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Curvature measures, normal cycles and asymptotic cones

Xiang Sun and Jean-Marie Morvan

Abstract

The purpose of this article is to give an overview of the theory of the *normal cycle* and to show how to use it to define a *curvature measures* on singular surfaces embedded in an (oriented) Euclidean space $\mathbb{E}^3$. In particular, we will introduce the notion of *asymptotic cone* associated to a Borel subset of $\mathbb{E}^3$, generalizing the *asymptotic directions* defined at each point of a smooth surface. For simplicity, we restrict our singular subsets to polyhedra of the 3-dimensional Euclidean space $\mathbb{E}^3$. The coherence of the theory lies in a convergence theorem: If a sequence of polyhedra (P_n) tends (for a suitable topology) to a smooth surface S, then the sequence of curvature measures of (P_n) tends to the curvature measures of S. Details on the first part of these pages can be found in [6].

1. Smooth surfaces and polyhedra

1.1. **Smooth surfaces.** Let us deal with the local *Riemannian* geometry of submanifolds. We endow $\mathbb{E}^3$ with its scalar product $< .,. >$ and its associated Levi-Civita connexion $\tilde{\nabla}$. In our context, an (oriented) *smooth surface* S of the (oriented) Euclidean space $\mathbb{E}^3$ means a 2-dimensional(oriented) C^2-manifold embedded in $\mathbb{E}^3$. We will only deal with closed oriented surfaces bounding a compact domain in $\mathbb{E}^3$. Such surfaces are endowed with a Riemannian structure induced by $< .,. >$. We still denote by $< .,. >$ their metric. With such a structure, the embedding of S in $\mathbb{E}^3$ becomes an isometric embedding. We denote by TS the tangent bundle of S and by ξ be the unit normal vector field compatible with the orientation. The *Weingarten tensor* $A : TS \to TS$, is the symmetric endomorphism defined for all X in TS by $A(X) = -\tilde{\nabla}_X \xi$. The second fundamental form of S is the symmetric tensor defined for every X, Y in TS by $h(X,Y) =< A(X), Y >$. The real function $H = \frac{1}{2}\text{trace } A$ is called the mean curvature of S, and the real function $G = \det A$ is called the Gauss curvature of S.

1.2. **Polyhedra.** We consider here (triangulated) polyhedra as 2-dimensional piecewise linear surfaces embedded in $\mathbb{E}^3$. These polyhedra will be closed, bounding a 3-dimensional domain. The area of the triangles, the length of the edges, the solid angle at vertices and the angle of incident triangles describe their geometry. If P is a polyhedron, we denote by $\mathbf{V}$, resp. $\mathbf{E}$, resp. $\mathbf{T}$ the set of its vertices, resp. edges, resp. triangles. Let us give precise definitions of angles.

Definition 1. Let P be a (triangulated) polyhedron in $\mathbb{E}^3$.

(1) The solid angle of a vertex p of P is defined by
$$\alpha_p = \sum_i \alpha_{i_p},$$
where the α_{i_p} are the angles at p of the triangles t_i incident to p.

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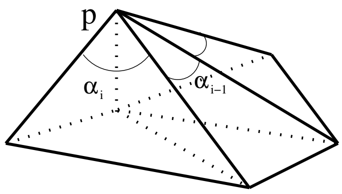
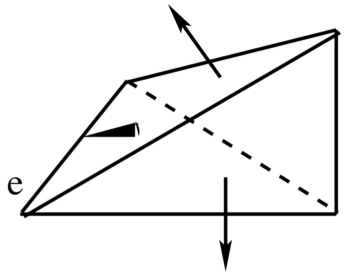


(a) The solid angle at p



(b) The angle between two oriented triangles incident to e

(2) The angle $\angle(e)$ between two oriented triangles t_1 and t_2 (with a common edge e) is the angle angle defined by their oriented normals.

2. A global definition of the mean curvature of a convex polyhedron

As an example, we discuss here a possible definition of the *global mean curvature* of a *convex* polyhedron, by using the well known *Steiner formula* of the volume of tubes (see for instance [6]). This formula gives the behavior of the volume $\mathrm{Vol}_3(\mathcal{K}_\epsilon)$ of the tube $\mathcal{K}_\epsilon$ of radius ϵ of a convex subset $\mathcal{K}$ of $\mathbb{E}^3$. By definition, $\mathcal{K}_\epsilon$ is the set of points at distance less or equal to ϵ of $\mathcal{K}$. Its volume is a polynomial in ϵ, whose coefficients (called the *Quermassintegrale* of Minkowski) depend only on $\mathcal{K}$.

Theorem 2. (1) *Let $\mathcal{K}$ be a convex subset of $\mathbb{E}^3$. Then for all $\epsilon \geq 0$, there exist constants $\Phi_k(\mathcal{K})$, $0 \leq k \leq 3$ such that*

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \sum_{k=0}^{3} \Phi_k(\mathcal{K})\epsilon^k. \tag{2.1}$$

In particular, the coefficients $\Phi_k(\mathcal{K})$'s can be explicitly computed in special situations :

(2) *If $\mathcal{K}$ is a convex domain with smooth boundary $S = \partial\mathcal{K}$, then*

$$Vol_3(\mathcal{K}_\epsilon) = Vol_3(\mathcal{K}) + A(S)\epsilon + (\int_S H da)\epsilon^2 + \frac{4}{3}\pi\epsilon^3. \tag{2.2}$$

(3) *If $\mathcal{K}$ is a convex domain with polyhedral boundary $P = \partial\mathcal{K}$ then,*

$$Vol_3(\mathcal{K}_\epsilon) = Vol_3(\mathcal{K}) + A(P)\epsilon + (\sum_a \angle(a)l(a))\epsilon^2 + \frac{4}{3}\pi\epsilon^3. \tag{2.3}$$

Here, $A(S)$ (resp. $A(P)$) denotes the area of S (resp. P).

Let us compare (2.2) and (2.3). By analogy with the smooth case, one can define the *global mean curvature* of a convex domain $\mathcal{K}$ with polyhedral boundary P as the coefficient $\Phi_2(\mathcal{K})$, and give the following definition, with the previous notations:

Definition 3. The global mean curvature of a convex domain with polyhedral boundary (or simply, if there is no possible confusion, the global mean curvature of a convex polyhedron) of $\mathbb{E}^3$ is the real number

$$\sum_{e \in \mathbf{E}} l(e)\angle(e).$$

We remark that it can be proved that all the Φ_k satisfy the following basic properties: If A and B are convex subsets such that $A \cup B$ and $A \cap B$ are convex, then for all $k \in \{0, 1, 2, 3\}$,

$$\Phi_k(A \cup B) = \Phi_k(A) + \Phi_k(B) - \Phi_k(A \cap B). \tag{2.4}$$

This implies that, in order to compute the curvatures of a complicated convex subset, on can decompose it into simple convex subsets and apply (2.4). Moreover, if C_n is a sequence of convex subsets tending to a convex subset C in the Hausdorff sense, then $\lim_n \Phi_k(C_n) = \Phi_k(C)$. In particular, if P_n is a sequence of convex polyhedra tending to a convex surface S in the Hausdorff sense,

$$\lim_{n} \sum_{e_{k_n} \in \mathbf{E}_n} l(e_{k_n}) \angle(e_{k_n}) = \int_S H da,$$

where da denotes the area form on S.

3. Invariant differential 2-forms and normal cycle

The theory of the *normal cycle* is still in progress. It has been introduced theory by P. Wintgen and M. Zahle to give a general method to define curvatures of a large class of objects, without any assumptions of smoothness or convexity, [7], [8]. It has been successfully developed by many authors, in particular [4], [5]. Curvature measures are defined as integrals of invariant differential forms on a *generalized unit normal bundle.* The *normal cycle* of a singular object is this generalized unit normal bundle. Let us be more precise.

(1) First of all, one the following differential 2-forms on $\mathbb{E}^3 \times \mathbb{E}^3$: If $(x_1, x_2, x_3, y_1, y_2, y_3)$ are the standard coordinates on $\mathbb{E}^3 \times \mathbb{E}^3$ identified with $T\mathbb{E}^3$, we put

$$\begin{aligned} \omega_G &= y_1 dy_2 \wedge dy_3 + y_2 dy_3 \wedge dy_1 + y_3 dy_1 \wedge dy_2; \\ \omega_H &= y_1(dx_2 \wedge dy_3 + dy_2 \wedge dx_3) + y_2(dx_3 \wedge dy_1 + dy_3 \wedge dx_1) \\ &\quad + y_3(dx_1 \wedge dy_2 + dy_1 \wedge dx_2), \end{aligned} \tag{3.1}$$

where $\wedge$ is the exterior product of differential forms.

(2) Then, one defines the normal cycle. The theory being very general, we describe here this construction in very special cases: closed smooth surfaces bounding a domain, compact convex domains and closed polyhedra bounding a domain of $\mathbb{E}^3$.

- Let S be a (closed) surface bounding a domain D of $\mathbb{E}^3$, the unit normal bundle of D is the manifold

$$ST^\perp D = \{(p, \xi_p), p \in S, \xi_p \text{ unit normal vector at } p, \}$$

endowed with the orientation induced by the one of D. The *normal cycle* $N(D)$ of D is nothing but the 2-current canonically associated to $ST^\perp D$: If ω is any 2-differential form defined on $\mathbb{E}^3 \times \mathbb{E}^3$, the duality bracket $<,>$ is given by $< ST^\perp D, \omega >= \int_{ST^\perp D} \omega$.

- Let C be a compact convex domain of $\mathbb{E}^3$. The normal cone $\mathcal{C}_p(C)$ of a point p of C is the set of unit vectors (p, ξ_p) such that

$$\forall q \in C, \overrightarrow{pq} \cdot \xi_p \leq 0.$$

The *normal cone* $\mathcal{C}(C)$ of C is the union of the $\mathcal{C}_p(C)$, when p runs over C. The *normal cycle* $N(C)$ of C is nothing but the 2-current associated to $\mathcal{C}(C)$ endowed with its canonical orientation.

- A crucial property of the normal cycle is its the additivity: if A and B are subsets of $\mathbb{E}^3$ admitting a normal cycle, and such that $A \cap B$ admits a normal cycle, then, $N(A \cup B)$ admits a normal cycle and

$$N(A \cup B) = N(A) + N(B) - N(A \cap B). \tag{3.2}$$

Since a compact 3-polyhedron is the union of (convex) tetrahedra, triangles, edges and vertices, we define the normal cycle of a polyhedron by decomposing it into (convex) simplices, and we apply (3.2). Of course, the result is independent of the decomposition into convex subsets.

We will now define curvature measures in an unified way, which will, in some sense, generalize our previous definitions in the pointwise and global situations, described in the previous paragraphs. These measures will be defined as measures on $\mathbb{E}^3$. We denote by $\mathcal{B}$ the set of Borel subsets of $\mathbb{E}^3$.

Definition 4. Let M be a compact smooth surface or a polyhedron of $\mathbb{E}^3$ bounding a domain D.

(1) The Gaussian curvature measure of M is defined for every $B \in \mathcal{B}$ by

$$\Phi^G_M(B) =< N(D), \chi_{B \times \mathbb{E}^3} \omega_G > .$$

(2) The mean curvature measure of M is defined for every $B \in \mathcal{B}$ by

$$\Phi_M^H(B) = < N(D), \chi_{B \times \mathbb{E}^3} \omega_H >,$$

where $\chi_{B \times \mathbb{E}^3}$ denotes the characteristic function of $B \times \mathbb{E}^3$ in $\mathbb{E}^3 \times \mathbb{E}^3$.

Let us give the explicit expression of these curvature measures for smooth surfaces and for polyhedra, (see [5] for instance).

Theorem 5. (1) *Let S be a smooth closed surface of $\mathbb{E}^3$.*

(a) *The* Gaussian curvature measure *of S is defined for every $B \in \mathcal{B}$ by*

$$\Phi_S^G(B) = \int_{B \cap S} G da.$$

(b) *The* mean curvature measure *of S is defined for every $B \in \mathcal{B}$ by*

$$\Phi_M^H(B) = \int_{B \cap S} H da.$$

(2) *Let P be a closed polyhedron of $\mathbb{E}^3$.*

(a) *The* Gaussian curvature measure *of P is defined for every $B \in \mathcal{B}$ by*

$$\Phi_S^G(B) = \sum_{v \in \mathbf{V} \cap B} \alpha_v.$$

(b) *The* mean curvature measure *of P is defined for every $B \in \mathcal{B}$ by*

$$\Phi_P^H(B) = \sum_{e \in \mathbf{E} \cap B} l(e \cap B) \beta(e),$$

where $\beta(e) \in [-\pi, \pi]$ is the angle between the normals to the triangles incident on e. The sign of $\beta(e)$ is positive if e is convex and negative if e is concave.

In [5], J. Fu proved a theorem related to the convergence of these curvatures. Details on the proof and extensions can be founded in [1], [3]. We cite one of these extensions. We recall that a polyhedron is said to be *closely inscribed* in a surface S if its vertices belong to S, and if the orthogonal projection of P to S is a bijection. If B be the relative interior of an union of triangles, we denote by $pr(B)$ the orthogonal projection of B on S.

Theorem 6. *Let P be a triangulated polyhedron closely inscribed in a smooth surface S. Let B be the relative interior of an union of triangles. Then,*

$$|\Phi_P^G(B) - \Phi_S^G(pr(B))| \leq C_S \mathbf{K} \epsilon; |\Phi_P^H(B) - \Phi_S^H(pr(B))| \leq C_S \mathbf{K} \epsilon,$$

where C_S is a constant depending on the geometry of S, and

$$\mathbf{K} = \sum_{t \subset \overline{B}} cr(t)^2 + \sum_{t \subset \overline{B}, t \cap \partial B \neq \emptyset} cr\ (t), \epsilon = \max\{\ cr(t), t \in \mathbf{T} \cap B\},$$

$cr(t)$ denoting the circumradius of the triangle t.

In other words, if P "approximates" S, the curvatures of P "approximate" the curvatures of S.

4. Asymptotic cones

We denote by $\Xi(\mathbb{E}^3)$ the $\mathcal{C}^\infty$-module of smooth vector fields on $\mathbb{E}^3$, and by $\Xi_S(\mathbb{E}^3)$, (resp. $\Xi_P(\mathbb{E}^3)$) the restriction to S (resp. P) of smooth vector fields on $\mathbb{E}^3$.

Definition 7. If $Z \in \Xi(\mathbb{E}^3)$, the asymptotic 2-form associated to Z is the differential 2-form of $\mathbb{E}^3 \times \mathbb{E}^3$ defined at any point $(p, n) \in \mathbb{E}^3 \times \mathbb{E}^3$ by

$$\mathbf{h}_{(p,n)}^Z = (n \times Z) \wedge Z, \tag{4.1}$$

where $n \times Z$ is the cross product in $\mathbb{E}^3$ of the vector field n (identified with the point n) and the vector field Z.

In this definition, we have used the usual identification of vectors and 1-forms by the standard scalar product. Remark that n is always in the kernel of $\mathbf{h}^Z_{(p,n)}$. Using the duality between 2-forms and 2-currents, we will evaluate $\mathbf{h}^Z$ on the normal cycle of Borel sets of any smooth surfaces S in $\mathbb{E}^3$. Let $\mathbf{G} : S \to \mathbb{E}^3 \times \mathbb{E}^3$ be its *Gauss map* defined for every $p \in S$ by $\mathbf{G}(p) = (p, \xi_p)$. The following is a simple computation, (see [1] or [3] for instance). We denote by pr_{TS} the orthogonal projection on the tangent space of S.

Theorem 8. (1) *Let S be a smooth surface in $\mathbb{E}^3$ bounding a domain D, and $p \in S$.*

(a) *For every vector field $Z \in \Xi_S(\mathbb{E}^3)$,*

$$\mathbf{G}_p^*(\mathbf{h}^Z) = h_p(pr_{TS}Z, pr_{TS}Z)da. \tag{4.2}$$

(b) *Let B be a Borel subset of $\mathbb{E}^3$. For any $Z \in \Xi_S(\mathbb{E}^3)$,*

$$< N(D), \chi_{B\times\mathbb{E}^3}\mathbf{h}^Z >= \int_{B\cap S} h(pr_{TS}Z, pr_{TS}Z)da. \tag{4.3}$$

(2) *Let P be a polyhedron in $\mathbb{E}^3$ bounding a domain D. For any Borel subset B in $\mathbb{E}^3$ and any* constant *vector field Z of $\mathbb{E}^3$,*

$$\begin{aligned} < N(D), \chi_{B\times\mathbb{E}^3}\mathbf{h}^Z > = \sum_{e\in\mathbf{E}} \frac{l(e\cap B)}{2}[(\angle(e) - \sin\angle(e)) < Z, e^+ >^2 \\ + (\angle(e) + \sin\angle(e)) < Z, e^- >^2], \end{aligned} \tag{4.4}$$

where e^+ (resp. e^-) is the normalized sum (resp. difference) of the unit outward normal vectors to the triangles incident to e.

Let M be any smooth surface or polyhedron bounding a domain D, and let $Z \in \Xi_M(\mathbb{E}^3)$. We give the following definition:

Definition 9. The asymptotic curvature measure $\mu_{X,M}$ associated to M and the vector field Z is the signed Borel measure defined for any Borel subset $B \subset \mathbb{E}^3$ by:

$$\mu_{Z,M}(B) =< N(D), \chi_{B\times\mathbb{E}^3}\mathbf{h}^Z > . \tag{4.5}$$

Now let us generalize the notion of asymptotic directions of smooth surfaces to integral currents. Let $D_2(\mathbb{E}^3 \times \mathbb{E}^3)$ denote the space of integral 2-currents. For every $C \in D_2(\mathbb{E}^3 \times \mathbb{E}^3)$, we define the following quadratic function ϕ on the module $\Xi(\mathbb{E}^3)$:

$$\begin{aligned} \phi_C : \Xi(\mathbb{E}^3) &\to \mathbb{R} \\ X &\mapsto < C, X \wedge (n \times X) > . \end{aligned} \tag{4.6}$$

Moreover, identifying the space of constant vector fields $Z \in \Xi_{B\cap S}(\mathbb{E}^3)$ with $\mathbb{E}^3$, we can define for every $C \in D_2(\mathbb{E}^3 \times \mathbb{E}^3)$, the quadratic function ϕ^c_C as the restriction to $\mathbb{E}^3$ of ϕ_C. This leads to the following

Definition 10. The asymptotic cones associated to the current $C \in D_2(\mathbb{E}^3 \times \mathbb{E}^3)$ are the subsets defined by:

$$\mathcal{C}_C = \{Z \in \Xi(\mathbb{E}^3), \phi_C(Z) = 0\}, \text{and } \mathcal{C}^c_C = \mathcal{C}_C \cap \mathbb{E}^3. \tag{4.7}$$

If M is a surface or a polyhedron bounding D, we put

$$\mathcal{C}_B(M) = \mathcal{C}_{N(B\cap D)} \text{ and } \mathcal{C}^c_B(M) = \mathcal{C}_B(D) \cap \mathbb{E}^3.$$

With the previous notations, using Theorem 8, we get

$$\mathcal{C}_B(S) = \{Z \in \Xi(\mathbb{E}^3), \int_{B\cap S} h(pr_{TS}Z, pr_{TS}Z)da = 0\}, \tag{4.8}$$

where h is the second fundamental form of S. Remember that classically, one says that a vector field X tangent to S is asymptotic if

$$h(X, X) = 0. \tag{4.9}$$

It is then natural to define the asymptotic cone over each point $p \in S$ as the cone in $T_p\mathbb{E}^3$ defined by:

$$\mathcal{C}_p(S) = \{Z \in \Xi_S(\mathbb{E}^3), h_p(pr_{T_pS}Z, pr_{T_pS}Z) = 0\}, \tag{4.10}$$

where h_p denotes the second fundamental form of S at the point p. Each $\mathcal{C}_p(S)$ is a degenerated cone (the union of two planes), each one being spanned by an asymptotic direction and by the normal of the surface at p. If P is a polyhedron and B is a Borel subset of $\mathbb{E}^3$, by Theorem 8, we characterize the asymptotic cone $\mathcal{C}^c_B(P)$ of P over B as follows:

$$\begin{aligned}\mathcal{C}^c_B(P) = \{Z \in \mathbb{E}^3, \sum_{e \text{ edge of T}} \frac{l(e \cap B)}{2}[(\angle(e) - \sin\angle(e)) < Z, e^+ >^2 \\ + (\angle(e) + \sin\angle(e)) < Z, e^- >^2] = 0\}.\end{aligned} \tag{4.11}$$

Another simpler cone associated to B demanding less computation can be given by the equation:

$$\sum_{e\in\mathbf{E}} l(e \cap B)\angle(e) < Z, e >^2 = 0. \tag{4.12}$$

Both of them can be used in different contexts. In the smooth case the second one (4.12) is obtained by replacing h by $h \circ j$ in (4.9) where j denotes a rotation of $\frac{\pi}{2}$ in the tangent plane.

With the same techniques as in the proof of Theorem 6, see [5], [1], [3], [2], we get the following result. In the assumptions of the theorem, we introduce the notion of *fatness* of a triangulation. If P is a triangulated polyhedron, the *fatness* of each of its triangle t is the real number $\frac{A(t)}{l^2}$ where $A(t)$ denotes the area of t and l the length of its longest edge. The *fatness* of P is the infimum of the fatness of its triangles. Roughly speaking, the fatness of P is not too small if the angles of its triangles are not too small...

Theorem 11. *Let S be a smooth surface of $\mathbb{E}^3$, let (P_n) be a sequence of (triangulated) polyhedra closely inscribed in S, and let Z be a constant vector field. Suppose that the Hausdorff limit of P_n is S when n tends to infinity, and the fatness of P_n is uniformly bounded from below by a strictly positive constant. Then, for any $Z \in \mathbb{E}^3$,*

$$\lim_{n\to\infty} \mu_{Z,P_n} = \mu_{Z,S}, \tag{4.13}$$

for the weak convergence of measures.

5. Examples

Let us consider the following hyperbolic surface S given by equation:

$$z = 1.1x^2 - y^2. \tag{5.1}$$

Let us draw a triangulation T on it. First of all we select 4 points $p_1, ..., p_4$ on S and build the points $q_1, ..., q_4$ on T such that $pr\ q_i = p_i$, $i = 1, 2, 3, 4$. Then we draw 4 balls $B_1, ..., B_4$ (with the same radius) centered at $q_1, ..., q_4$ respectively. These balls are the Borel sets from which we deduce 4 cones $C_1, ..., C_4$ computed by formula 4.11. These cones are centered at $q_1, ..., q_4$. We get figure 5.1.

Now we only select one point p on S, we build the point q on T such that $pr\ q = p$ and we draw the plane P spanned by the triangle containing q. Associated to some Borel subset, we build the cone C, centered at q. The intersection of P and C is reduced to 2 lines which approximated the asymptotic directions of S at p. We get figure 5.2.

Finally in figure 5.3, we plot some asymptotic lines of S and the approximated asymptotic directions computed by the previous process.

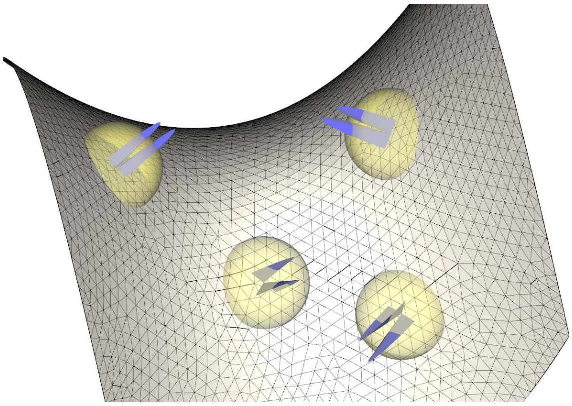
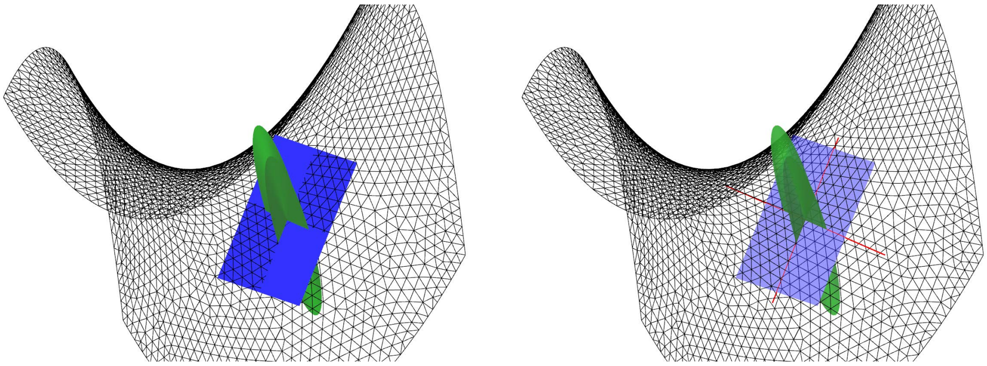


Figure 5.1. Four cones in blue in the center of their corresponding Borel sets in yellow, computed by using the triangulation.



(a) The blue plane is the plane spanned by a triangle of T. It approximated the tangent plane of the smooth surface.

(b) The red lines are the intersections of the cone with the blue plane. They approximate the asymptotic directions of the surface.

Figure 5.2

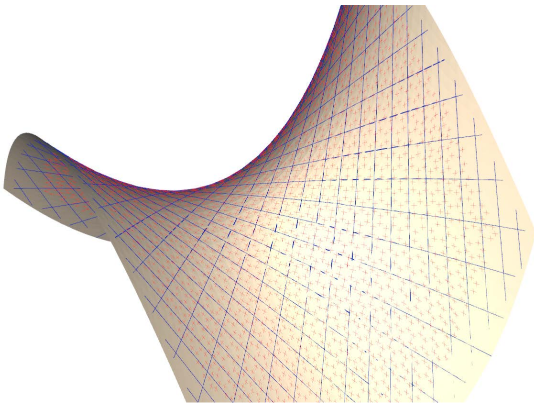


Figure 5.3. The blue lines are some asymtotic lines of S and the red lines are the approximation of asymptotic directions of S by the process described in 5.2.

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