_Q H_0(u-k)\{(j_h(u) - j_h(k))\xi_t \; dx \; dt - (\nabla b_h(u) \\ \qquad -(F_h(u) - F_h(k))).\nabla\xi \;\}dx \; dt \\ \qquad \geqslant -\displaystyle\int_{\mathbb{R}^N} (j_h(u_0) - j_h(k))^+ \xi(0) \; dx - \int\int_Q H_0(u-k)(f_h(u) \\ \qquad +G_h(u))\xi \; dx \; dt. \end{cases} \tag{3.13}$$

Let us take R large enough such that $supp(\xi) \subset B(0, R) \times (0, T) = Q_1$ and $\Omega = B(0, R)$.

The last inequality (3.13) is equivalent to

$$\begin{cases} \displaystyle\int\int_{Q_1} H_0(u-k)\{(j_h(u) - j_h(k))\xi_t \; dx \; dt \\ \qquad -(\nabla b_h(u) - (F_h(u) - F_h(k))).\nabla\xi \;\}dx \; dt \\ \qquad \geqslant -\displaystyle\int_{\Omega} (j_h(u_0) - j_h(k))^+ \xi(0) \; dx - \int\int_{Q_1} H_0(u-k)(f_h(u) \\ \qquad +G_h(u))\xi \; dx \; dt \end{cases} \tag{3.14}$$

which is the same as inequality (2.4) in [IW] ; therefore using the same method, we get the result (3.13). However we do not have to take the same function k because we are working in $\mathbb{R}^N$ and we do not have any boundary condition.

The second step to prove Proposition 3.2 is to get Kato's inequality for the auxiliary problem $(CP)_h$ i.e., if f_1, $f_2 \in L^1(Q)$, u_{01}, $u_{02} \in L^1(\mathbb{R}^N)$ and u_i is a renormalized solution of (CP) (b, F, f_i, u_{0i}) for $i = 1, 2$ then

$$\left\{ \begin{array}{l} \displaystyle\int\int_{Q_1} H_0(u_1 - u_2)\{(j_h(u_1) - j_h(u_2))\xi_t - (\nabla(b_h(u_1) - b_h(u_2))^+ \\ \qquad +F_h(u_1) - F_h(u_2)).\nabla\xi\} \; dx \; dt \\ \qquad \geqslant - \displaystyle\int_\Omega (j_h(u_{01}) - j_h(u_{02}))^+ \xi(0) \; dx - \int\int_{Q_1} \nu((f_h(u_1) + G_h(u_1)) \\ \qquad -(f_h(u_2) + G_h(u_2)))\xi \; dx \; dt \end{array} \right. \tag{3.15}$$

for any $\nu \in H(u_1 - u_2)$ a.e. and $\xi \in \mathcal{D}([0, T[\times\mathbb{R}^N)$, $\quad \xi \geqslant 0$.

The technique of Kružkhov's doubling variables enables us to deduce (3.15) from entropy inequalities (2.4)-(2.5) as in ([C1], [C2], [KR], ...).

Now given that we are working in the whole space, we cannot take the same test function $\xi \equiv 1$ as in [IW]. We have to use the technique of [MT3] to get the inequality (3.12) from Kato's inequality (3.15). To this end, we have to verify that the new functions j_h, b_h and F_h satisfy the hypothesis (H_2). As long as this is not true for all h, we have to make the appropriate choice of h to get the hypothesis (H_2) . Set $h_n(r) = inf((n+1) - |r|)^+, 1)$ in inequality (3.15) and let n large enough as in [IU2] ; the expressions of j_{h_n}, b_{h_n} take the following form :

$$j_{h_n}(r) = \begin{cases} b^{-1}(-n) + \displaystyle\int_{b^{-1}(-n)}^{b^{-1}(-n-1)} (n+1+b(s)) \; ds & \texttt{if} \;\; r \leqslant b^{-1}(-n-1) \\ b^{-1}(-n) + \displaystyle\int_{b^{-1}(-n)}^{r} (n+1+b(s)) ds & \texttt{if} \; b^{-1}(-n-1) \leqslant r \leqslant b^{-1}(-n) \\ r & \texttt{if} \;\; b^{-1}(-n) \leqslant r \leqslant b^{-1}(n) \\ b^{-1}(n) + \displaystyle\int_{b^{-1}(n)}^{r} (n+1-b(s)) \; ds & \texttt{if} \;\; b^{-1}(n) \leqslant r \leqslant b^{-1}(n+1) \\ b^{-1}(n) + \displaystyle\int_{b^{-1}(n)}^{b^{-1}(n+1)} (n+1+b(s)) \; ds & \texttt{if} \;\; r \geqslant b^{-1}(n+1), \end{cases} \tag{3.16}$$

$$b_{h_n}(r) = \begin{cases} -n-\frac{1}{2} & \texttt{if} \quad r \leqslant b^{-1}(-n-1) \\ -n+\displaystyle\int_{-n}^{b(r)}(n+1+s)ds & \texttt{if} \quad b^{-1}(-n-1) \leqslant r \leqslant b^{-1}(-n) \\ b(r) & \texttt{if} \quad b^{-1}(-n) \leqslant r \leqslant b^{-1}(n) \\ n+\displaystyle\int_{n}^{b(r)}(n+1-s)\ ds & \texttt{if} \quad b^{-1}(n) \leqslant r \leqslant b^{-1}(n+1) \\ n+\frac{1}{2} & \texttt{if} \quad r \geqslant b^{-1}(n+1). \end{cases} \tag{3.17}$$

For n large enough, only the set $[-(n+\frac{1}{2}), n+\frac{1}{2}]$ is important and the graph $b_{h_n}((j_{h_n})^{-1})$ can be extended by any regular increasing profile (for instance see [IU2]). Let us see b_{h_n} and j_{h_n} in this set.

$$j_{h_n}(r) = \begin{cases} b^{-1}(-n)+\displaystyle\int_{b^{-1}(-n)}^{r}(n+1+b(s))ds \ \texttt{if} \ b^{-1}(-n-\frac{1}{2}) \leqslant r \leqslant b^{-1}(-n) \\ r \qquad \texttt{if} \ b^{-1}(-n) \leqslant r \leqslant b^{-1}(n) \\ b^{-1}(n)+\displaystyle\int_{b^{-1}(n)}^{r}(n+1-b(s))\ ds \ \texttt{if} \ b^{-1}(n) \leqslant r \leqslant b^{-1}(n+\frac{1}{2}), \end{cases} \tag{3.18}$$

$$j'_{h_n}(r) = \begin{cases} n+1+b(r) & \texttt{if} \quad b^{-1}(-n-\frac{1}{2}) \leqslant r \leqslant b^{-1}(-n) \\ 1 & \texttt{if} \quad b^{-1}(-n) \leqslant r \leqslant b^{-1}(n) \\ n+1-b(r) & \texttt{if} \quad b^{-1}(n) \leqslant r \leqslant b^{-1}(n+\frac{1}{2}). \end{cases} \tag{3.19}$$

On the set $[-(n+\frac{1}{2}), n+\frac{1}{2}]$, j_{h_n} is an increasing Lipschitz function and $\frac{1}{2} \leqslant j'_{h_n} \leqslant 1$.

Consequently j_{h_n} is one to one, $j_{h_n}^{-1}$ exists and $|(j_{h_n}^{-1})'| \leqslant 2$. Then on the set $[-(n+\frac{1}{2}), n+\frac{1}{2}]$,

$$|b_{h_n}(r) - b_{h_n}(r')| = |b({j_{h_n}}^{-1})(r) - b({j_{h_n}}^{-1})(r')| \leqslant 2\omega(r-r')$$

with ω the modulus of continuity of b and

$$|F_{h_n}(r) - F_{h_n}(r')| = |F({j_{h_n}}^{-1})(r) - F({j_{h_n}}^{-1})(r')| \leqslant 2C\omega(r-r').$$

It is now clear that b_{h_n} and F_{h_n} satisfy (H2). We can use Theorem 2.3 to claim that for $h = h_n$ and n large enough we have the following result : if u_i is a renormalized solution of (CP) (b, F, f_i, u_{0i}) for $i = 1, 2$ then :

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} (\int_{u_2(t)}^{u_1(t)} h(b(s))\ ds)^+ dx \leqslant \int_{\mathbb{R}^N} (\int_{u_{02}}^{u_{01}} h(b(s))\ ds)^+ dx \\ \qquad\qquad \displaystyle + \int_0^t \int_{\mathbb{R}^N} \nu (f_1 h(b(u_1)) - f_2 h(b(u_2))) dx ds \\ \displaystyle + \int_0^t \int_{\mathbb{R}^N} \nu [(\nabla b(u_1) + F(b(u_1))).\nabla (h(b(u_1))) - (\nabla b(u_2) \\ \qquad\qquad + F(b(u_2))).\nabla (h(b(u_2)))]\ dx\ ds. \end{cases} \tag{3.20}$$

In the inequality (3.20), we take the limit when n goes to the infinity to get the comparison principle (3.12). In fact,

$$\int_{\mathbb{R}^N} (\int_{u_2(t)}^{u_1(t)} h(b(s))\ ds)^+\ dx \to \int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+\ dx,$$

$$\int_{\mathbb{R}^N} (\int_{u_{01}}^{u_{02}} h(b(s))\ ds)^+\ dx \to \int_{\mathbb{R}^N} (u_{01} - u_{02})^+\ dx,$$

$$\int_0^t \int_{\mathbb{R}^N} \nu (f_1 h(b(u_1)) - f_2 h(b(u_2)))\ dx\ ds \to \int_0^t \int_{\mathbb{R}^N} \nu (f_1 - f_2)\ dx\ ds.$$

The last term in (3.20), on the other hand, goes to 0 when $n \to \infty$ by definition of the renormalized solution (inequality (3.11)). $\square$

THEOREM 3.3. — *For all* u_0 *and* f *verifying* $(H3)$, *there exists a renormalized solution of the problem* $(CP)(b, F, u_0, f)$.

Proof of Theorem 3.3. — The proof of the existence of a renormalized solution is based on ideas and techniques developed in [IW], [MT2], [MT3]. By B_n, we denote the ball centered at 0 of radius n. We consider the following Cauchy problem in a bounded domain $(CP)(b, F, u_o, f)_n$:

$$(CP)_n \qquad \begin{cases} u_t - \Delta b(u) + div\ F(b(u)) = f & \text{in} \quad (0,T) \times B_n \\ b(u) = 0 & \text{on} \quad (0,T) \times \partial B_n \\ u(0,x) = u_0 & \text{in} \quad B_n. \end{cases}$$

Based on the results of [AI] and [IW], the Cauchy problem $(CP)_n$ has a unique renormalized solution u_n such that $u_n \in C([0,T), L^1(B_n))$ and satisfies the comparison principle. By u_n, we also denote the extension of u_n to $\mathbb{R}^N$ by 0 outside of B_n. Let $\Omega \subset \mathbb{R}^N$ a bounded open set with smooth boundary. We consider that n is large enough such that $\Omega \subset B_n$. Thus $u_n \in C([0,T), L^1(\Omega))$ and by comparison principle u_n is a Cauchy sequence. So when n goes to infinity,

$$u_n \to u \text{ in } \quad C([0,T), L^1(\Omega)). \tag{3.21}$$

We now prove that u is a renormalized solution of (CP).

Set $Q_1 = (0,T)\times\Omega$ and $Q_n = (0,T)\times B_n$. Let us take $h(b(u_n))\xi$ as a test function in the definition of u_n with $h \in w^{1,\infty}(\mathbb{R})$, $h \geqslant 0$ and $\xi \in C^1(B_n)$ such that $h(b(u_n))\xi \in L^2(0,T,H_0^1(B_n))$.

With Lemma 4 in [C1] one has

$$\begin{cases} -\displaystyle\int\int_{Q_n} \xi_t j_h(u_n)\ dx\ dt + \int\int_{Q_n} (\nabla b(u_n) - F(b(u_n))).\nabla(h(b(u_n))\xi)\ dx\ dt \\ \quad + \displaystyle\int_{B_n} \xi(T) j_h(u_n(T))\ dx = \int_{B_n} \xi(0) j_h(u_0)\ dx + \int\int_{Q_n} f h(b(u_n))\xi\ dx\ dt. \end{cases} \tag{3.22}$$

As first step, we prove that if n goes to infinity then

$$T_k b(u_n) \rightharpoonup T_k b(u) \text{ in } \quad L^2(0,T;H^1_{loc}(\mathbb{R}^N)) \text{ for any } \quad k>0. \tag{3.23}$$

If we choose $h(r) = T_k(r)$ on $\mathbb{R}$ and $\xi \equiv 1$ in (3.22), we get

$$\begin{cases} \displaystyle\int_{B_n} j_h(u_n(T))\ dx + \int\int_{Q_n} |\nabla T_k b(u_n)|^2\ dx\ dt \\ \qquad\qquad = \displaystyle\int\int_{Q_n} F(b(u_n)).\nabla T_k b(u_n)\ dx\ dt \\ \qquad\qquad + \displaystyle\int_{B_n} j_h(u_0)\ dx + \int\int_{Q_n} f T_k b(u_n)\xi\ dx\ dt. \end{cases} \tag{3.24}$$

One knows that $\displaystyle\int\int_{Q_n} F(b(u_n)).\nabla T_k b(u_n) = 0$ and $j_h \geqslant 0$. Thus we obtain

$$\int\int_{Q_1} |\nabla T_k b(u_n)|^2 dx\ dt \leqslant \int\int_{Q_n} |\nabla T_k b(u_n)|^2 dx\ dt$$
$$\leqslant k(\int\int_Q |f|\ dx\ dt + \int_{\mathbb{R}^N} |u_0|\ dx).$$

By Poincaré's inequality we deduce that $T_k b(u_n)$ is a bounded sequence in $L^2(0,T;H^1(\Omega))$.

We consider a subsequence denoted again by n such that

$$T_k b(u_n) \rightharpoonup G \quad \text{ in } L^2(0,T;H^1(\Omega)) \text{ for any } \quad k>0,$$

$$\begin{aligned} b(u_n) &\to b(u) \quad \text{in } C(0,T,L^1(\Omega)), \\ b(u_n) &\to b(u) \quad \text{in } Q_1 \ \texttt{a.e.}. \end{aligned}$$

Therefore one can easily get (3.23).

Let us now prove that u satisfies (3.11). We take $h(r) = T_{k+1}(r) - T_k(r)$, $\xi \equiv 1$ in (3.22) and by Lemma 4 in [C1] one has :

$$\left\{ \begin{aligned} &\int_{B_n} j_h(u_n(T))dx + \int\int_{[k\leqslant|b(u_n)|\leqslant k+1]} (|\nabla b(u_n)|^2 - F(b(u_n))\nabla b(u_n))dx\ dt \\ &\qquad\qquad = \int_{B_n} j_h(u_0)dx + \int\int_{Q_n} fh(b(u_n))dx\ dt. \end{aligned} \right.$$

Since $j_h \geqslant 0$ then

$$\left\{ \begin{aligned} &\int\int_{[k\leqslant|b(u_n)|\leqslant k+1]} (|\nabla b(u_n)|^2\ dx\ dt - \int\int_{[k\leqslant|b(u_n)|\leqslant k+1]} F(b(u_n))\nabla b(u_n))\ dx\ dt \\ &\qquad\qquad \leqslant \int_{[|u_0|\geqslant k]} j_h(u_0)\ dx + \int\int_{[|b(u_n)|\geqslant k]} fh(b(u_n))\ dx\ dt. \end{aligned} \right.$$

As $\displaystyle\int\int_{[k\leqslant|b(u_n)|\leqslant k+1]} F(b(u_n))\nabla b(u_n)\ dx\ dt = 0$, when n goes to ∞, one has :

$$\limsup_{n\to+\infty} \int\int_{[k\leqslant|b(u_n)|\leqslant k+1]} |\nabla b(u_n)|^2\ dx\ dt \leqslant \int\int_{[b(u)\geqslant k]} |f|\ dx\ dt + \int_{[|u_0|\geqslant k]} |u_0|\ dx.$$

With the fact that

$$T_{k+1}b(u_n) - T_k b(u_n) \rightharpoonup T_{k+1}b(u) - T_k b(u) \ \text{ in } \ L^2(0,T;H^1(\Omega)), \tag{3.25}$$

we get

$$\int\int_{[k\leqslant|b(u)|\leqslant k+1]} |\nabla b(u)|^2\ dx\ dt \leqslant \int\int_{[b(u)\geqslant k]} |f|\ dx\ dt + \int_{[|u_0|\geqslant k]} |u_0|\ dx.$$

Letting k goes to $+\infty$, we obtain (3.11).

The final step of the proof is to have (3.10). To this end, we have to verify that :

$$|\nabla T_k b(u_n)|^2 \to |\nabla T_k b(u)|^2 \quad \text{in} \quad L^1(Q_1) \quad \text{when} \quad n \to +\infty. \tag{3.26}$$

We will apply arguments developed in [AW] and [IW] to our case. We consider the regularization in time of $T_k b(u)$ defined by Landes's method in [L] :

$$v_m := m \int_{-\infty}^{t} e^{m(s-t)} T_k b(u(s,x))\ ds$$

for a.e (t,x) ; for $s<0$ we extend $b(u)$ by 0. Then we have :

$$v_m \in L^2(0,T;H^1(B_n)) \cap L^\infty(Q_n),\ v_m(0)=0;$$

v_m is differentiable for a.e. $t \in (0,T)$ and $\frac{\partial v_m}{\partial t} = m(T_k b(u) - v_m) \in L^2(0,T;H^1(B_n)) \cap L^\infty(Q_n)$.

Let $\sigma \in D^+(0,T)$ and $h(r) = h_l(r) = \inf((l+1-|r|)^+,1)$. One can see that $h \in w^{1,\infty}(\mathbb{R})$, $h \geqslant 0$ and $supp(h)$ is compact. So by inequality (3.9) in [IW], we deduce that

$$\liminf_{l\to\infty} \liminf_{m\to\infty} \lim_{n\to\infty} \int\int_{Q_n} \sigma(\nabla b(u_n) - F(b(u_n))) \\ .\nabla(h_l(b(u_n)(T_k(b(u_n)) - v_m))\ dx\ dt \leqslant 0. \tag{3.27}$$

Since $Q_1 \subset Q_n$ for n large enough, this last inequality implies that :

$$\liminf_{l\to\infty} \liminf_{m\to\infty} \lim_{n\to\infty} \int\int_{Q_1} \sigma(\nabla b(u_n) - F(b(u_n))) \\ .\nabla(h_l(b(u_n)(T_k(b(u_n)) - v_m))dx\ dt \leqslant 0 \tag{3.28}$$

and by using the same arguments as in [AW] and [IW], we deduce that :

$$\liminf_{m\to\infty} \lim_{n\to\infty} \int\int_{Q_1} \sigma|\nabla T_k b(u_n) - \nabla v_m|^2 dx\ dt = 0, \tag{3.29}$$

and then

$$\lim_{n\to\infty} \int\int_{Q_1} \sigma|\nabla T_k b(u_n) - \nabla b(u)|^2 dx\ dt = 0.$$

By using (3.23) the last equality implies (3.26) and therefore (3.10).

We then deduce that u satisfies the definition of renormalized solution which ends the proof of the existence of renormalized solution. $\square$

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