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The isoperimetric problem in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$

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Abstract – In this paper we study the isoperimetric problem in a class of x-spherically symmetric sets in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, $p > -h + 1$. First we prove the existence of weighted isoperimetric sets. Then we deduce that, up to a vertical translation, a dilation and a negligible set, the weighted isoperimetric set is only of the form $\left\{(x, y) \in \mathbb{R}^{h+1} : |y| < \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)dt,\ |x| < 1\right\}$.

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1. Introduction

Manifolds with density, a new category in geometry, arise naturally in geometry as quotients of Riemannian manifolds, in physics as spaces with different mediums, in probability as the famous Gauss space and in a number of other places as well (see [23, 25, 26]).

The isoperimetric problem in Euclidean spaces with density has been investigated with increasing interest in recent years. Given a positive function f on an n-dimensional Euclidean space, usually called "density", for any set E of locally finite perimeter, the weighted volume $V_f(E)$ and the weighted perimeter $P_f(E)$

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are defined by

$$V_f(E) := \int_E f(x)dx, \quad P_f(E) := \int_{\partial^* E} f(x)d\mathcal{H}^{n-1}(x),$$

where $\partial^* E$ denotes the reduced boundary of E, which coincides the usual boundary of E if it is a smooth or piecewise affine set. The theory of finite perimeter sets and functions of bounded variation in Euclidean spaces with density or in more general metric spaces with density are studied in [6, 1, 2, 3, 4, 5]. The isoperimetric problem in Euclidean spaces with density concerns the existence and characterization of minimizers of

$$\inf\{P_f(E): E \in \mathcal{A} \text{ such that } V_f(E) = v\}, \tag{1}$$

for a given volume $v > 0$ and for a given admissible sets $\mathcal{A}$. Minimizers in (1) are called weighted isoperimetric sets.

One of the first and most important examples is Gauss space, an Euclidean space with Gaussian density $\exp(-\pi|x|^2)$. The mathematical interest in the Gaussian density question comes from its wide range of applications in Probability Theory and Functional Analysis. Half-spaces are Gaussian isoperimetric sets. Then the isoperimetric problem in Euclidean spaces with density has been widely studied. However, in spite of the last advances, the characterization of the solutions has been achieved only for some densities having a special form or a nice behavior with respect to a certain subgroup of diffeomorphisms, see the related works [28, 11, 25, 7, 9, 15, 10, 14, 27, 8] and reference therein.

On the other hand, in the context of sub-Riemannian spaces, the perimeter of a Lebesgue measurable set $E \subset \mathbb{R}^n$ is defined via a system $X = \{X_1, \ldots, X_h\}$, $2 \le h \le n$, of self-adjoint vector fields in $\mathbb{R}^n$, $X_j = -X_j^*$,

$$P_X(E) = \sup\left\{\int_E \operatorname{div}_X \varphi dx : \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^h),\ \max|\varphi(x)| \le 1\right\}, \tag{2}$$

where $\operatorname{div}_X \varphi := -\sum_{i=1}^h X_i^* \varphi_i(x) = \sum_{i=1}^h X_i \varphi_i(x)$ is called X-divergence of the vector field $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^h)$. This definition is introduced and studied systematically in [16]. The perimeter P_X is known as X-perimeter (horizontal, subelliptic, or sub-Riemannian perimeter). One important example is the α-perimeter P_α in $\mathbb{R}^{h+k} = \{(x, y): x \in \mathbb{R}^h, y \in \mathbb{R}^k\}$ defined via Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_j = |x|^\alpha \frac{\partial}{\partial y_j}, \quad i = 1, \ldots, h,\ j = 1, \ldots, k,$$

where $\alpha > 0$, $|x|$ is the standard Euclidean norm.

The isoperimetric problem in the Grushin plane $\mathbb{R}^2$ was thoroughly resolved by Monti and Morbidelli [22]. Franceschi and Monti [13] studied the isoperimetric problem in a class of x-spherically symmetric sets in the high dimensional Grushin space $\mathbb{R}^{h+k}$, $h > 1$, $k \geq 1$. They proved that, up to a vertical translation and a null set, any isoperimetric set is of the form $\{(x, y) \in \mathbb{R}^{h+k} : |y| < f(|x|)\}$ for some decreasing function $f : (0, r_0) \to \mathbb{R}^+$ which satisfies a differential equation. Particularly, when $k = 1$, up to a vertical translation, a dilation and a null set, any x-spherically symmetric isoperimetric set is of the form

$$E_\alpha = \left\{(x, y) \in \mathbb{R}^{h+1} : |y| < \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)dt,\ |x| < 1\right\}. \tag{3}$$

In the respect of studying sub-Riemannian manifolds with density, the weighted Sobolev and Poincaré inequalities for Hörmander's vector fields were well studied in [12, 19, 18]. Recently, the weighted isoperimetric and Sobolev inequalities for hypersurfaces in Carnot groups have been obtained in [17]. As far as we know, there is very little about the isoperimetric problem in sub-Riemannian manifolds with density.

In this paper we will consider the Grushin space $\mathbb{R}^{h+1}$ with a certain density and study the isoperimetric problem in a class of x-spherically symmetric sets. It is well known that the existence of weighted isoperimetric sets depends on the form of the density. So the choice of the density is very important. Here we endow the Grushin space $\mathbb{R}^{h+1}$ with density $f = |x|^p$, $p > -h + 1$. Following the classical approach by De Giorgi, we define the weighted α-perimeter of a Lebesgue measurable set E in $\mathbb{R}^{h+1}$ with density $|x|^p$ as follows:

$$\begin{aligned} P_{\alpha,f}(E) = \sup\Bigg\{ \int_E \operatorname{div}_\alpha(|x|^p\varphi(x, y))dxdy : \varphi \in C_c^1(\mathbb{R}^{h+1}; \mathbb{R}^{h+1}), \\ \max|\varphi(x, y)| \leq 1\Bigg\}, \end{aligned} \tag{4}$$

where

$$\operatorname{div}_\alpha \varphi := X_1\varphi_1 + \cdots + X_h\varphi_h + Y\varphi_{h+1} = \frac{\partial\varphi_1}{\partial x_1} + \cdots + \frac{\partial\varphi_h}{\partial x_h} + |x|^\alpha \frac{\partial\varphi_{h+1}}{\partial y}$$

is called α-divergence of a vector valued function $\varphi \in C^1(\mathbb{R}^{h+1}; \mathbb{R}^{h+1})$.

The weighted volume of a measurable set E in $\mathbb{R}^{h+1}$ with density $|x|^p$ is defined as

$$V_f(E) = \int_E |x|^p dxdy. \tag{5}$$

We will study the existence and characterization of minimizers of

$$\inf\{P_{\alpha,f}(E)\colon E \subset \mathbb{R}^{h+1} \text{ is an } x\text{-spherically symmetric set with } V_f(E) = v_0\}. \tag{6}$$

Minimizers in (6) are called weighted x-spherically symmetric isoperimetric sets.

Though the density $|x|^p$ is simple, it has some interesting properties. Since $|x|^p$ is homogeneous, we can prove that the weighted perimeter measure and the weighted volume measure are $(d-1)$-homogeneous and d-homogeneous with respect to a dilation $\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$ $(\forall \lambda > 0)$, respectively, where $d = h + 1 + \alpha + p$. So the minimizers in (6) are just minimizers of

$$\inf\left\{\frac{P_{\alpha,f}(E)^d}{V_f(E)^{d-1}}\colon E \subset \mathbb{R}^{h+1} \text{ is an } x\text{-spherically symmetric set with } 0 < V_f(E) < +\infty\right\}. \tag{7}$$

On the other hand, for every x-spherically symmetric set E in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, whose generating set is $F \subset \mathbb{R}^+ \times \mathbb{R}$, we have the following reduction formulas:

$$P_{\alpha,f}(E) = h\omega_h Q(F), \quad V_f(E) = h\omega_h V(F),$$

where the definitions of $Q(F)$ and $V_f(E)$ are introduced in Section 2. Then (7) is equivalent to

$$h\omega_h \inf\left\{\frac{Q(F)^d}{V(F)^{d-1}}\colon F \subset \mathbb{R}^+ \times \mathbb{R} \text{ is a set with } 0 < V(F) < +\infty\right\}. \tag{8}$$

Minimizers in (8) are called Q-isoperimetric sets. Thus the isoperimetric problem in the class of x-spherically symmetric sets in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$ is turned into the Q-isoperimetric problem in $\mathbb{R}^+ \times \mathbb{R}$, which can be studied by using the argument in [20].

Our main result in this paper is as follows:

Theorem 1.1. *In the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, $p > -h + 1$, the infimum in (6) is attained. Up to a dilation, a vertical translation and a negligible set, the weighted x-spherically symmetric isoperimetric set is only of the from*

$$E_\alpha = \left\{(x, y) \in \mathbb{R}^{h+1}\colon |y| < \int_{\arcsin |x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt,\ |x| < 1\right\}. \tag{9}$$

Here, by a vertical translation we mean a mapping of the form $(x, y) \mapsto (x, y + y_0)$ for some $y_0 \in \mathbb{R}$.

REMARK 1.2. Using Theorem 1.1 we can get a weighted isoperimetric inequality in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$. Namely, let E be any measurable set with finite weighted volume in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, $p > -h+1$. Then we have

$$V_f(E) \leq \frac{d-1}{d(h+p)}(h\omega_h)^{-\frac{1}{d-1}}\left[2\int_0^{\frac{\pi}{2}}(\sin t)^{h-1+p+\alpha}dt\right]^{-\frac{1}{d-1}}[P_{\alpha,f}(E)]^{\frac{d}{d-1}}.$$

REMARK 1.3. When $h > 1$ and p is zero, Theorem 1.1 is exactly a result in [13].

REMARK 1.4. In the case of $\alpha = 1$, following the argument of Proposition 2.3 in [13], we also have $P_{H,f}(E) = P_{\alpha,f}(E)$ for any z-spherically symmetric set E where $P_{H,f}(E)$ denotes the weighted horizontal perimeter in the Heisenberg group $\mathbb{H}^n$ with a horizontal radial density $f(|z|) = |z|^p$. So by Theorem 1.1, we derive that up to a Heisenberg dilation, a vertical translation and a negligible set, any weighted z-spherically symmetric isoperimetric set in the Heisenberg group is a Bubble set, i.e.,

$$E_{\text{isop}} = \left\{(z,t) \in \mathbb{H}^n : |t| < \frac{1}{2}\left[\arccos|z| + |z|\sqrt{1-|z|^2}\,\right],\ |z| < 1\right\}.$$

The paper is organized as follows. In Section 2, we give some definitions and deduce reduced formulas for the weighted α-perimeter and the weighted volume of x-spherically symmetric sets in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$. In Section 3, we rearrange the generating sets in the half-plane $\mathbb{R}^+ \times \mathbb{R}$ by the rearrangement techniques in [20], and prove the existence of Q-isoperimetric sets. In Section 4, we use a variational method to obtain the equation of weighted isoperimetric sets.

2. The reduction formulas of the weighted α-perimeter and weighted volume

For any open set A and $m \in \mathbb{N}$, let us denote the family of test functions

$$\mathcal{F}_m(A) = \{\varphi = (\varphi_1, \ldots, \varphi_m) \in C_c^1(A;\mathbb{R}^m) : \|\varphi\|_\infty = \max_{(x,y)\in A}|\varphi(x,y)| \leq 1\}.$$

For a fixed real number $\alpha > 0$, the Grushin vector fields in $\mathbb{R}^{h+1}$ are given by

$$X_i = \frac{\partial}{\partial x_i}, \quad Y = |x|^\alpha\frac{\partial}{\partial y}, \quad i = 1, \ldots, h,$$

where $|x|$ is the standard Euclidean norm. For any function $\phi(x,y) \in C^1(\mathbb{R}^{h+1};\mathbb{R})$, the α-gradient of ϕ is given by

$$\nabla_\alpha\phi = (X_1\phi, \ldots, X_h\phi, Y\phi).$$

The α-divergence of a vector field $\varphi = (\varphi_1, \ldots, \varphi_{h+1}) \in C^1(\mathbb{R}^{h+1}; \mathbb{R}^{h+1})$ is defined as

$$\operatorname{div}_\alpha \varphi = X_1\varphi_1 + \cdots + X_h\varphi_h + Y\varphi_{h+1}.$$

Let $f = e^\phi$ be a smooth density on the Grushin space $\mathbb{R}^{h+1}$. The weighted volume measure on $\mathbb{R}^{h+1}$ is defined as $dv_f = e^\phi dxdy$. For all $\varphi \in C^1(\mathbb{R}^{h+1}; \mathbb{R}^{h+1})$, we define the weighted α-divergence of φ as

$$\operatorname{div}_{\alpha,f} \varphi = \operatorname{div}_\alpha \varphi + \langle \varphi, \nabla_\alpha \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product.

In the sense of De Giogi, the weighted α-perimeter of any Borel set E is defined as

$$P_{\alpha,f}(E) = \sup_{\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})} \left\{ \int_E \operatorname{div}_{\alpha,f} \varphi(x,y) dv_f \right\}. \tag{10}$$

Since $\operatorname{div}_{\alpha,f} \varphi(x,y) dv_f = \operatorname{div}_\alpha(e^\phi \varphi(x,y)) dxdy$, equation (10) is equivalent to

$$P_{\alpha,f}(E) = \sup_{\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})} \left\{ \int_E \operatorname{div}_\alpha(e^\phi \varphi(x,y)) dxdy \right\}. \tag{11}$$

Now we endow the Grushin space $\mathbb{R}^{h+1}$ with the density $f = |x|^p$, $p > -h+1$. By (11), the weighted α-perimeter of a measurable set $E \subset \mathbb{R}^{h+1}$ is given by

$$P_{\alpha,f}(E) = \sup_{\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})} \left\{ \int_E \operatorname{div}_\alpha(|x|^p \varphi(x,y)) dxdy \right\}. \tag{12}$$

If $P_{\alpha,f}(E) < +\infty$, the set E is said to have finite weighted α-perimeter. The weighted volume of a measurable set $E \subset \mathbb{R}^{h+1}$ is given by

$$V_f(E) = \int_E |x|^p dxdy. \tag{13}$$

For a set $E \subset \mathbb{R}^{h+1}$ with Lipschitz boundary, the outer unit normal $N^E = (N_x^E, N_y^E)$ is defined at $\mathcal{H}^h$-a.e. point of ∂E. The vector field $N_\alpha^E = (N_x^E, |x|^\alpha N_y^E)$ is called the α-normal to ∂E.

Following the argument of Proposition 3.1 in [21], we have the following proposition.

Proposition 2.1. *Let E be a bounded open set with Lipschitz boundary in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, then the weighted α-perimeter of E is*

$$P_{\alpha,f}(E) = \int_{\partial E} |N_\alpha^E| \, |x|^p d\mathcal{H}^h, \tag{14}$$

where $\mathcal{H}^h$ is the standard h-dimensional Hausdorff measure.

REMARK 2.2. The formula (14) holds also in the case of general density.

Setting $d = h + \alpha + 1 + p$, the weighted volume and the weighted α-perimeter are d-homogeneous and $(d-1)$-homogeneous with respect to the dilations $(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$ for any $\lambda > 0$, respectively.

PROPOSITION 2.3. *Let E be a measurable set in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$. Then we have*

(i) $V_f(\delta_\lambda(E)) = \lambda^d V_f(E)$,

(ii) $P_{\alpha,f}(\delta_\lambda(E)) = \lambda^{d-1} P_{\alpha,f}(E)$.

PROOF. (i) The formula is directly obtained by the definition of the weighted volume.

(ii) Assuming that $\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})$, we have

$$
\begin{aligned}
&\int_{\delta_\lambda(E)} \operatorname{div}_\alpha(|x|^p \varphi(x, y)) dx dy \\
&\quad = \int_{\delta_\lambda(E)} \Big[\sum_{i=1}^h \partial_{x_i}(|x|^p \varphi_i(x, y)) + |x|^\alpha \partial_y(|x|^p \varphi_{h+1}(x, y))\Big] dx dy \\
&\quad = \int_{\delta_\lambda(E)} \Big[\sum_{i=1}^h p|x|^{p-1} \frac{x_i}{|x|} \varphi_i(x, y) \\
&\qquad\qquad + \sum_{i=1}^h |x|^p \partial_{x_i} \varphi_i(x, y) + |x|^{\alpha+p} \partial_y \varphi_{h+1}(x, y)\Big] dx dy \\
&\quad = \int_E \Big[\sum_{i=1}^h p|\lambda\xi|^{p-1} \frac{\lambda \xi_i}{\lambda|\xi|} \varphi_i \circ \delta_\lambda(\xi, \eta) + \sum_{i=1}^h |\lambda\xi|^p (\partial_{x_i}\varphi_i) \circ \delta_\lambda(\xi, \eta) \\
&\qquad\qquad + |\lambda\xi|^{\alpha+p} (\partial_y \varphi_{h+1}) \circ \delta_\lambda(\xi, \eta)\Big] \lambda^{h+\alpha+1} d\xi d\eta \\
&\quad = \int_E \lambda^{h+\alpha+p} \Big[\sum_{i=1}^h p|\xi|^{p-1} \frac{\xi_i}{|\xi|} \varphi_i(\delta_\lambda(\xi, \eta)) + \sum_{i=1}^h |\xi|^p \partial_{\xi_i}(\varphi_i \circ \delta_\lambda(\xi, \eta)) \\
&\qquad\qquad + |\xi|^{p+\alpha} \partial_\eta(\varphi_{h+1} \circ \delta_\lambda(\xi, \eta))\Big] d\xi d\eta \\
&\quad = \lambda^{h+\alpha+p} \int_E \operatorname{div}_\alpha(|\xi|^p (\varphi \circ \delta_\lambda)(\xi, \eta)) d\xi d\eta.
\end{aligned}
$$

Obviously we have $\varphi \circ \delta_\lambda \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})$. Taking the supremum over test functions we have

$$P_{\alpha,f}(\delta_\lambda(E)) \leq \lambda^{d-1} P_{\alpha,f}(E).$$

The converse inequality is obtained in the same way. $\square$

We say a set $E \subset \mathbb{R}^{h+1}$ is x-spherically symmetric if there exists a set $F \subset \mathbb{R}^+ \times \mathbb{R}$ such that

$$E = \{(x, y) \in \mathbb{R}^{h+1} \colon (|x|, y) \in F\}.$$

The set $F \subset \mathbb{R}^+ \times \mathbb{R}$ is called the generating set of E.

Proposition 2.4. *Let E be a measurable x-spherically symmetric set in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$. Assume that E has finite weighted α-perimeter and let F be its generating set. Then we have*

$$P_{\alpha,f}(E) = h\omega_h \sup_{\psi \in \mathcal{F}_2(\mathbb{R}^+\times\mathbb{R})} \left\{ \int_F [\partial_r(r^{h-1+p}\psi_1) + r^{h-1+\alpha+p}\partial_y\psi_2] dr dy \right\}, \tag{15}$$

where ω_h denotes the Euclidean volume of a h-dimension unit ball.

In particular, if E has Lipschitz boundary then we have

$$P_{\alpha,f}(E) = h\omega_h \int_{\partial F} |(N_r^F, r^\alpha N_y^F)| r^{h-1+p} d\mathcal{H}^1(r, t), \tag{16}$$

where $N^F = (N_r^F, N_y^F)$ is the outer Euclidean unit normal to the boundary ∂F and $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

Proof. We call

$$Q(F) = \sup_{\psi \in \mathcal{F}_2(\mathbb{R}^+\times\mathbb{R})} \left\{ \int_{\partial F} [\partial_r(r^{h-1+p}\psi_1) + r^{h-1+\alpha+p}\partial_y\psi_2] dr dy \right\}$$

the Q-perimeter of the generating set F in $\mathbb{R}^+ \times \mathbb{R}$.

Step 1. We claim that if E is a x-spherically symmetric set generated by the set $F \subset \mathbb{R}^+ \times \mathbb{R}$ and has finite weighted α-perimeter, then we have $P_{\alpha,f}(E) \geq h\omega_h Q(F)$.

For any $\psi \in \mathcal{F}_2(\mathbb{R}^+ \times \mathbb{R})$, we define the test function $\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})$ as

$$\varphi(x, y) = \Big(\frac{x}{|x|}\psi_1(|x|, y), \psi_2(|x|, y)\Big), \quad \text{for } |x| \neq 0,$$

and

$$\varphi(0, y) = 0.$$

For any $i = 1, \dots, h$ and $x \neq 0$, we have

$$\partial_{x_i} \varphi_i(x, y) = \Big(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\Big)\psi_1(|x|, y) + \frac{x_i^2}{|x|^2}\partial_r \psi_1(|x|, y),$$
$$\partial_y \varphi_{h+1}(x, y) = \partial_y \psi_2(|x|, y).$$

Then we have

(17)
$$\begin{aligned}
\operatorname{div}_\alpha(|x|^p \varphi(x, y)) &= |x|^p \operatorname{div}_\alpha \varphi + \langle \varphi, \nabla_\alpha |x|^p\rangle \\
&= |x|^p \Big(\sum_{i=1}^{h} \partial_{x_i}\varphi_i(|x|, y) + |x|^p \partial_y \varphi_{h+1}(|x|, y)\Big) + p|x|^{p-1}\langle \varphi, \nabla_\alpha |x|\rangle \\
&= |x|^p \Big[\frac{h-1}{|x|}\psi_1(|x|, y) + \partial_r \psi_1(|x|, y) + |x|^\alpha \partial_y \psi_2(|x|, y)\Big] \\
&\quad + p|x|^{p-1}\psi_1(|x|, y) \\
&= (h-1+p)|x|^{p-1}\psi_1(|x|, y) + |x|^p \partial_r \psi_1(|x|, y) + |x|^{p+\alpha}\partial_y \psi_2(|x|, y).
\end{aligned}$$

For any $y \in \mathbb{R}$, we define the section $F_y = \{r > 0 \colon (r, y) \in F\}$. Using Fubini theorem, the symmetry of E, the Coarea formula and (17) we obtain

$$\begin{aligned}
&\int_E \operatorname{div}_\alpha(|x|^p \varphi) dx dy \\
&\quad= \int_{\mathbb{R}} \int_{F_y} \int_{|x|=r} \big[(h-1+p)|x|^{p-1}\psi_1(|x|, y) + |x|^p \partial_r \psi_1(|x|, y) \\
&\qquad\qquad + |x|^{p+\alpha}\partial_y \psi_2(|x|, y)\big] d\mathcal{H}^{2n-1}(x) dr dy \\
&\quad= \int_{\mathbb{R}} \int_{F_y} h\omega_h r^{h-1}\big[(h-1+p) r^{p-1}\psi_1(r, y) + r^p \partial_r \psi_1(r, y) \\
&\qquad\qquad + r^{p+\alpha}\partial_y \psi_2(r, y)\big] dr dy \qquad (18) \\
&\quad= h\omega_h \int_{\mathbb{R}} \int_{F_y} [\partial_r (r^{h-1+p}\psi_1 + r^{h-1+p+\alpha}\partial_y \psi_2] dr dy \\
&\quad= h\omega_h \int_F [\partial_r (r^{h-1+p}\psi_1 + r^{h-1+p+\alpha}\partial_y \psi_2] dr dy.
\end{aligned}$$

Taking the supremum over all $\psi \in \mathcal{F}_2(\mathbb{R}^+ \times \mathbb{R})$ of the right side of (18), we obtain

$$P_{\alpha, f}(E) \geq h\omega_h Q(F).$$

Step 2. We claim that when $E \subset \mathbb{R}^{h+1}$ is a x-spherically symmetric set with smooth boundary, it holds $P_{\alpha,f}(E) \le h\omega_h Q(F)$.

The unit outer normal $N^E = (N_x^E, N_y^E)$ is continuously defined on ∂E. At points $(0, y) \in \partial E$, we have $N_x^E(0, y) = 0$ and thus $N_\alpha^E(0, y) = 0$. For any $\varepsilon > 0$ we consider a compact set $K \subset \partial E \backslash Y$, where $Y = \{(0, y) \in \mathbb{R}^{h+1}, y \in \mathbb{R}\}$, such that

$$\int_{\partial E \backslash K} |N_\alpha^E|\, |x|^p \, d\mathcal{H}^h < \varepsilon. \tag{19}$$

Let $H \subset \mathbb{R}^+ \times \mathbb{R}$ be the generating set of K. By standard extension theorems, there exists $\psi \in \mathcal{F}_2(\mathbb{R}^+ \times \mathbb{R})$ such that

$$\psi(r, y) = \frac{(N_r^F, r^\alpha N_y^F)}{|(N_r^F, r^\alpha N_y^F)|}, \quad \text{for } (r, y) \in K.$$

Setting $\varphi(x, y) = \big(\frac{x}{|x|}\psi_1(|x|, y), \psi_2(|x|, y)\big)$ for $|x| \neq 0$ and $\varphi(0, y) = 0$, then we know $\varphi \in \mathcal{F}_{h+1}(\mathbb{R}^{h+1})$. And we have

$$\varphi(x, y) = \frac{N_\alpha^E(x, y)}{|N_\alpha^E(x, y)|}, \quad \text{for } (x, y) \in K. \tag{20}$$

By the definition of $Q(F)$, the identity (18), the divergence theorem, (19) and (20), we have

$$\begin{aligned}
h\omega_h Q(F) &\ge h\omega_h \int_F [\partial_r(r^{h-1+p}\psi_1) + r^{h-1+\alpha+p}\partial_t \psi_2] dr dy \\
&= \int_E \operatorname{div}_\alpha(|x|^p \varphi(x, y)) dx dy \\
&= \int_{\partial E} |x|^p \langle \varphi, N_\alpha^E \rangle d\mathcal{H}^h \\
&= \int_K |x|^p \Big\langle \frac{N_\alpha^E}{|N_\alpha^E|}, N_\alpha^E \Big\rangle d\mathcal{H}^h + \int_{\partial E \backslash K} |x|^p \langle \varphi, N_\alpha^E \rangle d\mathcal{H}^h \\
&\ge \int_{\partial E} |x|^p |N_\alpha^E| d\mathcal{H}^h - 2\varepsilon \\
&= P_{H,f}(E) - 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get $h\omega_h Q(F) \ge P_{H,f}(E)$.

Step 3. The general case $h\omega_h Q(F) \ge P_{H,f}(E)$ follows by the approximation argument as in [20].

Starting from (15), the formula (16) is obtained by using the divergence theorem and taking the supremum. □

Using the spherical coordinates, we obtain the reduction formula of the weighted volume for x-spherically symmetric sets E in the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, i.e.

$$\begin{aligned} V_f(E) &= \int_E |x|^p dxdy \\ &= \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{|x|=r} |x|^p d\mathcal{H}^{h-1}(z) drdy \\ &= \int_{-\infty}^{+\infty} \int_0^{+\infty} h\omega_h r^{h-1+p} drdy \\ &= h\omega_h \int_F r^{h-1+p} drdy. \end{aligned}$$

Letting $V(F) = \int_F r^{h-1+p} drdy$, then we have $V_f(E) = h\omega_h V(F)$. Moreover we have

$$\frac{P_f(E)^d}{V_f(E)^{d-1}} = h\omega_h \frac{Q(F)^d}{V(F)^{d-1}}.$$

Thus the weighted isoperimetric problem (7) is turned into

$$h\omega_h \inf\left\{\frac{Q(F)^d}{V(F)^{d-1}} : F \subset \mathbb{R}^+ \times \mathbb{R} \text{ with } 0 < V(F) < +\infty\right\}. \tag{21}$$

Minimizers (21) are called Q-isoperimetric sets.

It is easily verified that $Q(F)$ and $V(F)$ are invariant under Euclidean vertical translations of the form $(r, y) \to (r, y + y_0)$, for some $y_0 \in \mathbb{R}$. $Q(F)$ and $V(F)$ are homogeneous with respect to dilations $\delta_\lambda(r, t) = (\lambda r, \lambda^{\alpha+1} t)$ for any $\lambda > 0$. Namely, we have

$$Q(\delta_\lambda(F)) = \lambda^{d-1} Q(F), \quad V(\delta_\lambda(F)) = \lambda^d V(F).$$

3. Existence of Q-isoperimetric sets

In this section we first rearrange the generating sets F by using the rearrangement technique introduced in [20]. Then we prove the existence of Q-isoperimetric sets.

By the homogeneity of $Q(F)$ and $V(F)$, we can define the constant

$$C_I = \inf\{Q(F) : F \subset \mathbb{R}^+ \times \mathbb{R} \text{ with } V(F) = 1\}. \tag{22}$$

Theorem 3.1. *The infimum in* (22) *is attained and any* Q*-isoperimetric set* F *satisfies*:

i) *up to a* $\mathcal{L}^2$*-negligible set,* $F = F^\sharp := \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : r < g(y)\}$ *where* $g : \mathbb{R} \to [0, +\infty]$ *is a nonnegative function*;

ii) *the sections* F_r *are intervals for* $\mathcal{L}^1$*-a.e.* $r \in \mathbb{R}^+$;

iii) F *is bounded. In fact it holds*

$$F \subset \Big\{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : 0 \le r \le \Big(\frac{h + \alpha + p}{2} Q(F)\Big)^{\frac{1}{h+\alpha+p}}, \quad |y - y_0| \le \Big(\frac{2}{h + p}\Big)^{h-1+p} \frac{Q(F)^{h+p}}{V(F)^{h-1+p}}\Big\} \tag{23}$$

for some $y_0 \in \mathbb{R}$.

Proof. We will complete the proof by three steps.

Step 1. Since $V(F) < +\infty$, by Fubini theorem we know

$$h(r) = \frac{1}{2}\mathcal{L}^1(F_r) \in L^1_{\mathrm{loc}}(\mathbb{R}^+),$$

where $F_r = \{t \in \mathbb{R} : (r, t) \in F\}$ is a section. So we can rearrange the set F using the Steiner symmetrization in direction y. Namely, we let

$$F^* = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : |y| < h(r)\}.$$

By Theorem 3.2 in [20], we have

$$Q(F^*) \le Q(F),$$

and the equality $Q(F^*) = Q(F)$ implies that F_r are intervals for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$.

Moreover we have

$$\begin{aligned} V(F^*) &= \int_{F^*} r^{h-1+p} dr dy \\ &= \int_0^{+\infty} dr \int_{F_r^*} r^{h-1+p} dy \\ &= \int_0^{+\infty} r^{h-1+p} \mathcal{L}^1(F_r) dr \\ &= \int_0^{+\infty} r^{h-1+p} dr \int_{F_r} dy \\ &= V(F). \end{aligned} \tag{24}$$

Assume that $F = F^*$ and $Q(F) < +\infty$. We rearrange F in the coordinate r using the function $r^{h-1+\alpha+p}$. Namely, we define the function $g\colon \mathbb{R} \to [0, +\infty]$ via the identity

$$\frac{1}{h+\alpha+p} g(y)^{h+\alpha+p} = \int_0^{g(y)} r^{h-1+\alpha+p}\,dr = \int_{F_y} r^{h-1+\alpha+p}\,dr. \tag{25}$$

Let $F_1^\sharp = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : 0 < r < g(y)\}$. By Theorem 1.5 in [20], we know $Q(F_1^\sharp) \le Q(F)$ and moreover, if $Q(F_1^\sharp) = Q(F)$, then $F = F_1^\sharp$, up to a $\mathcal{L}^2$-negligible set.

Using Example 2.5 in [20], we know that for any measurable set $F_y \subset \mathbb{R}^+$, for any $p > -h + 1$, it holds that

$$\left[(h+\alpha+p)\int_{F_y} r^{h-1+\alpha+p}\,dr\right]^{\frac{1}{h-1+\alpha+p}} \ge \left[(h+p)\int_{F_y} r^{h-1+p}\,dr\right]^{\frac{1}{h+p}}. \tag{26}$$

By (25) and (26), we get

$$\int_{F_{1y}^\sharp} r^{h-1+p}\,dr \ge \int_{F_y} r^{h-1+p}\,dr \tag{27}$$

and (27) is an equality if and only if $F_y = F_{1y}^\sharp = (0, g(y))$, up to a $\mathcal{L}^1$-negligible set.

Using (27) and Fubini theorem we have $V(F_1^\sharp) \ge V(F)$, with equality if and only if $F = F_1^\sharp$, up to a $\mathcal{L}^2$-negligible set.

Setting $\lambda = \left(\frac{V(F)}{V(F_1^\sharp)}\right)^{\frac{1}{d}} \le 1$ and $F^\sharp = \delta_\lambda(F_1^\sharp)$, we have

$$V(F^\sharp) = V(F)$$

and

$$Q(F^\sharp) = \lambda^{d-1} Q(F_1^\sharp) \le Q(F_1^\sharp) \le Q(F^*) \le Q(F).$$

Moreover, if $Q(F^\sharp) = Q(F)$, then it must $\lambda = 1$ and $Q(F_1^\sharp) = Q(F)$. Thus we have $F = F^\sharp$, up to a $\mathcal{L}^2$-negligible set.

Step 2. Assume that $F = F^\sharp$ and the sections F_r are intervals. We claim that

$$\begin{aligned} F \subset \Big\{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : 0 \le r \le \left(\frac{h+\alpha+p}{2} Q(F)\right)^{\frac{1}{h+\alpha+p}}, \\ |y - y_0| \le \left(\frac{2}{h+p}\right)^{h-1+p} \frac{Q(F)^{h+p}}{V(F)^{h-1+p}}\Big\} \end{aligned}$$

for some $y_0 \in \mathbb{R}$.

Up to a $\mathcal{L}^2$-negligible set, the set F is of the form

$$F = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} \colon 0 < r < g(y), y \in \mathbb{R}\} \tag{28}$$

for some function $g\colon \mathbb{R} \to [0, +\infty]$ which is decreasing on $(y_0, +\infty)$ and increasing on $(-\infty, y_0)$ for some $y_0 \in \mathbb{R}$. Set

$$M = \sup_{y\in\mathbb{R}} g(y).$$

By the definition of $Q(F)$, we have

$$\begin{aligned}
Q(F) &\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+\times\mathbb{R})} \int_F r^{h-1+\alpha+p}\partial_y\psi(r, y)drdy \\
&\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R})} \int_F r^{h-1+\alpha+p}\partial_y\psi(y)drdy \\
&\geq \int_0^\infty r^{h-1+\alpha+p} \sup_{\psi\in\mathcal{F}_1(\mathbb{R})} \int_{\{g(y)>r\}} \partial_y(\psi(y))dydr \\
&= 2\int_0^\infty r^{h-1+\alpha+p}dr = \frac{2M^{h+\alpha+p}}{h+\alpha+p}.
\end{aligned}$$

Then we get the estimate

$$M \leq \Big[\frac{(h+\alpha+p)Q(F)}{2}\Big]^{\frac{1}{h+\alpha+p}}. \tag{29}$$

On the other hand, the set F in (28) is also of the form

$$F = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} \colon k(r) < y < h(r), r \in \mathbb{R}^+\}$$

for some functions $k, h\colon \mathbb{R}^+ \to [-\infty, +\infty]$ such that h and $-k$ are decreasing, thanks to $F = F^\sharp$. Moreover, we can assume that $h(r) = k(r) = y_0$ for all $r > M$. Thus by the definition of $Q(F)$, we have

$$\begin{aligned}
Q(F) &\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+\times\mathbb{R})} \int_F \partial_r(r^{h-1+p}\psi(r, y))drdy \\
&\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{\mathbb{R}^+} (h(r)-k(r))\partial_r(r^{h-1+p}\psi(r))dr \\
&\geq \lim_{r\to0^+} (h(r)-k(r))M^{2n-1+p}.
\end{aligned}$$

Then we get

$$|y - y_0| \leq \lim_{r\to0^+} (h(r)-k(r)) \leq \frac{Q(F)}{M^{h-1+p}}. \tag{30}$$

From (29) and (30), we know

$$F \subset R := [0, M] \times \left[y_0 - \frac{Q(F)}{M^{h-1+p}}, y_0 + \frac{Q(F)}{M^{h-1+p}} \right]. \tag{31}$$

Now from (31) we have

$$V(F) \leq V(R) = \int_R r^{h-1+p} dr dt = \frac{2Q(F)}{h+p} M.$$

Thus we get an estimate from below for M:

$$M \geq \frac{(h+p)V(F)}{2Q(F)}. \tag{32}$$

Finally from (29), (31), and (32) we get (23).

Step 3. We claim that the infimum in (22) is attained.
Let $\{F_j\}_{j\in\mathbb{N}}$ be a minimizing sequence for (22). Namely, let

$$V(F_j) = 1, \quad \lim_{j\to\infty} Q(F_j) = C_I > 0.$$

By Step 1, without loss of generality we can assume that every set satisfies $F_j = F_j^* = F_j^\sharp$. So we let $F_j = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : |y| < h_j(r)\}$ for some function $h_j : \mathbb{R}^+ \to [0, +\infty]$ which is decreasing on $(0, +\infty]$. By Step 2 functions h_j are uniformly bounded and moreover, there exists $r_0 > 0$ such $h_j(r) = 0$ for all $r > r_0$ and for all $j \in \mathbb{N}$. So there exists a subsequence of $\{h_j\}_{j\in\mathbb{N}}$, still denoted by $\{h_j\}$, such that $\lim_{j\to\infty} h_j(r) = h(r)$, for all $r > 0$ and the function $h(r)$ is also decreasing. Let

$$F = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R} : |y| < h(r), r \in \mathbb{R}^+\}.$$

By the dominated convergence theorem, we have

$$V(F) = \lim_{j\to+\infty} V(F_j) = 1.$$

Moreover, $\chi_{F_j} \to \chi_F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$. Then by the lower semicontinuity of the perimeter, we have

$$Q(F) \leq \liminf_{j\to\infty} Q(F_j) = C_I.$$

By the definition of C_I, we have $Q(F) = C_I$.

Now let F be any Q-isoperimetric set. By Step 1, F satisfies $F = F^\sharp$ and the section F_r are intervals for $\mathcal{L}^1$-a.e. $r > 0$. By Step 2, the set F satisfies (23). □

4. Characterization of weighted x-spherical symmetric isoperimetric sets

In this section we give the main result and use the variational principle to finish its proof.

Theorem 4.1. *In the Grushin space $\mathbb{R}^{h+1}$ with density $|x|^p$, $p > -h+1$, the infimum in (6) is attained. Up to a dilation, a vertical translation and a negligible set, any weighted x-spherically symmetric isoperimetric set is of the from*

$$E_\alpha = \left\{(x,y) \in \mathbb{R}^{h+1} \colon |y| < \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)dt,\ |x| < 1\right\}. \tag{33}$$

Here, by a vertical translation we mean a mapping of the form $(x,y) \mapsto (x, y+y_0)$ for some $y_0 \in \mathbb{R}$.

Proof. By Theorem 3.1, the infimum in (22) is attained at a set $F \subset \mathbb{R}^+ \times \mathbb{R}$. By the homogeneity of $Q(F)$ and $V(F)$, the set F is also a minimizer for (21). In the following we will compute the Q-isoperimetric set F by the variational method.

Let $F \subset \mathbb{R}^+ \times \mathbb{R}$ be a minimizer for (21). By Theorem 3.1, up to a negligible set and a vertical translation, the set F is of the form

$$F = \{(r,y) \in \mathbb{R}^+ \times \mathbb{R} \colon |y| < h(r), r \in (0, r_0)\} \tag{34}$$

where $h\colon (0,r_0) \to (0,+\infty)$ is a decreasing function, for some $r_0 \in (0,+\infty]$. By the regularity theory of isoperimetric hypersurfaces (see [24]), the boundary $\partial F \cap (\mathbb{R}^+ \times \mathbb{R})$ is a curve of class C^∞. By (16) we have

$$Q(F) = \int_{\partial F} |(N_r^F, r^\alpha N_y^F)| r^{h-1+p} d\mathcal{H}^1(r,y),$$

where (N_r^F, N_y^F) is the outer unit normal to ∂F. Thus by (34) we get

$$Q(F) = 2\int_0^{r_0} \sqrt{h'(r)^2 + r^{2\alpha}}\, r^{h-1+p} dr,$$

$$V(F) = \int_F r^{h-1+p} dr dy = 2\int_0^{r_0} h(r) r^{h-1+p} dr.$$

For any $\psi \in C_c^\infty(0,r_0)$ and $\varepsilon \in \mathbb{R}$, we consider perturbation $h+\varepsilon\psi$, and define the set $F_\varepsilon = \{(r,y) \in \mathbb{R}^+ \times \mathbb{R} \colon |y| < h(r) + \varepsilon\psi(r)\}$. Then we have

$$Q(\varepsilon) := Q(F_\varepsilon) = 2\int_0^{r_0} \sqrt{(h' + \varepsilon\psi')^2 + r^{2\alpha}}\, r^{h-1+p} dr,$$

$$V(\varepsilon) := V(F_\varepsilon) = 2\int_0^{r_0} (h+\varepsilon\psi) r^{h-1+p} dr.$$

Since F is a Q-isoperimetric set, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{Q(F_\varepsilon)^d}{V(F_\varepsilon)^{d-1}} = \frac{dQ(\varepsilon)^{d-1}Q'(\varepsilon)V(\varepsilon)^{d-1} - (d-1)Q(\varepsilon)^d V(\varepsilon)^{d-2}V'(\varepsilon)}{V(\varepsilon)^{2(d-1)}}\Big|_{\varepsilon=0}$$
$$= 0.$$

It follows that

$$Q'(0) - KV'(0) = 0, \quad \text{where } K = \frac{d-1}{d}\frac{Q(F)}{V(F)}.$$

i.e.

$$\int_0^{r_0} \frac{h'\psi'}{\sqrt{(h')^2 + r^{2\alpha}}} r^{h-1+p} dr = \int_0^{r_0} K r^{h-1+p}\psi dr.$$

Using the divergence theorem and the fact that $\psi \in C_c^\infty(0, r_0)$ is arbitrary, we deduce that h solves the following second order ordinary differential equation:

$$\Big(\frac{h' r^{h-1+p}}{\sqrt{h'^2 + r^{2\alpha}}}\Big)' = -K r^{h-1+p}. \tag{35}$$

We integrate the equation (35) on the interval $[0, r]$ and get

$$\frac{h'}{\sqrt{h'^2 + r^{2\alpha}}} = -\frac{K}{h+p} r + \frac{D}{r^{h-1+p}}, \tag{36}$$

where D is some constant.

Because of $p > -h+1$, we deduce that $D = 0$. In fact, the left-hand side of (36) is bounded as $r \to 0^+$, while the right-hand side diverges to $\pm\infty$ according to the sign $D \neq 0$. So (36) is

$$\frac{h'}{\sqrt{h'^2 + r^{2\alpha}}} = -\frac{K}{h+p} r. \tag{37}$$

By a dilation of the set F, we can assume that $K = h + p$. Noting that $h' \le 0$, we have

$$h'(r) = -\frac{r^{\alpha+1}}{\sqrt{1-r^2}}. \tag{38}$$

From (38) we know the function h' is defined on the interval $(-1, 1)$.

On the other hand, because $\partial F \cap (\mathbb{R}^+ \times \mathbb{R})$ is of class C^∞, it must be $h(1) = 0$. With this condition, the solution of equation (38) is

$$h(r) = \int_r^1 \frac{t^{\alpha+1}}{\sqrt{1-t^2}} ds = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t) dt, \quad r < 1.$$

So the weighted x-spherically symmetric isoperimetric set E generated by F is the form

$$E_\alpha = \left\{(x, y) \in \mathbb{R}^{h+1} \colon |y| < \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)dt,\ |x| < 1\right\}. \qquad \square$$

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References

[1] L. Ambrosio – S. Maniglia – M. Miranda – D. Pallara, *BV functions in abstract Wiener spaces*, J. Funct. Anal. **258** (2010), pp. 785–813.

[2] L. Ambrosio – M. Miranda – D. Pallara, *Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, pp. 591–606.

[3] L. Ambrosio – G. Da Prato – D. Pallara, *BV functions in a Hilbert space with respect to a Gaussian measure*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **21** (2010), no. 4, 405–414.

[4] L. Ambrosio – A. Figalli – E. Runa, *On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **24** (2013), no. 1, pp. 111–122.

[5] L. Ambrosio – M. Miranda – D. Pallara, *Some fine properties of BV functions on Wiener spaces*, Anal. Geom. Metr. Spaces **3** (2015), pp. 212–230.

[6] A. Baldi, *Weighted BV functions*, Houston J. Math. **27** (2001), no. 3, pp. 683–705.

[7] A. Cañete – M. Jr. Miranda – D. Vittone, *Some isoperimetric problems in planes with density*, J. Geom. Anal. **20** (2010), no. 2, pp. 243–290.

[8] A. Cañete – C. Rosales, *Compact stable hypersurfaces with free boundary in convex solid cones with homogeneous densities*, Calc. Var. Partial Differ. Equ. **51** (2014), no. 3-4, pp. 887–913.

[9] J. Dahlberg – A. Dubbs – E. Newkirk – H. Tran, *Isoperimetric regions in the plane with density r^p*, New York J. Math. **16** (2010), pp. 31–51.

[10] A. Díaz – N. Harman – S. Howe – D. Thompson, *Isoperimetric problems in sectors with density*, Adv. Geom. **12** (2012), pp. 589–619.

[11] M. Engelstein – A. Marcuccio – Q. Maurmann – T. Pritchard, *Isoperimetric problems on the sphere and on surfaces with density*, New York J. Math. **15** (2009), pp. 97–123.

[12] B. Franchi – G. Lu – R. L. Wheeden, *Representation formulas and weighted Poincaré inqualities for Hörmander vector fields*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, pp. 577–604.

[13] V. Franceschi – R. Monti, *Isoperimetric problem in H-type groups and Grushin spaces*, Rev. Mat. Iberoam. **32** (2016), no. 4, pp. 1227–1258.

[14] A. Figalli – F. Maggi, *On the isoperimetric problem for radial log-convex densities*, Calc. Var. Partial Differential Equations, **48** (2013), no. 3-4, pp. 447–489.

[15] N. Fusco – F. Maggi – A. Pratelli, *On the isoperimetric problem with respect to a fixed Euclidean–Gaussian density*, J. Funct. Anal. **260** (2011), no. 12, pp. 3678–3717.

[16] N. Garofalo – D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the exentence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), pp. 1081–1144.

[17] G. He – P. Zhao, *The weighted isoperimetric-type and sobolev-type inequalities for hypersurfaces in Carnot groups,* Nonlinear Anal. **135** (2016), pp. 35–56.

[18] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoam **8** (1992), no. 3, pp. 367–439.

[19] G. Lu, *Local and global interpolation inequalities on the Folland-Stein Sobolev spaces and polynomials on straighted groups*, Math. Resercher Letters **4** (1997), pp. 777–790.

[20] R. Monti, *Heisenberg isoperimetric problem. The axial case*, Adv. Calc. Var. **1** (2008), no. 1, pp. 93–121.

[21] R. Monti, *Isoperimetric problem and minimal surfaces in the Heisenberg group,* in L. Ambrosio (ed.), *Geometric measure theory and real analysis,* lecture notes from the school held at the Centro De Giorgi, Pisa, 2013, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, 17, Edizioni della Normale, Pisa, 2014, pp. 57–130.

[22] R. Monti – D. Morbidelli, *Isoperimetric inequality in the Grushin plane*, J. Geom. Anal. **14** (2004), no. 2, pp. 355–368.

[23] F. Morgan, *Manifolds with denstiy*, Notices Amer. Math. Soc., **5**(8) (2005), pp. 853–858.

[24] F. Morgan, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 12, pp. 5041–5052.

[25] F. Morgan, *Manifolds with density and Perelman's proof of the Poincaré conjecture*, Amer. Math. Monthly **116** (2009), no. 2, pp. 134–142.

[26] F. Morgan, *Geometric measure theory,* A beginner's guide, 4^{th} ed., Elsevier/Academic Press, Amsterdam, 2009.

[27] F. Morgan – A. Pratelli, *Existence of isoperimetric regions in $\mathbb{R}^n$ with density*, Ann. Global Anal. Geom. **43** (2013), pp. 331–365.

[28] C. Rosales – A. Cañete – V. Bayle – F. Morgan, *On the isoperimetric problem in Euclidean space with density*, Calc. Var. Partial Differential Equations **31** (2008), pp. 27–46.

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