Rend. Sem. Mat. Univ. Padova, Vol. 138 (2017), 257–263
DOI 10.4171/RSMUP/138-13

A note on S-semipermutable subgroups of finite groups

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Abstract – In this note, we obtain some criteria for p-supersolvablity of a finite group and extend some known results concerning weakly S-semipermutable subgroups.

Mathematics Subject Classification (2010). 20D10.

Keywords. p-supersolvablity, weakly S-semipermutable subgroups.

1. Introduction

Throughout the paper, we suppose G is a finite group and p is a prime. Let $\pi(G)$ be the set of all the prime divisors of $|G|$. Let $O^p(G) = \bigcap\{N \mid N \trianglelefteq G$ and G/N is a p-group$\}$. To state our results, we need to recall some notation. According to Kegel (see [6]), let H be a subgroup of a finite group G; then H is called an S-permutable subgroup of G if H permutes with every Sylow subgroup of G. According to Chen (see [2]), let H be a subgroup of a finite group G; then H is said to be S-semipermutable in G if $HQ = QH$ for all Sylow q-subgroups Q of G for all primes q not dividing $|H|$. According to Li et al. (see [7]), let H be a subgroup of a finite group G; then H is called a weakly S-semipermutable subgroup of G if there exist $T \trianglelefteq\trianglelefteq G$ and $H_1 \leq G$ such that $G = HT$, $H \cap T \leq H_1 \leq H$ and H_1 is S-semipermutable in G. Following Yakov Berkovich and I. M. Isaacs (see [1]), if G is a finite group and p is a prime divisor of $|G|$, we write G_p^* to denote the unique smallest normal subgroup of G for which the corresponding factor group is abelian of exponent dividing $p-1$. It is well known that G is p-supersolvable if and only if G_p^* is p-nilpotent (see Lemma 3.6 of [1]).

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Recently, we proved the following theorem.

THEOREM 1.1 (Theorem 1.2, [9]). *Let p be a prime dividing the order of a finite group G, e be a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then G is p-supersolvable if and only if $|P \cap O^p(G_p^*)| \leq p^e$ and $P_1 \cap O^p(G_p^*)$ is S-permutable in G for all subgroups $P_1 \trianglelefteq P$ with $|P_1| = p^e$.*

In this note, at first, we generalize Theorem 1.2 of [9] as follows.

THEOREM 1.2. *Let p be a prime dividing the order of a finite group G, e be a positive integer, $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$ and $L \trianglelefteq G$ with $O^p(G_p^*) \leq L \leq G$. Suppose that $|P \cap L| \leq p^e$ and $P_1 \cap L$ is S-semipermutable in G for all subgroups $P_1 \trianglelefteq P$ with $|P_1| = p^e$. Then G is p-supersolvable.*

Using Theorem 1.2, we prove the following results which generalize Theorem 1.3 and Theorem 1.4 of [9].

THEOREM 1.3. *Let p be a prime dividing the order of a finite group G, $e \geq 2$ be an integer, $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$ and $L \trianglelefteq G$ with $O^p(G_p^*) \leq L \leq G$. Suppose that $P_1 \cap L$ is S-semipermutable in G for all subgroups $P_1 \leq P$ with $|P_1| = p^e$. Then G is p-supersolvable.*

Let p be a prime and P be a nonidentity p-group with $|P| = p^n$. We define the set $\Omega(P)$. If $p = 2$ and P is non-abelian, let

$$\Omega(P) = \{P_1 \mid P_1 \leq P \text{ and } |P_1| = 2\} \\ \cup \{P_2 \mid P_2 \leq P \text{ and } P_2 \text{ is a cyclic subgroup of order } 4\}.$$

Otherwise, let $\Omega(P) = \{P_1 \mid P_1 \leq P \; and \; |P_1| = p\}$.

THEOREM 1.4. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $L \trianglelefteq G$ with $O^p(G_p^*) \leq L \leq G$. Suppose that $P_1 \cap L$ is S-semipermutable in G for all subgroups $P_1 \in \Omega(P)$. Then G is p-supersolvable.*

Note that Theorem 1.3 and Theorem 1.4 also generalize Theorem 3.5 of [7] and Theorems 3.1 and 3.4 of [8].

2. Preliminaries

LEMMA 2.1. *Let p be a prime dividing the order of a finite group G. Then the following results hold.*

(a) (*Lemma* 3.1, [1]) *Let $P_1 \leq G$ be a p-group and $N \trianglelefteq G$. If P_1 is S-semipermutable in G, then P_1N/N is S-semipermutable in G/N.*

(b) (*Lemma* 3.2, [1]) *Let $P_1 \leq G$ be a p-group and N be a normal p-subgroup of G. If P_1 is S-semipermutable in G, then $P_1 \cap N$ is normalized by $O^p(G)$. In particular, if $P_1 \leq N$, then P_1 is normalized by $O^p(G)$.*

(c) (*Lemma* 3.3, [1]) *Let $X \leq H \leq G$. If X is S-semipermutable in G, then X is S-semipermutable in H.*

Lemma 2.2 (Lemma 2.1, [9]). *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$, $N \trianglelefteq G$ and let e be a positive integer. Write $P_1 = P \cap N$. Assume that $P_1 \trianglelefteq N$ and N is not p-nilpotent. Also assume that $|P_1| \leq p^e$ and $|P| \geq p^{e+1}$. Then P has a normal subgroup P_2 of order p^e with $[P_1 : P_1 \cap P_2] = p$.*

Lemma 2.3 (Lemma 2.2, [9]). *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Write $\widehat{P} = P \cap O^p(G_p^*)$. Assume that $\widehat{P} > 1$ and $\widehat{P}$ has a maximal subgroup T with $T \trianglelefteq G$. Then $\widehat{P} \ntrianglelefteq G$.*

Lemma 2.4 (Lemma 2.8, [9]). *Let p be a prime dividing the order of a finite group G and P_1 be a p-subgroup of G. Let $L \trianglelefteq G$ and N be a normal p'-subgroup of G. Then $P_1N/N \cap LN/N = (P_1 \cap L)N/N$.*

Lemma 2.5 (Lemma 2.9, [9]). *Let p be a prime dividing the order of a finite group G and $N \trianglelefteq G$. Then $(G/N)_p^* = G_p^*N/N$, $O^p(G/N) = O^p(G)N/N$ and $O^p((G/N)_p^*) = O^p(G_p^*)N/N$.*

Recently, I. M. Isaacs proved the following significant theorem.

Lemma 2.6 (Theorem A, [5]). *Let p be a prime dividing the order of a finite group G and P_1 be an S-semipermutable p-subgroup of G. Then P_1^G is solvable.*

Recently, Yakov Berkovich and I. M. Isaacs proved the following powerful results.

Lemma 2.7 (Yakov Berkovich and I. M. Isaacs). *Let p be a prime and P be a nonidentity finite p-group. Let A act on P via automorphisms.*

(a) (*Lemma* 2.1(a), [1]) *If P is cyclic, then $O^p(A_p^*)$ acts trivially on P.*

(b) (*Theorem* A, [1]) *Fix an integer $e \geq 3$. If P is a noncyclic p-group with $|P| \geq p^{e+1}$ and every noncyclic subgroup of P with order p^e is stabilized by $O^p(A)$, then $O^p(A_p^*)$ acts trivially on P.*

(c) (*Corollary* B, [1]) *If P is a noncyclic p-group with $|P| \geq p^3$ and every subgroup of P with order p^2 is stabilized by $O^p(A)$, then $O^p(A_p^*)$ acts trivially on P.*

Lemma 2.8 (Yakov Berkovich and I. M. Isaacs). *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$.*

(1) (*Lemma* 3.8, [1]) *If P is cyclic and some nonidentity subgroup $U \leq P$ is S-semipermutable in G, then G is p-supersolvable.*

(2) (*Theorem* D, [1]) *Fix an integer $e \geq 3$. If P is a noncyclic p-group with $|P| \geq p^{e+1}$ and every noncyclic subgroup of P with order p^e is S-semipermutable in G, then G is p-supersolvable.*

(3) (*Corollary* E, [1]) *If P is a noncyclic p-group with $|P| \geq p^3$ and every subgroup of P with order p^2 is S-semipermutable in G, then G is p-supersolvable.*

Lemma 2.9 (Lemma 2.12, [9]). *Let p be a prime and P be a nonidentity finite p-group. Let A act on P via automorphisms. Assume that for all $P_1 \in \Omega(P)$, P_1 is stabilized by $O^p(A)$. Then $O^p(A_p^*)$ acts trivially on P.*

3. Main results

Proof of Theorem 1.2. Suppose that G is a counterexample with minimal order; we complete the following steps to obtain a contradiction. Since G is not p-supersolvable, it follows that $O^p(G_p^*)$ is not p-nilpotent.

Step 1. $P \cap L \geq P \cap O^p(G_p^*) > 1$.

Since $O^p(G_p^*)$ is not p-nilpotent, it follows that $P \cap O^p(G_p^*) > 1$. Since $L \geq O^p(G_p^*)$, it follows that $P \cap L \geq P \cap O^p(G_p^*) > 1$.

Step 2. $O_{p'}(G) = 1$.

By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-supersolvable, and thus G is p-supersolvable. This is a contradiction. Hence $O_{p'}(G) = 1$.

Step 3. *Let* $\widehat{P} = P \cap O^p(G_p^*)$. *Then* $\widehat{P} \trianglelefteq G$.

Let $U = P \cap L$. Then $U \trianglelefteq P$ and $|U| \le p^e$. Hence P has a normal subgroup P_1 of order p^e with $U \le P_1$. By the hypotheses, $U = P_1 \cap L$ is S-semipermutable in G. By Lemma 2.6, U^G is solvable. By Step 1, $U^G \ge U > 1$. By Step 2, $O_{p'}(U^G) = 1$. Since U^G is solvable, it follows that $O_p(U^G) > 1$. Let $N = O_p(U^G)$. Since $U^G \le L$, it follows that $N \le U$. Hence $|N| \le |U| \le p^e$.

Assume that $1 < |N| < p^e$. Recall that $N \le P \cap L$. By Lemma 2.1(a) and Lemma 2.5, the hypotheses are inherited by G/N. Hence G/N is p-supersolvable. By Lemma 2.5, it follows that $O^p(G_p^*)N/N = O^p((G/N)_p^*)$ is a p'-group, and thus $N \cap O^p(G_p^*)$ is the normal Sylow p-subgroup of $O^p(G_p^*)$. Hence $\widehat{P} = N \cap O^p(G_p^*)$, and thus $\widehat{P} \trianglelefteq G$.

Assume that $|N| = p^e$. Since $N \le U$ and $|U| \le p^e$, it follows that $U = N$ is a normal subgroup of G. Hence $\widehat{P} = P \cap O^p(G_p^*) = U \cap O^p(G_p^*) \trianglelefteq G$.

Step 4. *The final contradiction.*

Recall that $\widehat{P}$ is the normal Sylow p-subgroup of $O^p(G_p^*)$ (Step 3), $O^p(G_p^*)$ is not p-nilpotent, $|P| \ge p^{e+1}$ and $1 < |\widehat{P}| \le |U| \le p^e$. By Lemma 2.2, P has a normal subgroup P_2 of order p^e with $[\widehat{P} \colon \widehat{P} \cap P_2] = p$. Note that $\widehat{P} \cap P_2 \trianglelefteq P$. By the hypotheses, $P_2 \cap L$ is S-semipermutable in G. By Lemma 2.1(b), $\widehat{P} \cap P_2 = \widehat{P} \cap P_2 \cap L$ is normalized by $O^p(G)$. Hence $\widehat{P} \cap P_2 \trianglelefteq G$. By Lemma 2.3, $\widehat{P} \ntrianglelefteq G$. This is a contradiction since $\widehat{P} \trianglelefteq G$. Hence we obtain the final contradiction. □

Proof of Theorem 1.3. We proceed by induction on $|G|$. By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, by induction, $G/O_{p'}(G)$ is p-supersolvable, and thus G is p-supersolvable. So we can assume $O_{p'}(G) = 1$. Let $U = P \cap L$. If $|U| \le p^e$, by Theorem 1.2, G is p-supersolvable. Assume that $|U| \ge p^{e+1}$. For any subgroup $P_1 \le U$ with $|P_1| = p^e$, P_1 is S-semipermutable in G. By Lemma 2.1(c), P_1 is S-semipermutable in L. By Lemma 2.8, L is p-supersolvable, and thus L is p-solvable with p-length 1. Since $O_{p'}(G) = 1$, it follows that U is the normal Sylow p-subgroup of L, and thus $U \trianglelefteq G$. Note that for all subgroups $P_1 \le U$ with $|P_1| = p^e$, P_1 is S-semipermutable in G. By Lemma 2.1(b), it follows that P_1 is normalized by $O^p(G)$. By Lemma 2.7, U is centralized by $O^p(G_p^*)$. Let $\widehat{P} = P \cap O^p(G_p^*)$. Note that $\widehat{P} = U \cap O^p(G_p^*)$, and thus $\widehat{P} \le Z(O^p(G_p^*))$. By Burnside's Theorem (see Theorem 5.13 of [4]), $O^p(G_p^*)$ is p-nilpotent, i.e., G_p^* is p-nilpotent. Hence G is p-supersolvable. □

Proof of Theorem 1.4. By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, it is no loss to assume that $O_{p'}(G) = 1$. Assume G is not p-supersolvable; we work to obtain a contradiction. Since G is not p-supersolvable, it follows that $O^p(G_p^*)$ is not p-nilpotent. Let $\widehat{P} = P \cap O^p(G_p^*)$. Then $\widehat{P} > 1$. Since $O^p(G_p^*) \leq L$, it follows that for any $P_1 \in \Omega(\widehat{P})$, P_1 is S-semipermutable in G. By Lemma 2.6, P_1^G is solvable. Let $M = \prod_{P_1 \in \Omega(\widehat{P})} P_1^G$. Then M is a solvable normal subgroup of G and $M \leq O^p(G_p^*)$. Since $\widehat{P} > 1$ and for any $P_1 \in \Omega(\widehat{P})$, $P_1 \leq M$, it follows that $M > 1$. Since $O_{p'}(G) = 1$ and $M \trianglelefteq G$, we have $O_{p'}(M) = 1$. Since $M > 1$ is solvable and $O_{p'}(M) = 1$, we have $O_p(M) > 1$. Note that $O_p(M) \leq P \cap L$. Hence for any $P_2 \in \Omega(O_p(M))$, P_2 is S-semipermutable in G. By Lemma 2.1(b), P_2 is normalized by $O^p(G)$. By Lemma 2.9, $O_p(M)$ is centralized by $O^p(G_p^*)$. Recall that $M \leq O^p(G_p^*)$. Hence $O_p(M) \leq Z(M)$. Since M is solvable and $O_{p'}(M) = 1$, by Hall–Higman's Lemma (see Theorem 3.21 of [4]), it follows that $M = O_p(M)$. Hence M is centralized by $O^p(G_p^*)$. Recall that for any $P_1 \in \Omega(\widehat{P})$, $P_1 \leq M$. Hence for any $P_1 \in \Omega(\widehat{P})$, P_1 is centralized by $O^p(G_p^*)$. By Satz IV.5.5 of [3], $O^p(G_p^*)$ is p-nilpotent. This is the desired contradiction. Hence G is p-supersolvable. □

4. Final remarks

Theorem 1.3 and Theorem 1.4 have the following corollaries.

Corollary 4.1. *Let p be a prime dividing the order of a finite group G, $e \geq 2$ be an integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Suppose that $P_1 \cap O^p(G)$ is S-semipermutable in G for all subgroups $P_1 \leq P$ with $|P_1| = p^e$. Then G is p-supersolvable.*

Corollary 4.2. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Suppose that $P_1 \cap O^p(G)$ is S-semipermutable in G for all subgroups $P_1 \in \Omega(P)$. Then G is p-supersolvable.*

Note that Corollary 4.1 and Corollary 4.2 generalize Theorem 3.5 of [7].

Remark 4.3. For any odd prime p and any positive integer e, there exists a finite group G with p an odd prime divisor of $|G|$, $P \in \mathrm{Syl}_p(G)$ and $|P| \geq p^{e+1}$ such that for every subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is S-semipermutable in G, but P has a subgroup P_3 of order p^e such that P_3 is not weakly S-semipermutable in G. Hence our Corollary 4.1 and Corollary 4.2 are stronger than Theorem 3.5 of [7]. See the following example.

Example 4.4. Let n be an integer with $n > e$, p be an odd prime and $T = \langle a, b \mid a^{p^n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle \cong D_{2p^n}$. There exists $c \in \mathrm{Aut}(T)$ such that $a^c = a$ and $b^c = ba$. Consider $G = T \rtimes \langle c \rangle \cong \langle a, b, c \mid a^{p^n} = b^2 = c^{p^n} = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = ba \rangle$.

Let $P = \langle a \rangle \times \langle c \rangle$. Then P is the normal Sylow p-subgroup of G with order p^{2n}. Then $b \in O^p(G)$, so $ba = b^c \in O^p(G)$. Hence $a \in O^p(G)$, so $O^p(G) = T$. Hence for any subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is normal in G, and thus S-semipermutable in G. Let $\tilde{c} = c^{p^{n-e}}$. Consider $\langle \tilde{c} \rangle$. Note that $|\langle \tilde{c} \rangle| = p^e$. Then $\langle \tilde{c} \rangle$ is not weakly S-semipermutable in G. To see this, assume that $\langle \tilde{c} \rangle$ is weakly S-semipermutable in G, since $\langle \tilde{c} \rangle \leq \Phi(\langle c \rangle)$, it follows that $\langle \tilde{c} \rangle$ is S-semipermutable in G. Hence $\langle \tilde{c} \rangle$ normalizes $\langle b \rangle$. This is a contradiction since $\tilde{c}^{-1} b \tilde{c} = ba^{p^{n-e}}$. Hence $\langle \tilde{c} \rangle$ is not weakly S-semipermutable in G.

Acknowledgment. The author is grateful to the referee who provided his/her valuable suggestions and detailed report.

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Manoscritto pervenuto in redazione il 7 aprile 2016.