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Computing r-removed P-orderings and P-orderings of order h

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Computing r-removed P-orderings and P-orderings of order h

Keith Johnson

Abstract

We develop a recursive method for computing the r-removed P-orderings and P-orderings of order h, the characteristic sequences associated to these and limits associated to these sequences for subsets S of a Dedekind domain D. This method is applied to compute these objects for $S = \mathbb{Z}$ and $S = \mathbb{Z}\backslash p\mathbb{Z}$.

1. Introduction

Let D be a Dedekind domain, P a prime ideal of D, K the quotient field of D and q the cardinality of D/P. Also, for $x \in D$, let $\nu_P(x)$ denote the largest integer k for which $x \in P^k$. If S is a subset of D then a P-ordering of S, as introduced in [2], is a sequence $\{a_i, i = 0, 1, 2, \ldots\}$ of elements of S with the property that for each $i \geq 1$ the element a_i minimizes $\sum_{j<i} \nu_P(s - a_j))$ over all $s \in S$. Such orderings play a central role in the study of polynomials which are integer valued on subsets of D ([2], [3]), giving a method of constructing a regular basis for this algebra.

In two recent papers, ([4], [5]), two variations on the idea of a P-ordering have been introduced. For r a positive integer, an r-removed P-ordering is a sequence $\{a_i, i = 0, 1, 2, \ldots\}$ in which the first $r + 1$ elements are chosen arbitrarily and then for $i \geq r + 1$ a_i is chosen to minimize $\sum_{j \in A} \nu_p(s - a_j)$ over all $s \in S$ and over all subsets A of $\{a_0, a_1, \ldots, a_{i-1}\}$ of cardinality $i - r$. Similarly for h a positive integer a P-ordering of order h is a sequence $\{a_i, i = 0, 1, 2, \ldots\}$ in which a_i is chosen to minimize $\sum_{j<i} \min(\nu_p(s - a_j, h))$. Just as a P-ordering gives a regular basis for the ring of polynomials integer valued on S, an r removed P-ordering gives a regular basis for the ring of polynomials all of whose divided differences of order less than or equal to r are integer valued on S and a P-ordering of order h gives a regular basis for the ring of polynomials which are integer valued of modulus P^h on S.

For any algebra of polynomials the n-th characteristic ideal is the fractional ideal consisting of 0 and the leading coefficients of polynomials in the algebra of degree less than or equal to n. The sequence of P-adic values of these ideals is called the characteristic sequence of the algebra and an important property of P-orderings is that the sequence of sums $\{\sum_{j<i} \nu_P(a_i - a_j)\}$, called the P-sequence of S, gives the characteristic sequence for the algebra of polynomials integer valued on S, independent of the choice of P-ordering. The corresponding result holds for r-removed P-orderings and P-orderings of order h also.

In [9] a recursive method of constructing P-orderings was used to establish some combinatorial results about them. The sets, S, considered in that case were finite however the construction applies to infinite sets also and it was used in [8] to calculate certain limits associated to the characteristic sequences of infinite sets describing the asymptotic behavior of those characteristic

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sequences. The purpose of the present paper is to show that these methods apply equally well to r-removed P-orderings and to P-orderings of order h and to explore some of the consequences of this.

In more detail the paper is organized as follows: section 2 contains the extensions of lemmas 3.3 and 3.5 of [9] to the cases of r-removed P-orderings and P-orderings of order h. A similar result is also given in ([5], 2.10) for P-orderings of order h as the key step in showing that those algebras are finitely generated. The extension of proposition 7 of [8] is also given in that section. In section 3 these results are applied to obtain orderings and characteristic sequences in the case $S = D = \mathbb{Z}$. The results are:

Theorem 1.1. *The r-removed p-sequence of $\mathbb{Z}$ is, for $n \geq rp$, given by*

$$\alpha(n) = \sum_{i=1}^{k} \lfloor n/p^i \rfloor - kr = \nu_p(n!) - \nu_p(\lfloor n/p^k \rfloor !) - kr$$

where $k = \lfloor log(n/r)/log(p) \rfloor$. If $n < rp$ its value is 0. The p-ordering of $\mathbb{Z}$ of order h is given by

$$\alpha(n) = \sum_{i=1}^{h} \lfloor n/p^i \rfloor = \nu_p(n!) - \nu_p(\lfloor n/p^h \rfloor !).$$

In both cases the usual increasing order on $\mathbb{Z}^{\geq 0}$ gives a p-ordering.

In spite of the similarity of names the algebra of polynomials whose divided differences of order $\leq r$ are integer valued on $\mathbb{Z}$ does not coincide with the algebra of polynomials whose finite differences of order $\leq r$ are integer valued on $\mathbb{Z}$ as studied in [1], [7], and [6] unless $r = 0$ or 1. For higher values of r the former is a proper subalgebra [5] and in section 3 we compare the characteristic sequences of these two algebras and give a formula for the difference between them. This difference is bounded by $\nu_p(r!)$, although that can be shown directly. In section 4 we apply the results of sections 2 and 3 to describe an algorithm for computing the r-removed p-sequences and p-sequences of order h and their limits for $\mathbb{Z}$ and for homogeneous subsets of $\mathbb{Z}$.

2. P-Sequences, Shuffles and Limits

We begin by establishing the analogs of Lemma 3.3 (c) of [9]:

Proposition 2.1. (a) *For any $c \in D$ and any subset $S \subseteq D$ the r-removed P-sequences of S and of $S + c$ are equal.*

(b) *If $\alpha(n)$ is the r-removed P-sequence of $S \subseteq D$ then the r-removed P-sequences of $p \cdot S$ is given by*

$$\begin{cases} \alpha(n) + n - r & \text{if } n \geq r \\ 0 & \text{if } n < r \end{cases}$$

(c) *For any $c \in D$ and any subset $S \subseteq D$ the P-sequences of order h of S and of $S + c$ are equal.*

(d) *If $\alpha(n)$ is the P-sequence of order h of $S \subseteq D$ then the P-sequence of order $h - 1$ of $p \cdot S$ is equal to $\alpha(n) + n$.*

Proof. Since $\sum \nu_p(s - a_j) = \sum \nu_p((s + c) - (a_j + c))$ the map $\phi : S \to c + S$ given by $\phi(x) = c + x$ preserves the value of the sums $\sum \nu_p(s - a_j))$ and so maps P-orderings, r-removed P-orderings and P-orderings of order h bijectively from S to $c + S$. Parts (a) and (c) follow.

Similarly $\sum \nu_p(ps - pa_j) = m + \sum \nu_p(s - a_j)$ where m is the number of terms in the sum. Since $m = n - r$ is constant at the n-th stage in the construction of an r-removed P-ordering, the map $\phi : S \to p \cdot S$ given by $\phi(x) = px$ gives a bijective map between r-removed P-orderings of S and of $p \cdot S$. Part (b) follows since $\sum \nu_p(pa_n - pa_j) = n - r + \sum \nu_p(a_n - a_j)$.

Since $\min(\nu_p(px - py), h) = 1 + \min(\nu_p(x - y), h - 1)$ it follows that $\sum_{j<n} \min(\nu_p(ps - pa_j), h) = n + \sum_{j<n} \min(\nu_p(s - a_j), h - 1)$ and so that the map $\phi : S \to p \cdot S$ given by $\phi(x) = px$ gives a

bijective map between P-orderings of order $h-1$ of S and P-orderings of order h of $p \cdot H$. Part (d) follows. □

Recall from [9] that a shuffle is a finite set of maps $(\phi_1, \ldots, \phi_s)$ with $\phi_i : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ strictly increasing for each i such that the images of the ϕ_i's are disjoint and together cover $\mathbb{Z}^{\geq 0}$. If $\alpha_1(n), \alpha_2(n), \ldots, \alpha_s(n)$ are s sequences then the $(\phi_1, \ldots, \phi_s)$ shuffle of these sequences is the unique sequence $\alpha(n)$ for which $\alpha(n) = \alpha_i(m)$ if $\phi_i(m) = n$. By proposition 4 of [9] if the α_i's are nondecreasing,unbounded sequences in $\mathbb{Z}^{\geq 0}$ then there is a unique sequence $\alpha(n)$ which is the shuffle of the α_i's and is nondecreasing.

Proposition 2.2. (a) *If S is a subset of D which is not contained in a single residue class modulo P then the r-removed P-sequence of S is the nondecreasing shuffle of the r-removed P-sequences of the sets $S_c = S \cap (c + PD)$ where c ranges over those residue classes modulo P for which this intersection is nonempty. The shuffle which combines these P-sequences to give that of S will, when applied to a set of r-removed P-orderings for the sets S_c give an r-removed P-ordering of S.*

(b) *If S is a subset of D which is not contained in a single residue class modulo P then the P-sequence of order h of S is the nondecreasing shuffle of the P-sequences of order h of the sets $S_c = S \cap (c + PD)$ where c ranges over those residue classes modulo P for which this intersection is nonempty. The shuffle which combines these P-sequences to give that of S will, when applied to a set of P-orderings of order h for the sets S_c give a P-ordering of order h of S.*

Proof. If $\{a_i\}$ is an r-removed P-ordering and $a_n \in S_c$ then

$$\sum \nu_p(a_n - a_j) = \sum_{a_j \in S_c} \nu_p(a_n - a_j)$$

and so the subsequence of entries in the r-removed P-sequence of S which are in S_c give an r-removed P-ordering of S_c. The entries in the r-removed P-sequence of S corresponding to a_i's which are in S_c are equal to the entries in the r-removed P-sequence of S_c. Since this is true for each residue class c the r-removed P-sequence of S is a shuffle of those of the S_c's and since it must be nondecreasing as are those of the S_c's it must be the unique nondecreasing shuffle. The same argument applies in the case of P-orderings of order h. □

This result allows us to apply corollary 10 of [8] to obtain a recursive formula for computing the limits $\lim_{n\to\infty} \alpha(n)/n$ for either the r-removed P-sequence or the P-sequence of order h of a set S.

Proposition 2.3. *If $\alpha(n)$ is either the r-removed P-sequence or the P-sequence of order h of a set S which is not contained in a single residue class modulo P and if $\alpha_c(n)$ are the corresponding sequences for the sets $S_c = S \cap c + PD$ and if $L = \lim_{n\to\infty} \alpha(n)/n$ and $L_c = \lim_{n\to\infty} \alpha_c(n)/n$ then*

$$L = (\sum (L_c)^{-1})^{-1}$$

where the sum is over the residue classes for which S_c is nontrivial.

In order to make use of these formulas we need to know the value of the r-removed P-sequences and P-sequences of order h for some basic set S. We compute this in the next section.

3. Orderings and p-sequences for $S = D = \mathbb{Z}$

If $S = D = \mathbb{Z}$ then the sets $i + p\mathbb{Z}$ for $i = 0, 1, 2, \ldots, p-1$ all have the same r-removed p-sequence and the same p-sequence of order h by parts (a) and (c) of proposition 2.1 and so the shuffle that combines these to make the p-sequence of $\mathbb{Z}$ in either case can be taken to be the one that uniformly interleaves them, i.e. the shuffle $(\phi_0, \phi_1, \ldots, \phi_{p-1})$ with $\phi_i(m) = pm + i$. Since the relation between the p-sequence of the set $i + p\mathbb{Z}$ and that of $\mathbb{Z}$ is given by parts (b) and (d) of proposition 2.1 we have:

Proposition 3.1. (a) *The r-removed p-sequence, $\alpha(n)$, of $\mathbb{Z}$ satisfies the recurrence*

$$\alpha(n) = \alpha(\lfloor n/p \rfloor) + \lfloor n/p \rfloor - r$$

provided $n \geq pr$. If $n < pr$ then $\alpha(n) = 0$.

(b) *The p-sequence of order h, $\alpha(n)$, of $\mathbb{Z}$ is related to the p-sequence $\alpha'(n)$ of order $h-1$ by*

$$\alpha(n) = \alpha'(\lfloor n/p \rfloor) + \lfloor n/p \rfloor$$

The recurrence in part (a) of this proposition can be applied repeatedly to give the following result:

Proposition 3.2. *The r-removed p-sequence, $\alpha(n)$, of $\mathbb{Z}$ is given by*

$$\alpha(n) = \sum_{i=1}^{k} \lfloor n/p^i \rfloor - kr = \nu_p(n!) - \nu_p(\lfloor n/p^k \rfloor !) - kr$$

where $k = \lfloor log(n/r)/log(p) \rfloor$.

Proof. The formula in part (a) of the previous proposition can be applied repeatedly to give

$$\alpha(n) = \alpha(\lfloor n/p^j \rfloor) + \sum_{i=1}^{j} \lfloor n/p^i \rfloor - jr$$

as long as $n/p^{j-1} \geq pr$. If k, as in the statement of the proposition, is such that $p^{k+1}r > n \geq p^k r$ then $\alpha(\lfloor n/p^k \rfloor) = 0$. The second part of the proposition follows from the Legendre formula for $\nu_p(n!)$. □

Corollary 3.3. *If $\alpha(n)$ is the r-removed p-sequence of $\mathbb{Z}$ then*

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = \frac{1}{p-1}.$$

Proof. This follows from the corresponding calculation for the p-sequence of $\mathbb{Z}$ since

$$\lim_{n\to\infty} \log(n/r)/n = 0$$

□

Corollary 3.4. *For $r = 1$ the r-removed p-sequence for $\mathbb{Z}$ is given by*

$$\alpha(n) = \nu_p(n!) - \lfloor \log(n)/\log(p) \rfloor.$$

This agrees with the formula for the p-sequence of the algebra of integer valued polynomials whose first finite difference is also integer valued as given in [6, IX 3.2].

Proposition 3.5. *For any prime p and any nonnegative integer r the usual order on $\mathbb{Z}^{\geq 0}$ gives an r-removed P-ordering of $\mathbb{Z}$.*

Proof. First note that the sequence $(0, 1, 2, \ldots, rp-1)$ satisfies the condition for being an r-removed p-ordering of $\mathbb{Z}$ since it contains r representatives of each residue class modulo p and so the p-adic norm of the relevant products are all 0. For any i the map $\psi : \mathbb{Z} \to (i + p\mathbb{Z})$ given by $\psi(x) = px + i$ will map an r-removed p-ordering of $\mathbb{Z}$ to one of $i + p\mathbb{Z}$ and the shuffle $(\phi_0, \ldots, \phi_{p-1})$ will combine r-removed p-orderings of the sets $i + p\mathbb{Z}$ to give one of $\mathbb{Z}$. Composing these two operations gives a construction of the beginning portion of an r-removed p-ordering of $\mathbb{Z}$ of length $p^{m+1}r$ from one of length $p^m r$. Repeatedly applying this, starting with the portion of length pr given above, gives the result. □

Definition 3.6. *For $n \in \mathbb{Z}^{\geq 0}$ let $\mu_r(n) = \nu_p(\lfloor n/p^k \rfloor !) + kr$ where $k = \lfloor log(n/r)/log(p) \rfloor$.*

Corollary 3.7. *An integer valued polynomial $\sum_{n=0}^{\ell} c_n \binom{x}{n}$ with $c_n \in \mathbb{Z}$ for all n is an r-removed integer valued polynomial if and only if $\nu_p(c_n) \geq \mu_p(n)$ for all n and all primes p.*

Applying the recurrence in part (b) of proposition 3.1 repeatedly gives the following formula for the p-sequence of order h of $\mathbb{Z}$:

Proposition 3.8. *The p-sequence of order h of $\mathbb{Z}$ is given by*

$$\alpha(n) = \sum_{i=1}^{h} \lfloor n/p^i \rfloor = \nu_p(n!) - \nu_p(\lfloor n/p^h \rfloor !).$$

Proof. First note that the p-sequence of order 0 is the constant 0 sequence (and any ordering of $\mathbb{Z}$ can serve as a p-sequence of order 0). Proceeding by induction on h we have by proposition 3.1 part (b) that

$$\alpha(n) = \alpha'(\lfloor n/p \rfloor) + \lfloor n/p \rfloor$$

and by induction

$$\alpha'(\lfloor n/p \rfloor) = \sum_{i=1}^{h-1} \lfloor n/p^{i+1} \rfloor.$$

□

Corollary 3.9. *If $\alpha(n)$ is the p-sequence of order h of $\mathbb{Z}$ then*

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = \frac{1-p^h}{p^h(p-1)}.$$

Corollary 3.10. *For any prime p and any nonnegative integer h the usual order on $\mathbb{Z}^{\geq 0}$ gives a p-ordering of order h of $\mathbb{Z}$.*

Proof. It suffices to note that

$$\sum_{j=1}^{n} \min(\nu_p(j), h) = \sum_{i=1}^{h} \lfloor n/p^i \rfloor.$$

□

As remarked in the introduction, the algebra of polynomials with integral valued divided differences of order $\leq r$ to which r-removed P-orderings provide a basis does not, for $r > 1$, coincide with the algebra of polynomials with integral valued finite differences of order $\leq r$ which is studied in [1] and [7] and in chapter 9 of [6]. According to those sources the p-sequence of that algebra for the prime p is $\alpha(n) = \nu_p(n!) - \lambda_r(n)$ where

$$\lambda_r(n) = \max\{\sum_{i=1}^{k} \nu_p(j_i) : k \leq r, \sum_{i=1}^{k} j_i \leq n, (j_i \geq 1)\}$$

and an integer valued polynomial $\sum_{n=0}^{\ell} c_n \binom{x}{n}$ has integer valued finite differences of order $\leq r$ if and only if $\nu_p(c_n) \geq \lambda_r(n)$ for all n and all primes p.

Lemma 3.11. *If $n \geq rp$ then there is a sequence $(j_1, \ldots, j_r)$ realizing the value of $\lambda_r(n)$ in which $j_i \geq p$ for each i.*

Proof. Suppose $(j_1, \ldots, j_r)$ realizes the value of $\lambda_r(n)$ and that $j_i < p$ for some i. If for some ℓ it is the case that $j_\ell \geq p^2$ then replacing j_ℓ by $j_\ell - (p - j_i)$ and j_i by p alters neither the sum of the sequence nor the sum of its p-adic values. This can be repeated for any j_i's which are less than p. If there are no $j_\ell \geq p^2$ then $\sum_{i=1}^{r} \nu_p(j_i) \leq r$ however the sequence $(p, \ldots, p)$ has sum rp and $\sum_{i=1}^{r} \nu_p(p) = r$ hence must realize the value of $\lambda_r(n)$ in this case. □

Lemma 3.12. *For any $r > 0$ the function λ_r satisfies the condition*

$$\lambda_r(pn) = \lambda_r(n) + r$$

for any $n \geq rp$.

Proof. If $(j_1, \ldots, j_r)$ is a sequence of positive integers with $\sum_{i=1}^{k} j_i \leq n$ then $\sum_{i=1}^{r} pj_i \leq pn$ and $\sum_{i=1}^{r} \nu_p(pj_i) = r + \sum_{i=1}^{r} j_i$. Thus $\lambda_r(pn) \geq \lambda_r(n) + r$. On the other hand if $(j_1, \ldots, j_r)$ is a sequence of positive integers with $\sum_{i=1}^{r} j_i \leq pn$ realizing the maximum value of $\sum_{i=1}^{r} \nu_p(j_i)$ and $j_i \geq p$ for each i then we may assume that each j_i is divisible by p since otherwise $\nu_p(j_i) = 0$ and replacing it by $p\lfloor j_i/p \rfloor$ which is nonzero would not increase the sum but would increase the sum of the p-adic norms. Thus $(j_1/p, \ldots, j_k/p)$ is a sequence with $\sum_{i=1}^{k} j_i/p \leq n$ and $\sum_{i=1}^{k} \nu_p(j_i/p) = \lambda_r(pn) - r$. Thus $\lambda_r(pn) - r \leq \lambda_r(n)$. □

Corollary 3.13. *The characteristic sequence, $\alpha(n)$, of the algebra of polynomials whose finite differences of order up to r are integer valued satisfies the recurrence*

$$\alpha(n) = \alpha(\lfloor n/p \rfloor) + \lfloor n/p \rfloor - r$$

provided $n \geq rp^2$.

Corollary 3.14. *If $k = \lfloor log(n/r)/log(p) \rfloor$ then*

$$\mu_r(n) - \lambda_r(n) = \mu_p(\lfloor n/p^{k-1} \rfloor) - \lambda_r(\lfloor n/p^{k-1} \rfloor)$$

and so the values of $\mu_r(n) - \lambda_r(n)$ for all n are determined by those for $n < rp^2$.

Proof. From 3.1 and 3.13 it follows that the difference at n is equal to that at $\lfloor n/p \rfloor$ provided $n \geq rp^2$. □

To determine the difference in the range $0 \leq n \leq rp^2$ we compute $\lambda_r(n)$ for in this range:

Lemma 3.15.

$$\lambda_r(n) = \begin{cases} \lfloor n/p \rfloor & \text{if } n \leq rp \\ r + \lfloor \dfrac{n - rp}{p(p-1)} \rfloor & \text{if } rp < n < rp^2 \end{cases}$$

Proof. If $n \leq rp$ then taking

$$j_i = \begin{cases} p \text{ if } i \leq \lfloor n/p \rfloor \\ 0 \text{ if } i > \lfloor n/p \rfloor \end{cases}$$

gives a set $(j_1, \ldots, j_r)$ realizing the maximal value $\lambda_r(n) = \lfloor n/p \rfloor$.

If $rp < n \leq rp^2$ then taking

$$j_i = \begin{cases} p^2 \text{ if } i \leq \lfloor \dfrac{n - rp}{p(p-1)} \rfloor \\ p \text{ if } \lfloor \dfrac{n - rp}{p(p-1)} \rfloor < i \leq rp^2 \end{cases}$$

gives a set $(j_1, \ldots, j_r)$ realizing the maximal value $\lambda_r(n) = r + \lfloor \dfrac{n - rp}{p(p-1)} \rfloor$. □

Corollary 3.16.

$$\mu_r(n) - \lambda_r(n) = \begin{cases} \nu_p(\lfloor n/p \rfloor !) & \text{if } n \leq rp \\ \nu_p(\lfloor n/p \rfloor !) - \lfloor \dfrac{n - rp}{p(p-1)} \rfloor & \text{if } rp < n \leq rp^2 \end{cases}$$

Proof. For $n \leq rp$ the difference is

$$\nu_p(n!) - \lambda_r(n) = \nu_p(n!) - \lfloor n/p \rfloor = \nu_p(\lfloor n/p \rfloor !)$$

while if $rp < n \leq rp^2$ it is

$$\begin{aligned} \nu_p(n!) - (\lfloor \frac{n}{p} \rfloor - r) - \lambda_r(n) &= \nu_p(n!) - \lfloor \frac{n - rp}{p(p-1)} \rfloor - \lfloor \frac{n}{p} \rfloor \\ &= \nu_p(\lfloor \frac{n}{p} \rfloor !) - \lfloor \frac{n - rp}{p(p-1)} \rfloor \end{aligned}$$

□

It is clear from this formula that this difference is bounded above by $\nu_p(r!)$ and in fact achieves that value when $n = rp^2$ at which point

$$\nu_p(\lfloor \frac{rp^2}{p} \rfloor !) - \lfloor \frac{rp^2 - rp}{p(p-1)} \rfloor = \nu_p(r!)$$

This bound may also be found by noting that the r-th derivative of a polynomial can be expressed in terms of $r!$ times the r-th divided difference and that the ring of polynomials whose derivatives up to order r are integer valued is contained in the algebra of polynomials whose finite differences up to order r are integer valued.

4. Orderings and p-sequences of Homogeneous Subsets

Homogeneous sets $S \subseteq D$ are sets with the property that there exists $\ell > 0$ such that $S = S + P^\ell D$. Such sets can be expressed as unions of cosets

$$S = \cup_i (d_i + P^\ell D)$$

where the d_i's are a collection of representatives of some of the distinct cosets modulo P^ℓ. The p-sequence of each of the sets $d_i + P^\ell D$ can be computed using proposition 2.1 and these can be combined to obtain the p-sequence of S by repeated use of proposition 2.2.

The set $\mathbb{Z}\backslash p\mathbb{Z}$ is equal to the union $\cup_{i=1}^{p-1}(i + p\mathbb{Z})$ in which each of the components has, by proposition 2.1(b) and proposition 3.2, the r-removed p-sequence

$$\begin{aligned}\nu_p(n!) - \nu_p(\lfloor n/p^k \rfloor!) - kr + (n-r) \\ &= (n + \nu_p(n!)) - \nu_p(\lfloor n/p^k \rfloor!) - (k+1)r \\ &= \nu_p((pn)!) - \nu_p(\lfloor pn/p^{k+1} \rfloor!) - (k+1)r\end{aligned}$$

where $k = \lfloor log(n/r)/log(p) \rfloor$. Since these are the same for all nonzero residue classes the shuffle which combines them to give the r-removed p-sequence of S repeats each entry $p-1$ times, yielding the sequence

$$\alpha(n) = \nu_p(p(\lfloor n/(p-1) \rfloor)!) - \nu_p((\lfloor (p\lfloor n/(p-1) \rfloor)/p^{k+1} \rfloor)!) - (k+1)r$$

Since an r-removed p-ordering for each of the cosets $i + p\mathbb{Z}$ is the set of positive elements with the increasing order and the shuffle that combined the p-sequences can be taken to be the one that uniformly interleaves them the increasing order on the positive elements of S gives an r-removed p-ordering.

Similarly the p-sequence of order h of $i + p\mathbb{Z}$ will, by proposition 2.1(d) and proposition 3.8, be

$$\nu_p(n!) - \nu_p(\lfloor n/p^{h+1} \rfloor!) + n = \nu_p((pn)!) - \nu_p(\lfloor n/p^{h+1} \rfloor!)$$

and the p-sequence of order h of $S = \mathbb{Z}\backslash p\mathbb{Z}$ will be the shuffle of $(p-1)$ copies of this, hence

$$\alpha(n) = \nu_p((p\lfloor n/p - 1 \rfloor)!) - \nu_p((\lfloor \lfloor n/(p-1) \rfloor / p^{h+1} \rfloor)!).$$

As in the r-removed case the usual increasing order on the positive elements of S provides a p-ordering of order h in this case.

For $m > 1$ the set $S = \mathbb{Z}\backslash p^m\mathbb{Z}$ has the decomposition

$$\mathbb{Z}\backslash p^m\mathbb{Z} = \mathbb{Z}\backslash p^{m-1}\mathbb{Z} \cup \cup_{i=1}^{p-1}(i + p^m\mathbb{Z}).$$

The p-sequences of the sets $i + p^m\mathbb{Z}$ can be determined using Proposition 2.1 and so if those of $\mathbb{Z}\backslash p^{m-1}\mathbb{Z}$ are known by induction then those of S can be determined. This scheme will determine any finite portion of these sequences however the shuffles involved cannot be deduced by symmetry as in the case $m = 1$ so a closed form formula for these sequences is not available.

Proposition 4.1. *If S is a homogeneous subset of $\mathbb{Z}$ with p-sequence $\alpha(n)$ and r-removed p-sequence $\alpha'(n)$ then*

$$\lim_{n\to\infty} \alpha'(n)/n = \lim_{n\to\infty} \alpha(n)/n.$$

Proof. It follows from Proposition 3.3 that for $S = \mathbb{Z}$ these limits agree and, by Proposition 2.1, that they agree for any coset of the form $S = d_i + P^\ell\mathbb{Z}$. Corollary 10 of [8] gives a formula for these limits for sets formed as unions of cosets in terms of the limits for the cosets. Since these limits agree for such cosets they must agree for unions of these also, i.e. for all homogeneous sets. □

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