Semisimple groups that are quasi-split
over a tamely ramified extension

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Abstract – Let K be a discretly henselian field whose residue field is separably closed. Answering a question raised by G. Prasad, we show that a semisimple K-group G is quasi-split if and only if it quasi–splits after a finite tamely ramified extension of K.

1. Introduction

Let K be a discretly valued henselian field with valuation ring $\mathcal{O}$ and residue field k. We denote by K_{nr} the maximal unramified extension of K and by K_t its maximal tamely ramified extension. If G/K is a semisimple simply connected group, Bruhat–Tits theory is available in the sense of [14, 15] and the Galois cohomology set $H^1(K_{\mathrm{nr}}/K, G)$ can be computed in terms of the Galois cohomology of special fibers of Bruhat–Tits group schemes [6]. This permits to compute $H^1(K, G)$ when the residue field k is perfect.

On the other hand, if k is not perfect, "wild cohomology classes" occur, that is $H^1(K_t, G)$ is non-trivial. Such examples appear for example in the study of bad unipotent elements of semisimple algebraic groups [11]. Under some restrictions on G, we would like to show that $H^1(K_t/K_{\mathrm{nr}}, G)$ vanishes (see Corollary 3.3). This is related to the following quasi-splitness result.

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THEOREM 1.1. *Let G be a semisimple simply connected K-group which is quasi-split over K_t.*

(1) *If the residue field k is separably closed, then G is quasi-split.*

(2) *$G \times_K K_{\mathrm{nr}}$ is quasi-split.*

This theorem answers a question raised by Gopal Prasad who found another proof by reduction to the inner case of type A [15, Theorem 4.4]. Our first observation is that the result is quite simple to establish under the following additional hypothesis:

(∗) *If the variety of Borel subgroups of G carries a 0-cycle of degree one, then it has a K-rational point.*

Property (∗) holds away of E_8 for an arbitrary base field (Section 2). It is an open question if (∗) holds for groups of type E_8. For the E_8 case (and actually for any strongly inner K-group G) of Theorem 1.1, our proof is a Galois cohomology argument using Bruhat–Tits buildings (Section 3).

We can make at this stage some remarks about the statement. Since K_{nr} is a discretely valued henselian field with residue field k_s, we observe that $(1) \implies (2)$. Also a weak approximation argument [8, Proposition 3.5.2] reduces to the complete case. If the residue field k is separably closed of characteristic zero, we have then $\mathrm{cd}(K) = 1$, so that the result follows from Steinberg's theorem [16, §4.2, Corollary 1]. In other words, the main case to address is that of characteristic exponent $p > 1$.

2. The variety of Borel subgroups and 0-cycle of degree one

Let k be a field, let k_s be a separable closure and let $\mathrm{Gal}(k_s/k)$ be the absolute Galois group of k. Let q be a nonsingular quadratic form. A celebrated result of Springer states that the Witt index of q is insensitive to odd degree field extensions. In particular the property to have a maximal Witt index is insensible to odd degree extensions and this can be rephrased by saying that the algebraic group $\mathrm{SO}(q)$ is quasi-split if and only if it is quasi-split over an odd degree field extension of k. This fact generalizes for all semisimple groups without type E_8.

THEOREM 2.1. *Let G be a semisimple algebraic k-group without quotient of type E_8. Let $k_1, \ldots, k_r$ be finite field extensions of k with coprime degrees. Then G is quasi-split if and only if G_{k_i} is quasi-split for $i = 1, \ldots, r$.*

The proof is far to be uniform hence gathers several contributions [1, 9]. Note that the split version (in the absolutely almost simple case) is [10, Theorem C]. We remind the reader that a semisimple k-group G is isomorphic to an inner twist of a quasi-split group G^q and that such a G^q is unique up to isomorphism. Denoting by G^q_{ad} the adjoint quotient of G^q, this means that there exists a Galois cocycle $z \colon \mathrm{Gal}(k_s/k) \to G^q_{\mathrm{ad}}(k_s)$ such that G is isomorphic to ${}_z G^q$. We denote by $\pi \colon G^{\mathrm{sc},q} \to G^q_{\mathrm{ad}}$ the simply connected cover of G^q_{ad}. Then ${}_z G^{\mathrm{sc},q}$ is the simply connected cover of ${}_z G^q \cong G$.

LEMMA 2.2. *The following are equivalent*:

(i) *G is quasi-split*;

(ii) *$[z] = 1 \in H^1(k, G^q_{\mathrm{ad}})$*;

If furthermore $[z] = \pi_*[z^{\mathrm{sc}}]$ for a 1-cocycle $z^{\mathrm{sc}} \colon \mathrm{Gal}(k_s/k) \to G^{\mathrm{sc},q}(k_s)$, then (i) *and* (ii) *are also equivalent to*

(iii) *$[z^{\mathrm{sc}}] = 1 \in H^1(k, G^{\mathrm{sc},q})$*.

PROOF. The isomorphism class of G is encoded by the image of $[z]$ under the map $\mathrm{int}_* \colon H^1(k, G^q_{\mathrm{ad}}) \to H^1(k, \mathrm{Aut}(G^q))$. The map int_* has trivial kernel since the exact sequence

$$1 \longrightarrow G^q_{\mathrm{ad}} \xrightarrow{\ \mathrm{int}\ } \mathrm{Aut}(G^q) \longrightarrow \mathrm{Out}(G^q) \longrightarrow 1$$

is split ([7, XXIV.3.10] or [13, 31.4]), whence the implication (ii) $\implies$ (i). The reverse inclusion (i) $\implies$ (ii) is obvious.

Now we assume that z lifts to a 1-cocycle z^{sc}. The implication (iii) $\implies$ (ii) is then obvious. The point is that the map $H^1(k, G^{\mathrm{sc},q}) \to H^1(k, G^q)$ has trivial kernel [10, III.2.6] whence the implication (ii) $\implies$ (iii). $\qquad\square$

We proceed to the proof of Theorem 2.1.

PROOF. Let X be the variety of Borel subgroups of G [7, XXII.5.8.3], a projective k-variety. The k-group G is quasi-split if and only if X has a k-rational point. Thus we have to prove that if X has a 0-cycle of degree one, then X has a k-point.

Without loss of generality, we can assume that G is simply connected. According to [7, XXIV.5] we have that $G \xrightarrow{\sim} \prod_{j=1,\ldots,s} R_{l_j/k}(G_j)$ where G_j is an absolutely almost simple simply connected group defined over a finite separable field extension l_j of k (the notation $R_{l_j/k}(G_j)$ stands as usual for the Weil restriction to l_j to k). The variety of Borel subgroup X of G is then isomorphic to $\prod_{j=1,\ldots,s} R_{l_j/k}(X_j)$ where X_j is the l_j-variety of Borel subgroups of G_j.

REDUCTION TO THE ABSOLUTELY ALMOST SIMPLE CASE. Our assumption is that $X(k_i) \neq \emptyset$ for $i = 1, \ldots, r$ hence $X_j(k_i \otimes l_j) \neq \emptyset$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Since l_j/k is separable, $k_i \otimes l_j$ is an étale l_j-algebra for $i = 1, \ldots, r$ and it follows that X_j carries a 0-cycle of degree one. If we know to prove the case of each X_j, we have $X_j(k_j) \neq \emptyset$ hence $X(k) \neq \emptyset$. From now on, we assume that G is absolutely almost simple. We denote by G_0 the Chevalley group over $\mathbb{Z}$ such that G is a twisted form of $G_0 \times_{\mathbb{Z}} k$.

REDUCTION TO THE CHARACTERISTIC ZERO CASE. If k is of characteristic $p > 0$, let $\mathcal{O}$ be a Cohen ring for the residue field k, that is a complete discrete valuation ring such that its fraction field K is of characteristic zero and for which p is an uniformizing parameter [3, IX.41]. The isomorphism class of G is encoded by a Galois cohomology class in $H^1(k, \mathrm{Aut}(G_0))$. Since $\mathrm{Aut}(G_0)$ is a smooth affine $\mathbb{Z}$-group scheme [7, XXIV.1.3], we can use Hensel's lemma [7, XXIV.8.1] i.e. the bijection $H^1_{\text{étale}}(\mathcal{O}, \mathrm{Aut}(G_0)) \xrightarrow{\sim} H^1(k, \mathrm{Aut}(G_0))$. It implies that G lifts in a (unique) semisimple simply connected group scheme $\mathfrak{G}$ over $\mathcal{O}$. Let $\mathfrak{X}$ be the $\mathcal{O}$-scheme of Borel subgroups of $\mathfrak{G}$ [7, XXII.5.8.3]. It is smooth and projective. For $i = 1, \ldots, r$, let K_i be an unramified field extension of K of degree $[k_i : k]$ and of residue field k_i. Denoting by $\mathcal{O}_i$ its valuation ring, we consider the maps

$$\mathfrak{X}(K_i) = \mathfrak{X}(\mathcal{O}_i) \longrightarrow X(k_i).$$

The left equality come from the projectivity and the right surjectivity is Hensel's lemma. It follows that $\mathfrak{X}(K_i) \neq \emptyset$ for $i = 1, \ldots, r$ so that $\mathfrak{X}_K$ has a 0-cycle of degree one. Assuming the result in the characteristic zero case, it follows that $\mathfrak{X}(K) = \mathfrak{X}(\mathcal{O}) \neq \emptyset$ whence $X(k) \neq \emptyset$.

We may assume from now that k is of characteristic zero. We denote by μ the center of G and by $t_G \in H^2(k, \mu)$ the Tits class of G [13, §31]. Since the Tits class of the quasi-split form G^q of G is zero, the classical restriction-corestriction argument yields that $t_G = 0$. In other words G is a strong inner form of its quasi-split form G^q. It means that there exists a Galois cocycle z with value in $G^q(k_s)$ such that $G \cong {}_z G^q$, that is the twist by inner conjugation of G by z. Lemma 2.2

shows that our problem is rephrased in Serre's question [17, §2.4] on the triviality of the kernel of the map

$$H^1(k, G^q) \longrightarrow \prod_{i=1,\dots,r} H^1(k_i, G^q)$$

That kernel is indeed trivial in our case [2, Theorem 0.4], whence the result. $\square$

We remind the reader that one can associate to a semisimple k-group G its set $S(G)$ of torsion primes which depends only of its Cartan–Killing type [17, §2.2]. Since an algebraic group splits after an extension of degree whose primary factors belong to $S(G)$, see [18], we get the following refinement.

COROLLARY 2.3. *Let G be a semisimple algebraic k-group without quotient of type E_8. Let $k_1, \dots, k_r$ be finite field extensions of k such that g.c.d.$([k_1 : k], \dots, [k_r : k])$ is prime to $S(G)$. Then G is quasi-split if and only if G_{k_i} is quasi-split for $i = 1, \dots, r$.*

Lemma 2.2 together with the Corollary implies the following statement.

COROLLARY 2.4. *Let G be a semisimple simply connected quasi-split algebraic k-group without factors of type E_8. Let $k_1, \dots, k_r$ be finite field extensions of k such that g.c.d.$([k_1 : k], \dots, [k_r : k])$ is prime to $S(G)$. Then the maps*

$$H^1(k, G) \longrightarrow \prod_{i=1,\dots,r} H^1(k_i, G)$$

and

$$H^1(k, G_{\mathrm{ad}}) \longrightarrow \prod_{i=1,\dots,r} H^1(k_i, G_{\mathrm{ad}})$$

have trivial kernels.

We can proceed now on the proof of Theorem 1.1.(1) away of E_8 since Theorem 2.1 shows that the condition ($*$) is fullfilled in that case.

PROOF OF THEOREM 1.1.(1) UNDER ASSUMPTION ($*$). Here K is a discretly valued henselian field whose residue field is separably closed. We are given a semisimple K-group G satisfying assumption ($*$), and such that G becomes quasi-split after a finite tamely ramified extension L/K. Note that $[L : K]$ is prime to p. We denote by X the K-variety of Borel subgroups of G. We want to show that $X(K) \neq \emptyset$. We are then reduced to the following cases:

(i) K is perfect and the absolute Galois group $\mathrm{Gal}(K_s/K)$ is a pro-l-group for a prime $l \neq p$.

(ii) $\mathrm{Gal}(K_s/K)$ is a pro-p-group.

By weak approximation [8, Proposition 3.5.2], we may assume that K is complete. Note that this operation does not change the absolute Galois group (*ibid*, 3.5.1).

CASE (i). We have that $\mathrm{cd}_l(K) \leq \mathrm{cd}_l(k) + 1 = 1$ [16, §II.4.3] so that $\mathrm{cd}(K) \leq 1$. Since K is perfect, Steinberg's theorem [16, §4.2, Corollary 1] yields that G is quasi-split.

CASE (ii). The extension K has no proper tamely ramified extension hence our assumption implies that G is quasi-split. $\qquad\qquad\square$

REMARKS 2.5. a) In case (i) of the proof, there is no need to assume that K is perfect and l can be any prime different from p. The point is that if $\mathrm{Gal}(K_s/K)$ is a pro-l-group, then the separable cohomological dimension of K is less than or equal to 1, and then any semi-simple K-group is quasi-split, see [14, §1.7]

b) It an open question whether a k-group of type E_8 is split if it is split after coprime degree extensions k_i/k. A positive answer to this question would imply Serre's vanishing conjecture II for groups of type E_8 [12, §9.2].

c) Serre's injectivity question has a positive answer for an arbitrary classical group (simply connected or adjoint) and holds for certain exceptional cases [2].

3. Cohomology and buildings

The field K is as in the introduction.

PROPOSITION 3.1. *Assume that k is separably closed. Let G be a split semisimple connected K-group. Then $H^1(K_t/K, G) = 1$.*

PROOF. We can reason at finite level and shall prove that $H^1(L/K, G) = 1$ for a given finite tamely ramified extension of L/K. We put $\Gamma = \mathrm{Gal}(L/K)$, it is a cyclic group whose order n is prime to the characteristic exponent p of k.

Let $\mathcal{B}(G_L)$ be the Bruhat–Tits building of G_L. It comes equipped with an action of $G(L) \rtimes \Gamma$ [5, §4.2.12]. Let (B, T) be a Killing couple for G. The split K-torus T defines an apartment $\mathcal{A}(T_L)$ of $\mathcal{B}(G_L)$ which is preserved by the action of $N_G(T)(L) \rtimes \Gamma$.

We are given a Galois cocycle $z \colon \Gamma \to G(L)$; it defines a section $u_z \colon \Gamma \to G(L) \rtimes \Gamma, \sigma \mapsto z_\sigma \sigma$ of the projection map $G(L) \rtimes \Gamma \to \Gamma$. This provides an action of Γ on $\mathcal{B}(G_L)$ called the twisted action with respect to the cocycle z. The Bruhat–Tits fixed point theorem [4, §3.2] provides a point $y \in \mathcal{B}(G_L)$ which is fixed by the twisted action. This point belongs to an apartment and since $G(L)$ acts

transitively on the set of apartments of $\mathcal{B}(G_L)$ there exists a suitable $g \in G(L)$ such that $g^{-1}.y = x \in \mathcal{A}(T_L)$. We observe that $\mathcal{A}(T_L)$ is fixed pointwise by Γ (for the standard action), so that x is fixed under Γ. We consider the equivalent cocycle $z'_\sigma = g^{-1} z_\sigma \sigma(g)$ and compute

$$
\begin{aligned}
z'_\sigma . x = z'_\sigma . \sigma(x) \\
= (g^{-1} z_\sigma \sigma(g))(\sigma(g^{-1}).\sigma(y)) \\
= g^{-1} . ((z_\sigma \sigma).y) \\
= g^{-1} . y \quad [y \text{ is fixed under the twisted action}] \\
= x.
\end{aligned}
$$

Without loss of generality, we may assume that $z_\sigma.x = x$ for each $\sigma \in \Gamma$. We put $P_x = \mathrm{Stab}_{G(L)}(x)$; since x is fixed by Γ, the group P_x is preserved by the action of Γ. Let $\mathcal{P}_x$ be the Bruhat–Tits $\mathcal{O}_L$-group scheme attached to x. We have $\mathcal{P}_x(\mathcal{O}_L) = P_x$ and we know that its special fiber $\mathcal{P}_x \times_{\mathcal{O}_L} k$ is smooth connected, that its quotient $M_x = (\mathcal{P}_x \times_{\mathcal{O}_L} k)/U_x$ by its split unipotent radical U_x is split reductive.

An important point is that the action of Γ on $\mathcal{P}_x(\mathcal{O}_L)$ arises from a semilinear action of Γ on the $\mathcal{O}_L$-scheme $\mathcal{P}_x$ as explained in the beginning of §2 of [15]. It induces then a k-action of the group Γ on $\mathcal{P}_x \times_{\mathcal{O}_L} k$, on U_x and on M_x. Since x belongs to $\mathcal{A}(T_L)$, $\mathcal{P}_x$ carries a natural maximal split $\mathcal{O}_L$-torus $\mathcal{T}_x$; the k-torus $T_x = \mathcal{T}_x \times_{\mathcal{O}_L} k$ is a maximal k-split torus of $\mathcal{P}_x \times_{\mathcal{O}_L} k$ and its image in M_x still denoted by T_x is a maximal k-split torus of M_x. Once again we use that Γ acts trivially on $\mathcal{A}(T_L)$ by observing that Γ acts trivially on the k-torus T_x. But $T_x/C(M_x) = \mathrm{Aut}(M_x, \mathrm{id}_{T_x})$ [7, XXIV.2.11], so that Γ acts on M_x by means of a group homomorphism $\phi \colon \Gamma \to T_{x,\mathrm{ad}}(k)$ where $T_{x,\mathrm{ad}} = T_x/C(M_x) \subseteq M_x/C(M_x) = M_{x,\mathrm{ad}}$. For each $m \in M_x(k)$ and for each $\sigma \in \Gamma$, we have $\sigma(m) = \mathrm{int}(\phi(\sigma)).m$.

Now we take a generator σ of Γ and denote by a_σ the image in $M_x(k)$ of $z_\sigma \in P_x$ and by $\underline{a}_\sigma$ its image in $(M_x/C(M_x))(k)$. The cocycle relation yields $\underline{a}_{\sigma^2} = \underline{a}_\sigma \sigma(\underline{a}_\sigma) = \underline{a}_\sigma \phi(\sigma) \underline{a}_\sigma \phi(\sigma)^{-1}$ and more generally (observe that $\phi(\sigma)$ is fixed by Γ) we have

$$
\begin{aligned}
\underline{a}_{\sigma^j} &= \underline{a}_\sigma \phi(\sigma) \underline{a}_\sigma \phi(\sigma)^{-1} \ldots \phi(\sigma)^{j-1} \underline{a}_\sigma \phi(\sigma)^{1-j} \, \phi(\sigma)^j \underline{a}_\sigma \phi(\sigma)^{-j} \\
&= (\underline{a}_\sigma \phi(\sigma))^j \, \phi(\sigma)^{-j}
\end{aligned}
$$

for $j = 2, \ldots, n$. Since $\phi(\sigma)^n = 1$, we get the relation

$$
1 = (\underline{a}_\sigma \phi(\sigma))^n.
$$

Then $\underline{a}_\sigma \phi(\sigma)$ is an element of order n of $M_{x,\mathrm{ad}}(k)$ so is semisimple. But k is separably closed so that $\underline{a}_\sigma \phi(\sigma)$ belongs to a maximal k-split torus $^m T_{x,\mathrm{ad}}$ with $m \in M_x(k)$. It follows that

$$m^{-1} \underline{a}_\sigma \phi(\sigma) m \in T_{x,\mathrm{ad}}(k).$$

Since $\phi(\sigma)$ belongs to $T_{x,\mathrm{ad}}(k)$, we have that $m^{-1} \underline{a}_\sigma \phi(\sigma) m \phi(\sigma)^{-1} \in T_{x,\mathrm{ad}}(k)$ hence $m^{-1} \underline{a}_\sigma \sigma(m) \in T_{x,\mathrm{ad}}(k)$. It follows that $m^{-1} a_\sigma \sigma(m) \in T_x(k)$. Since the map $\mathcal{P}_x(\mathcal{O}_L) \to M_x(k)$ is surjective we can then assume that $a_\sigma \in T_x(k)$ without loss of generality so that the cocycle $a = (a_\tau)_{\tau \in \Gamma}$ takes value in $T_x(k)$. But $T_x(k)$ is a trivial Γ-module so that a is given by a homomorphism $f_a \colon \Gamma \to T_x(k)$. This homomorphism lifts (uniquely) to a homomorphism $\tilde{f}_a \colon \Gamma \to \mathcal{T}_x(\mathcal{O}_L)^\Gamma$. The main technical step is

CLAIM 3.2. *The fiber of $H^1(\Gamma, P_x) \to H^1(\Gamma, M_x(k))$ at $[f_a]$ is $\{[\tilde{f}_a]\}$.*

Using the Claim, we have $[z] = [\tilde{f}_a] \in H^1(\Gamma, P_x)$. Its image in $H^1(\Gamma, G(L))$ belongs to the image of the map $H^1(\Gamma, \mathcal{T}_x(L)) \to H^1(\Gamma, G(L))$. But $0 = H^1(\Gamma, \mathcal{T}_x(L))$ (Hilbert 90 theorem) thus $[z] = 1 \in H^1(\Gamma, G(L))$ as desired.

It remains to establish the Claim. We put $P_x^\star = \ker(P_x \to M_x(k))$ and this group can be filtered by a Γ-stable decreasing filtration by normal subgroups $\mathcal{U}^{(i)}{}_{i \geq 0}$ such that for each $i \leq j$ there is a split unipotent k-group $U^{(i,j)}$ equipped with an action of Γ such that $\mathcal{U}^{(i)}/\mathcal{U}^{(j)} = U^{(i,j)}(k)$, see [15, p. 6]. We denote by $_{\tilde{f}_a} P_x^\star$ the Γ-group $P_x^\star$ twisted by the cocycle $\tilde{f}_a$; the set $H^1(\Gamma, {}_{\tilde{f}_a} P_x^\star)$ maps onto on the fiber at $[f_a]$ of the map $H^1(\Gamma, P_x) \to H^1(\Gamma, M_x(k))$, see [16, I.5.5, Corollary 2]. It is then enough to show that $H^1(\Gamma, {}_{\tilde{f}_a} P_x^\star) = 1$. It happens fortunately that the filtration is stable under the adjoint action of the image of $\tilde{f}_a$. By using the pro-unipotent k-group $U = \varprojlim U^{(0,j)}$ and Lemma A.1 in the next subsection, we have that $H^1(\Gamma, {}_{\tilde{f}_a} P_x^\star) = H^1(\Gamma, ({}_{\tilde{f}_a} U)(k)) = 1$. The Claim is established. $\qquad\square$

This permits to complete the proof of Theorem 1.1.

PROOF OF THEOREM 1.1.(1). By the usual reductions, the question boils down to the semisimple simply connected case and even to the absolutely almost K-simple semisimple simply connected case. Taking into account the cases established in Section 2, it remains to deal with the case of type E_8. Denote by G_0 the split group of type E_8, we have $G_0 = \mathrm{Aut}(G_0)$. It follows that $G \cong {}_z G_0$ with $[z] \in H^1(K, G_0)$. Our assumption is that G_{K_t} is quasi-split so that $[z] \in H^1(K_t/K, G_0)$. Proposition 3.1 states that $H^1(K_t/K, G_0) = 1$, whence G is split. $\qquad\square$

We record the following cohomological application.

COROLLARY 3.3. *Let G be a semisimple algebraic K-group which is quasi-split over K_t. We assume that G is simply connected or adjoint. Then $H^1(K_t/K_{nr}, G) = 1$.*

PROOF. Theorem 1.1 permits to assume that G is quasi-split. We denote by $\pi: G \to G_{ad}$ the adjoint quotient of G. Since the map $H^1(K, G) \to H^1(K, G_{ad})$ has trivial kernel [10, lemme III.2.6], we can assume that G is adjoint. Let $[z] \in H^1(K_t/K_{nr}, G)$. We consider the twisted K_{nr}-form $G' = {}_z G$ of G. Since G'_{K_t} is isomorphic to G_{K_t}, G'_{K_t} is quasi-split and Theorem 1.1 shows that G' is quasi-split hence isomorphic to G. It means that z belongs to the kernel of the map $\mathrm{int}_*: H^1(K, G) \to H^1(K, \mathrm{Aut}(G))$. But the exact sequence of K-groups

$$1 \longrightarrow G \xrightarrow{\mathrm{int}} \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1$$

splits (see [7, XXIV.3.10]) so that the above kernel is trivial. Thus, $[z] = 1 \in H^1(K_{nr}, G)$. $\square$

Appendix A. Galois cohomology of pro-unipotent groups

Let k be a separably closed field. Let U be a pro-unipotent algebraic k-group equipped with an action of a finite group Γ, that is U admits a decreasing filtration $U = U_0 \supset U_1 \supset U_2 \supset \cdots$ by normal pro unipotent k-groups which are stabilized by Γ and such that U_i/U_{i+1} is an unipotent algebraic k-group for $i = 1, \ldots, n$.

LEMMA A.1. *We assume that $\sharp\Gamma$ is invertible in k and that the U_i/U_{i+1}'s are smooth and connected. Then $H^1(\Gamma, U(k)) = 1$.*

PROOF. We start with the algebraic case, that is of a smooth connected unipotent k-group. According to [7, XVII.4.11], U admits a central characteristic filtration $U = U_0 \supset U_1 \supset \cdots \supset U_n = 1$ such that U_i/U_{i+1} is a twisted form of a k-group $\mathbb{G}_a^{n_i}$. Since U_{i+1} is smooth and k is separably closed, we have the following exact sequence of Γ-groups

$$1 \longrightarrow U_{i+1}(k) \longrightarrow U_i(k) \longrightarrow (U_i/U_{i+1})(k) \longrightarrow 1.$$

The multiplication by $\sharp\Gamma$ on the abelian group $(U_i/U_{i+1})(k)$ is an isomorphism so that $H^1(\Gamma, (U_i/U_{i+1})(k)) = 0$. The exact sequence above shows that the map $H^1(\Gamma, U_{i+1}(k)) \to H^1(\Gamma, U_i(k))$ is onto. By induction it follows that $1 = H^1(\Gamma, U_n(k))$ maps onto $H^1(\Gamma, U(k))$ whence $H^1(\Gamma, U(k)) = 1$.

We consider now the pro-unipotent case. Since the U/U_i's are smooth, we have that $U(k) = \varprojlim(U/U_i)(k)$. Therefore by successive approximations the kernel of the map

$$H^1(\Gamma, U(k)) \longrightarrow \varprojlim H^1\big(\Gamma, (U/U_i)(k)\big)$$

is trivial. But according to the first case, the right handside is trivial thus we get $H^1(\Gamma, U(k)) = 1$.   □

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