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David F. Anderson, Said El Baghdadi, and Muhammad Zafrullah

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Star operations in extensions of integral domains

David F. Anderson, Said El Baghdadi, and Muhammad Zafrullah

Abstract

An extension $D \subseteq R$ of integral domains is *strongly t-compatible* (resp., *t-compatible*) if $(IR)^{-1} = (I^{-1}R)_v$ (resp., $(IR)_v = (I_vR)_v$) for every nonzero finitely generated fractional ideal I of D. We show that strongly t-compatible implies t-compatible and give examples to show that the converse does not hold. We also indicate situations where strong t-compatibility and its variants show up naturally. In addition, we study integral domains D such that $D \subseteq R$ is strongly t-compatible (resp., t-compatible) for every overring R of D.

SUMMARY[1]

Throughout this summary, let D be an integral domain with quotient field K. Let $F(D)$ be the set of nonzero fractional ideals of D, $f(D)$ the set of nonzero finitely generated fractional ideals of D, and $I(D)$ the set of nonzero integral ideals of D. Recall that a *star operation* $*$ on D is a function $I \mapsto I^*$ on $F(D)$ with the following properties:

If $I, J \in F(D)$ and $0 \neq x \in K$, then

(i) $D^* = D$ and $(xI)^* = xI^*$;
(ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$; and
(iii) $(I^*)^* = I^*$.

For a quick review of properties of star operations, the reader may consult [23, Sections 32 and 34]. An $I \in F(D)$ is said to be a *$*$-ideal* if $I^* = I$, and a $*$-ideal I has *finite type* if $I = J^*$ for some $J \in f(D)$. A star operation $*$ is of *finite type* if $I^* = \bigcup\{J^* \mid J \in f(D) \text{ and } J \subseteq I\}$. To any star operation $*$, we can associate a star operation $*_s$ of finite type by defining $I^{*_s} = \bigcup\{J^* \mid J \in f(D)$ and $J \subseteq I\}$. Clearly $I^{*_s} \subseteq I^*$, and if I is finitely generated, then $I^* = I^{*_s}$.

Recall that for $I \in F(D)$, we have $I^{-1} = D :_K I = \{x \in K \mid xI \subseteq D\}$. The functions defined on $F(D)$ by $I \mapsto I_v = (I^{-1})^{-1}$ and $I \mapsto I_t = \bigcup\{J_v \mid J \in f(D) \text{ and } J \subseteq I\}$ are well known star operations, known as the v- and t-operations. An $I \in F(D)$ is *divisorial* or a *v-ideal* (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$). By definition, the t-operation is the finite-type star operation associated to the v-operation.

Let D be a subring of an integral domain R. We call $D \subseteq R$ an *extension of integral domains* and call R an *overring* of D if $R \subseteq K$. We shall use the v- and t-operations extensively, and we shall assume a working knowledge of these operations. Following [15], an integral domain R is said to be *t-linked* over its subring D if $I^{-1} = D$ implies that $(IR)^{-1} = R$ for every $I \in f(D)$. One reason for writing this article is the following comment in [42, page 443]. "We note that in each of the extensions $D \subseteq R$, discussed above, R is t-linked over D, i.e., for every $I \in f(D), I^{-1} = D$ implies $(IR)^{-1} = R$ ([15]). So in each case, there is a homomorphism $\theta : Cl_t(D) \longrightarrow Cl_t(R)$ defined by $\theta([I]) = [(IR)_t]$ ([3]). However, if R is t-linked over D, the extension $D \subseteq R$ may not satisfy

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[1]This summary was presented by Said El-Baghdadi

any of (a)-(d) and may not satisfy any of the equivalent conditions. (These facts will be included in a detailed account in the promised article.)” The “equivalent conditions" mentioned in the quote are the equivalent conditions of [42, Proposition 2.6]. (The third author thanks Jesse Elliott for reminding him of that promise.) We provide the example(s) hinted at in the above quote. The rest of the plan will be presented after we have given sufficient introduction.

Let v_X- (resp., t_X-) denote the v- (resp., t-) operation on an integral domain X. We have the following theorem.

Theorem 0.1. *Let R be an integral domain with quotient field L, and let D be a subring of R with quotient field K. Then the following statements are equivalent. Moreover, if $R :_L IR = ((D :_K I)R)_{v_R}$ for every $I \in f(D)$, then the following statements hold.*

(1) $I_{v_D}R \subseteq (IR)_{v_R}$ *for every* $I \in f(D)$.
(2) $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in f(D)$.
(3) $I_{t_D}R \subseteq (IR)_{t_R}$ *for every* $I \in F(D)$.
(4) $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ *for every* $I \in F(D)$.
(5) $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
(6) *If I is an integral t-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*
(7) *If I is a principal fractional ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*

According to [8, Proposition 1.1], via [42, Proposition 2.6], conditions (1)-(6) are all equivalent and an extension $D \subseteq R$ of integral domains is called *t-compatible* if it satisfies any of (1)-(6) (e.g., $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ for every $I \in F(D)$). (These are the equivalent conditions hinted at in the quote above.) More generally, as in [4], given star operations $*_D$ and $*_R$ on integral domains $D \subseteq R$, we say that $*_D$ and $*_R$ are *compatible* if $(IR)^{*_R} = (I^{*_D}R)^{*_R}$ for every $I \in F(D)$. Note that v-compatibility implies t-compatibility. We prove that the statements (1)-(7) are all equivalent and that all of them are implied by the hypothesis of Theorem 0.1. We give examples (i) that show that none of (1)-(7) implies the hypothesis of the theorem and examples (ii) that give t-linked overrings that do not satisfy any of (1)-(7) and the conditions (a)-(d) of [42, page 443] which are listed below.

(a) $I^{-1}R = (IR)^{-1}$ for every $I \in f(D)$.
(b) $(I^{-1}R)_{v_R} = (IR)^{-1}$ for every $I \in f(D)$.
(c) $I^{-1}R = (IR)^{-1}$ for every $I \in F(D)$.
(d) $(I^{-1}R)_{v_R} = (IR)^{-1}$ for every $I \in F(D)$.

Clearly $(c) \Rightarrow (a) \Rightarrow (b)$ and $(c) \Rightarrow (d) \Rightarrow (b)$. We determine the overrings of D that are characterized by condition (b) (resp., condition (d)). If D is integrally closed, then (a) holds for every overring R of D if and only if D is a Prüfer domain.

Let us call an extension $D \subseteq R$ of integral domains *strongly t-compatible* if $D \subseteq R$ satisfies the hypothesis of Theorem 0.1 (i.e., if $(IR)^{-1} = (I^{-1}R)_{v_R}$ for every $I \in f(D)$, or equivalently, condition (b) above holds). By Theorem 0.1, strong t-compatibility implies t-compatibility. We indicate situations in which strong t-compatibility and some of its variants appear naturally, and we characterize the domain extensions where strong t-compatibility holds. Finally, we study integral domains D such that $D \subseteq R$ is t-compatible for every overring R of D and relevant notions.

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Department of Mathematics, University of Tennessee
Knoxville, TN 37996, USA • anderson@math.utk.edu

Department of Mathematics, Faculté des Sciences et Techniques
P.O. Box 523, Beni Mellal, Morocco • baghdadi@fstbm.ac.ma

57 Colgate Street, Pocatello, ID 83201, USA • mzafrullah@usa.net