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Aldous' conjecture on a killed branching random walk

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Abstract

Consider a branching random walk on the real line with an killing barrier at zero: starting from a nonnegative point, particles reproduce and move independently, but are killed when they touch the negative half-line. The population of the killed branching random walk dies out almost surely in both critical and subcritical cases, where by subcritical case we mean that the rightmost particle of the branching random walk without killing has a negative speed and by critical case when this speed is zero. We investigate the total progeny of the killed branching random walk and give its precise tail distribution both in the critical and subcritical cases, which solves an open problem of D.Aldous.

We consider a one-dimensional discrete-time branching random walk V on the real line $\mathbb{R}$. At the beginning, there is a single particle located at the origin 0. Its children, who form the first generation, are positioned according to a certain point process $\mathscr{L}$ on $\mathbb{R}$. Each of the particles in the first generation independently gives birth to new particles that are positioned (with respect to their birth places) according to a point process with the same law as $\mathscr{L}$; they form the second generation. And so on. For any $n \geq 1$, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation.

Clearly, the particles of the branching random walk V form a Galton–Watson tree, which we denote by $\mathcal{T}$. Call $\varnothing$ the root. For every vertex $u \in \mathcal{T}$, we denote by $|u|$ its generation (then $|\varnothing| = 0$) and by $(V(u), |u| = n)$ the positions of the particles in the n-th generation. Then $\mathscr{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$. The tree $\mathcal{T}$ will encode the genealogy of our branching random walk.

It will be more convenient to consider a branching random walk V starting from an arbitrary $x \in \mathbb{R}$ [namely, $V(\varnothing) = x$], whose law is denoted by $\mathbf{P}_x$ and the corresponding expectation by $\mathbf{E}_x$. For simplification, we write $\mathbf{P} \equiv \mathbf{P}_0$ and $\mathbf{E} \equiv \mathbf{E}_0$. Let $\nu := \sum_{|u|=1} 1$ be the number of particles in the first generation and denote by $\nu(u)$ the number of children of $u \in \mathcal{T}$.

Assume that $\mathbf{E}[\nu] > 1$, namely the Galton–Watson tree $\mathcal{T}$ is supercritical, then the system survives with positive probability $\mathbf{P}\big(\mathcal{T} = \infty\big) > 0$. Let us define the logarithmic generating function for the branching walk:

$$\psi(t) := \log \mathbf{E}\Big[\sum_{|u|=1} e^{tV(u)}\Big] \in (-\infty, +\infty], \qquad t \in \mathbb{R}.$$

We shall assume that ψ is finite on an open interval containing 0 and that $\mathrm{supp}\mathscr{L} \cap (0, \infty) \neq \emptyset$ [the later condition is to ensure that V can visit $(0, \infty)$ with positive probability, otherwise the problem that we shall consider becomes trivial]. Assume that there exists $\varrho^* > 0$ such that

$$\psi(\varrho^*) = \varrho^* \psi'(\varrho^*). \tag{0.1}$$

We also assume that ψ is finite on an open set containing $[0, \varrho^*]$. The condition (0.1) is rather mild, roughly saying, if we denote by $m^* = \mathrm{esssup}\,\mathrm{supp}\mathscr{L}$, then (0.1) is satisfied either $m^* = \infty$ or $m^* < \infty$ and $\mathbf{E} \sum_{|u|=1} 1_{(V(u)=m^*)} < 1$.

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Recall that (Kingman, Hammersley, Biggins (1974, 1975)) conditioned on the survival of the system,

$$\lim_{n\to\infty} \frac{1}{n} \max_{|u|=n} V(u) = \psi'(\varrho^*), \qquad \text{a.s.,} \tag{0.2}$$

where ϱ^* is defined in (0.1). According to $\psi'(\varrho^*) = 0$ or $\psi'(\varrho^*) < 0$, we call the critical case or the subcritical case. Conditioned on $\{\mathcal{T} = \infty\}$, the rightmost particle in the branching random walk without killing has a negative speed in the subcritical case, while in the critical case it converges almost surely to $-\infty$ in the logarithmical scale (see Addario-Berry and Reed (2009) and Hu and Shi (2009) for the precise statement of the rate of almost sure convergence).

We now place a killing barrier at zero, hence at every generation $n \geq 0$, survive only the particles that always stayed nonnegative up to time n. Denote by $\mathscr{Z}$ the set of all living particles of the killed branching walk:

$$\mathscr{Z} := \Big\{ u \in \mathcal{T} \,:\, V(v) \geq 0, \qquad \forall\, v \in [\varnothing, u] \Big\},$$

where $[\varnothing, u]$ denotes the shortest path relating u from the root $\varnothing$. We are interested in the total progeny

$$Z := \#\mathscr{Z},$$

on which David Aldous made the following conjecture:

Conjecture (D.Aldous:

- **(i) (critical case):** If $\psi'(\varrho^*) = 0$, then $\mathbf{E}[Z] < \infty$ and $\mathbf{E}[Z \log Z] = \infty$.
- **(ii) (subcritical case):** If $\psi'(\varrho^*) < 0$, then there exists some constant $b > 1$ such that $\mathbf{P}(Z > n) = n^{-b+o(1)}$ as $n \to \infty$.

Let us call iid case if $\mathscr{L}$ is of form: $\mathscr{L} = \sum_{i=1}^{\nu} \delta_{\{X_i\}}$ with $(X_i)_{i\geq 1}$ a sequence of i.i.d. real-valued variables, independent of ν. There are several previous works on the critical and iid case: when (X_i) are Bernoulli random variables, Pemantle (1998) obtained the precise asymptotic of $\mathbf{P}(Z = n)$ as $n \to \infty$, where the key ingredient of his proof is the recursive structure of the system inherited from the Bernoulli variables (X_i). For general random variables (X_i), Addario-Berry and Broutin (2009) recently confirmed Aldous' conjecture (i) under some integrability hypothesis; This was improved later by Aïdékon (2009+) who proved that for a regular tree $\mathcal{T}$ (namely when ν equals some integer), for any fixed $x \geq 0$,

$$n (\log n)^2 \mathbf{P}_x(Z > n) \approx R(x) e^x,$$

where $R(x)$ is a renewal function which will be defined later.

In this paper, we aim at the exact tail behavior of Z both in critical and subcritical cases and for a general point process $\mathscr{L}$.

Before the statement of our result, we remark that in the subcritical case ($\psi'(\varrho^*) < 0$), there are two real numbers ϱ_- and ϱ_+ such that $0 < \varrho_- < \varrho^* < \varrho_+$ and

$$\psi(\varrho_-) = \psi(\varrho_+) = 0,$$

[the existence of ϱ_+ follows from the assumption that $\text{supp}\mathscr{L} \cap (0, \infty) \neq \emptyset$].

Assume that

$$\mathbf{E}[\nu^\alpha] < \infty, \qquad \text{for some } \begin{cases} \alpha > 2, & \text{in the critical case;} \\ \alpha > 2\frac{\varrho_+}{\varrho_-}, & \text{in the subcritical case.} \end{cases} \tag{0.3}$$

In the critical case, we suppose that

$$\mathbf{E}\Big[\nu^{1+\delta^*}\Big] < \infty, \qquad \sup_{\theta \in [-\delta^*, \varrho^* + \delta^*]} \psi(\theta) < \infty, \qquad \text{for some } \delta^* > 0. \tag{0.4}$$

In the subcritical case, we suppose that

$$\mathbf{E}\left[\sum_{|u|=1} (1 + e^{\varrho_- V(u)}) \right]^{\frac{\varrho_+}{\varrho_-} + \delta^*} < \infty, \qquad \sup_{\theta \in [-\delta^*, \varrho_+ + \delta^*]} \psi(\theta) < \infty, \tag{0.5}$$

for some $\delta^* > 0$. In both cases, we always assume that there is no lattice that supports $\sum_{|u|=1} \delta_{V(u)}$ almost surely.

Our main result reads as follows.

Theorem 1 (Tail of the total progeny)**.** *Assume* (0.1)*,* (0.3)*.*

(i) (Critical case) If $\psi'(\varrho^*) = 0$ and (0.4) holds, then there exists a constant $c_{crit} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \,\sim\, c_{crit}\, R(x)\, e^{\varrho^* x}\, \frac{1}{n(\log n)^2}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function.

(ii) (Subcritical case) If $\psi'(\varrho^*) < 0$ and (0.5) holds, then there exists a constant $c_{sub} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \,\sim\, c_{sub} R(x) e^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function.

The values of c_{crit} and c_{sub} are given in Lemma 1. To explain the strategy of the proof of Theorem 1, we introduce at first some notations: for any vertex $u \in \mathcal{T}$ and $a \in \mathbb{R}$, we define

$$\tau_a^+(u) \;:=\; \inf\{0 \leq k \leq |u| \,:\, V(u_k) > a\}, \tag{0.6}$$

$$\tau_a^-(u) \;:=\; \inf\{0 \leq k \leq |u| \,:\, V(u_k) < a\}, \tag{0.7}$$

with convention $\inf \emptyset := \infty$ and for $n \geq 1$ and for any $|u| = n$, we write $\{u_0 = \varnothing, u_1, ..., u_n\} = [\varnothing, u]$ the shortest path relating u from the root $\varnothing$ (u_k is the ancestor of k-th generation of u).

By using these notations, the living set $\mathscr{Z}$ of the killed branching random walk can be represented as follows:

$$\mathscr{Z} = \{u \in \mathcal{T} : \tau_0^-(u) > |u|\}.$$

For $a \leq x$, we define $\mathcal{L}[a]$ as the set of individuals which lives below a for its first time:

$$\mathcal{L}[a] := \{u \in \mathcal{T} \,:\, |u| = \tau_a^-(u)\}, \qquad a \leq x, \tag{0.8}$$

Since the whole system goes to $-\infty$, $\mathcal{L}[a]$ is well defined. In particular, $\mathcal{L}[0]$ is the set of leaves of the progeny of the killed branching walk. As an application of a general fact for a wide class of graphs, we can compare the set of leaves $\mathcal{L}[0]$ with $\mathscr{Z}$. Then it is enough to investigate the tail asymptotics of $\#\mathcal{L}[0]$.

To state the result for $\#\mathcal{L}[0]$, we shall need an auxiliary random walk S, under a probability $\mathbf{Q}$, which depend on the parameter $\varrho = \varrho^*$ in the critical case, and $\varrho = \varrho^+$ in the subcritical case. We mention that under $\mathbf{Q}$, S is recurrent in the critical case and transient in the subcritical case. Let us also consider the renewal function $R(x)$ associated to S and τ_0^- the first time when S becomes negative.

Theorem 2 (Tail of the set of leaves)**.** *Assume (0.1).*

(i) Critical case : if $\psi'(1) = 0$ and (0.4) holds, then for any $x \geq 0$, we have when $n \to \infty$

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{crit} R(x) e^{\varrho^* x} \frac{1}{n(\log n)^2},$$

where $c'_{crit} := (\mathbf{Q}[e^{-S_{\tau_0^-}}] - 1)$.

(ii) Subcritical case : If $\psi'(1) < 0$ and (0.5) holds, then we have for any $x \geq 0$ when $n \to \infty$,

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{sub} R(x) \mathrm{e}^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}},$$

for some constant $c'_{sub} > 0$.

If $\sum_{|u|=1}(1 + e^{\varrho_- V(u)})$ has some larger moments, then we can give, as in the critical case (i), a probabilistic interpretation of the constant c'_{sub} for the subcritical case.

The next lemma establishes the relation between $\#\mathcal{L}[0]$ and the total progeny $Z = \#\mathscr{Z}$. Recall that $\mathbf{E}(\nu) > 1$.

Lemma 1. *Assume (0.3). Then Theorem 2 implies Theorem 1 with*
(i) in the critical case: $c_{crit} = (\mathbf{E}(\nu) - 1)^{-1} c'_{crit}$,
(ii) in the subcritical case: $c_{sub} = (\mathbf{E}(\nu) - 1)^{-\varrho_+/\varrho_-} c'_{sub}$.

The proof of Theorem 2 relies on an analysis of the maximum of the killed branching random walk and its progeny. We need to establish some Yaglom-type results. The main tool will be a spinal decomposition for the killed branching random walk.