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Continuity of the bending map^(*)

CYRIL LECUIRE⁽¹⁾

ABSTRACT. — The bending map of a hyperbolic 3-manifold maps a convex cocompact hyperbolic metric on a 3-manifold with boundary to its bending measured geodesic lamination. As proved in [KeS] and [KaT], this map is continuous. In the present paper we study the extension of this map to the space of geometrically finite hyperbolic metrics. We introduce a relationship on the space of measured geodesic laminations and show that the quotient map obtained from the bending map is continuous.

RÉSUMÉ. — L'application de plissage d'une variété hyperbolique de dimension 3 associée à une métrique hyperbolique convexe cocompacte sur une variété compacte à bord sa lamination géodésique mesurée de plissage. Il a été démontré dans [KeS] et [KaT] que cette application est continue. Dans ce texte, on étudie l'extension de cette application à l'espace des métriques hyperboliques géométriquement finies. On introduit une relation d'équivalence dans l'espace des laminations géodésiques mesurées et on montre que l'application quotient de l'application de plissage est continue.

Introduction

Let M be a compact, orientable 3-manifold with boundary. Assume that M is hyperbolic, namely that the interior of M is endowed with a complete metric σ of constant sectional curvature -1 . Assume also that ∂M contains a surface with genus greater than 1. A fundamental subset of (M, σ) is its convex core $N(\sigma)$. This core $N(\sigma)$ is defined as the smallest non-empty closed subset of the interior of M which is locally convex and homotopically equivalent to M . Its boundary $\partial N(\sigma)$ endowed with the intrinsic metric (given by rectifiable path length) is isometric to a hyperbolic surface of finite volume and can be embedded in a natural way into ∂M . This surface

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is bent along a geodesic lamination and the amount of bending is described by a measured geodesic lamination called the *bending measured geodesic lamination* of σ (cf. [Th] or [CEG]). This yields a bending map b which to a complete hyperbolic metric associates its bending measured geodesic lamination.

In [Bo2], F. Bonahon considers quasi-isometric deformations of a given metric σ on $\text{int}(M)$, namely hyperbolic metrics σ' on $\text{int}(M)$ for which there exists a diffeomorphism $(\text{int}(M), \sigma) \rightarrow (\text{int}(M), \sigma')$ whose differential is uniformly bounded. Let $\mathcal{QD}(\sigma)$ be the space of quasi-isometric deformations of a given metric σ , where we identify two deformations $(\text{int}(M), \sigma) \rightarrow (\text{int}(M), \sigma')$ and $(\text{int}(M), \sigma) \rightarrow (\text{int}(M), \sigma'')$ if they are isotopic. The continuity of the bending map $b : \mathcal{QD}(\sigma) \rightarrow \mathcal{ML}(\partial M)$ is proved in [KaT], using ideas of Thurston. Its differentiability (in a weak sense) is proved in [Bo2].

A complete hyperbolic metric σ on $\text{int}(M)$ is said to be *convex cocompact* if $N(\sigma)$ is compact. If σ is convex cocompact, then $\mathcal{QD}(\sigma)$ is the set of convex cocompact metrics on $\text{int}(M)$. In this case, the continuity of $b : \mathcal{QD}(\sigma) \rightarrow \mathcal{ML}(\partial M)$ has been proved in [KeS] and its image has been described in [BoO] and [Le1]. A complete hyperbolic metric σ on $\text{int}(M)$ is said to be *geometrically finite* if $N(\sigma)$ has finite volume. When σ is a geometrically finite metric, $\mathcal{QD}(\sigma)$ is the set of geometrically finite metrics having the same parabolic subgroups as σ . In the present paper, we are addressing the question of the continuity of the bending map on the whole set of geometrically finite metrics. In particular, we are interested in sequences converging to a limit with some new parabolics.

Since we want to consider metrics which do not have the same parabolic subgroups, we have to allow deformations which are not quasi-isometric. We will consider the set of isotopy classes of hyperbolic metrics on the interior of M . Two metrics σ_1 and σ_2 are identified if there exists a diffeomorphism $f : M \rightarrow M$ isotopic to the identity such that $\sigma_1 = f^*\sigma_2$. We will consider the set $\mathcal{GF}(M)$ of isotopy classes of geometrically finite hyperbolic metrics which are not fuchsian. We will topologise $\mathcal{GF}(M)$ in the following way. Let us choose a point x in $\text{int}(M)$. A metric σ_2 lies in a (k, r) -neighbourhood of σ_1 if there exists a diffeomorphism $g : M \rightarrow M$ isotopic to the identity such that the restriction of g to the ball $B(x, r) \subset (M, \sigma_1)$ is a k -quasi-isometry into its image in (M, σ_2) . We obtain a basis of neighbourhoods of σ_1 by letting k tend to 1 and r tend to $+\infty$. The topology defined in this way does not depend on the choice of the point x . For a metric $\sigma \in \mathcal{GF}(M)$, $\mathcal{QD}(\sigma)$ can be viewed as a subset of $\mathcal{GF}(M)$. The topology of $\mathcal{QD}(\sigma)$ considered as a subset of $\mathcal{GF}(M)$ coincides with the topology given by quasi-isometric

deformations.

The bending map $b_{\mathcal{GF}}(M) : \mathcal{GF}(M) \rightarrow \mathcal{ML}(\partial M)$ maps an isotopy class of geometrically finite metrics to its bending measured geodesic lamination. The image of $b_{\mathcal{GF}}$ has been described in [BoO] and [Le1], it is the set $\mathcal{P}(M)$ of measured geodesic laminations satisfying the following conditions:

- a) no closed leaf of λ has a weight greater than π ;
- b) $\exists \eta > 0$ such that, for any essential annulus E , $i(\partial E, \lambda) \geq \eta$;
- c) $i(\lambda, \partial D) > 2\pi$ for any essential disc D .

Taking a careful look at the behaviour of the map $b_{\mathcal{GF}}$, we notice that it is not a continuous map. If a metric σ lies in $\mathcal{GF}(M)$ but has some rank one cusps, its bending measured geodesic lamination λ has some compact leaves with a weight equal to π . Let us denote by $\lambda^{(p)}$ the union of the leaves of λ which have a weight equal to π . Using the result of [BoO], it is not hard to construct a sequence of metrics $\sigma_n \in \mathcal{GF}(M)$ with measured geodesic laminations λ_n such that the sequence (λ_n) converges to a measured geodesic lamination λ_∞ which differs from λ only on $\lambda^{(p)}$ and which has some leaves with a weight greater than π . Using some arguments of [Le1], we get that a subsequence of (σ_n) converges to a geometrically finite metric σ_∞ . Since λ_∞ does not satisfies condition a), it is not the bending measured geodesic lamination of σ_∞ . Thus we get that $b_{\mathcal{GF}}$ is not continuous on any neighbourhood of a metric with some rank one cusps. To overcome this difficulty, we will quotient the space $\mathcal{ML}(\partial M)$ of measured geodesic laminations by the following relationship :

Let $\lambda, \mu \in \mathcal{ML}(\partial M)$ be two measured geodesic laminations and let us denote by λ' (resp. μ') the measured geodesic laminations obtained by replacing by π the weights of the leaves of λ (resp. μ) which have a weight greater than π ; we set $\lambda \mathcal{R} \mu$ if and only if $\lambda' = \mu'$. We denote by $\dot{\lambda}$ the class of λ modulo \mathcal{R} .

Let us endow $\mathcal{ML}(\partial M)$ with the weak* topology and $\mathcal{ML}(\partial M)/\mathcal{R}$ with the quotient topology. From $b_{\mathcal{GF}}(M)$ we obtain a quotient map $b_{\mathcal{R}} : \mathcal{GF}(M) \rightarrow \mathcal{P}(M)/\mathcal{R}$. We will prove the following result :

THEOREM 0.1. — *The map $b_{\mathcal{R}}$ from $\mathcal{GF}(M)$ to $\mathcal{ML}(\partial M)/\mathcal{R}$ is a continuous map.*

In [Le2], we show the reverse of this theorem. This gives rise to a criterion for the strong convergence of a sequence of geometrically finite representations $\rho_n \in U(M)$.

The paper is organised as follows. In section 1, we state some definitions and we prove some facts about $\mathcal{ML}(\partial M)/\mathcal{R}$. In section 2, we study convex pleated surfaces and prove the continuity of the bending measured geodesic laminations of a converging sequence of convex pleated surfaces. In section 3 we use the results of section 2 to prove the continuity of $b_{\mathcal{R}}$.

1. Definitions

Let σ be a hyperbolic metric (up to isotopy) on $\text{int}(M)$. Given an isometry from the interior of \tilde{M} to \mathbb{H}^3 , the covering transformations yield a discrete faithful representation $\rho : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$. The representations that appear in this way will be called *representations associated to σ* . The set of representations associated to σ is the set of all representations conjugated to ρ . The image $\rho(\pi_1(M))$ is a finitely generated torsion free Kleinian group and $\text{int}(M)$ endowed with σ is isometric to $\mathbb{H}^3/\rho(\pi_1(M))$. The *Nielsen core* of $\mathbb{H}^3/\rho(\pi_1(M))$ is the quotient by $\rho(\pi_1(M))$ of the convex hull $C(\rho)$ of the limit set L_ρ of $\rho(\pi_1(M))$ (see [Th, chap 8] for details). This set $N(\rho)$ and the convex core defined in the introduction are isometric and from now on we will identify them. The *thick part*, $N(\rho)^{ep}$ of the Nielsen core is the complementary of the cuspidal part of $\mathbb{H}^3/\rho(\pi_1(M))$ in $N(\rho)$. The representation ρ is *geometrically finite* when $N(\rho)^{ep}$ is compact (here it is equivalent to say that $N(\rho)$ has finite volume) and *convex cocompact* when $N(\rho)$ is compact.

When ρ is geometrically finite and not fuchsian, the natural retraction from $\mathbb{H}^3/\rho(\pi_1(M))$ to $N(\rho)$ associates to σ a homeomorphism (defined up to isotopy) $h : M \rightarrow N(\rho)^{ep}$. Such a homeomorphism will be said to be *associated to σ* .

Let $\sigma \in \mathcal{QD}(M)$ be a metric whose only cusps are rank 2 cusps and let ρ and h be respectively a representation and a homeomorphism associated to σ . Let $\tilde{h} : \tilde{M} \rightarrow \mathbb{H}^3$ be a lift of h , we will define the Floyd-Gromov compactification $\overline{\tilde{M}}$ of \tilde{M} as being the closure of $\tilde{h}(\tilde{M})$ in the usual compactification of \mathbb{H}^3 by the unit ball.

Let (σ_n) be a sequence of isotopy classes of complete hyperbolic metrics on the interior of M . The sequence (σ_n) *converges algebraically* when there is a sequence of representations $\rho_n : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$ associated to σ_n (as above) that converges algebraically; namely $\rho_n(g)$ converges for any $g \in \pi_1(M)$. We obtain a new representation $\rho_\infty : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$ defined by $\rho_\infty(g) = \lim_{n \rightarrow \infty} \rho_n(g)$ for any $g \in \pi_1(M)$. This representation ρ_∞ is discrete and faithful (cf. [Jor]). This representation defines a met-

ric on a manifold homotopy equivalent to M . This manifold might not be homeomorphic to M (examples are given in [AnC]).

The sequence (σ_n) *converges geometrically* if there is a sequence of representations $\rho_n : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$ associated to the σ_n such that $(\rho_n(\pi_1(M)))$ converges geometrically. The sequence of groups $(\rho_n(\pi_1(M)))$ converges geometrically to a group $\Gamma_\infty \subset \text{Isom}(\mathbb{H}^3)$ if and only if :

- for any sequence $a_n \in \rho_n(\pi_1(M))$, any accumulation point a_∞ of $\{a_n | n \in \mathbb{N}\}$ lies in Γ_∞ ;
- any element a_∞ of Γ_∞ is the limit of a sequence $a_n \in \Gamma_n$.

The sequence $\rho_n(\pi_1(M))$ *converges strongly* if there is a sequence of representations ρ_n associated to σ_n such that (ρ_n) converges algebraically to a representation ρ_∞ and that $(\rho_n(\pi_1(M)))$ converges geometrically to $\rho_\infty(\pi_1(M))$. If (σ_n) converges to σ for the topology defined in the introduction, then (σ_n) converges strongly to σ_∞ (cf. [CEG]).

2. Geodesic laminations and the relationship \mathcal{R}

A *geodesic lamination* L on \mathbb{H}^2 is a closed subset which is the disjoint union of complete geodesics. A complete geodesic lying in L is a *leaf* of L .

A *measured geodesic lamination* λ is a transverse measure for some geodesic lamination $|\lambda|$. Any arc $k \approx [0, 1]$ embedded in S transversely to $|\lambda|$, such that $\partial k \subset S - \lambda$, is endowed with an additive measure $d\lambda$ such that :

- the support of $d\lambda|_k$ is $|\lambda| \cap k$;
- if an arc k can be homotoped into k' by a homotopy respecting $|\lambda|$ then $\int_k d\lambda = \int_{k'} d\lambda$.

We will denote by $\mathcal{ML}(\mathbb{H}^2)$ the space of measured geodesic lamination topologised with the topology of the weak* convergence.

Let S be a surface (which may not be compact) endowed with a complete hyperbolic metric with finite area. A geodesic lamination in S is a compact subset which is the disjoint union of simple complete geodesics. Using the fact that two complete hyperbolic metrics with finite area on S are quasi-isometric, this definition can be made independent of the chosen metric on S (see for example [Ot1]). A measured geodesic lamination is a transverse measure for some geodesic lamination as defined above. Let γ be a weighted simple closed geodesic with support $|\gamma|$ and weight w and let λ be a measured geodesic lamination. The intersection number of γ and λ is

defined by $i(\gamma, \lambda) = w \int_{|\gamma|} d\lambda$. If $|\gamma|$ is a leaf of λ , then we define the intersection number by $i(\gamma, \lambda) = 0$. The weighted simple closed curves are dense in $\mathcal{ML}(S)$ and this intersection number extends continuously to a function $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \rightarrow \mathbb{R}$ (cf. [Bo1]).

A measured geodesic lamination $\lambda \in \mathcal{ML}(S)$ is *arational* if for any essential simple closed curve c which is not homotopic to a cusp, we have $i(c, \lambda) = \int_c d\lambda > 0$.

Let us recall the definition of \mathcal{R} given in the introduction.

Let $\lambda, \mu \in \mathcal{ML}(S)$ be two measured geodesic laminations. Let us denote by λ' (resp. μ') the measured geodesic lamination obtained by replacing by π the weights of the compact leaves of λ (resp. μ) which have a weight greater than π . We will say that λ is related to μ by the relationship \mathcal{R} , and we will write $\lambda \mathcal{R} \mu$, if and only if $\lambda' = \mu'$. We will denote by $\dot{\lambda}$ the class of λ modulo \mathcal{R} and we endow $\mathcal{ML}(S)$ with the quotient topology of the weak* topology on $\mathcal{ML}(S)$.

Let $\partial_{\chi < 0} M$ be the union of the connected components of ∂M with negative Euler characteristic. To simplify the notations, we will note $\mathcal{ML}(\partial M)$ for $\mathcal{ML}(\partial_{\chi < 0} M)$.

Let $\lambda, \mu \in \mathcal{ML}(\partial M)$ be two measured geodesic laminations. If $\lambda \mathcal{R} \mu$, then λ and μ share the same support. Thus we can define the support $|\dot{\lambda}|$ of an element $\dot{\lambda}$ of $\mathcal{ML}(\partial M)$ as being the support of any representative of $\dot{\lambda}$.

Let (λ_n) be a sequence of measured geodesic laminations such that $(\dot{\lambda}_n)$ converges to $\dot{\lambda}$ in $\mathcal{ML}(\partial M)/\mathcal{R}$ and let $\lambda \in \mathcal{ML}(\partial M)$ be a representative of $\dot{\lambda}$. Let us denote by $\lambda^{(p)}$ the union of the compact leaves of λ with a weight at least π . We have the following :

CLAIM 2.1. — *Let $\lambda \in \mathcal{ML}(\partial M)$ and let λ' be the representative of $\dot{\lambda}$ whose compact leaves have all a weight at most π . If $k \subset \partial M$ is a simple arc such that the points of $k \cap |\dot{\lambda}|$ and of $k \cap |\dot{\lambda}_n|$ are transverse intersections, then we have $\int_k d\lambda' \leq \liminf \int_k d\lambda_n$. Furthermore, if k does not intersect $\lambda^{(p)}$, then $\int_k d\lambda_n$ converges to $\int_k d\lambda$.*

Proof. — Let $k \subset \partial M$ be a simple arc such that the points of $k \cap |\dot{\lambda}|$ and of $k \cap |\dot{\lambda}_n|$ are transverse intersections. By definition of λ' , up to cutting k into finitely many sub-arcs, we may assume that we have $\int_k d\lambda' \leq \pi$. The set $\mathcal{V}_{k, \varepsilon}(\lambda) = \{\gamma \in \mathcal{ML}(\partial M) \mid \int_k d\gamma - \int_k d\lambda < \varepsilon\}$ is a neighbourhood of λ in $\mathcal{ML}(\partial M)$. Since $\dot{\lambda}_n$ converges to $\dot{\lambda}$, for any ε , there is n_ε such that

for any $n \geq n_\varepsilon$, $\exists \mu_n, \alpha_n$ with $\mu_n \mathcal{R} \lambda$, $\alpha_n \mathcal{R} \lambda_n$ and $\alpha_n \in \mathcal{V}_{k, \varepsilon}(\mu_n)$. So, for $n \geq n_\varepsilon$, we have $|\int_k d\alpha_n - \int_k d\mu_n| < \varepsilon$. Since λ' is the representative of $\dot{\lambda}$ whose compact leaves have all a weight at most π the measure of any $\mu \in \mathcal{ML}(\partial M)$ satisfying $\mu \mathcal{R} \lambda$ is at least the measure of λ' , namely we have $\int_k d\lambda' \leq \int_k d\mu$. It follows that we have $\int_k d\alpha_n \geq \int_k d\mu_n - \varepsilon \geq \int_k d\lambda' - \varepsilon$. Thus we get $\int_k d\lambda' \leq \liminf \int_k d\alpha_n$.

If we have $\int_k d\lambda_n < \pi$, then k does not intersect any closed leaf of λ_n with a weight at least π . Therefore we have $\int_k d\lambda_n = \int_k d\alpha_n$ and the inequality $\int_k d\lambda' \leq \liminf \int_k d\lambda_n$ follows from the paragraph above. Otherwise we have $\int_k d\lambda_n > \pi \geq \int_k d\lambda'$ by assumption and the inequality is obvious.

If k does not intersect any closed leaf of λ' with a weight equal to π , we may assume, up to cutting k into finitely many sub-arcs that we have $\int_k d\lambda' < \pi$. Since k does not intersect any closed leaf of λ' with a weight equal to π , we have $\int_k d\mu = \int_k d\lambda$ for any $\mu \in \mathcal{ML}(\partial M)$ satisfying $\mu \mathcal{R} \lambda$. Especially we have $\int_k d\alpha_n \rightarrow \int_k d\lambda' < \pi$. Hence we have $\int_k d\alpha_n < \pi$ for n large enough. It follows that k does not intersect any closed leaf of α_n with a weight at least π . So we have $\int_k d\lambda_n = \int_k d\alpha_n \rightarrow \int_k d\lambda' = \int_k d\lambda$. \square

An arc k is *generic* if it is transverse to every simple geodesic of S . Especially a generic arc is transverse to every geodesic lamination. By [BiS], the union of all simple geodesics of S has Hausdorff dimension 1. It follows that almost every geodesic arc is generic and that every arc can be approximated by a generic arc.

CLAIM 2.2. — *Let $(\lambda_n) \in \mathcal{ML}(\partial M)$ be a sequence of measured geodesic laminations such that (λ_n) converges to $\dot{\lambda}$ and that $|\lambda_n|$ converges to some geodesic lamination L in the Hausdorff topology. We have $|\dot{\lambda}| \subset L$.*

Proof. — Let x be a point of $|\dot{\lambda}|$, let $\varepsilon > 0$ be a real number and let k be a geodesic generic arc intersecting $|\dot{\lambda}|$, with length ε such that x lies in the interior of k . Since we have $\int_k d\lambda' > 0$, we deduce from 2.1 that, for n large enough, we have $\int_k d\lambda_n \geq \frac{\int_k d\lambda'}{2} > 0$. Therefore k intersects λ_n and x lies in an ε -neighbourhood of $|\lambda_n|$. Considering a covering of $|\dot{\lambda}|$ by discs with radius ε and with centres lying in $|\dot{\lambda}|$, we get that, for n large enough, $|\dot{\lambda}|$ lies in an ε -neighbourhood of $|\lambda_n|$. Letting ε tend to 0, we get $|\dot{\lambda}| \subset L$. \square

Claim 2.1 can also be used to prove that the quotient space $\mathcal{ML}(\partial M)/\mathcal{R}$ is Hausdorff.

LEMMA 2.3. — *The space $\mathcal{ML}(\partial M)/\mathcal{R}$ is Hausdorff.*

Proof. — Let $\dot{\lambda}$ and $\dot{\mu}$ be two elements of $\mathcal{ML}(\partial M)/\mathcal{R}$ such that any neighbourhood of $\dot{\lambda}$ intersects any neighbourhood of $\dot{\mu}$. So there is a sequence of measured geodesic laminations λ_n such that $\dot{\lambda}_n$ converges simultaneously to $\dot{\lambda}$ and to $\dot{\mu}$. Let us denote by $\lambda^{(p)}$ (resp. $\mu^{(p)}$) the union of the compact leaves of λ (resp. μ) with a weight at least π . Let λ' (resp. μ') be the representative of $\dot{\lambda}$ (resp. $\dot{\mu}$) whose compact leaves have all a weight at most π . Let $k \subset \partial M - \lambda^{(p)}$ be a generic arc intersecting $|\lambda|$ and $|\mu|$ transversely so that $\int_k d\lambda < \pi$. By Claim 2.1, we have $\int_k d\mu' = \lim \int_k d\lambda_n < \pi$. It follows that $\mu^{(p)} \subset \lambda^{(p)}$. Reversing the roles of λ and of μ , we get $\mu^{(p)} = \lambda^{(p)}$. It follows also from Claim 2.1 that we have the equality $\int_k d\mu = \int_k d\lambda$ for any arc $k \subset \partial M - \lambda^{(p)}$. This yields the conclusion $\dot{\lambda} = \dot{\mu}$. \square

The following variation of Claim 2.1 will be used in the present paper.

CLAIM 2.4. — *If c is a simple closed curve that does not intersect $\lambda^{(p)}$ transversely, then the sequence $i(c, \lambda_n)$ converges to $i(c, \lambda)$.*

Proof. — If the points of c intersects $|\dot{\lambda}|$ transversely, then by Claim 2.2, c intersects $|\dot{\lambda}|$ transversely for n large enough. In this case, we get the conclusion by cutting c into two arcs and by applying Claim 2.1 to these two arcs.

Let us now consider the case where c does not intersect $|\dot{\lambda}|$ transversely. Let $\lambda \in \mathcal{ML}(S)$ be a representative of $\dot{\lambda}$. Consider a simple closed curve c' which is disjoint from $|\lambda|$, which is freely homotopic to c and which is the union of 2 generic arcs. We have $\int_{c'} d\lambda = 0$. By Claim 2.1, the sequence $\int_{c'} d\lambda_n$ converges to $\int_{c'} d\lambda = 0$. Furthermore for any measured geodesic lamination $\gamma \in \mathcal{ML}(S)$, we have $i(c, \gamma) \leq \int_{c'} d\gamma$. Thus we can conclude that $i(c, \lambda_n)$ converges to $0 = i(c, \lambda)$. \square

3. Convex pleated surfaces

A *pleated surface* in a complete hyperbolic 3-manifold M is a map $f : S \rightarrow M$ from a surface S to M with the following properties :

- the path metric obtained by pulling back the hyperbolic metric of M by f is a hyperbolic metric s on S ;
- every point of S lies in the interior of some s -geodesic arc which is mapped into a geodesic arc in M ;

– if $c \subset S$ is a simple closed curve lying in a cusp of S and if c does not bound a disc in S , then $f(c)$ does not bound a disc in M .

A map $\hat{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is a pleated map if any point of \mathbb{H}^2 lies in the interior of a geodesic arc which is mapped by \hat{f} into a geodesic arc.

The *pleating locus* of a pleated map is the set of points of \mathbb{H}^2 where the map fails to be an isometry. The pleating locus of a pleated map is a geodesic lamination (cf. [Th]).

An *abstract pleated surface* is a triple (\hat{f}, Γ, ρ) where $\hat{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is a pleated map, Γ is a lattice in $Isom(\mathbb{H}^2)$ and $\rho : \Gamma \rightarrow Isom(\mathbb{H}^3)$ is a discrete representation (which may not be faithful) such that for any $a \in \Gamma$, we have $\hat{f} \circ a = \rho(a) \circ \hat{f}$ and that if $a \in \Gamma$ is a parabolic isometry then $\rho(a)$ is also a parabolic isometry.

Abstract pleated surfaces and pleated surfaces are related as follows :

When $\rho(\Gamma)$ has no torsion, the abstract pleated surface (\hat{f}, Γ, ρ) induces a pleated surface $f : S \rightarrow M$ where $S \approx \mathbb{H}^2/\Gamma$, $M \approx \mathbb{H}^3/\rho(\Gamma)$ and where f is the quotient map from \hat{f} .

If $f : S \rightarrow M$ is a pleated surface, consider isometric covering maps $\mathbb{H}^2 \rightarrow S$ and $\mathbb{H}^3 \rightarrow M$. These maps yield representations $r : \pi_1(S) \rightarrow Isom(\mathbb{H}^2)$ and $R : \pi_1(M) \rightarrow Isom(\mathbb{H}^3)$ and by lifting f to a map $\hat{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ we get an abstract pleated surface $(\hat{f}, r(\pi_1(S)), R \circ f_*)$.

In the following we will omit the adjective abstract and assume that our (abstract) pleated surfaces are torsion free.

We will consider the following topology on the space of abstract pleated surfaces.

A sequence $(\hat{f}_n, \Gamma_n, \rho_n)$ of pleated surfaces converge to a pleated surface (\hat{f}, Γ, ρ) if and only if :

- (Γ_n) converges geometrically to Γ ;
- for any sequence $a_n \in \Gamma_n$ converging to $a \in \Gamma$, $\rho_n(a_n)$ converges to $\rho(a)$;
- \hat{f}_n converges to \hat{f} on any compact set of \mathbb{H}^2 .

Let (\hat{f}, Γ, ρ) be a (abstract) pleated surface (without torsion), let $\hat{L} \subset \mathbb{H}^2$ be the pleating locus of \hat{f} and let P be a connected component of $\mathbb{H}^2 - \hat{L}$. The surface $\hat{f}(P)$ lies in a geodesic plane Π_P . Given an orientation of \mathbb{H}^2 ,

Π_P inherits a natural orientation and we denote by H_P^+ (resp. H_P^-) the half-space bounded by Π_P such that the union of a direct frame of Π_P and of the inward normal vector to Π_P is a direct frame of \mathbb{H}^3 (resp. indirect).

A pleated surface (\hat{f}, Γ, ρ) with pleating locus \hat{L} is a *convex pleated surface* if :

- 1) there is $\epsilon \in \{+, -\}$ such that for any component P of $\mathbb{H}^2 - \hat{L}$, $\hat{f}(\mathbb{H}^2)$ lies in H_P^ϵ ;
- 2) the interior of $C_{\hat{f}} = \bigcap \{H_{P_i}^\epsilon | P_i \text{ is a connected component of } \mathbb{H}^2 - \hat{L}\}$ is not empty.

If a pleated surface (\hat{f}, Γ, ρ) satisfies 1) but not 2), namely if $C_{\hat{f}}$ as empty interior, we will call it an *even pleated surface*.

In the following, for any convex pleated surface (\hat{f}, Γ, ρ) , we will choose the orientation of \mathbb{H}^2 so that for any component P of $\mathbb{H}^2 - L$, $\hat{f}(\mathbb{H}^2)$ lies in H_P^+ .

LEMMA 3.1. — *The set of the pleated surfaces which are either convex or even is a closed subset of the set of pleated surfaces.*

Proof. — Let $(\hat{f}_n, \Gamma_n, \rho_n)$ be a sequence of convex and even pleated surfaces converging to $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$.

Let us follow [BoO, Lemmas 20 and 21], to show that \hat{f}_∞ is either a convex pleated surface or an even pleated surface. Let us denote by \hat{L}_n the pleating locus of $(\hat{f}_n, \Gamma_n, \rho_n)$ and let us consider a geodesic lamination \hat{L}_∞ which is a limit point of $\{\hat{L}_n/n \in \mathbb{N}\}$ in the Hausdorff topology. By [CEG] the pleating locus of \hat{f}_∞ lies in \hat{L}_∞ . Let us consider a component P_∞ of $\mathbb{H}^2 - \hat{L}_\infty$ and a component P_n of $\mathbb{H}^2 - \hat{L}_n$ such that (P_n) tends to P_∞ . Since $\hat{f}_n(P_n)$ converges to $\hat{f}_\infty(P_\infty)$, up to extracting a subsequence, $H_{P_n}^+$ converges to a half-space $H_{P_\infty}^+$ such that $\hat{f}_\infty(P_\infty) \subset \partial H_{P_\infty}^+$. Since $\hat{f}_n(\mathbb{H}^2)$ converges to $\hat{f}_\infty(\mathbb{H}^2)$, we have $\hat{f}_\infty(\mathbb{H}^2) \subset H_{P_\infty}^+$. Doing this for any component of $\mathbb{H}^2 - L_\infty$, we conclude that \hat{f}_∞ satisfies 1). \square

Let \hat{x} be a point of \mathbb{H}^2 , a *support plane* of $\hat{f}(\mathbb{H}^2)$ at $\hat{f}(\hat{x})$ is a hyperbolic plane $\Pi_{\hat{f}(\hat{x})}$ containing $\hat{f}(\hat{x})$ and such that $\hat{f}(\mathbb{H}^2)$ lies entirely in one of the two half-spaces bounded by $\Pi_{\hat{f}(\hat{x})}$. We will denote this half-space by $H_{\hat{f}(\hat{x})}^+$.

Let $\hat{k} \subset \mathbb{H}^2$ be a compact geodesic segment, a *polygonal approximation to $\hat{f}(\hat{k})$* is a finite family $\mathfrak{P} = \{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)}) | i = 1, \dots, p\}$ such that :

- $\partial\hat{k} = \{\hat{x}_1, \hat{x}_p\}$;
- the \hat{x}_i are ordered points of \hat{k} ;
- $\Pi_{\hat{f}(\hat{x}_i)}$ is a support plane at $\hat{f}(\hat{x}_i)$;
- $\Pi_{\hat{f}(\hat{x}_i)} \cap \Pi_{\hat{f}(\hat{x}_{i+1})} \neq \emptyset$ for any $i = 1, \dots, p-1$;
- if $\hat{x}_{i-1} = \hat{x}_i = \hat{x}_{i+1}$, then either $\Pi_{\hat{f}(\hat{x}_i)} = \Pi_{\hat{f}(\hat{x}_{i-1})}$ or $\Pi_{\hat{f}(\hat{x}_i)} = \Pi_{\hat{f}(\hat{x}_{i+1})}$ or $\Pi_{\hat{f}(\hat{x}_i)}$ intersects the interior of $H_{\hat{f}(\hat{x}_{i+1})}^+ - H_{\hat{f}(\hat{x}_{i-1})}^+$ (the planes $\Pi_{\hat{f}(\hat{x}_i)}$ are “ordered”);
- $\Pi_{\hat{f}(\hat{x}_i)} \cap \Pi_{\hat{f}(\hat{x}_{i+1})}$ contains a geodesic \hat{d} such that the nearest point retraction from \hat{d} to $\hat{f}(\mathbb{H}^2)$ intersects the sub-arc of $\hat{f}(\hat{k})$ joining $\hat{f}(\hat{x}_i)$ to $\hat{f}(\hat{x}_{i+1})$.

The integer $p-1$ (the number of components of $\hat{k} - \{\hat{x}_i\}$) is the *length* of the polygonal approximation. We will denote by $\theta(\Pi_{\hat{f}(\hat{x}_i)}, \Pi_{\hat{f}(\hat{x}_{i+1})})$ the internal angle of $H_{\hat{f}(\hat{x}_i)}^+ - H_{\hat{f}(\hat{x}_{i+1})}^+$.

The existence of a polygonal approximation to any arc $\hat{f}(\hat{k})$ intersecting at most once any leaf of $\hat{f}(\hat{L})$ is proved in [CEG].

The *bending measure* $\int_{\hat{k}} d\hat{\lambda}$ along \hat{k} is defined by $\int_{\hat{k}} d\hat{\lambda} = \inf_{\mathfrak{P}} \sum_{i=1}^{p-1} \theta(\Pi_{\hat{f}(\hat{x}_i)}, \Pi_{\hat{f}(\hat{x}_{i+1})})$ where \mathfrak{P} runs over all polygonal approximations to $\hat{f}(\hat{k})$.

It is shown in [EpM, section 1.11] that this defines a transverse measure $\hat{\lambda}$ on the pleating locus of \hat{f} .

A polygonal approximation $\mathfrak{P} = \{(\hat{x}_i, \Pi_i)\}$ to a path $\hat{f}(\hat{k})$ is an (α, s) -*approximation* if

- $\max_{1 \leq i \leq p-1} \theta(\Pi_i, \Pi_{i+1}) < \alpha$ and
- $\max d_{\mathbb{H}^2}(\hat{x}_i, \hat{x}_{i+1}) < s$.

The existence of a (δ, ε) -approximation for any (δ, ε) , and any arc $\hat{f}(\hat{k})$ is shown in [EpM]. In the sequel we will need to have (δ, ε) -approximations with bounded length. The following lemma shows their existence.

LEMMA 3.2. — *Let δ and ε be two positive numbers such that $0 < \varepsilon < \frac{\log 3}{2}$. Let (\hat{f}, Γ, ρ) be a convex pleated surface and let $\hat{k} \subset \mathbb{H}^2$ be an arc that*

intersects the pleating locus $|\hat{\lambda}|$ transversely so that $\hat{f}(\hat{k})$ intersects at most once any leaf of $\hat{f}(|\hat{\lambda}|)$. Then, there is a (δ, ε) -approximation to $\hat{f}(\hat{k})$ with length at most $4(\frac{l(\hat{k})}{\varepsilon} + 1)(\frac{\pi}{\delta} + 1) = B(\varepsilon, \delta, l(\hat{k}))$.

Proof. — Consider the integer p satisfying $p - 1 \leq \frac{l(\hat{k})}{\varepsilon} \leq p$ and choose $p + 1$ ordered points $\hat{x}_1, \dots, \hat{x}_{p+1}$ in $\hat{k} - |\hat{\lambda}|$ such that we have $\{\hat{x}_1, \hat{x}_{p+1}\} \subset \partial \hat{k}$ and $d(\hat{x}_i, \hat{x}_{i+1}) \leq \varepsilon$ for any $i \in \{1, \dots, p + 1\}$. Choose also a support plane $\Pi_{\hat{f}(\hat{x}_i)}$ at $\hat{f}(\hat{x}_i)$ for each $i \in \{1, \dots, p + 1\}$. The first step of the proof will be to extend this family of support planes to obtain a polygonal approximation. There are three possible configuration for the positions of $\Pi_{\hat{f}(\hat{x}_i)}$ and $\Pi_{\hat{f}(\hat{x}_{i+1})}$. Let \hat{k}_i be the sub-arc of \hat{k} joining \hat{x}_i to \hat{x}_{i+1} .

– First configuration : $\Pi_{\hat{f}(\hat{x}_i)}$ intersects $\Pi_{\hat{f}(\hat{x}_{i+1})}$ and $\Pi_{\hat{f}(\hat{x}_i)} \cap \Pi_{\hat{f}(\hat{x}_{i+1})}$ contains a geodesic \hat{d} such that the nearest point retraction from \hat{d} to $\hat{f}(\mathbb{H}^2)$ intersects the arc $\hat{f}(\hat{k}_i) \subset \hat{f}(\hat{k})$. In this configuration, $\{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)}); (\hat{x}_{i+1}, \Pi_{\hat{f}(\hat{x}_{i+1})})\}$ is already a polygonal approximation to $\hat{f}(\hat{k}_i)$.

– Second configuration : $\Pi_{\hat{f}(\hat{x}_i)}$ does not intersect $\Pi_{\hat{f}(\hat{x}_{i+1})}$. Let y be a point of \hat{k}_i and let $\Pi_{\hat{f}(\hat{y})}$ be a support plane at $\hat{f}(\hat{y})$. The 3 half-spaces $H_{\hat{f}(\hat{y})}^-$, $H_{\hat{f}(\hat{x}_i)}^-$ and $H_{\hat{f}(\hat{x}_{i+1})}^-$ intersect the ball $B(\hat{f}(\hat{y}), \varepsilon) \subsetneq B(\hat{f}(\hat{y}), \frac{\log 3}{2})$. By [Ga] (see also [Br]) these 3 half-spaces are not disjoint, hence $\Pi_{\hat{f}(\hat{y})}$ intersects either $\Pi_{\hat{f}(\hat{x}_i)}$ or $\Pi_{\hat{f}(\hat{x}_{i+1})}$. So any support plane at a point of $\hat{f}(\hat{k}_i)$ intersects either $\Pi_{\hat{f}(\hat{x}_i)}$ or $\Pi_{\hat{f}(\hat{x}_{i+1})}$. The arc $\hat{f}(\hat{k}_i)$ can be extended to an arc into the set of all the support planes at $\hat{f}(\hat{k}_i)$. Therefore there is a point $\hat{y} \subset \text{int}(\hat{k}_i)$ and a support plane $\Pi_{\hat{f}(\hat{y})}$ at $\hat{f}(\hat{y})$ intersecting both $\Pi_{\hat{f}(\hat{x}_i)}$ and $\Pi_{\hat{f}(\hat{x}_{i+1})}$ (cf. [KeS, §3.4]). Furthermore when we follow this arc $\Pi(t)$ in the set of support planes joining $\Pi_{\hat{f}(\hat{x}_i)}$ to $\Pi_{\hat{f}(\hat{x}_{i+1})}$, the nearest point retraction from $\Pi(t) \cap \Pi_{\hat{f}(\hat{x}_i)}$ ($\Pi(t) \cap \Pi_{\hat{f}(\hat{x}_i)}$ is a geodesic for $t > 0$ small enough) to $\hat{f}(\mathbb{H}^2)$ moves from $\hat{f}(\hat{x}_i)$ in the direction of $\hat{f}(\hat{x}_{i+1})$ transversely to $\hat{f}(\hat{k}_i)$. It follows that the nearest point retraction from $\Pi_{\hat{f}(\hat{x}_i)} \cap \Pi_{\hat{f}(\hat{y})}$ to $\hat{f}(\mathbb{H}^2)$ intersects the sub-arc of $\hat{f}(\hat{k}_i)$ joining $\hat{f}(\hat{x}_i)$ to $\hat{f}(\hat{y})$. With a similar argument, we get that the nearest point retraction from $\Pi_{\hat{f}(\hat{y})} \cap \Pi_{\hat{f}(\hat{x}_{i+1})}$ to $\hat{f}(\mathbb{H}^2)$ intersects the sub-arc of $\hat{f}(\hat{k}_i)$ joining $\hat{f}(\hat{y})$ to $\hat{f}(\hat{x}_{i+1})$. Therefore $\{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)}); (\hat{y}, \Pi_{\hat{f}(\hat{y})}); (\hat{x}_{i+1}, \Pi_{\hat{f}(\hat{x}_{i+1})})\}$ is a polygonal approximation to $\hat{f}(\hat{k}_i)$.

– Third configuration : $\Pi_{\hat{f}(\hat{x}_i)}$ intersects $\Pi_{\hat{f}(\hat{x}_{i+1})}$ but for any geodesic $\hat{d} \subset \Pi_{\hat{f}(\hat{x}_i)} \cap \Pi_{\hat{f}(\hat{x}_{i+1})}$, the nearest point retraction from \hat{d} to $\hat{f}(\mathbb{H}^2)$ does not inter-

sect $\hat{f}(\hat{k}_i)$. If any support plane at a point of $\hat{f}(\hat{k}_i)$ intersects either $\Pi_{\hat{f}(\hat{x}_i)}$ or $\Pi_{\hat{f}(\hat{x}_{i+1})}$ then we can find, as in the preceding case, a point \hat{y} and a support plane $\Pi_{\hat{f}(\hat{y})}$ such that $\{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)}); (\hat{y}, \Pi_{\hat{f}(\hat{y})}); (\hat{x}_{i+1}, \Pi_{\hat{f}(\hat{x}_{i+1})})\}$ is a polygonal approximation to $\hat{f}(\hat{k}_i)$. Otherwise, there is a point $\hat{y} \in \hat{k}$ and a support plane $\Pi_{\hat{f}(\hat{y})}$ at $\hat{f}(\hat{y})$ such that $\Pi_{\hat{f}(\hat{y})}$ does not intersect $\Pi_{\hat{f}(\hat{x}_i)}$ nor $\Pi_{\hat{f}(\hat{x}_{i+1})}$. The planes $\Pi_{\hat{f}(\hat{x}_i)}$ and $\Pi_{\hat{f}(\hat{y})}$ are in the second configuration, so there is a point $\hat{z} \subset \hat{k}_i$ between \hat{x}_i and \hat{y} such that $\{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)}); (\hat{z}, \Pi_{\hat{f}(\hat{z})}); (\hat{y}, \Pi_{\hat{f}(\hat{y})})\}$ is a polygonal approximation to the sub-arc of \hat{k} joining \hat{x}_i to \hat{y} . Doing the same for $\Pi_{\hat{f}(\hat{y})}$ and $\Pi_{\hat{f}(\hat{x}_{i+1})}$ we get a polygonal approximation to $\hat{f}(\hat{k}_i)$.

Let us do the construction above for all the components of $\hat{k} - \{\hat{x}_1, \dots, \hat{x}_p\}$. In each component of $\hat{k} - \{\hat{x}_1, \dots, \hat{x}_p\}$, we have added at most 3 points. So the resulting polygonal approximation has a length smaller than $4p$. Let us denote by $\{(\hat{x}_i, \Pi_{\hat{f}(\hat{x}_i)})/i = 1, \dots, 4p+1\}$ this polygonal approximation and let us denote by $\hat{k}_i \subset \hat{k}$ the geodesic arc joining \hat{x}_i to \hat{x}_{i+1} . Consider $q \in \mathbb{N}$ satisfying $q - 1 \leq \frac{\theta(\Pi_{\hat{f}(\hat{x}_i)}, \Pi_{\hat{f}(\hat{x}_{i+1})})}{\delta} \leq q$. Since $\theta(\Pi_{\hat{f}(\hat{x}_i)}, \Pi_{\hat{f}(\hat{x}_{i+1})}) \leq \pi$ we have $q \leq \frac{\pi}{\delta} + 1$. We have already seen that we can extend an arc $\hat{f}(\hat{k}_i)$ to an arc in the set of all the support planes at $\hat{f}(\hat{k}_i)$. This implies that there are points $\hat{y}_j \in \hat{k}_i$, $1 \leq j \leq q+1$, such that we have $\hat{y}_1 = \hat{x}_i$, $\hat{y}_{q+1} = \hat{x}_{i+1}$ and $\theta(\Pi_{\hat{f}(\hat{y}_j)}, \Pi_{\hat{f}(\hat{y}_{j+1})}) \leq \delta$. Choosing such points for each arc \hat{k}_i , $1 \leq i \leq 4p$, we get a (δ, ε) -approximation with length smaller than $4pq$. Since $p \leq \frac{l(\hat{k})}{\varepsilon} + 1$ and $q \leq \frac{\pi}{\delta} + 1$, this polygonal approximation satisfies the conclusion of Lemma 3.2. \square

The following proposition of [KeS] gives an estimate of the error which is made when approximating the bending measure :

PROPOSITION 3.3 (KeS, Proposition 4.8). — *There is a universal constant K , and a function $s(\alpha)$, $0 < s(\alpha) < 1$, such that if $\mathfrak{P} = \{(\hat{x}_i, \pi_i)\}$ is an $(\alpha, s(\alpha))$ -approximation to a path $\hat{f}(\hat{k})$, where $\alpha < \frac{\pi}{2}$, then we have*

$$\left| \sum_{\mathfrak{P}} \theta(\Pi_i, \Pi_{i+1}) - \int_k d\gamma \right| < K\alpha l(k).$$

Now we will use this proposition to prove the continuity of the bending measured geodesic lamination of a converging sequence of convex pleated surfaces.

LEMMA 3.4. — *Let $(\hat{f}_n, \Gamma_n, \rho_n)$ be a sequence of convex pleated surfaces converging to a pleated surface $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ and let $\hat{\lambda}_n$ be the bending mea-*

asured geodesic lamination of \hat{f}_n . The sequence $(\hat{\lambda}_n)$ converges for the weak* topology to a measured geodesic lamination $\hat{\lambda}_\infty$ and we have one of the following two situations :

- $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is a convex pleated surface and $\hat{\lambda}_\infty$ is its bending measured geodesic lamination ;
- $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is an even pleated surface, $|\hat{\lambda}_\infty|$ is the pleating locus of \hat{f}_∞ and $\hat{\lambda}_\infty$ is obtained by endowing each leaf of $|\hat{\lambda}_\infty|$ with a Dirac mass with a weight equal to π .

Proof. — Notice that when $\mathbb{H}^3/\rho_\infty(\pi_1(M))$ is quasi-isometric to $\mathbb{H}^3/\rho_n(\pi_1(M))$ this result is a consequence of results of [Bo3]. By lemma 3.1, $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is an even or convex pleated surface. We will show that any subsequence of $(\hat{f}_n, \Gamma_n, \rho_n)$ contains a subsequence satisfying the conclusions of lemma 3.4, the conclusion follows from this fact.

Let us choose a subsequence such that $(|\hat{\lambda}_n|)$ converge in the Hausdorff topology to a geodesic lamination \hat{L}_∞ . By [CEG, §5.2], the pleating locus of \hat{f}_∞ lies in \hat{L}_∞ . Let $\hat{k} \subset \mathbb{H}^2$ be an arc intersecting \hat{L}_∞ transversely such that $\hat{f}_\infty(\hat{k})$ intersects at most once any leaf of $\hat{f}_\infty(\hat{L}_\infty)$. Since $(\hat{f}_n(|\hat{\lambda}_n|))$ tends to $\hat{f}_\infty(\hat{L}_\infty)$ in the Hausdorff topology and since (\hat{f}_n) converge to \hat{f}_∞ , for n large enough $\hat{f}_n(\hat{k})$ intersects at most once any leaf of $\hat{f}_n(|\hat{\lambda}_n|)$. Fix $\delta < \frac{\pi}{2}$ and $\varepsilon < s(\delta)$ and choose for each n a (δ, ε) -approximation $(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})$ to $\hat{f}_n(\hat{k})$ whose length is the number $B(\delta, \varepsilon, l(\hat{k}))$ appearing in Lemma 3.2. Extract a subsequence such that for any $i \leq B(\delta, \varepsilon, l(\hat{k}))$, the sequence $(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})$ converges and denote by $(\hat{x}_{i,\infty}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})})$ its limit. Since \hat{f}_n converges to \hat{f}_∞ and since $(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})$ converges to $(\hat{x}_{i,\infty}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})})$, the nearest point retraction from $\Pi_{\hat{f}_n(\hat{x}_{i,n})}$ to $\hat{f}_n(\mathbb{H}^2)$ converges to the nearest point retraction from $\Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}$ to $\hat{f}_\infty(\mathbb{H}^2)$. It follows that $\{(\hat{x}_{i,\infty}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})})\}$ satisfies all the requirements for being a (δ, ε) -approximation to $\hat{f}_\infty(\hat{k})$. Furthermore, the length of $\{(\hat{x}_{i,\infty}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})})\}$ is $B(\delta, \varepsilon, l(\hat{k}))$.

CLAIM 3.5. — *If $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is a convex pleated surface, then $(\hat{\lambda}_n)$ converge to the bending measured geodesic lamination $\hat{\lambda}_\infty$ of $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$.*

Proof. — Assume that $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is a convex pleated surface and let us denote by $\hat{\lambda}_\infty$ its bending measured geodesic lamination. Let us recall that $\hat{\lambda}_n$ tends to $\hat{\lambda}_\infty$ if for any arc $\hat{k} \subset \mathbb{H}^2$ transverse to $|\hat{\lambda}_\infty|$, $(\int_{\hat{k}} d\hat{\lambda}_n)$ tends to $\int_{\hat{k}} d\hat{\lambda}_\infty$.

Let $\hat{k} \subset \mathbb{H}^2$ be an arc transverse to $|\hat{\lambda}_\infty|$ and let $\{(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})\}$ be the (δ, ε) -approximations defined above. For any $i \leq p$, $\Pi_{\hat{f}_n(\hat{x}_{i,n})}$ tends to $\Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}$. Therefore, for n large enough, we have

$\sum_{i=1}^{p-1} |\theta(\Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}, \Pi_{\hat{f}_\infty(\hat{x}_{i+1,\infty})}) - \theta(\Pi_{\hat{f}_n(\hat{x}_{i,n})}, \Pi_{\hat{f}_n(\hat{x}_{i+1,n})})| \leq \delta$. It follows from Proposition 3.3 that we have

$|\sum_{i=1}^{p-1} \theta(\Pi_{\hat{f}_n(\hat{x}_{i,n})}, \Pi_{\hat{f}_n(\hat{x}_{i+1,n})}) - \int_{\hat{k}} d\hat{\lambda}_n| < K\delta l(\hat{k})$ for any $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Hence we have $|\int_{\hat{k}} d\hat{\lambda}_\infty - \int_{\hat{k}} d\hat{\lambda}_n| < 2K\delta l(\hat{k}) + \delta$. Letting δ tend to 0 yields $\int_{\hat{k}} d\hat{\lambda}_n \rightarrow \int_{\hat{k}} d\hat{\lambda}_\infty$. The same is true for any arc \hat{k} transverse to $|\hat{\lambda}_\infty|$, hence $\hat{\lambda}_n$ converges to $\hat{\lambda}_\infty$. \square

Next we will consider the case where $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is an even pleated surface. Let us first show that all the leaves of the pleating locus $|\hat{\lambda}_\infty|$ of $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ are isolated leaves, namely that their projections to $\mathbb{H}^2/\Gamma_\infty$ are isolated leaves.

CLAIM 3.6. — *When $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is an even pleated surface, the pleating locus $|\hat{\lambda}_\infty|$ of \hat{f}_∞ contains only isolated leaves.*

Proof. — Choose two distinct successive points $\hat{x}_{i,\infty}$ and $\hat{x}_{i+1,\infty}$ and denote by \hat{k}_i the sub-arc of \hat{k} joining $\hat{x}_{i,\infty}$ to $\hat{x}_{i+1,\infty}$. Let \hat{y} be a point of $\text{int}(\hat{k}_i)$ and let $(\Pi_{\hat{f}_n(\hat{y})})$ be a sequence of support planes at $\hat{f}_n(\hat{y})$ converging to a support plane $\Pi_{\hat{f}_\infty(\hat{y})}$ at $\hat{f}_\infty(\hat{y})$. For n large enough, \hat{y} lies in the sub-arc of \hat{k} joining $\hat{x}_{i,n}$ to $\hat{x}_{i+1,n}$. Since $\{(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})\}$ is a (δ, ε) -approximation, $\Pi_{\hat{f}_n(\hat{y})}$ intersects both $\Pi_{\hat{f}_n(\hat{x}_{i,n})}$ and $\Pi_{\hat{f}_n(\hat{x}_{i+1,n})}$ and we have $\theta(\Pi_{\hat{f}_n(\hat{x}_{i,n})}, \Pi_{\hat{f}_n(\hat{y})}) \leq \delta$ and $\theta(\Pi_{\hat{f}_n(\hat{y})}, \Pi_{\hat{f}_n(\hat{x}_{i+1,n})}) \leq \delta$. Letting n tend to ∞ yields $\theta(\Pi_{\hat{f}_\infty(\hat{y})}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}) \leq \delta$ and $\theta(\Pi_{\hat{f}_\infty(\hat{y})}, \Pi_{\hat{f}_\infty(\hat{x}_{i+1,\infty})}) \leq \delta$. Since $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is an even pleated surface, the dihedral angle between two adjacent support planes lies in $\{0, \pi\}$. Hence we have $\theta(\Pi_{\hat{f}_\infty(\hat{y})}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}) = 0$ and $\theta(\Pi_{\hat{f}_\infty(\hat{y})}, \Pi_{\hat{f}_\infty(\hat{x}_{i+1,\infty})}) = 0$. It follows that for any arc $\hat{\kappa} \subset \text{int}(\hat{k}_i)$, $\int_{\hat{\kappa}} d\hat{\lambda}$ tends to 0. From the proof of [Bo3, Prop 27] we deduce that $\text{int}(\hat{k}_i)$ does not intersect the pleating locus of \hat{f}_∞ .

So we have shown that the intersection between \hat{k} and the pleating locus of \hat{f}_∞ lies in $\{\hat{x}_i \mid i = 1 \dots p\}$, in particular this intersection has a finite cardinal. \square

Endowing each leaf of $|\hat{\lambda}_\infty|$ with a Dirac measure whose weight is equal to π yields a measured geodesic lamination $\hat{\lambda}_\infty$.

Let r be a lower bound for the set $\{d(\hat{x}_1, \hat{x}_2)/\hat{x}_1$ and \hat{x}_2 are two different points of $\hat{k} \cap |\hat{\lambda}_\infty|\}$. By the proof of Claim 3.6, we can choose $r > 0$.

If $\{(\hat{x}_{i,\infty}, \Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})})\}$ is a (δ, ε) -approximation satisfying $\varepsilon < r$, we have
$$\sum_{i=1}^{q-1} \theta(\Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}, \Pi_{\hat{f}_\infty(\hat{x}_{i+1,\infty})}) = \int_{\hat{k}} d\hat{\lambda}_\infty.$$

In the proof of Claim 3.5, we can use this equality instead of the inequality of proposition 3.3. Thus we get the following claim which concludes the proof of Lemma 3.4 :

CLAIM 3.7. — *The sequence $(\hat{\lambda}_n)$ tends to $\hat{\lambda}_\infty$ in $\mathcal{ML}(\mathbb{H}^2)$.*

Let us notice that in the case where $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ is even, we did not show that the projection of $|\hat{\lambda}_\infty|$ to \mathbb{H}/Γ_∞ is compact. We will see further in the text that this is actually true.

We will now improve the description of $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$ when it is an even pleated surface.

LEMMA 3.8. — *Let $(\hat{f}_n, \Gamma_n, \rho_n)$ be a sequence of convex pleated surfaces converging to an even pleated surface $(\hat{f}_\infty, \Gamma_\infty, \rho_\infty)$. There is a surface S with geodesic boundary, such that $\mathbb{H}^2/\Gamma_\infty$ is the double of S and such that the pleating locus $|\hat{\lambda}_\infty|$ of \hat{f}_∞ project to $\partial S \subset \mathbb{H}/\Gamma_\infty$. Furthermore ∂S is compact.*

Proof. — First the following claim will define S .

CLAIM 3.9. — *All the connected components of $\mathbb{H}^2 - |\hat{\lambda}_\infty|$ have the same image under \hat{f}_∞ .*

Proof. — Recall that since \hat{f}_∞ is a pleated surface, the restriction of \hat{f}_∞ to each connected component of $\mathbb{H}^2 - |\hat{\lambda}_\infty|$ is one to one.

Assume that Claim 3.9 is not true. There exist two connected components \hat{P} and \hat{P}' of $\mathbb{H}^2 - |\hat{\lambda}_\infty|$ whose closures intersect and whose images under \hat{f}_∞ are different. There is a point \hat{y} lying in the boundary of the closure of \hat{P} such that $f_\infty(\hat{y})$ lies in $\hat{f}_\infty(\hat{P}')$ or a point \hat{y} lying in the boundary of the closure of \hat{P}' such that $\hat{f}_\infty(\hat{y})$ lies in $f_\infty(\hat{P})$. This two cases are similar and we will only deal with the first one. Let us denote by \hat{l} the leaf of $|\hat{\lambda}_\infty|$ containing \hat{y} . By the proof of Claim 3.6, \hat{l} is an isolated leaf. Let $\hat{k} \subset \mathbb{H}^2$ be a geodesic arc intersecting \hat{l} transversally so that $\hat{k} \cap |\hat{\lambda}_\infty| = \{\hat{y}\}$. For $n \in \overline{\mathbb{N}}$, let $(\hat{x}_{i,n}, \Pi_{\hat{f}_n(\hat{x}_{i,n})})$ be the (δ, ε) -approximation to $\hat{f}_n(\hat{k})$ constructed at the beginning of the proof of Lemma 3.4 and let us denote by $p = B(\delta, \varepsilon, l(\hat{k}))$ its length. By Claim 3.7, the bending measure on \hat{k} tends to a Dirac measure whose weight is equal to π and whose support is $\{\hat{y}\}$. It follows that there

is i such that $(\hat{x}_{i,n})$ and $(\hat{x}_{i+1,n})$ converge to \hat{y} and such that we have the inequality $\frac{\pi}{p} \leq \theta(\Pi_{\hat{f}_n(\hat{x}_{i,n})}, \Pi_{\hat{f}_n(\hat{x}_{i+1,n})}) \leq \delta$ for n large enough.

Let \hat{z} be the point of \hat{P}' such that we have $\hat{f}_\infty(\hat{z}) = \hat{f}_\infty(\hat{y})$. For any n , \hat{z} lies either in a connected component \hat{P}'_n of $\mathbb{H}^2 - |\hat{\lambda}_n|$ or in the closures of two connected components of $\mathbb{H}^2 - |\hat{\lambda}_n|$, in this second case we will denote by \hat{P}'_n the interior of the union of these two closures. Let us extract a subsequence such that \hat{P}'_n converges to an open subsurface \hat{P}'_∞ of \hat{P}' (for example a subsequence such that $|\hat{\lambda}_n|$ converges in the Hausdorff topology). Since \hat{f}_n converge to \hat{f}_∞ on any compact set, the sequence $\hat{f}_n(\hat{P}'_n)$ converges to $\hat{f}_\infty(\hat{P}'_\infty) \subset \hat{f}_\infty(\hat{P}')$.

The point $\hat{f}_\infty(\hat{y})$ lies in $\hat{f}_\infty(\hat{P}'_\infty)$, in $\Pi_{\hat{f}_\infty(\hat{x}_{i,\infty})}$ and in $\Pi_{\hat{f}_\infty(\hat{x}_{i+1,\infty})}$ and we have the inequality $\frac{\pi}{p} \leq \theta(\Pi_{\hat{f}_n(\hat{x}_{i,n})}, \Pi_{\hat{f}_n(\hat{x}_{i+1,n})}) \leq \delta$. Therefore, for n large enough, one plane among $\Pi_{\hat{f}_n(\hat{x}_{i,n})}$ and $\Pi_{\hat{f}_n(\hat{x}_{i+1,n})}$ intersects $\hat{f}_n(\hat{P}'_n)$ transversely. This contradicts the convexity of $(\hat{f}_n, \Gamma_n, \rho_n)$ and concludes this proof. \square

Next we will show that the quotient of $\mathbb{H}^2 - |\hat{\lambda}_\infty|$ by Γ_∞ has two connected components.

Let us extract a subsequence such that $(|\hat{\lambda}_n|)$ converges in the Hausdorff topology to a geodesic lamination \hat{L}_∞ and let $\hat{x}, \hat{y} \subset \mathbb{H}^2 - \hat{L}_\infty$ be two points whose images $\hat{f}_\infty(\hat{x})$ and $\hat{f}_\infty(\hat{y})$ coincide. Denote by $\hat{P}'_{\hat{x}}$ and $\hat{P}'_{\hat{y}}$ the connected components of $\mathbb{H}^2 - |\hat{\lambda}_\infty|$ containing \hat{x} and \hat{y} and assume that $H_{\hat{f}_\infty(\hat{P}'_{\hat{x}})}^+$ is equal to $H_{\hat{f}_\infty(\hat{P}'_{\hat{y}})}^+$. We will show the existence of an element g_∞ of Γ_∞ such that $\hat{x} = g_\infty \hat{y}$.

The half-spaces $H_{\hat{f}_n(\hat{P}'_{\hat{x}})}^+$ and $H_{\hat{f}_n(\hat{P}'_{\hat{y}})}^+$ converge to $H_{\hat{f}_\infty(\hat{P}'_{\hat{x}})}^+$ and we have $\hat{f}_n(\hat{P}'_{\hat{y}}) \subset H_{\hat{f}_n(\hat{P}'_{\hat{x}})}^+$ and $\hat{f}_n(\hat{P}'_{\hat{x}}) \subset H_{\hat{f}_n(\hat{P}'_{\hat{y}})}^+$. It follows that the planes $\Pi_{\hat{f}_n(\hat{x})}$ and $\Pi_{\hat{f}_n(\hat{y})}$ intersect each other for n large enough and that the sequence $\theta(\Pi_{\hat{f}_n(\hat{x})}, \Pi_{\hat{f}_n(\hat{y})})$ tends to 0. This implies that the distance $d_{(\Pi_{\hat{f}_n(\hat{x})} \cup \Pi_{\hat{f}_n(\hat{y})})}(\hat{f}_n(\hat{x}), \hat{f}_n(\hat{y}))$ measured on $\Pi_{\hat{f}_n(\hat{x})} \cup \Pi_{\hat{f}_n(\hat{y})}$ tends to 0. Since we have $d_{(\Pi_{\hat{f}_n(\hat{x})} \cup \Pi_{\hat{f}_n(\hat{y})})}(\hat{f}_n(\hat{x}), \hat{f}_n(\hat{y})) \geq d_{\hat{f}_n(\mathbb{H}^2)}(\hat{f}_n(\hat{x}), \hat{f}_n(\hat{y}))$ and since \hat{f}_∞ is a local homeomorphism on the complementary regions of $|\hat{\lambda}_\infty|$, there is, for n large enough, an isometry $a_n \in \Gamma_n$ such that $d_{\mathbb{H}^2}(a_n \hat{y}, \hat{x}) \rightarrow 0$. The distance $d_{\mathbb{H}^2}(\hat{y}, a_n \hat{y})$ is bounded, hence, up to extracting a subsequence, (a_n) converges to some $a_\infty \in \Gamma_\infty$. We have then $a_\infty \hat{y} = \hat{x}$.

This implies that any point of $\hat{f}_n(\mathbb{H}^2 - |\hat{\lambda}_\infty|)$ has at most two preimages in $\mathbb{H}^2/\Gamma_\infty$. Combining this with Claim 3.9, we get the existence of a surface S such that $\mathbb{H}^2/\Gamma_\infty$ is the double of S and that the pleating locus $|\hat{\lambda}_\infty|$ of \hat{f}_∞ projects to $\partial S \subset \mathbb{H}^2/\Gamma_\infty$.

It remains to show that ∂S is compact. Assume the contrary, then ∂S contains two asymptotic half geodesics. Let $\hat{k} \subset S$ be a geodesic arc joining these two half geodesics. The double of \hat{k} is a simple closed curve $c \subset \mathbb{H}^2/\Gamma_\infty$ bounding a cusp of $\mathbb{H}^2/\Gamma_\infty$. The image of c under f_∞ is $f_\infty(\hat{k})$ covered twice. It follows that the curve $f_\infty(c)$ is homotopic to a point in $\mathbb{H}^3/\rho_\infty(\Gamma_\infty)$. This contradicts the assumption that the parabolic elements of Γ_∞ are mapped to parabolic isometries by ρ_∞ . \square

We will conclude this section with two lemmas which will be used in the next sections. They are proved in [Le1] (see also [Se]). We will call them slight bending Lemmas.

LEMMA 3.10. — *Let (\hat{f}, Γ, ρ) be a convex pleated surface, let \hat{x}_1 and $\hat{x}_2 \in \mathbb{H}^2$, let $\hat{c}_1 \subset \mathbb{H}^2$ be the geodesic segment joining \hat{x}_1 to \hat{x}_2 and let $\tilde{c}_2 \subset \mathbb{H}^3$ be the geodesic segment joining $\hat{f}(\hat{x}_1)$ to $\hat{f}(\hat{x}_2)$. If there exists $\varepsilon < \frac{\pi}{2}$ such that the bending measure of $\hat{f}(\hat{c}_1)$ is smaller than ε , then $\exists C_\varepsilon$ such that $l(\hat{c}_1) \leq C_\varepsilon l(\tilde{c}_2)$. Furthermore $\lim_{\varepsilon \rightarrow 0} C_\varepsilon = 1$ and the sum of the exterior angles that $\hat{f}(\hat{c}_1)$ and \tilde{c}_2 make at their vertices is smaller than ε .*

LEMMA 3.11 (Slightly bent curves are quasi-geodesics). — *Let (\hat{f}, Γ, r) be a convex pleated surface, let $\hat{\lambda}$ be its bending measured geodesic lamination, let c be a simple closed geodesic of \mathbb{H}^2/Γ and let c^* be the geodesic of $\mathbb{H}^3/r(\Gamma_n)$ in the homotopy class of $f(c)$. For any $\varepsilon < \frac{\pi}{2}$ there exists C_ε and A_ε such that if $i(c, \lambda) \leq \varepsilon$ then $l(c) \leq C_\varepsilon l(c^*) + A_\varepsilon$. Moreover $\lim_{\varepsilon \rightarrow 0} C_\varepsilon = 1$, and $\lim_{\varepsilon \rightarrow 0} A_\varepsilon = 0$.*

4. The continuity of $b_{\mathcal{R}}$

Next we will use the results of the previous section to show the continuity of $b_{\mathcal{R}}$. But first let us precise the definition of the bending measured lamination of a geometrically finite metric $\sigma \in \mathcal{GF}(M)$. Let $\rho : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$ be a representation associated to σ and let $N(\rho)$ be the Nielsen core of ρ . There are a multi-curve $\lambda^{(p)}$ and a natural (relative to σ) homeomorphism $f : \partial_{\chi < 0} M - \lambda^{(p)} \rightarrow \partial N(\rho)$. This homeomorphism f is constructed by using the retraction map from the domain of discontinuity to the boundary of the convex core of the limit set. It is well-defined up to isotopy. The surface $\partial N(\rho)$ is the image of a convex pleated surface and therefore, it has a

bending measured lamination. Let us denote by $\lambda \in \mathcal{ML}(\partial M)$ the image under f^{-1} of this bending measured lamination. Adding the leaves of $\lambda^{(p)}$ endowed with Dirac masses with weights equal to π we get the bending measured geodesic lamination of σ . This measured lamination λ does not depend on the choice of ρ (among the representations associated to σ).

Given a simple closed geodesic c and a hyperbolic metric s on $\partial_{\chi < 0} M$, let $l_s(c)$ be the length of the corresponding s -geodesic. If γ is a weighted simple closed geodesic with a weight $w(\gamma)$ we define $l_s(\gamma)$ by $l_s(\gamma) = w(\gamma)l_s(|\gamma|)$. This function l_s extends continuously to a function $l_s : \mathcal{ML}(\partial M) \rightarrow \mathbb{R}$.

Given a simple closed curve $c \in \partial_{\chi < 0} M$, and a hyperbolic metric σ on $\text{int}(M)$, we denote by c^* the closed σ -geodesic in the free homotopy class of c and by $l_\sigma(c^*)$ its σ -length.

The following proposition shows the continuity of the bending map $b_{\mathcal{R}}$. It is essentially a rephrasing of the statement of Theorem 1. Thus Theorem 1 follows from Proposition 4.1.

PROPOSITION 4.1. — *Let (σ_n) be a sequence of geometrically finite metrics on the interior of M converging to a non fuchsian geometrically finite metric σ_∞ and let λ_n be the bending measured geodesic lamination of σ_n . The sequence λ_n converges to $\lambda \in \mathcal{P}(M)/\mathcal{R}$ and the bending measured geodesic lamination of σ_∞ is the representative λ' of λ that lies in $\mathcal{P}(M)$.*

Proof. — We are going to show that any subsequence of (σ_n) contains a subsequence satisfying the conclusion of the proposition. Let us begin by a result about the curve whose length tends to 0 when n tends to ∞ .

LEMMA 4.2. — *Let (σ_n) be a sequence of geometrically finite metrics on the interior of M converging algebraically and let (s_n) be the sequence of metrics induced on ∂M by a homeomorphism $h_n : M \rightarrow N(\rho_n)^{ep}$ associated to σ_n . Then, there is $K > 0$ such that if $c \subset \partial M$ is a simple closed curve that bounds an essential disc, we have $l_{s_n}(c) \geq K$.*

Proof. — Let $D \subset M$ be an essential disc and let $e \subset \partial M$ be a simple closed curve which does not bound an essential disc. We will say that e intersects ∂D essentially if the ends of any lift of e to \tilde{M} are separated by a lift of ∂D . As in the introduction $\mathcal{P}(M)$ consists of the measured geodesic laminations satisfying conditions a), b) and c).

CLAIM 4.3. — *Let $\gamma \subset \mathcal{P}(M)$ be a weighted multi-curve and let $D \subset M$ be an essential disc; then at least one leaf of γ intersects ∂D essentially.*

Proof. — Let $\partial\tilde{D}$ be a lift of ∂D to $\partial\tilde{M}$. This lift separates $\partial\tilde{M} \approx S^2$ into two discs C and C' . Let us denote by $I_i, 1 \leq i \leq p$ (resp. $I'_i, 1 \leq i \leq p'$) the connected components of $p^{-1}(\gamma) - \partial\tilde{D}$ lying in C (resp. C') whose endpoints lie in $\partial\tilde{D}$. These arcs I_i (resp. I'_i) cut C (resp. C') in $p+1$ discs $C_i, 0 \leq i \leq p$ (resp. in $p'+1$ discs $C'_i, 0 \leq i \leq p'$). The curves ∂C_i and $\partial C'_i \subset \partial\tilde{M}$ bound essential discs in \tilde{M} . It follows from conditions *a*) and *c*) that for any i , we have $\#\{\partial C_i \cap p^{-1}(\gamma)\} \geq 3$ and $\#\{\partial C'_i \cap p^{-1}(\gamma)\} \geq 3$. If no leaf of γ intersects ∂D essentially, then, for any i , each point of $\partial C_i \cap \partial\tilde{D} \cap p^{-1}(\gamma)$ is the endpoint of an I'_j and each point of $\partial C'_i \cap \partial\tilde{D} \cap p^{-1}(\gamma)$ is the endpoint of an I_j . This implies that $3(p+1) \leq 2p'$ and that $3(p'+1) \leq 2p$. We get $p+p' \leq -6$. This contradiction shows that at least one leaf of γ intersects ∂D essentially. \square

Let (c_n) be a sequence of simple closed curves such that each c_n bounds an essential disc and that $l_{s_n}(c_n) \rightarrow 0$. For large n , c_n is the core of a wide Margulis tube $T_n \subset \partial M$. Let $c_{1,n}$ and $c_{2,n}$ be the boundary components of T_n and let us choose T_n so that we have $l_{s_n}(c_{1,n}) = l_{s_n}(c_{2,n}) \rightarrow 0$ and $d_{s_n}(c_{1,n}, c_n) \rightarrow \infty$. Let e be a simple closed curve supporting a transverse measure $\gamma \in \mathcal{P}(M)$. By Claim 4.3, e intersects all the c_n essentially. For each n , let e_n^* be the σ_n -geodesic in the homotopy class of e , let $\tilde{e}_n^*, \tilde{c}_n$ and \tilde{T}_n be lifts of e_n^*, c_n, T_n respectively. Let $\tilde{c}_{1,n}$ and $\tilde{c}_{2,n}$ be the components of $\partial\tilde{T}_n$. Let Π be a geodesic plane containing \tilde{e}_n^* . Let \tilde{k}_n and \tilde{k}'_n be the two connected components of $\tilde{T}_n \cap \Pi$ and let $\tilde{\kappa}_n$ and $\tilde{\kappa}'_n$ be the geodesic segment of Π joining the endpoints of \tilde{k}_n and \tilde{k}'_n , see figure 1.

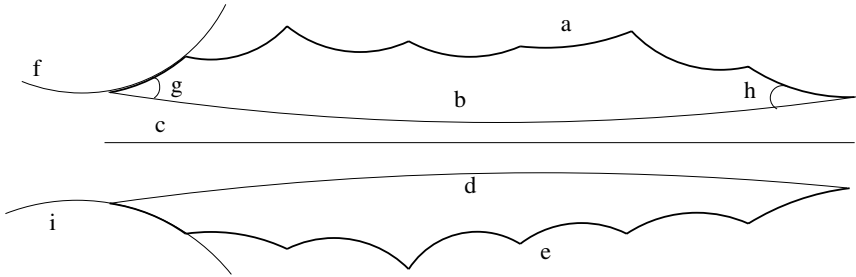


Figure 1. — The section $\Pi \cap C(\rho_n)$

Assume that $\partial\tilde{k}_n$ and $\partial\tilde{k}'_n$ do not lie in the preimage of λ_n . Let \tilde{d}_n and \tilde{d}'_n be the intersections of Π and of support planes at $\tilde{k}_n \cap \tilde{c}_{1,n}$ and at $\tilde{k}'_n \cap \tilde{c}_{1,n}$ respectively. The section $C(\rho_n) \cap \Pi$ lies between \tilde{d}_n and \tilde{d}'_n . Since \tilde{e}_n^* lies in $C(\rho_n)$, \tilde{e}_n^* does not intersect \tilde{d}_n nor \tilde{d}'_n . The arcs $\tilde{\kappa}_n$ and $\tilde{\kappa}'_n$ are very close, this implies that the angle α_1 between \tilde{d}_n and $\tilde{\kappa}_n$ is small. The same

considerations apply for the angle α_2 on the other vertex of $\tilde{\kappa}_n$. Let $\alpha < \frac{\pi}{2}$ be an upper bound for α_1 and α_2 and set $r_n = \max\{d(\tilde{x}, \tilde{\kappa}_n) | \tilde{x} \in \tilde{k}_n\} \leq \max\{d(\tilde{x}, \tilde{e}_n^*) / \tilde{x} \in \tilde{k}_n\}$. Any point \tilde{x} lying in \tilde{k}_n lies in a simple closed curve $\tilde{c}_{x,n} \subset \tilde{T}_n$ homotopic to \tilde{c}_n such that $l_{s_n}(c_{x,n}) \leq l_{s_n}(c_{1,n})$. This curve $\tilde{c}_{x,n}$ bounds a disc with diameter less than $l_{s_n}(c_{x,n})$ which intersects \tilde{e}_n^* . It follows that $r_n \leq l_{s_n}(c_{1,n}) \rightarrow 0$. A computation using Fermi coordinates (compare with [Le1, Lemme A.2]) yields $l_{\sigma_n}(\tilde{k}_n) \leq \sqrt{1 + \tan^2 \alpha} (\text{chr}_n)^2 l_{\sigma_n}(\tilde{\kappa}_n)$. Since the width of T_n tends to infinity, we have $l_{\sigma_n}(\tilde{k}_n) \rightarrow \infty$. So we get $l_{\sigma_n}(\tilde{\kappa}_n) \rightarrow \infty$. Since any point of κ_n is close to the geodesic e_n^* , we have $l_{\sigma_n}(e_n^*) \rightarrow \infty$. But this contradicts the algebraic convergence of (σ_n) . \square

Since (σ_n) converges to σ_∞ , we may choose representations $\rho_n : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^3)$ associated to σ_n such that (ρ_n) converges algebraically to a representation ρ_∞ associated to σ_∞ and that $(\rho_n(\pi_1(M)))$ converges geometrically to $\rho_\infty(\pi_1(M))$. For $n \in \mathbb{N}$, let λ_n be the bending measured geodesic lamination of σ_n . Denote by $\lambda_n^{(p)}$ the union of the leaves of λ_n which have a weight equal to π . Let $M_n = \mathbb{H}^3 / \rho_n(\pi_1(M))$ and let $h_n : M - \lambda_n^{(p)} \rightarrow N(\rho_n)$ be a homeomorphism associated to σ_n .

Let x_∞ be a point in $\partial N(\rho_\infty)$, let $\partial_{x_\infty} N(\rho_\infty)$ be the connected component of $\partial N(\rho_\infty)$ containing x_∞ and let S be a connected component of $\partial M - \lambda_\infty^{(p)}$ satisfying $h_\infty(S) = \partial_{x_\infty} N(\rho_\infty)$. Let $\tilde{x}_\infty \in C(\rho_\infty)$ be a lift of x_∞ and let $\partial_{\tilde{x}_\infty} C(\rho_\infty)$ be the connected component of $\partial C(\rho_\infty)$ containing \tilde{x}_∞ . By [Ta], the convex hull $C(\rho_\infty)$ of L_{ρ_∞} is the limit of $(C(\rho_n))$ with respect to the Hausdorff topology. Hence there exists $\tilde{x}_n \subset \partial C(\rho_n)$ such that (\tilde{x}_n) converges to \tilde{x}_∞ . Let us denote by $\partial_{\tilde{x}_n} C(\rho_n)$ the connected component of $\partial C(\rho_n)$ containing \tilde{x}_n and by $\partial_{\tilde{x}_n} N(\rho_n)$ its projection to $N(\rho_n)$. There is a connected component S_n of $\partial M - \lambda_n^{(p)}$ and a pleated map $g_n : S_n \rightarrow M$ onto $\partial_{\tilde{x}_n} N(\rho_n)$. Let $(\hat{g}_n, \Gamma_n, r_n)$ be the pleated surface that lifts g_n , namely the map $\hat{g}_n : \mathbb{H}^2 \rightarrow \partial_{\tilde{x}_n} C(\rho_n)$ is onto, $r_n : \Gamma_n \rightarrow \rho_n(\pi_1(M))$ is onto the stabiliser of $\partial_{\tilde{x}_n} C(\rho_n)$ and the quotient map $\mathbb{H}^2 / \ker(r_n) \rightarrow \partial_{\tilde{x}_n} C(\rho_n)$ is a homeomorphism. By [CEG, Theorem 5.2.2] and Lemma 4.2 a subsequence of the sequence of pleated surfaces $(\hat{g}_n, \Gamma_n, r_n)$ converges to a pleated surface $(\hat{g}_\infty, \Gamma_\infty, r_\infty)$.

By Lemma 3.1, $(\hat{g}_\infty, \Gamma_\infty, r_\infty)$ is a convex or even pleated surface. Since $\tilde{x}_\infty \in \hat{g}_\infty(\mathbb{H}^2)$, $\hat{g}_\infty(\mathbb{H}^2) \subset \partial_{\tilde{x}_\infty} C(\rho_\infty)$ and $r_\infty(\Gamma_\infty) \subset \rho_\infty(\pi_1(M))$ is a subgroup of the stabiliser of $\partial_{\tilde{x}_\infty} C(\rho_\infty)$.

LEMMA 4.4. — *The quotient map $g_\infty : \mathbb{H}^2 / \Gamma_\infty \rightarrow \partial_{x_\infty} N(\rho_\infty)$ is a homeomorphism.*

Proof. — Set $C_{\hat{g}_n} = \bigcap \{H_{P_i}^\epsilon / P_i \mid \text{is a connected component of } \mathbb{H}^2 - \hat{L}\}$. For any $n \in \mathbb{N}$, we have $C(\rho_n) \subset C_{\hat{g}_n}$. Since σ_n converges to σ_∞ , it follows from [Ta] that $C(\rho_n)$ converges to $C(\rho_\infty)$ in the Hausdorff topology. So we get $C(\rho_\infty) \subset C_{\hat{g}_\infty}$. Since ρ_∞ is not fuchsian, we have $\text{int}(C(\rho_\infty)) \neq \emptyset$. Therefore $\text{int}(C_{\hat{g}_\infty}) \neq \emptyset$ and $(\hat{g}_\infty, \Gamma_\infty, r_\infty)$ is a convex pleated surface. By [EpM], this implies that g_∞ is a covering map.

Let $\hat{\gamma}_n$ be the bending geodesic lamination of \hat{g}_n . Extract a subsequence such that $(|\hat{\gamma}_n|)$ converges to some geodesic lamination \hat{L}_∞ in the Hausdorff topology.

Assume that g_∞ is not a homeomorphism. There are two points $z \neq y \in (\mathbb{H}^2 - \hat{L}_\infty) / r_\infty(\Gamma_\infty)$ such that $g_\infty(z) = g_\infty(y)$. If we lift the situation to \mathbb{H}^2 , we get $\hat{z}, \hat{y} \in \mathbb{H}^2 - \hat{L}_\infty$ and $\rho_\infty(a) \in \rho_\infty(\pi_1(M))$ such that $\hat{z} \notin \Gamma_\infty \hat{y}$ and that $\hat{g}_\infty(\hat{z}) = \rho_\infty(a) \circ \hat{g}_\infty(\hat{y})$. The sequences $\tilde{z}_n = \hat{g}_n(\hat{z})$ and $\tilde{y}_n = \rho_n(a) \circ \hat{g}_n(\hat{y})$ converge simultaneously to $\hat{g}_\infty(\hat{z})$. Let $\Pi_{\tilde{z}_n}$ and $\Pi_{\tilde{y}_n}$ be support planes of $C_n(\rho_n)$ at \tilde{z}_n and \tilde{y}_n . The half-spaces $(H_{\Pi_{\tilde{z}_n}}^+)$ and $(H_{\Pi_{\tilde{y}_n}}^+)$ tend to half-spaces $H_{\tilde{z}}^+$ and $H_{\tilde{y}}^+$ respectively. These two half-spaces $H_{\tilde{z}}^+$ and $H_{\tilde{y}}^+$ are bounded by the support plane $\Pi_{\hat{g}_\infty(\hat{z})}$ at $\hat{g}_\infty(\hat{z})$. Thus we have either $H_{\tilde{z}}^+ \cap H_{\tilde{y}}^+ = \Pi_{\hat{g}_\infty(\hat{z})}$, or $H_{\tilde{z}}^+ = H_{\tilde{y}}^+$.

Since $C(\rho_\infty) \subset C_{\hat{g}_\infty} \subset H_{\tilde{z}_n}^+ \cap H_{\tilde{y}_n}^+$ and since $\rho_\infty(\pi_1(M))$ is not fuchsian, we have $H_{\tilde{z}}^+ \cap H_{\tilde{y}}^+ \neq \Pi_{\hat{g}_\infty(\hat{z})}$. So we have $H_{\tilde{z}}^+ = H_{\tilde{y}}^+$. This implies that for large n , if $\Pi_{\tilde{z}_n}$ and $\Pi_{\tilde{y}_n}$ are disjoint then we have either $H_{\Pi_{\tilde{z}_n}}^+ \subsetneq H_{\Pi_{\tilde{y}_n}}^+$ or $H_{\Pi_{\tilde{y}_n}}^+ \subsetneq H_{\Pi_{\tilde{z}_n}}^+$. We get a contradiction with the fact that we have $\tilde{y}_n \subset C(\rho_n) \subset H_{\Pi_{\tilde{z}_n}}^+$ and $\tilde{z}_n \subset C(\rho_n) \subset H_{\Pi_{\tilde{y}_n}}^+$. We deduce that, up extracting a subsequence, $\Pi_{\tilde{z}_n}$ intersects $\Pi_{\tilde{y}_n}$. It follows that, for n large enough, $\hat{g}_n(\mathbb{H}^2)$ intersects $\rho_n(a) \circ \hat{g}_n(\mathbb{H}^2)$. Therefore $\hat{g}_n(\mathbb{H}^2)$ and $\rho_n(a) \circ \hat{g}_n(\mathbb{H}^2)$ coincide. Since $H_{\Pi_{\tilde{y}_n}}^+$ is equal to $H_{\Pi_{\tilde{z}_n}}^+$, the dihedral angle $\theta(\Pi_{\tilde{z}_n}, \Pi_{\tilde{y}_n})$ tends to 0. This implies that the distance between \tilde{z}_n and \tilde{y}_n measured on $\hat{g}_n(\mathbb{H}^2)$ tends to 0. Since \hat{g}_∞ is a covering map, there is a neighbourhood $\mathcal{V}(\hat{z}) \subset \mathbb{H}^2$ of \hat{z} such that, for $n \in \bar{\mathbb{N}}$ large enough, the map $\hat{g}_n|_{\mathcal{V}(\hat{z})} : \mathcal{V}(\hat{z}) \rightarrow \hat{g}_n(\mathcal{V}(\hat{z}))$ is a homeomorphism. For n large enough, \hat{y}_n lies in $\hat{g}_n(\mathcal{V}(\hat{z}))$. Hence there is $\hat{y}'_n \in \mathcal{V}(\hat{z})$ such that $\hat{g}_n(\hat{y}'_n) = \hat{y}_n$. Since $\hat{g}_n(\hat{y}'_n) = \rho_n(a) \circ \hat{g}_n(\hat{y})$ and since $r_n(\Gamma_n)$ is the stabiliser of $\hat{g}_n(\mathbb{H}^2)$ in $\rho_n(\pi_1(M))$, there is $a_n \in \Gamma_n$ such that $a_n \hat{y} = \hat{y}'_n$. The point \hat{y}'_n lies in $\mathcal{V}(\hat{z})$, so a_n moves \hat{y} a bounded distance. Therefore there is a subsequence such that (a_n) converge to an isometry $a_\infty \in \Gamma_\infty$. Moreover $\hat{g}_n(\hat{y}'_n)$ tends to $\hat{g}_\infty(\hat{z})$, hence (\hat{y}'_n) tend to \hat{z} . Thus we get $a_\infty \hat{y} = \hat{z}$. This yields a contradiction with the assumption that $\hat{z} \notin \Gamma_\infty \hat{y}$ and concludes the proof of Lemma 4.4. \square

Let F be a compact subset of S such that the connected components of $S - F$ are infinite annuli. Since Γ_n converges geometrically to Γ_∞ , there are $\varepsilon > 0$ and maps $l_n : F \rightarrow \mathbb{H}^2/\Gamma_n$ with the following properties :

- l_n is a homeomorphism onto its image ;
- $l_n(F)$ is a connected component of the ε -thick part of \mathbb{H}^2/Γ_n ;
- the induced representations $l_{n*} : \pi_1(F) \rightarrow \Gamma_n$ converge to a faithful representation $l_{\infty*} : \pi_1(F) \rightarrow \Gamma_\infty$;
- $g_\infty \circ l_\infty$ coincides with h_∞ on $F \subset \partial M$.

Let us show that, for large n , $g_n \circ l_n : F \rightarrow \partial N(\rho_n)$ is isotopic to h_n . Let \hat{F} be the universal cover of F and let $\hat{l}_n : \hat{F} \rightarrow \mathbb{H}^2$ be a lift of l_n . Since (l_{n*}) converges algebraically, we can choose the \hat{l}_n such that they converge to \hat{l}_∞ on compact sets. Since (\hat{g}_n) converges to \hat{g}_∞ , the sequence $(\hat{g}_n \circ \hat{l}_n)$ converges to $\hat{g}_\infty \circ \hat{l}_\infty$ on compact sets.

Let $x \in \text{int}(M)$. Since σ_n converges to σ_∞ , there are diffeomorphisms $u_n : M \rightarrow M$ isotopic to the identity which satisfy the following : for any $k > 1$ and $r > 0$, there exists $n(k, r)$ such that for $n \geq n(k, r)$, the restriction of u_n to $B(x, r) \subset (M, \sigma_\infty)$ is a k -quasi-isometry into its image in (M, σ_∞) . For $n \in \bar{\mathbb{N}}$, set $M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ and let $x_n \in M_n$ be the projection of the origin $o \in \mathbb{H}^3$. For $n \in \bar{\mathbb{N}}$, the metric σ_n yields an identification between $(\text{int}(M), \sigma_n)$ and M_n such that x is identified with x_n . Thus we can consider the restriction of u_n to $\text{int}(M)$ as an homeomorphism $u_n : M_\infty \rightarrow M_n$.

Let $\tilde{u}_n : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ be a lift of u_n such that $\tilde{u}_n(o) = o$. Since $(\rho_n(\pi_1(M)))$ converges geometrically to $\rho_\infty(\pi_1(M))$, $\tilde{u}_n : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ converges to the identity on compact sets (see [BeP]).

Since $(\hat{g}_n \circ \hat{l}_n)$ converges to $\hat{g}_\infty \circ \hat{l}_\infty$ on compact sets, the sequence $\tilde{u}_n^{-1} \circ \hat{g}_n \circ \hat{l}_n$ converges to $\hat{g}_\infty \circ \hat{l}_\infty$ on compact sets. Furthermore, F has a compact fundamental domain. Hence $u_n^{-1} \circ g_n \circ l_n : F \rightarrow M_\infty$ converges uniformly to $g_\infty \circ l_\infty$ which coincides with h_∞ . It follows that for large n , $u_n^{-1} \circ g_n \circ l_n$ is isotopic to $h_\infty|_F$. Therefore $g_n \circ l_n$ is isotopic to $u_n \circ h_\infty|_F$. Since u_n is isotopic to the identity, $u_n \circ h_\infty|_F$ is isotopic to h_n . Thus we can change l_n by an isotopy such that $g_n \circ l_n$ coincides with h_n on F .

Let $\hat{\gamma}_n$ be the bending measured geodesic lamination of \hat{g}_n and let $\gamma_n \in \mathcal{ML}(\mathbb{H}^2/\Gamma_n)$ be its projection. Let $k \subset F$ be an arc such that $l_\infty(k)$ is transverse to $|\gamma_\infty|$ and let $\hat{k} \subset \hat{F}$ be a lift of k . The arcs $\hat{l}_n(\hat{k})$

converge to $\hat{l}_\infty(\hat{k})$. Since the convex pleated surfaces $(\hat{g}_n, \Gamma_n, r_n)$ converge to $(\hat{g}_\infty, \Gamma_\infty, r_\infty)$, Lemma 3.4 yields $\int_{\hat{l}_n(\hat{k})} d\hat{\gamma}_n \longrightarrow \int_{\hat{l}_\infty(\hat{k})} d\hat{\gamma}_\infty$. Since, for any $n \in \overline{\mathbb{N}}$, $g_n \circ l_n$ coincides with h_n on F , we have $\int_k d\lambda_n = \int_{\hat{l}_n(\hat{k})} d\hat{\gamma}_n$ where λ_n is the bending measured geodesic lamination of σ_n . Thus we get $\int_k d\lambda_n \longrightarrow \int_k d\lambda_\infty$.

Doing the same for each component of $\partial M - \lambda_\infty^{(p)}$, we conclude that for any arc $k \subset \partial M - \lambda_\infty^{(p)}$ transverse to $|\lambda_\infty|$, we have $\int_k d\lambda_n \longrightarrow \int_k d\lambda_\infty$. It follows that any subsequence of $(\dot{\lambda}_n)$ contains a subsequence converging in $\mathcal{ML}(\partial M)/\mathcal{R}$ and that the limit $\dot{\lambda}$ differs from $\dot{\lambda}_\infty$ only on $\lambda_\infty^{(p)}$, namely if λ is a representative of $\dot{\lambda}$ then removing from λ the closed leaves which lie in $\lambda_\infty^{(p)}$ yields the same measured geodesic lamination as the one obtained by removing $\lambda_\infty^{(p)}$ from λ_∞ . Thus, if we show that any leaf of $\lambda_\infty^{(p)}$ is a leaf of λ with a weight at least equal to π , we can conclude that $\dot{\lambda}_n$ converges to $\dot{\lambda}_\infty$.

CLAIM 4.5. — *Let λ be a representative of $\dot{\lambda}$; any leaf c of $\lambda^{(p)}$ is a leaf of λ and has a weight greater than or equal to π (as a leaf of λ).*

Proof. — Let c be a leaf of $|\lambda_\infty^{(p)}| \subset |\lambda_\infty|$. Since $\dot{\lambda}_n$ converges to $\dot{\lambda}$ which satisfies $|\dot{\lambda}| \subset |\lambda_\infty|$, then either c is a leaf of λ and we will denote its weight by $w(c)$, or c is a simple closed curve disjoint from λ and we will take $w(c) = 0$.

We will prove the claim by assuming that $w(c) < \pi$ and by ending in a contradiction.

Let S be a component of $\partial M - \lambda_\infty^{(p)}$ whose closure contains c . From the fact that $r_n(\pi_1(S))$ converges geometrically to $r_\infty(\pi_1(S))$ we deduce that $l_{s_n}(c) \longrightarrow 0$.

In the case where, up to extracting a subsequence, $i(c, \lambda_n)$ is equal to 0 for all n , it follows from [BoO, Lemma 19] that c is a leaf of λ with a weight equal to π . Now we can assume that we have $i(c, \lambda_n) > 0$ for any n large enough. Especially c is not a leaf of λ_n .

Let $\eta > 0$ be a number such that $\pi - \eta$ is greater than $w(c)$. The curve c lies in the boundaries of two surfaces F^1 and F^2 (which may coincide) such that the following holds : $c \subset \text{int}(F^1 \cup F^2)$ and either F^1 (resp. F^2) is a pair of pants satisfying $\text{int}(F^1) \cap |\lambda| = \emptyset$ or there is connected component of λ which is an arational geodesic lamination in F^1 (resp. F^2). We will only deal with the case where F^1 and F^2 are distinct, the other case is handled in the same way. Approximating $|\lambda| \cap F_i$ by an arc with endpoints in c , we can construct a simple closed geodesic d that intersects c in two points and

satisfies $i(d, \lambda - c) \leq \frac{\eta}{8}$. Consider a point x_n^1 (respectively x_n^2) of $d \cap F^1 - \lambda_n$ (respectively $d \cap F^2 - \lambda_n$) lying in the thick part of (F^1, s_n) (respectively (F^2, s_n)). Let k_n^1 and k_n^2 be the connected components of $d - \{x_n^1, x_n^2\}$. We have $\limsup \int_{k_n^i} d\lambda_n \leq i(d, \lambda - c) + w(c)$.

Now we want to find 2 points $y_n^1 \subset k_n^1$ and $y_n^2 \subset k_n^2$ that cut k_n^1 and k_n^2 into 4 arcs κ_n^j , $1 \leq j \leq 4$, such that we have $\limsup \int_{\kappa_n^j} d\lambda_n < \frac{\pi}{2}$ for any j . If $w(c) = 0$ then we have $\limsup \int_{\kappa_n^i} d\lambda_n \leq i(d, \lambda) = i(d, \lambda - c) \leq \frac{\eta}{4} < \frac{\pi}{4}$. Hence any points $y_n^1 \subset k_n^1$ and $y_n^2 \subset k_n^2$ would be suitable. If we have $w(c) > 0$ then c is a leaf of λ . Since we have assumed that c is not a leaf of λ_n , λ_n spirals more and more around c . More precisely, in a neighbourhood $\mathcal{V}(c)$ of c , $\lambda_n \cap \mathcal{V}(c)$ is a family of compact arcs each one spiralling many times toward c and carrying a small measure. It follows that we can find points $y_n^j \subset k_n^j$ such that if κ_n^j , $1 \leq j \leq 4$, are the closure of the components of $d - \{x_n^1, x_n^2, y_n^1, y_n^2\}$ (cf. figure 2), we have $\frac{w(c)}{2} - \frac{\eta}{8} \leq \limsup \int_{\kappa_n^j} d\lambda_n \leq i(d, \lambda - c) + \frac{w(c)}{2} + \frac{\eta}{8} \leq \frac{\pi}{2} - \frac{\eta}{4}$. Roughly, the y_n^j are chosen so that λ_n spirals as many time before y_n^j as it does after.

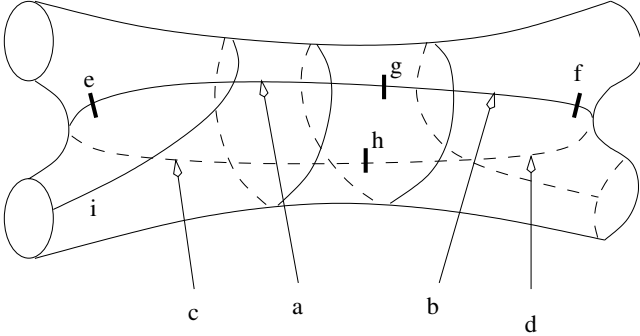


Figure 2. — Cutting d

For $j \in \{1, 2, 3, 4\}$, consider a lift $\tilde{\kappa}_n^j \subset C(\rho_n) \subset \mathbb{H}^3$ of κ_n^j such that $\bigcup_j \tilde{\kappa}_n^j$ is connected. Let $\tilde{d}_n^j \subset \mathbb{H}^3$ be the geodesic segment joining the vertices of $\tilde{\kappa}_n^j$.

If there is j_0 such that $l_{s_n}(\kappa_n^{j_0})$ is bounded, then $\kappa_n^{j_0}$ is entirely contained in the thick part of $(F^1 \cup F^2, s_n)$. Since $g_n(F^1)$ and $f_n(F^2)$ tend to pleated surfaces whose bending measured geodesic laminations are $\lambda \cap F^j$, we have the following inequality $\lim \int_{\kappa_n^{j_0}} d\lambda_n \leq i(d, \lambda - c) \leq \frac{\eta}{8}$. Hence we have $\frac{w(c)}{2} \leq \frac{\eta}{4}$. Thus we get $i(d, \lambda) \leq 2w(c) + \frac{\eta}{4} \leq 2\eta$. Since d intersects c transversally,

and since $l_{s_n}(c) \rightarrow 0$, $l_{s_n}(d)$ tends to ∞ . Taking $\eta < \frac{\pi}{4}$, we get from the slight bending lemmas (Lemma 3.11) that $l_{\sigma_n}(d^*) \geq C_{2\eta}(l_{s_n}(d) + A_{5\eta})$. Thus $l_{\sigma_n}(d^*)$ tends to ∞ contradicting the algebraic convergence of ρ_n .

Thus we have $\forall j, l_{s_n}(\kappa_n^j) \rightarrow \infty$. The slight bending Lemmas (Lemma 3.10) says that there is $C > 0$ such that $l_{s_n}(\tilde{\kappa}_n^j) \leq Cl_{\sigma_n}(\tilde{d}_n^j)$ and that the interior angle between two adjacent segments \tilde{d}_n^j is greater than $\frac{\eta}{4}$. Let a be the element of $\pi_1(M)$ such that $\rho_n(a)$ fixes d^* and let $\tilde{d}_n = \bigcup_{i \in \mathbb{Z}; j=1,2,3,4} a^i(\kappa_n^j)$. The curve \tilde{d}_n is the union of long geodesic segments with interior angles greater than $\frac{\eta}{4}$. It follows from a classical result (see [Ot2] for example) that there is K such that $l(\tilde{d}_n/a)$ is smaller than $Kl_{\sigma_n}(d^*)$. Thus we get $l_{s_n}(d) \leq Cl(\tilde{d}_n/a) \leq CKl_{\sigma_n}(d^*)$. Since $l_{s_n}(d) \rightarrow \infty$, this yields a contradiction with the algebraic convergence of σ_n .

It follows that $\dot{\lambda} = \dot{\lambda}_\infty$. Thus Proposition 4.1 is proved. \square

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