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An overview of curvature bounds and spectral theory of planar tessellations

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An overview of curvature bounds and spectral theory of planar tessellations

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Abstract

We give a survey about the spectral consequences of upper bounds on the curvature on planar tessellating graphs. We first discuss spectral bounds and then put a particular focus on uniformly decreasing curvature. This case is characterized by purely discrete spectrum and we further present eigenvalue asymptotics and exponential decay of eigenfunctions. We then discuss absence of compactly supported eigenfunctions and dependence of the spectrum of the Laplacian on the underlying ℓ^p space.

1. Introduction

In this article we survey results relating curvature bounds and spectral theory. We focus on infinite planar tessellations which can be considered as discrete analogues of non compact surfaces. The tiles of the tessellations are all treated as regular polygons.

We study a curvature function which arises as an angular defect. This notion of curvature is justified by the Gauß Bonnet formula. This idea goes back at least to Descartes, see [16], and appeared since then independently at various places, see e.g. [37, 20, 25, 40]. A substantial amount of research was conducted to study the geometric property of the tessellation in dependence of the curvature, see e.g. [3, 4, 5, 9, 10, 11, 22, 23, 24, 26, 29, 34, 37, 39, 40, 42]. The operator under investigation is the graph Laplacian on the tessellation with constant edge weights. First, we show spectral bounds resulting from curvature bounds. Here, the quantitative bounds result from estimates in [29]. Secondly, we take a closer look at the case of uniformly unbounded negative curvature. This case characterizes discreteness of spectrum, [26], for which we present eigenvalue asymptotics, [6, 7], and decay properties of eigenfunctions, [28]. Thirdly, unique continuation properties of eigenfunctions are discussed, [7, 30], and, finally we summarize results on the p-dependence of the spectrum of the Laplacian as an operator on ℓ^p, $p \in [1, \infty]$.

2. Set up and definitions

In this section we introduce planar tessellations, notions of curvature and the graph Laplacian.

2.1. **Planar tessellations.** Let (V, E) be a simple planar graph embedded into an orientable topological surface $\mathcal{S}$ that is homeomorphic to $\mathbb{R}^2$. We assume the embedded graph is *locally finite*, i.e., for every compact $K \subseteq \mathcal{S}$, one has

$$\#\{e \in E \mid e \cap K \neq \emptyset\} < \infty.$$

In particular, this excludes the situation that a vertex has infinitely many neighbors. In the following we do not distinguish between the graph and its embedding. Nevertheless, we stress that we only use the combinatorial properties of the graph which do not depend on the embedding.

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The set of *faces* F has the connected components of

$$\mathcal{S} \setminus \bigcup E$$

as elements. For $f \in F$, we denote by $\overline{f}$ the closure of f in $\mathcal{S}$. We denote $G = (V, E, F)$. Following [3, 4], we call $G = (V, E, F)$ a *tessellation* if the following three assumptions are satisfied:

(T1) Every edge is contained in two faces.
(T2) Two faces are either disjoint or intersect in a vertex or an edge.
(T3) Every face is homeomorphic to the unit disc.

There are related definitions such as semi-planar graphs see [23, 24] and locally tessellating graphs [27]. Indeed, most of the results presented here hold for general planar graphs on surfaces of finite genus. However, the definition of curvature becomes more involved and some of the estimates turn out to me more technical. Thus, we stick to the 'tame' special case of tessellations.

2.2. **Curvature.** In order to define a curvature function, we first introduce the notation for the vertex degree and the face degree. We denote the *vertex degree* of a vertex $v \in V$ by

$$|v| = \#\text{edges emanating from } v$$

and the *face degree* of a face $f \in F$ by

$$|f| = \#\text{boundary edges of } f = \#\text{boundary vertices of } f.$$

The *vertex curvature* $\kappa : V \to \mathbb{R}$ is defined as

$$\kappa(v) = 1 - \frac{|v|}{2} + \sum_{f \in F, v \in \overline{f}} \frac{1}{|f|}.$$

The idea traces back at least to Descartes [16] and was later introduced in the above form by Stone in [37] referring to ideas of Alexandrov. Since then this notion of curvature was widely used, see e.g. [3, 4, 11, 22, 24, 26, 27, 29, 34, 40, 42]. The notion of curvature is motivated as an angular defect: Assume a face f is a regular polygon. Then, the inner angles of f are all equal to

$$\beta(f) = 2\pi \frac{|f| - 2}{2|f|}.$$

This formula is easily derived as walking around f once results in an angle of 2π, while going around the $|f|$ corners of f one takes a turn by an angle of $\pi - \beta(f)$ each time. In this light the vertex curvature might be rewritten as

$$2\pi\kappa(v) = 2\pi - \sum_{f \in F, v \in \overline{f}} \beta(f), \quad v \in V.$$

Nevertheless, it should be stressed that the mathematical nature of κ is purely combinatorial while assuming a particular nice embedding it allows for a geometric interpretation. The notion has its further justification in the Gauß Bonnet formula which is mathematical folklore and may for instance be found in [3] or [27].

A finer notion of curvature arises when one asks which contribution to the total curvature at a vertex v comes from a corner at a face f with $v \in \overline{f}$. Precisely, the set of *corners* of a tessellation G is given by

$$C(G) = \{(v, f) \in V \times F \mid v \in \overline{f}\}.$$

Define the *corner curvature* $\kappa_C : C(G) \to \mathbb{R}$ by

$$\kappa_C(v, f) = \frac{1}{|v|} - \frac{1}{2} + \frac{1}{|f|}.$$

One immediately infers

$$\kappa(v) = \sum_{f \in F, v \in \overline{f}} \kappa_C(v, f).$$

This notion of curvature was first introduced in [3] and further studied in [4, 27].

2.3. **The Laplacian.** Next, we come to the graph Laplacian. Consider the quadratic form $Q : C(V) \to [0,\infty]$

$$Q(f) = \frac{1}{2}\sum_{v\sim w}|f(v)-f(w)|^2,$$

where $C(V)$ is the space of complex valued functions on V. Choosing the counting measure on V yields a Hilbert space $\ell^2(V)$ of complex valued functions whose absolute value square is summable. The scalar product on $\ell^2(V)$ is given by

$$\langle f,g\rangle = \sum_{v\in V}\overline{f}(v)g(v), \quad f,g\in\ell^2(V),$$

and the norm by $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$. Restricting Q to the subspace

$$\{f\in\ell^2(V) \mid Q(f)<\infty\}$$

yields a positive quadratic form which can seen to be closed. By general theory, there is a positive selfadjoint operator Δ associated to Q which acts as

$$\Delta f(v) = \sum_{w\sim v}(f(v)-f(w))$$

and has the domain

$$D(\Delta) = \{f\in\ell^2(V) \mid \Delta f\in\ell^2(V)\}.$$

The complex valued functions of compact support $C_c(V)$ are dense in $D(\Delta)$, see [41]. It is not hard to see that the operator Δ is bounded if and only if

$$\sup_{v\in V}|v|<\infty.$$

Remark 2.1. There is another common choice of a measure on V, namely the vertex degree function. This way one determines the volume of a set by counting the number of edges (twice the number of edges with both end vertices in the set and ones the number of edges having only one vertex in the set) rather than by the number of vertices in the case of the counting measure. Restricting the quadratic form Q to the corresponding ℓ^2 space yields a different positive selfadjoint operator Δ_n which is always bounded and is often referred to as the normalized Laplacian.

3. The bottom of the spectrum

In this section we turn to the bottom of the spectrum of Δ. In the case where the bottom of the spectrum is strictly positive one speaks of existence of a spectral gap. We discuss that non-negative curvature implies absence of a spectral gap while in the case of negative curvature we show existence of a spectral gap for which we then present quantitative estimates.

Since the operator Δ is a positive selfadjoint operator on the Hilbert space $\ell^2(V)$, its spectrum $\sigma(\Delta)$ is included in the positive half axis $[0,\infty)$. We consider the bottom of the spectrum of Δ

$$\lambda_0(\Delta) = \inf\sigma(\Delta)$$

which, by the Rayleigh-Ritz variational characterization, is equal to

$$\lambda_0(\Delta) = \inf_{f\in D(\Delta),\|f\|=1}\langle f,\Delta f\rangle = \inf_{f\in C_c(V),\|f\|=1}\langle f,\Delta f\rangle,$$

where the second equality follows from the density of $C_c(V)$ in $D(\Delta)$.

3.1. **Non-negative curvature.** The following theorem is a rather immediate consequence of considerations of Jost/Hua/Liu in [24, Theorem 4.1], that every non-negatively curved planar graph has at most quadratic volume growth, and [13, 17, 35], that the bottom of spectrum of graphs with subexponential volume is zero which holds for general graphs. An important technicality is that a lower curvature bound implies boundedness of Δ and, thus, [13, 17, 35] are applicable by [26, Theorem 1].

Theorem 3.1. *Assume $\kappa\geq 0$, then $\lambda_0(\Delta)=0$.*

3.2. **Average negative curvature.** In [40], Woess showed that the isoperimetric constant of a planar tessellation is strictly positive whenever the curvature is negative on average on large sets. He defined

$$\overline{\kappa} = \limsup_{n\to\infty} \inf_{W\subseteq V, n\leq |W|<\infty} \frac{1}{|W|}\sum_{v\in W}\kappa(v).$$

From [40, Theorem 1] and [26, Theorem 1] we infer the next result.

Theorem 3.2. *Assume $\overline{\kappa} < 0$, then $\lambda_0(\Delta) > 0$.*

Remark 3.3. Dodziuk proved in [12] that planar graphs with $|v| \geq 7$, $v \in V$, satisfy $\lambda_0(\Delta) > 0$. In particular, this assumption implies $\kappa < 0$. Independently to the theorem above, but somewhat later, Higuchi [22] showed that $\kappa < 0$ implies $\lambda_0(\Delta) > 0$.

3.3. **Negative curvature.** The theorem above only yields positivity of the bottom of the spectrum. In the case of negative curvature, we get the following quantitative result which is a direct consequence of [29, Theorem 1], [18, Proposition 1] and estimate [26, Theorem 1].

Theorem 3.4. *Assume $\kappa < 0$ and let*

$$K = -\sup_{v\in V}\frac{\kappa(v)}{|v|} \quad \text{and} \quad d = \min_{v\in V}|v|.$$

Then there is a constant $C \geq 1$ specified below such that

$$\lambda_0(\Delta) \geq d(1-\sqrt{1-4C^2K^2}) \geq 2dC^2K^2,$$

where for $p = \sup_{v\in V}|v|$ and $q = \sup_{f\in F}|f|$ the constant C is given by

$$C := \begin{cases} 1 & : \text{ if } q=\infty, \\ 1+\frac{2}{q-2} & : \text{ if } q<\infty \text{ and } p=\infty, \\ (1+\frac{2}{q-2})(1+\frac{2}{(p-2)(q-2)-2}) & : \text{ if } p,q<\infty. \end{cases}$$

Remark 3.5. (a) The second inequality in the theorem follows simply by the Taylor expansion of the square root.
(b) The theorem above can be considered as a discrete analogue to a theorem of McKean [32] who proves for a n-dimensional complete Riemannian manifold M with upper sectional curvature bound $-k$ that the bottom of the spectrum of the Laplace-Beltrami $\Delta_M \geq 0$ satisfies

$$\lambda_0(\Delta_M) \geq (n-1)^2 k/4.$$

(c) A curious fact noted by Higuchi [22], see also [43], is that if $\kappa < 0$, then already $\kappa \leq -1/1806$ which is the case that a triangle, a heptagon and a 43-gon meet in a vertex. Indeed, this implies that $\kappa < 0$ yields for the constant in our theorem $K > 0$ and, therefore, $\lambda_0(\Delta) > 0$.

4. Discrete spectrum, eigenvalue asymptotics and decay of eigenfunctions

In this section we study the case of uniformly decreasing curvature. That is if the quantity

$$\kappa_\infty = \inf_{K\subseteq V \text{finite}} \sup_{v\in V\setminus K} \kappa(v)$$

equals $-\infty$. In this case, we discuss below that the spectrum of Δ consists only of discrete eigenvalues which accumulate at ∞. We denote the eigenvalues in increasing order counted with multiplicity by $\lambda_n(\Delta)$, $n \geq 0$. Moreover, we discuss the asymptotics of the eigenvalues and the exponential decay of eigenfunctions.

4.1. **Discrete spectrum.** The next theorem which is characterizing pure discrete spectrum is found in [26, Theorem 3].

Theorem 4.1. *The spectrum of Δ is purely discrete if and only if $\kappa_\infty = -\infty$.*

Remark 4.2. (a) The theorem above can be considered as a discrete analogue of a theorem of Donnelly/Li [15] which states that a negatively curved, complete Riemannian manifold M with sectional curvature bound decaying uniformly to $-\infty$ the Laplace-Beltrami operator Δ_M has pure discrete spectrum.

(b) In [18] Fujiwara proved a related statement for the normalized Laplacian on trees, namely that spectrum is discrete except for the point 1, where the discrete eigenvalues accumulate.

(c) Wojciechowski [41] showed also discreteness of the spectrum of Δ on general graphs in terms of a different quantity which is sometimes referred to as a mean curvature.

4.2. **Eigenvalue asymptotics.** An important observation in the proof of the theorem above is the following estimate

$$-\frac{|v|}{2} \leq \kappa(v) \leq 1 - \frac{|v|}{6}, \quad v \in V.$$

This inequality implies that $|\cdot|$ and $-\kappa$ go simultaneously to ∞.

In particular, if $\kappa_\infty = -\infty$, then there is a bijective map $\mathbb{N}_0 \to V$, $n \mapsto v_n$, such that

$$|v_n| \leq |v_{n+1}|, \qquad n \geq 0.$$

The following eigenvalue asymptotics are found in the recent work [7] improving the results for planar graphs which were obtained in [19] for trees and for general graphs in [6] .

Theorem 4.3. *If $\kappa_\infty = -\infty$, then*

$$|v_n| - 2\sqrt{|v_n|} \lesssim \lambda_n \lesssim |v_n| + 2\sqrt{|v_n|},$$

that is

$$\lim_{n\to\infty} \frac{\lambda_n}{|v_n|} = 1$$

and

$$-1 \leq \liminf_{n\to\infty} \frac{\lambda_n - |v_n|}{2\sqrt{|v_n|}} \leq \limsup_{n\to\infty} \frac{\lambda_n - |v_n|}{2\sqrt{|v_n|}} \leq 1.$$

Remark 4.4. The only related results we are aware of are found in [33] for the adjacency matrix on sparse finite graphs.

4.3. **Decay of eigenfunctions.** After having treated eigenvalues, we next come to the decay of eigenfunctions. It turns out that eigenfunctions decay exponentially in an ℓ^2 sense, see [28].

Theorem 4.5. *If $\kappa_\infty - \infty$ and $\varphi_n \in D(\Delta)$, $n \geq 0$, are eigenfunctions, i.e.,*

$$\Delta\varphi_n = \lambda_n \varphi_n$$

then, for any $\beta < e^{-1}$ and $o \in V$,

$$|\kappa|^{\frac{1}{2}} e^{\beta d(o,\cdot)} \varphi_n \in \ell^2(V),$$

where $d(\cdot,\cdot)$ is the natural graph distance.

Remark 4.6. The proof is based on ideas based on the work of Agmon for Schrödinger operators in $\mathbb{R}^d$. The somewhat curious constant e^{-1} comes in via an optimization that is caused by the non-locality of the graph Laplacian in contrast to the strongly local Laplace operator on $\mathbb{R}^n$.

5. Unique continuation of eigenfunctions

In Riemannian manifolds very strong unique continuation properties of eigenfunctions hold. Often very subtle quantitative statements can be obtained which all have the basic corollary that there are no eigenfunctions of compact support. However, for graphs such eigenfunctions can be produced rather easily, see e.g. [1, 8, 14, 31] for examples. In this section we discuss that having non-positive corner curvature excludes such eigenfunctions.

5.1. **Absence of compactly supported eigenfunctions.** Klassert/Lenz/Peyerimhoff/Stollmann [30] proved the following unique continuation result for tessellations with non-positive corner curvature, $\kappa_C \leq 0$, which was later generalized to planar graphs in [27] with a different proof.

Theorem 5.1. *If $\kappa_C \leq 0$, then there are no eigenfunctions of compact support.*

We stress that the assumption $\kappa_C \leq 0$ can not be relaxed to $\kappa \leq 0$ (or $\kappa < 0$). This can be seen by the example that contains a $2n$-gon f and has triangles attached at every edge f. Now, given a function which takes the values ± 1 alternating around the vertices f and zero otherwise, can be seen to be a compactly supported eigenfunction to the eigenvalue 6.

5.2. **Finitely many compactly supported eigenfunctions.** One can now ask whether the exponentially decaying eigenfunctions of Theorem 4.5 can be compactly supported. As it can be seen from the theorem above if $\kappa_C \leq 0$ then there are no such eigenfunctions at all. However, $\kappa_\infty = -\infty$ only implies $\kappa_C \leq 0$ outside of a finite set. The following theorem, found in [7], tells us that in this case there can be at most finitely many linearly independent compactly supported eigenfunctions.

Theorem 5.2. *If $\kappa_\infty = -\infty$, then there is a finite set such that every eigenfunction of compact support is supported in this set.*

It can be seen from the proof in [7] that $\kappa_\infty = -\infty$ is not necessary but it is sufficient to have sufficiently negative curvature outside of a finite set. On the other hand, it is also shown in [7] that $\kappa_C \leq 0$ outside of a finite set is not enough.

6. The ℓ^p spectrum

Finally we come to the spectrum of the Laplacian as an operator on $\ell^p(V)$, $p \in [1, \infty]$. For these Banach spaces the Laplacian Δ_p acts as Δ and has the domain

$$D(\Delta_p) = \{\varphi \in \ell^p(V) \mid \Delta\varphi \in \ell^p(V)\}.$$

The operators Δ_p are the generators of the extension of the semigroup $e^{-t\Delta_2}$, $t > 0$, to $\ell^p(V)$, $p \in [1, \infty)$, and Δ_∞ is the adjoint of Δ_1. An important question which was initially brought up by Simon [36] and answered by Hempel/Voigt [21] for Schrödinger operators is whether the spectrum depends on the underlying Banach space. Sturm, [38], addressed this question in the manifold settings in terms of uniform subexponential volume growth. A special case he considers is the case of curvature bounds. An analogous result for graphs is proven in [2]. As a consequence of this theorem and some geometric and functional analytic ingredients one can derive the following theorem which is also found in [2].

Theorem 6.1. (a) *If $\kappa \geq 0$, then $\sigma(\Delta_2) = \sigma(\Delta_p)$ for all $p \in [1, \infty]$.*
(b) *If $-K \leq \kappa < 0$, then $\lambda_0(\Delta_2) \neq \lambda_0(\Delta_1)$.*
(c) *If $\kappa_\infty = -\infty$, then $\sigma(\Delta_2) = \sigma(\Delta_p)$ for all $p \in (1, \infty)$.*

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