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Weak local-global compatibility in the p-adic Langlands program for $U(2)$

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Abstract – We study the completed cohomology $\widehat{H}^0$ of a definite unitary group G in two variables associated with a CM-extension $\mathcal{K}/F$. When the prime p splits, we prove that (under technical asumptions) the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ occurs in $\widehat{H}^0$. As an application, we obtain a result towards the Fontaine–Mazur conjecture over $\mathcal{K}$. If x is a point on the eigenvariety such that ρ_x is geometric (and satisfying additional hypotheses which we suppress), then x must be a classical point. Thus, not only is ρ_x modular, but the weight of x defines an accessible refinement. This follows from a recent result of Colmez (which describes the locally analytic vectors in p-adic unitary principal series), knowing that ρ_x admits a triangulation compatible with the weight.

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Contents

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1. Introduction

In the last decade, the p-adic Langlands program has exploded with activity. In a nutshell, it predicts a close relationship between continuous representations $\Gamma_F = \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$, where F and E are both finite extensions of $\mathbb{Q}_p$, and unitary representations of $\mathrm{GL}_n(F)$ on Banach E-spaces. For $\mathrm{GL}_2(\mathbb{Q}_p)$, this correspondence was pioneered by Breuil, and in the last few years one has achieved a more complete understanding for $\mathrm{GL}_2(\mathbb{Q}_p)$, due to the work of many people (notably Berger, Colmez, Emerton, Kisin, Paskunas, and others). These results have already had astounding applications to various notoriously difficult problems in number theory.

This article takes its point of departure in Emerton's progress on the Fontaine–Mazur conjecture. In [Em2], and its predecessor [Em1], he explains how the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the completed cohomology $\widehat{H}^1$ of the tower of modular curves. From the "de Rham" condition, one gets the existence of locally algebraic vectors, and Emerton deduces that a "promodular" representation $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$, which is de Rham (with distinct Hodge–Tate weights), must in fact be modular; again, under weak technical assumptions, which we will not record here.

In this paper we look at the tower of arithmetic manifolds of a definite unitary group in two variables, $G = \mathrm{Res}_{F/\mathbb{Q}}(U)$; an inner form of $\mathrm{GL}(2)_{/\mathcal{K}}$ over a CM extension $\mathcal{K}/F$. The arithmetic manifolds are in fact just (arithmetically rich) finite sets, occasionally called Hida varieties[1]. Thus, instead of $\widehat{H}^1$, we are looking at the Banach space $\widehat{H}^0$, which can be realized as the space of continuous functions on a profinite set (endowed with the sup-norm). When the prime p splits, we relate the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\widehat{H}^0$ (see Theorem 1.1 below for a precise statement). As a result thereof, we obtain a corollary towards the Fontaine–Mazur conjecture for representations $\Gamma_{\mathcal{K}} \to \mathrm{GL}_2(E)$ associated with p-adic modular forms on G, in the vein of Emerton (see Theorem 1.2 below). In fact we prove a little more than modularity; we prove classicality (that is, we keep track of refinements and triangulations). This is our main theorem.

To orient the reader, we briefly point out the major differences with [Em2].

(1) We work with a fixed tame level K^p throughout, to surmount the difficulty with the non-uniqueness of hyperspecials in ℓ-adic $U(2)$, at primes $\ell \neq p$, and find it convenient to formulate our results in terms of the eigenvariety $X = X_{K^p}$.

[1] This can be somewhat misleading. They are just finite sets (zero-dimensional manifolds), with no natural structure of a variety, and more importantly no Galois action.

(2) We allow $F \neq \mathbb{Q}$, but must assume p splits in $\mathcal{K}$. Thus we really embed a tensor product (over places $v|p$ of F) of p-adic local Langlands correspondents in $\widehat{H}^0$. Consequently, we make progress on Fontaine–Mazur for two-dimensional representations of $\Gamma_{\mathcal{K}}$, for CM-fields $\mathcal{K}$, as opposed to $\Gamma_{\mathbb{Q}}$.

(3) We work with $\widehat{H}^0$, which we find to be more hands-on than $\widehat{H}^1$. The arithmetic manifolds of G are not Shimura varieties; so $\widehat{H}^0$ carries no Galois-action (as opposed to $\widehat{H}^1$ of modular curves). This simplifies some of the arguments.

(4) We prove classicality, not "just" modularity. That is, if the representation ρ_x at $x \in X$ is de Rham (with distinct Hodge–Tate weights etc.) then x is a classical point. Here the crux of the matter lies in relating weights and refinements (which makes critical use of recent work of Colmez, and results of Hellmann and others on triangulinity).

(5) On the flip-side, for now, we must make the rather bold assumption that the mod p reduction $\bar{\rho}_x$ is irreducible at all places of $\mathcal{K}$ above p. Emerton gets by with much weaker hypotheses at p, using [BE].

In order to state our main results, we must briefly set up the notation in use throughout the paper. Once and for all, we fix a prime number p. To be safe, we will always assume $p > 3$. We let $F/\mathbb{Q}$ be a totally real field, and $\mathcal{K}/F$ a CM extension, in which p *splits* completely. Places of F are usually denoted by v, and those of $\mathcal{K}$ by w. For each place $v|p$ of F, we choose a place $\tilde{v}|v$ of $\mathcal{K}$ above it (note that $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$, canonically). Given an algebraic isomorphism $\iota\colon \mathbb{C} \longrightarrow \overline{\mathbb{Q}}_p$, the choice of a collection $\{\tilde{v}\}$ amounts to choosing a CM-type, which is ordinary for ι, in the sense of Katz.

Let D be a quaternion algebra over $\mathcal{K}$, endowed with an F-linear anti-involution $\star$ of the second kind ($\star|_{\mathcal{K}} = c$). This pair defines a unitary group $U = U(D, \star)_{/F}$, an inner form of GL(2) over $\mathcal{K}$. Indeed, $U \times_F \mathcal{K} \simeq D^\times$. We find it convenient to work over $\mathbb{Q}$, and introduce $G = \mathrm{Res}_{F/\mathbb{Q}}(U)$. We will always assume $G(\mathbb{R})$ is compact, and that D splits above p. Using our choices $\{\tilde{v}\}$, we identify

$$G(\mathbb{R}) \xrightarrow{\sim} U(2)^{\mathrm{Hom}(F,\mathbb{R})}, \quad G(\mathbb{Q}_p) \xrightarrow{\sim} \prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}}).$$

(Of course, $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$, but we wish to incorporate $\tilde{v}$ in our notation to emphasize how our identification depends on this choice. Hence we stick to the somewhat cumbersome notation $\mathcal{K}_{\tilde{v}}$.)

Our main occupation will be Galois representations ρ associated with p-adic modular forms on G. We will show a "Fontaine–Mazur" like result of the following form. If ρ is geometric, with distinct weights (and satisfies additional technical hypotheses), then in fact ρ arises from a *classical* modular form on G.

Eigensystems arising from p-adic modular forms of finite slope, and their Galois representations, are parametrized by rigid analytic spaces called eigenvarieties. For $U(n)$, they have been constructed in [BC] by Bellaiche-Chenevier (for $F = \mathbb{Q}$), and in [Che2] by Chenevier (for any F) using Buzzard's "eigenvariety machine" from [Buz]. A more general construction was given by Emerton in [Em4] (which certainly covers the case where $G(\mathbb{R})$ is compact; and it does not require Iwahori-level at p). Yet another construction, in the style of Chenevier, was given by Loeffler in [Loe], only assuming $G(\mathbb{R})$ is compact (and curiously dealing with any parabolic, as opposed to just the Borel). By the uniqueness, shown in [BC] for example, all these constructions are compatible, and define the same eigenvariety. In the special case where $F = \mathbb{Q}$, the eigenvariety for our two-variable unitary group G, is formally reminiscent of the mother of all eigenvarieties; the celebrated "eigencurve" of Coleman and Mazur [CM], which in turn has its origin in Hida theory (the slope zero case).

We will not use much about eigenvarieties, besides their definition and basic structural properties. We work with a fixed tame level $K^p = \prod_{v \nmid p} K_v$ throughout the paper. Given K^p, let Σ_0 denote the set of places $v \nmid p$ for which K_v is not a hyperspecial subgroup, and then introduce $\Sigma = \Sigma_0 \sqcup \Sigma_p$, where Σ_p are places above p. The eigenvariety $X = X_{K^p}$ parametrizes eigensystems of $\mathcal{H}(K^p)^{\mathrm{sph}}$ (the Hecke algebra of $G(\mathbb{A}_f^\Sigma)$ relative to K^Σ - restricted product away from Σ), which are associated with p-adic modular forms. To be more precise, we first introduce weight space

$$\widehat{T} = \mathrm{Hom}_{\mathrm{loc.an.}}(T(\mathbb{Q}_p), \mathbb{G}_m^{\mathrm{rig}}), \quad T(\mathbb{Q}_p) \simeq \prod_{v|p} T_{\mathrm{GL}(2)}(\mathcal{K}_{\tilde{v}}).$$

Here T is a torus of G and $T_{\mathrm{GL}(2)}$ is the diagonal torus of GL(2). This is a rigid analytic space over $\mathbb{Q}_p$, endowed with a universal locally analytic character $\widehat{T} \to \mathcal{O}(\widehat{T})^\times$. See paragraph 8 of [Buz] for more details. Similarly, $\mathcal{W}$ represents locally analytic characters of $T(\mathbb{Z}_p)$, a disjoint union of open balls. Moreover, $\widehat{T}$ is non-canonically isomorphic to the direct product $\mathcal{W} \times (\mathbb{G}_m^{\mathrm{rig}})^d$ for some integer $d \geq 1$.

Now, the eigenvariety $X = X_{K^p}$ is a reduced rigid analytic variety over $\mathbb{Q}_p$ (since $\mathcal{K}_{\tilde{v}} = \mathbb{Q}_p$), which comes equipped with a finite morphism $\chi: X \to \widehat{T}$, and additional structure (see Theorem 1.6 on p. 5 in [Che2], for example):

- $\lambda\colon \mathcal{H}(K^p)^{\mathrm{sph}} \to \mathcal{O}(X)$, a homomorphism of $\mathbb{Q}_p$-algebras,
- $t\colon \Gamma_{\mathcal{K}} \to \mathcal{O}(X)$, a pseudo-character,
- $X_{\mathrm{cl}} \subset X(\overline{\mathbb{Q}}_p)$, a Zariski-dense subset.

Here t is associated with λ, in a natural sense (cf. Theorem 2.1 in the main text). Furthermore,

$$X(\overline{\mathbb{Q}}_p) \longrightarrow (\widehat{T} \times \operatorname{Spec} \mathcal{H}(K^p)^{\mathrm{sph}})(\overline{\mathbb{Q}}_p), \quad x \longmapsto (\chi_x, \lambda_x),$$

restricts to a bijection between X_{cl} and the set of "classical" points. That is, those (χ, λ), for which $\chi = \psi\theta$ is locally algebraic (where ψ is an algebraic character of the torus and θ is the smooth part), and there exists an automorphic representation π of $G(\mathbb{A})$, of weight ψ and tame level K^p, such that π_p embeds into the (unnormalized) principal series representation $i_B(\theta)$ (consisting of all smooth functions $f\colon G \to \overline{\mathbb{Q}}_p$ with the transformation property $f(bg) = \theta(b)f(g)$, and with G acting via right translations).

Finally, for each $x \in X(\overline{\mathbb{Q}}_p)$, we let $\rho_x\colon \Gamma_{\mathcal{K}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ denote the unique semisimple continuous representation with $\operatorname{tr} \rho_x = t_x$. When $x \in X_{\mathrm{cl}}$, it is known that ρ_x is "geometric" in the sense of Fontaine–Mazur (unramified almost everywhere, and de Rham above p). Our goal here is a partial converse. We first state the weak local-global compatibility, referred to in the title, which roughly says the p-adic local Langlands correspondence occurs in the completed cohomology of the tower of arithmetic manifolds of G (as announced on p. 8 in [So2]).

In a companion paper, [CS], we prove a version of strong local-global compatibility, which gives a more precise description of the completed cohomology, which involves "local Langlands in families" away from p (following Emerton-Helm [EH]). We emphasize that [CS] does not make this paper obsolete. The classicality of geometric points on the egenvariety (which is where the novelty of this paper lies) is not implied by [CS]. Furthermore, for our timid progress on the Fontaine–Mazur conjecture (in this context), the weak version of local-global compatibility treated here suffices. The focus of [CS] is on the structural properties of completed cohomology, and their relation to Ihara's lemma.

THEOREM 1.1. *Let $E/\mathbb{Q}_p$ be a finite extension, and let $x \in X(E)$ be a point on the eigenvariety, of tame level K^p, which satisfies the following assumptions*:

(1) *ρ_x is defined over E*;

(2) *$\bar{\rho}_{x,w}$ is absolutely **irreducible** for all $w|p$.*

Then there is a nonzero continuous $G(\mathbb{Q}_p)$-equivariant map ($\mathfrak{p}_x = \ker \lambda_x$),

$$\widehat{\bigotimes}_{v|p} B(\rho_{x,\tilde{v}}) \longrightarrow \hat{H}^0(K^p)_E[\mathfrak{p}_x],$$

where $B(\cdot)$ is the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

The flaring assumption here is (2); the mod p irreducibility at p. This simplifies many of the arguments a great deal, see the proof of the key Lemma 3.15 in the main text. In particular, we do not need any results on Serre weights (although available) to carry out the proof.

We currently do not know how to deal with the reducible case. It would require an analogue of [BE], which is based on a geometrically intricate construction of "overconvergent" companion forms in order to show that, for ordinary eigenforms f, $B(\rho_{f,p}|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)})$ occurs in the completed cohomology of the tower of modular curves. (See Theorem 1.1.2 of [BE].) Recently Y. Ding has made progress along the same lines for ordinary Hilbert modular forms arising from quaternionic Shimura curves, cf. [Din].

Bergdall has given recently a simpler proof of the geometric part of [BE] in [Berg] and it seems possible to do similar things in our unitary setting. We hope to come back to this issue elsewhere.

Guided by [Em2], one would expect that (2) can be replaced with the following two (much weaker) assumptions:

(3) $\bar{\rho}_x$ is absolutely irreducible (as a representation of the full $\Gamma_{\mathcal{K}}$),

(4) $\bar{\rho}_{x,w} \nsim \chi \otimes \begin{pmatrix} 1 & * \\ & \epsilon \end{pmatrix}$, for all places $w|p$ of $\mathcal{K}$.

We wish to stress that the proof of Theorem 1.1 hinges upon local-global compatibility at $p = \ell$, now known (even for $U(n)$ with n arbitrary) due to the work of Barnet-Lamb, Gee, Geraghty, and Taylor (see [BLGGT1] and its follow-up [BLGGT2]), and that of Caraiani [Car].

Our main motivation for proving this weak local-global compatibility, is its application to the Fontaine–Mazur conjecture, along the following lines:

Theorem 1.2. *Let $x \in X(E)$ be a point as in Theorem 1.1, and assume moreover that $\rho_{x,\tilde{v}}$ is potentially semistable, with distinct Hodge–Tate weights, for all $v|p$. Then x is a classical point, $x \in X_{\mathrm{cl}}$.*

This is more than a modularity result. To conclude $x \in X_{\mathrm{cl}}$, one needs to keep careful track of refinements. This involves the existence of specific triangulations of $\rho_{x,\tilde{v}}$, proved in many instances by Hellmann, and a description of the locally analytic vectors in unitary principal series; a recent result due to Colmez [Co2]. We remark also that a similar theorem (but only with modularity as a result) should follow from the techniques used by Kisin, cf. [Ki2]. Results for $U(n)$ analogous to Theorem 1.2 have very recently been announced in [BHS2].

Acknowledgments. The tremendous debt this paper owes to Emerton's work [Em2] will be clear to the experts. We wish to acknowledge its impact on both of us, as well as that of Breuil's beautiful Bourbaki survey [Bre], which helped us enter this circle of ideas. We thank James Newton for useful remarks as well as the anonymous referee for comprehensive feedback. Finally, we wish to thank the Fields Institute in Toronto for hosting the p-adic Langlands workshop in April, 2012, where this collaboration began.

2. Automorphic Galois representations

Galois representations associated with regular polarized cusp forms on $\mathrm{GL}(n)_{/K}$ are now almost completely understood, thanks to the "book project" and its spin-offs. This is the culmination of collective efforts of many people, initiated by Clozel, Harris, Kottwitz, Taylor, and others. Here we wish to briefly give the lay of the land for unitary groups in two variables.

Thus let π be an automorphic representation of $G(\mathbb{A})$, whose infinity component $\pi_\infty = \bigotimes_{v|\infty} \pi_v$ is an irreducible algebraic representation of $G(\mathbb{C})$, restricted to the compact Lie group $G(\mathbb{R})$. We assign weights to π_∞ as follows. Each factor π_v is a representation of $U(F_v)$, which we may "complexify" to a representation of $\mathrm{GL}_2(\mathcal{K}_w)$, where $w|v$ is unique. Upon choosing an embedding $\tau\colon \mathcal{K} \hookrightarrow \mathbb{C}$, among the pair corresponding to w (i.e. we fix $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_p$ and write $\mathcal{K} \hookrightarrow \mathcal{K}_w \hookrightarrow^{\tau} \overline{\mathbb{Q}}_p \to^{\iota^{-1}} \mathbb{C}$), we identify this with a representation of $\mathrm{GL}_2(\mathbb{C})$, which is irreducible algebraic of highest weight ψ_τ, relative to the lower-triangular Borel. In other words, there are integers $\kappa_{1,\tau} < \kappa_{2,\tau}$ such that

$$\psi_\tau(t) = t_1^{\kappa_{1,\tau}} t_2^{\kappa_{2,\tau}-1}, \quad t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T_{\mathrm{GL}(2)}(\mathbb{C}).$$

(Note that $\kappa_{1,\bar\tau} = 1 - \kappa_{2,\tau}$ and $\kappa_{2,\bar\tau} = 1 - \kappa_{1,\tau}$.) Another way to think of this is in terms of the local base change $BC_{w|v}(\pi_v)$, which has infinitesimal character given by ψ_τ, upon identifying it with a representation of $\mathrm{GL}_2(\mathbb{C})$, via τ.

Theorem 2.1. *Choose an isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. With π as above, we may associate a unique continuous semisimple Galois representation,*

$$\rho = \rho_{\pi,\iota}\colon \Gamma_{\mathcal{K}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p),$$

satisfying the following list of desiderata:

- *For every finite place $w|v$, for which the local base change $BC_{w|v}(\pi_v)$ is defined* (*even those above p*)*, one has*

$$WD(\rho|_{\Gamma_{\mathcal{K}_w}})^{F-ss} \simeq \iota\,\mathrm{rec}(BC_{w|v}(\pi_v) \otimes |\det|^{-1/2}).$$

(*Here the classical local Langlands correspondence* rec$(\cdot)$ *is normalized as in* [HT]. *At $w|p$, the Weil–Deligne representation is defined by Fontaine.*)

- *Indeed, for every place $w|p$, the restriction $\rho_w = \rho|_{\Gamma_{\mathcal{K}_w}}$ is potentially semistable, with labelled Hodge–Tate weights determined by π_∞,*

$$HT_\tau(\rho_w) = \{\kappa_{1,\tau} < \kappa_{2,\tau}\},$$

for $\tau: \mathcal{K}_w \hookrightarrow \overline{\mathbb{Q}}_p$, tacitly identified with an embedding $\mathcal{K} \hookrightarrow \mathbb{C}$ via ι.

- $\rho^\vee \simeq \rho^c \otimes \epsilon$ (*where $\epsilon = \epsilon_{cyc}$ is the cyclotomic character of $\Gamma_{\mathcal{K}}$*).
- $\det\rho \circ \mathrm{Art}_{\mathcal{K}} = BC_{\mathcal{K}/F}(\chi_\pi) \cdot \epsilon^{-1}$.

(*The local base change $BC_{w|v}(\pi_v)$ is defined when v splits in $\mathcal{K}$, or π_v is unramified*; *for some hyperspecial maximal subgroup.*)

PROOF. Let us quickly sketch the argument, and list the key references. Using Rogawski's book [ROG] (Theorem 11.5.1, part (b), on p. 166), we first base change π to $\mathrm{GL}(2)_{/\mathcal{K}}$, resulting in an "isobaric" automorphic representation $\Pi = BC_{\mathcal{K}/F}(\pi)$, whose infinity component Π_∞ has the same infinitesimal character as an algebraic representation. Moreover, $\Pi^\vee \simeq \Pi^c$. If Π is cuspidal, we take $\rho = \rho_{\Pi,\iota}$, where the latter is given by Theorem 1.2 in [CY], for instance. It satisfies the desired properties by Theorem 1.1 in [BLGGT2], and Caraiani's sequel [CAR], which removes the Shin-regularity assumption. At last, if Π is non-cuspidal, it is an isobaric sum of algebraic Hecke characters $\chi_1 \boxplus \chi_2$, with which we associate Galois characters $\rho_{\chi_i,\iota}$ via class field theory. Then we let $\rho = \rho_{\chi_1,\iota} \oplus \rho_{\chi_2,\iota}$, for which it is straightforward to verify the properties. □

REMARK 2.2. The analogous result holds for unitary groups in n variables, except that a certain "regularity" condition creeps in (ruling out $\pi_\infty = 1$ for example), which ensures that the base change is an isobaric sum of cusp forms (as opposed to just discrete automorphic representations). The lucky circumstance for $n = 2$, which we rely on, is that "discrete" is the same as "cuspidal" for GL(1).

For later use, we spell out what happens at the places v where π is unramified. First off, $U_{/F_v}$ must be unramified (quasi-split and split over an unramified extension), in which case it has two conjugacy-classes of hyperspecial subgroups. Thus we should really pick one, say K_v, and specify that $\pi_v^{K_v} \neq 0$. Via the Satake parametrization, local base change is given by pulling back eigensystems along the natural algebra homomorphism (defined over $\mathbb{Z}[p^{\frac{1}{2}}]$),

$$b_{w|v}\colon \mathcal{H}(\mathrm{GL}_2(\mathcal{K}_w), \mathrm{GL}_2(\mathcal{O}_w)) \twoheadrightarrow \mathcal{H}(U(F_v), K_v).$$

For the surjectivity[2] of this map, and a thorough discussion of the salient facts, see [Min] (especially Corollary 4.2). For each w, we let T_w be the Hecke operator for $\mathrm{GL}_2(\mathcal{K}_w)$, which acts on $\Pi_w^{\mathrm{GL}_2(\mathcal{O}_w)}$ by the sum of the integral Satake parameters of Π_w. Similarly, S_w acts by their product. Then, the target Hecke algebra $\mathcal{H}(K_v)$ is generated by $t_w = b_{w|v}(T_w)$ and $s_w = b_{w|v}(S_w)$. Furthermore,

$$\operatorname{tr} \rho_{\pi,\iota}(\mathrm{Frob}_w) = \iota\lambda_{\pi_v}(t_w), \quad \det \rho_{\pi,\iota}(\mathrm{Frob}_w) = \iota\lambda_{\pi_v}(s_w),$$

where $\lambda_{\pi_v}\colon \mathcal{H}(K_v) \to \mathbb{C}$ is the eigensystem of $\pi_v^{K_v}$, when π_v is K_v-unramified.

3. Proof of Theorem 1.1

Here we prove our main Theorem 1.1 from the introduction. The overall strategy follows that of [Em2] closely. In section 3.8 below we introduce a Hecke module $\mathcal{X}$ which interpolates the homomorphisms in Theorem 1.1. To show the $\mathfrak{p}$-torsion $\mathcal{X}[\mathfrak{p}]$ is nonzero, it suffices to do so for the dense subset of crystalline points. This special case boils down to local-global compatibility at p (due to Caraiani and others) and the key fact for GL(2) that the invariant norm is unique for irreducible crystalline representations.

3.1 – *Finite level Hecke algebras*

For any compact open subgroup $K \subset G(\mathbb{A}_f)$, the arithmetic manifold of G,

$$Y(K) = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K,$$

is a finite set. (Recall that $G(\mathbb{R})$ is compact.) For any commutative ring A, we let $H^0(K)_A$ be the set of functions $Y(K) \to A$. It is naturally a module for the Hecke algebra $\mathcal{H}(K)_A$ of compactly supported K-biinvariant functions $G(\mathbb{A}_f) \to A$, equipped with convolution. We will often assume K factors as a direct product $K = \prod_{v<\infty} K_v$, where $K_v \subset U(F_v)$ is hyperspecial for almost all places v of F. We introduce the following finite sets of finite places of F,

[2] Since we are in the GL(2)-case, surjectivity can also be shown directly by writing out explicitly what $b_{w|v}$ does in terms of Satake parameters. We leave this to the reader.

- $\Sigma_p = \{v|p\}$,
- $\Sigma_0 = \{v \nmid p\colon K_v \text{ is } \textit{not} \text{ hyperspecial}\}$,
- $\Sigma = \Sigma_0 \sqcup \Sigma_p$.

Then $\mathcal{H}(K)_A$ factors as a tensor product $\bigotimes_{v<\infty} \mathcal{H}(K_v)_A$. We will be interested in the action of the spherical part. That is, the central subalgebra below,

$$\mathcal{H}(K^p)^{\mathrm{sph}} = \mathcal{H}(K^\Sigma)_A = \bigotimes_{v\notin\Sigma} \mathcal{H}(K_v)_A.$$

We let $\mathbb{T}(K)_A$ denote the quotient of $\mathcal{H}(K^p)^{\mathrm{sph}}$ which acts faithfully on $H^0(K)_A$. In other words, the A-subalgebra of $\operatorname{End} H^0(K)_A$ generated by all Hecke operators from $\mathcal{H}(K_v)_A$, where $v \notin \Sigma$. As an A-module, $\mathbb{T}(K)_A = A \otimes \mathbb{T}(K)_{\mathbb{Z}}$ is finite free (since this holds for $A = \mathbb{Z}$, and hence for all rings). In particular, when A is a *field*, for dimension reasons $\mathbb{T}(K)_A$ is therefore Artinian. (Consequently, prime ideals are maximal, there are only finitely many of them, and $\mathbb{T}(K)_A$ is Noetherian and semi-local; the direct product of all its localizations. When $\mathbb{T}(K)_A$ is reduced, these localizations can be identified with its residue fields.)

As a Hecke-module, $H^0(K)_{\mathbb{C}}$ breaks up as a direct sum of (finitely many) simple modules π_f^K, where π runs over automorphic representations of $G(\mathbb{A})$, with $\pi_\infty = 1$. As a result, $H^0(K)_{\mathbb{C}}$ is a sum of simultaneous eigenspaces for $\mathcal{H}(K^p)^{\mathrm{sph}}$, and $\mathbb{T}(K)_{\mathbb{C}}$ is a semisimple algebra $\mathbb{C}\times\cdots\times\mathbb{C}$, where the direct factors correspond to automorphic Hecke eigensystems $\mathbb{T}(K)_{\mathbb{C}} \to \mathbb{C}$ (giving the action on some π_f^K). The same holds over $\bar{\mathbb{Q}}_p$, by transferring via an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_p$.

We will usually work over a finite extension $E/\mathbb{Q}_p$, with integers $\mathcal{O}$, uniformizer ϖ, and residue field k. We are primarily interested in the $\mathcal{O}$-algebra $\mathbb{T}(K)_{\mathcal{O}}$, which is known to factor as a direct product of localizations,

$$\mathbb{T}(K)_{\mathcal{O}} \xrightarrow{\sim} \prod_{\mathfrak{m}} \mathbb{T}(K)_{\mathcal{O},\mathfrak{m}},$$

with $\mathfrak{m}$ ranging over its maximal ideals (which correspond to maximal ideals of $\mathbb{T}(K)_k$ via the reduction map). Here each factor $\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}}$ is a complete local Noetherian $\mathcal{O}$-algebra. Furthermore, after extending scalars to E,

$$\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}} \otimes E \xrightarrow{\sim} \prod_{\mathfrak{p}\subset\mathfrak{m}} \mathbb{T}(K)_{E,\mathfrak{p}},$$

where $\mathfrak{p} \subset \mathfrak{m}$ runs over the minimal primes of $\mathbb{T}(K)_{\mathcal{O}}$ (which correspond to maximal ideals of $\mathbb{T}(K)_E$ via the inclusion map; which in turn, after fixing $E \hookrightarrow \bar{\mathbb{Q}}_p$, correspond to Galois-conjugacy classes of eigensystems $\mathbb{T}(K)_E \to \bar{\mathbb{Q}}_p$).

3.2 – *Infinite level Hecke algebras*

Emerton developed the theory of completed cohomology for any reductive group in [Em4]. For a fixed tame level K^p, the tower of locally symmetric spaces of level K_pK^p, where we let K_p shrink, has completed cohomology spaces $\widehat{H}^i(K^p)$. In this paper, our unitary group G is compact at infinity, so we will only use this machinery in degree $i = 0$, where everything can be done explicitly by hand. We recall the definitions. For each choice of tame level K^p, there are two modules,

- $\widehat{H}^0(K^p)_{\mathcal{O}} = (\varinjlim_{K_p} H^0(K_pK^p)_{\mathcal{O}})^\wedge$ (where $\wedge$ is p-adic completion),
- $\tilde{H}^0(K^p)_{\mathcal{O}} = \varprojlim_{s}(\varinjlim_{K_p} H^0(K_pK^p)_{\mathcal{O}/\varpi^s\mathcal{O}})$.

There is a natural isomorphism $\widehat{H}^0(K^p)_{\mathcal{O}} \xrightarrow{\sim} \tilde{H}^0(K^p)_{\mathcal{O}}$, and both are naturally identified with the lattice of all continuous functions $Y(K^p) \to \mathcal{O}$. Similarly we define $\widehat{H}^0(K^p)_E$, which thus becomes a Banach E-space (for the sup-norm), endowed with a unitary $G(\mathbb{Q}_p)$-action (via right translations). Moreover, $\widehat{H}^0(K^p)_E$ becomes a Banach-module for the completed Hecke algebra $\widehat{\mathcal{H}}(K^p)$ of biinvariant functions, which vanish at infinity (and the $G(\mathbb{Q}_p)$-action is Hecke-linear). Here $\widehat{\mathcal{H}}(K^p) = \varprojlim_{K_p} \mathcal{H}(K_pK^p)$ where the limit runs over compact open subgroups K_p of $G(\mathbb{Q}_p)$.

The p-levels K_p are ordered by reverse-inclusion; if $K_p' \subset K_p$, there is a surjective transition map $\mathbb{T}(K_p'K^p)_{\mathcal{O}} \twoheadrightarrow \mathbb{T}(K_pK^p)_{\mathcal{O}}$, which makes the collection of all $\mathbb{T}(K_pK^p)_{\mathcal{O}}$ into a projective system, as K_p varies. We define the $\mathcal{O}$-algebra

$$\mathbb{T}(K^p)_{\mathcal{O}} = \varprojlim_{K_p} \mathbb{T}(K_pK^p)_{\mathcal{O}},$$

with its projective limit topology. Thus $\mathbb{T}(K^p)_{\mathcal{O}}$ is a reduced, compact, complete $\mathcal{O}$-algebra, equipped with a natural map $\mathcal{H}(K^p)^{\mathrm{sph}} \to \mathbb{T}(K^p)_{\mathcal{O}}$ having dense image. Moreover, the action of $\mathcal{H}(K^p)^{\mathrm{sph}}$ on the completed cohomology $\widehat{H}^0(K^p)_{\mathcal{O}}$ extends naturally to a faithful action of $\mathbb{T}(K^p)_{\mathcal{O}}$, as follows. Say $h = (h_{K_p})$ is a compatible sequence in $\mathbb{T}(K^p)_{\mathcal{O}}$, and f is a p-smooth function $Y(K^p) \to \mathcal{O}$. Then $h(f) = h_{K_p}(f)$, if f is K_p-invariant. (One verifies the right-hand side is independent of the choice of K_p.) This defines a continuous action on the p-smooth functions, which extends to the completion. Thus, one has maps

$$\mathcal{H}(K^p)^{\mathrm{sph}} \xrightarrow{\text{dense}} \mathbb{T}(K^p)_{\mathcal{O}} \xrightarrow{\text{faithful}} End^{\mathrm{cts}}\widehat{H}^0(K^p)_{\mathcal{O}}.$$

(A short argument shows that $\mathbb{T}(K^p)_{\mathcal{O}}$ is weakly closed.)

As in the finite-level case, $\mathbb{T}(K^p)_{\mathcal{O}}$ is semi-local. If $K'_p \lhd K_p$ are pro-p groups, an eigensystem $\mathcal{H}(K^p)^{\mathrm{sph}} \to k$ occurs in $H^0(K)_k$ if and only if it occurs in $H^0(K')_k$. Therefore $\mathbb{T}(K')_{\mathcal{O}} \twoheadrightarrow \mathbb{T}(K)_{\mathcal{O}}$ identifies the maximal ideals of $\mathbb{T}(K)_{\mathcal{O}}$ with those of $\mathbb{T}(K')_{\mathcal{O}}$. Passing to the limit over all K_p, therefore yields

$$\mathbb{T}(K^p)_{\mathcal{O}} \xrightarrow{\sim} \prod_{\mathfrak{m}} \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}, \quad \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}} = \varprojlim_{K_p} \mathbb{T}(K_p K^p)_{\mathcal{O},\mathfrak{m}},$$

with $\mathfrak{m}$ ranging over its maximal ideals (corresponding to Galois-conjugacy classes of eigensystems occurring in $H^0(K^p)_k$). Each of the (finitely many) factors $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$ is a complete local $\mathcal{O}$-algebra, and they play a key role in deformation theory of Galois representations. Correspondingly,

$$\widehat{H}^0(K^p)_{\mathcal{O}} \xrightarrow{\sim} \bigoplus\nolimits_{\mathfrak{m}} \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}},$$
$$\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}} = \widehat{H}^0(K^p)_{\mathcal{O}} \otimes_{\mathbb{T}(K^p)_{\mathcal{O}}} \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}.$$

These summands $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$ are tightly connected to the p-adic local Langlands correspondence, as we will see below in the course of the proof of our main Theorem 1.1. Again, we wish to emphasize that these constructions (where K_p shrinks) are due to Emerton; see Section 5.2, p. 46, in [Em2], for instance.

3.3 – *Locally algebraic vectors in completed cohomology*

Inside the Banach space $\widehat{H}^0(K^p)_E$, we have the dense subspace of locally analytic vectors, $\widehat{H}^0(K^p)_E^{\mathrm{an}}$, on which the Lie algebra $\mathfrak{g} = \mathrm{Lie}G(\mathbb{Q}_p)$ acts. In turn, inside the locally analytic vectors, we have the locally algebraic vectors,

$$\widehat{H}^0(K^p)_E^{\mathrm{alg}} = \bigoplus\nolimits_{\xi} \widehat{H}^0(K^p)_E^{\xi-\mathrm{alg}}.$$

Here ξ runs over the absolutely irreducible algebraic E-representations of $G_{/E}$, up to equivalence, and the superscript $\xi - \mathrm{alg}$ means we take the subspace of locally ξ-algebraic vectors; that is, those in the image of the evaluation map

$$\xi \otimes \mathrm{Hom}_{K_p}(\xi, \widehat{H}^0(K^p)_E) \hookrightarrow \widehat{H}^0(K^p)_E,$$

for some sufficiently small K_p. Equivalently, those vectors in the image of

$$\xi \otimes \mathrm{Hom}_{\mathfrak{g}}(\xi, \widehat{H}^0(K^p)_E^{\mathrm{an}}) \hookrightarrow \widehat{H}^0(K^p)_E^{\mathrm{an}},$$

see Proposition 4.2.10 in [Em3]. Here the tensor product is over $\mathrm{End}_G(\xi)$, a priori a finite extension of E. However, in our case $G_{/E}$ is a product of copies of GL_2, in which case it is easy to see $\mathrm{End}_G(\xi) = E$ (using the standard polynomial model of ξ, say).

Each ξ defines a local system $\mathcal{V}_\xi$ on $Y(K)$, and its space of sections $H^0(K, \mathcal{V}_\xi)_E$ is naturally identified with the space of algebraic modular forms on G, of level K, and weight ξ. That is, all functions

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow \xi, \quad f(gk) = \xi(k_p^{-1}) f(g), \quad \text{for all } k \in K.$$

(This is straightforward; for details, see Proposition 3.2.2 in [Em4], for example.) We let $H^0(K^p, \mathcal{V}_\xi)_E$ denote the collection of all such f, where we allow K_p to shrink to the identity, that is $H^0(K^p, \mathcal{V}_\xi)_E = \varinjlim_{K_p} H^0(K_p K^p, \mathcal{V}_\xi)_E$. It carries an action of $G(\mathbb{Q}_p)$, and as is well-known,

$$H^0(K^p, \mathcal{V}_\xi)_{\mathbb{C}} \simeq \bigoplus\nolimits_{\pi : \pi_\infty = \xi^\vee} m_G(\pi) \cdot \pi_p \otimes (\pi_f^p)^{K^p},$$

where π runs over all automorphic representations of $G(\mathbb{A})$ with $\pi_\infty = \xi^\vee$. Most likely, $m_G(\pi) = 1$, but we will not need that. (We tacitly move between complex coefficients and p-adic coefficients via a choice of isomorphism ι as above.)

One of the purposes of this section is to remind the reader of the following result.

Proposition 3.1. *Let ξ be an irreducible algebraic E-representation ξ of $G_{/E}$.*

(*a*) $H^0(K^p, \mathcal{V}_{\xi^\vee})_E \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\xi, \hat{H}^0(K^p)_E^{\mathrm{an}})$.

(*b*) $\xi \otimes_E H^0(K^p, \mathcal{V}_{\xi^\vee})_E \xrightarrow{\sim} \hat{H}^0(K^p)_E^{\xi\text{-alg}}$.

Proof. Clearly (b) follows from (a), in conjunction with the preceding remarks. Part (a) can really be proved by hand, so to speak (which was done in 2.2 of [So1], for instance), but can more conveniently be explained as a very special case of the general machinery developed in [Em4]. Indeed the map in (a) is the edge map of a certain spectral sequence (given by Corollary 2.2.18 in [Em4]). Since $Y(K)$ is zero-dimensional, most terms vanish, and the edge map is an isomorphism. See Corollary 2.2.25 in loc. cit., which deduces the isomorphism in (b). □

Corollary 3.2. $\hat{H}^0(K^p)_E^{\xi\text{-alg}} \simeq \bigoplus\nolimits_{\pi : \pi_\infty = \xi} m_G(\pi) \cdot (\xi \otimes \pi_p) \otimes (\pi_f^p)^{K^p}$.

Proof. Follows from (b) and the automorphic description of $H^0(K^p, \mathcal{V}_{\xi^\vee})$. □

3.4 – *Universal modular deformations*

Suppose $\bar{\rho}\colon \Gamma_{\mathcal{K}} \to \mathrm{GL}_2(k)$ is an absolutely irreducible representation, which we assume to be modular of level $K = K_p K^p$, in the sense that there exists a maximal ideal $\mathfrak{m} \subset \mathbb{T}(K)_k$ with which $\bar{\rho}$ is associated; by which we mean that $\mathfrak{m}$ contains

$$(t_w)_K - \operatorname{tr} \bar{\rho}(\mathrm{Frob}_w), \quad (s_w)_K - \det \bar{\rho}(\mathrm{Frob}_w),$$

for all $w|v$ with $v \notin \Sigma$. By $(t_w)_K$ we mean the operator on $H^0(K)_k$ defined by t_w, and similarly for s_w. Note that $k = \mathbb{T}(K)_k/\mathfrak{m}$.

Remark 3.3. Eventually we will take $\bar{\rho} = \bar{\rho}_x$, for a point $x \in X(E)$ as in Theorem 1. Then for K_p deep enough (pro-p suffices), the reduced eigensystem $\bar{\lambda}_x$ factors through $\mathbb{T}(K)_k$, and we may take $\mathfrak{m} = \mathfrak{m}_x = \ker(\bar{\lambda}_x)$ above.

The goal of this section is to introduce the so-called universal modular deformation of $\bar{\rho}$ (of level K), defined as follows.

Proposition 3.4. *Up to equivalence, there is a unique continuous representation*

$$\rho_{\mathfrak{m}} = \rho_{\mathfrak{m},K}\colon \Gamma_{\mathcal{K}} \longrightarrow \mathrm{GL}_2(\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}})$$

such that for every place $w|v$, with $v \notin \Sigma$,

$$\operatorname{tr} \rho_{\mathfrak{m}}(\mathrm{Frob}_w) = (t_w)_K, \quad \det \rho_{\mathfrak{m}}(\mathrm{Frob}_w) = (s_w)_K.$$

Moreover,

- $\bar{\rho}_{\mathfrak{m}} \simeq \bar{\rho}$ *and*
- *for every modular deformation $\rho_{\pi,\iota}\colon \Gamma_{\mathcal{K}} \to \mathrm{GL}_2(A)$ of $\bar{\rho}$ (where A is a complete local Noetherian $\mathcal{O}$-algebra, with reside field k), there is a unique local morphism $\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}} \to A$ with respect to which $\rho_{\pi,\iota}$ is the specialization of $\rho_{\mathfrak{m}}$.*

(*In this sense, $\rho_{\mathfrak{m}}$ is the universal level-K modular deformation of $\bar{\rho}$.*)

Proof. First, for each minimal prime $\mathfrak{p} \subset \mathfrak{m}$ of $\mathbb{T}(K)_{\mathcal{O}}$, we define a representation

$$\rho_{\mathfrak{p}}\colon \Gamma_{\mathcal{K}} \longrightarrow \mathrm{GL}_2(\mathbb{T}(K)_{E,\mathfrak{p}}), \quad \mathbb{T}(K)_{E,\mathfrak{p}} = \mathbb{T}(K)_E/\mathfrak{p},$$

as follows. Pick an eigensystem $\lambda\colon \mathbb{T}(K)_E \to \bar{\mathbb{Q}}_p$ with $\mathfrak{p} = \ker(\lambda)$, extending our chosen embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$. Its restriction to $\mathbb{T}(K)_{\mathbb{Q}}$ then arises from an automorphic representation π, of level K, and weight $\pi_\infty = 1$. That is, $\lambda = \iota\lambda_\pi$. After possibly enlarging E, we may assume $\rho_{\pi,\iota}$ takes values in $\mathrm{GL}_2(\mathcal{O})$. We then let $\rho_{\mathfrak{p}} = \rho_{\pi,\iota}$, and observe (by Chebotarev) that this only depends on $\mathfrak{p}$; not the choice of π. By construction, for all $w|v$ with $v \notin \Sigma$,

$$\operatorname{tr}\rho_{\mathfrak{p}}(\mathrm{Frob}_w) = (t_w)_K + \mathfrak{p}, \quad \det\rho_{\mathfrak{p}}(\mathrm{Frob}_w) = (s_w)_K + \mathfrak{p}.$$

By the factorization at the end of 3.1, we obtain a representation

$$\varrho = \prod_{\mathfrak{p}\subset\mathfrak{m}} \rho_{\mathfrak{p}} \colon \Gamma_{\mathcal{K}} \longrightarrow \mathrm{GL}_2(\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}} \otimes E)$$

with similar properties. In particular, $\operatorname{tr}\varrho(\mathrm{Frob}_w) = (t_w)_K$, which shows that the pseudo-character $\operatorname{tr}\varrho$ takes values in the complete local ring $\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}}$. Its reduction equals $\operatorname{tr}\bar{\rho}$; the trace of an absolutely irreducible representation. By results of Nyssen, Rouquier, and Procesi, there is a unique representation

$$\rho_{\mathfrak{m}} = \rho_{\mathfrak{m},K} \colon \Gamma_{\mathcal{K}} \longrightarrow \mathrm{GL}_2(\mathbb{T}(K)_{\mathcal{O},\mathfrak{m}})$$

with trace $\operatorname{tr}\varrho$. (We refer to Chapter 1 of [BC] for an in-depth treatment of pseudocharacters and a thorough discussion of when they originate from genuine representations.) □

Finally, we will let K_p shrink. When K_p varies, the representations $\rho_{\mathfrak{m},K_pK^p}$ are compatible (because of their unicity), and we may pass to the limit, resulting in

$$\rho_{\mathfrak{m}} = \rho_{\mathfrak{m},K^p} = \varprojlim_{K_p} \rho_{\mathfrak{m},K_pK^p} \colon \Gamma_{\mathcal{K}} \longrightarrow \mathrm{GL}_2(\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}),$$

the universal modular deformation of $\bar{\rho}$, of tame level K^p. Below, we shall link its local restrictions $\rho_{\mathfrak{m},\tilde{v}}$ to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, using the deformation-theoretic approach of Kisin [Ki1].

3.5 – *Colmez's Montreal functor*

It turns out it is easier to describe the inverse of the p-adic local Langlands correspondence $B(\cdot)$. At a 2005 workshop in Montreal, Colmez gave an elegant definition of a functor $\mathbb{V}$, from representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to those of $\Gamma_{\mathbb{Q}_p}$, which he subsequently studied in detail [Co1]. More precisely, $\mathbb{V}$ is an exact covariant functor,

$$\mathbb{V} \colon \mathrm{Rep}_{\mathcal{O}}(\mathrm{GL}_2(\mathbb{Q}_p)) \longrightarrow \mathrm{Rep}_{\mathcal{O}}(\Gamma_{\mathbb{Q}_p}),$$

from the category of smooth, admissible, finite length representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ on torsion $\mathcal{O}$-modules (which admit a central character) to the category of representations of $\Gamma_{\mathbb{Q}_p}$ on finite length $\mathcal{O}$-modules.

Theorem 3.5. *The functor $\mathbb{V}$ enjoys the following properties.*

(1) $\mathbb{V}(\pi) = 0$ *if and only if π is finite over $\mathcal{O}$.*

(2) $\mathbb{V}$ *is compatible with twisting by a character.*

(3) *If V is a two-dimensional k-representation of $\Gamma_{\mathbb{Q}_p}$, which is **not** a twist of*

$$\begin{pmatrix} 1 & * \\ & \bar{\epsilon} \end{pmatrix},$$

then there exists a unique representation $\bar{\pi} \in \mathrm{Rep}_k(\mathrm{GL}_2(\mathbb{Q}_p))$ such that

(a) $\mathbb{V}(\bar{\pi}) \simeq V$, *and $\bar{\pi}$ has central character (corresponding to)* $\det(V)\bar{\epsilon}^{-1}$;

(b) $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\bar{\pi}, \bar{\pi}) \hookrightarrow \mathrm{Ext}^1_{\Gamma_{\mathbb{Q}_p}}(V, V)$;

(c) $\bar{\pi}^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$.

(*Thus $\bar{\pi}$ has no finite-dimensional submodules or quotients.*)

Furthermore, $\mathbb{V}$ realizes the mod p local Langlands correspondence: $\bar{\pi}^{\mathrm{ss}} \leftrightarrow V^{\mathrm{ss}}$.

Proof. This is taken from [Ki1], where it occurs as Theorem 2.1.1. It summarizes some of the most important results from [Co1]. □

The representation $\bar{\pi}$ can be written down explicitly when V is reducible; see Remark 3.3.3 in [Em2]. For instance, if V is a sum of characters (in general position), $\bar{\pi}$ is a sum of two irreducible principal series. In general, $\bar{\pi}$ sits in an extension of such. On the other hand, when V is irreducible, $\bar{\pi}$ is supersingular.

$\mathbb{V}$ can be defined for more general coefficient rings A, namely for a local Artinian $\mathcal{O}$-algebra A whose residue field is a finite extension of k. If A is a complete local Noetherian $\mathcal{O}$-algebra (with residue field finite over k), and π is a "suitable" A-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, one defines $\mathbb{V}(\pi)$ as the inverse limit of $\mathbb{V}(\pi/\mathfrak{m}_A^s\pi)$. See Section 3.2 of [Em2] for a more thorough discussion of this.

3.6 – *Deformation theory and p-adic Langlands*

We go back to our absolutely irreducible, modular, representation $\bar{\rho}$ from Section 3.4. At each place $w|p$, we look at its restriction $\bar{\rho}_w$ to the Galois group of $\mathcal{K}_w \simeq \mathbb{Q}_p$. One of our standing hypotheses is that (for all χ and extensions $*$),

$$\bar{\rho}_w \nsim \chi \otimes \begin{pmatrix} 1 & * \\ & \bar{\epsilon} \end{pmatrix}.$$

Therefore, $\bar{\rho}_w \leftrightarrow \bar{\pi}_w$, a smooth, admissible, finite length k-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, via mod p local Langlands. Since $\mathbb{V}$ is exact, it takes a deformation of $\bar{\pi}_w$ to a deformation of $\bar{\rho}_w$. Recall that a deformation of $\bar{\rho}_w$ to a complete local Noetherian $\mathcal{O}$-algebra A (with residue field k) is a free rank two A-module ρ_w, with a continuous $\Gamma_{\mathcal{K}_w}$-action, such that $\rho_w \otimes_A k \simeq \bar{\rho}_w$. Deformations of $\bar{\pi}_w$ are defined similarly; we refer to Definition 3.2 in [Bre].

For simplicity only, we will assume $\mathrm{End}_{\Gamma_{\mathcal{K}_w}}(\bar{\rho}_w) = k$ (for example, this holds if $\bar{\rho}_w$ is absolutely irreducible). Equivalently, $\mathrm{End}_{\mathrm{GL}_2(\mathcal{K}_w)}(\bar{\pi}_w) = k$. This forces a certain rigidity into our deformation problems, and Schlessinger's criterion guarantees they are representable, by complete local Noetherian $\mathcal{O}$-algebras $R(\bar{\rho}_w)$ and $R(\bar{\pi}_w)$, respectively. One can relax the "endomorphism-condition" and employ more advanced deformation theory (groupoid-valued functors, framings etc.). We refrain from doing so. The necessary modifications are made exactly as on p. 25 in [Em2].

Since $\mathbb{V}$ is exact, it defines a morphism of local $\mathcal{O}$-algebras,

$$\mathbb{V}\colon R(\bar{\rho}_w) \longrightarrow R(\bar{\pi}_w),$$

which we will continue to denote $\mathbb{V}$. Let $R(\bar{\pi}_w)^{\mathrm{det}}$ denote the quotient of $R(\bar{\pi}_w)$, which parametrizes deformations π_w, which admit a central character χ_{π_w} corresponding to $\det \mathbb{V}(\pi_w)\epsilon$, via local class field theory. We say π_w satisfies the "determinant-condition." In [Col], VII.5.3, Colmez showed the composition

$$R(\bar{\rho}_w) \xrightarrow{\mathbb{V}} R(\bar{\pi}_w) \longrightarrow R(\bar{\pi}_w)^{\mathrm{det}}$$

is *onto*. Geometrically, $\operatorname{Spec} R(\bar{\pi}_w)^{\mathrm{det}}$ is a closed subset of the deformation space $\operatorname{Spec} R(\bar{\rho}_w)$. We will intersect it with another closed subset; the Zariski-closure of the crystalline points. More precisely, consider the quotient

$$R(\bar{\rho}_w)^{\mathrm{cris}} = R(\bar{\rho}_w)/\bigcap \mathfrak{p},$$

where $\mathfrak{p}$ runs over all prime ideals of the form $\mathfrak{p} = \ker(R(\bar{\rho}_w) \xrightarrow{\alpha} \bar{\mathbb{Z}}_p)$, where α is a homomorphism such that the α-specialization is a crystalline representation $\Gamma_{\mathcal{K}_w} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ with distinct Hodge–Tate weights. We say $\mathfrak{p}$ runs over the "crystalline-regular" points of $\operatorname{Spec} R(\bar{\rho}_w)$. Thus we may think of $\operatorname{Spec} R(\bar{\rho}_w)^{\mathrm{cris}}$ as their Zariski-closure. We look at the "intersection,"

$$\operatorname{Spec} R(\bar{\pi}_w)^{\mathrm{cris}} = \operatorname{Spec} R(\bar{\rho}_w)^{\mathrm{cris}} \times_{\operatorname{Spec} R(\bar{\rho}_w)} \operatorname{Spec} R(\bar{\pi}_w)^{\mathrm{det}}.$$

In other words, the tensor product $R(\bar{\pi}_w)^{\text{cris}}$ fits in a Cartesian square,

$$\begin{array}{ccc} R(\bar{\rho}_w) & \longrightarrow & R(\bar{\pi}_w)^{\text{det}} \\ \downarrow & & \downarrow \\ R(\bar{\rho}_w)^{\text{cris}} & \longrightarrow & R(\bar{\pi}_w)^{\text{cris}} \end{array}$$

Here the bottom map turns out to be an isomorphism, by a key result of Kisin.

Theorem 3.6. $R(\bar{\rho}_w)^{\text{cris}} \xrightarrow{\sim} R(\bar{\pi}_w)^{\text{cris}}$.

Proof. This is Proposition 2.3.3 in [Ki1], and its Corollary 2.3.4. □

Intuitively, this says that Spec $R(\bar{\rho}_w)^{\text{cris}}$ is contained in Spec $R(\bar{\pi}_w)^{\text{det}}$. Hence, what goes into the proof is to first show that crystalline-regular deformations ρ_w lie in the image of $\mathbb{V}$, say $\rho_w = \mathbb{V}(\pi_w)$; and thereafter that such π_w automatically satisfies the determinant-condition.

We now apply this to our universal modular deformation $\rho_{\mathfrak{m}}$ over $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$. Its restriction $\rho_{\mathfrak{m},w}$ is a deformation of $\bar{\rho}_w$, so it arises from the universal deformation over $R(\bar{\rho}_w)$ via a unique morphism of local $\mathcal{O}$-algebras,

$$R(\bar{\rho}_w) \xrightarrow{\alpha} \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}.$$

We wish to show α factors through $R(\bar{\rho}_w)^{\text{cris}}$. This, combined with the previous Theorem, would show the existence of a unique deformation $\pi_{\mathfrak{m},w}$ of $\bar{\pi}_w$ over $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$, which satisfies the determinant-condition, and such that

$$\rho_{\mathfrak{m},w} = \mathbb{V}(\pi_{\mathfrak{m},w}).$$

Here we will show that α factors, granted that the crystalline points are Zariski dense; which is the main result of the next section.

Lemma 3.7. *The morphism* $R(\bar{\rho}_w) \xrightarrow{\alpha} \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$ *factors through* $R(\bar{\rho}_w)^{\text{cris}}$.

Proof. Suppose $r \in \bigcap \mathfrak{p}$. We must show $\alpha(r)$ acts trivially on $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$. By the main Proposition of the next section, Proposition 3.10, it suffices to show $\alpha(r)$ acts trivially on each λ-eigenspace of $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}} \otimes E$, with $\lambda = \lambda_\pi$ as in 3.10 (i.e. the kernel of an eigensystem $\lambda: \mathbb{T}(K^p)_{\mathcal{O}} \to \bar{\mathbb{Q}}_p$ is in $\mathcal{C}_{\mathfrak{m}}$). For the remainder of this proof, let $\mathfrak{q} = \ker(\lambda)$, viewed as a prime ideal in $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$. We need to show its pullback $\alpha^{-1}(\mathfrak{q})$ is among the $\mathfrak{p}$'s in the intersection $\bigcap \mathfrak{p}$. However, the $\mathfrak{q}$-specialization $\rho_{\mathfrak{m}}(\mathfrak{q})$ of the universal modular deformation, can be identified with $\rho_{\pi,\iota}$, once we fix an embedding $\mathbb{T}(K^p)_{\mathcal{O}}/\mathfrak{q} \hookrightarrow \overline{\mathbb{Q}}_p$. Since $\rho_{\pi,\iota}$ is a deformation of $\bar{\rho}$, which is crystalline-regular at every $w|p$, the morphism

$$R(\bar{\rho}_w) \longrightarrow \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}} \longrightarrow \mathbb{T}(K^p)_{\mathcal{O}}/\mathfrak{q} \hookrightarrow \overline{\mathbb{Q}}_p$$

factors through $R(\bar{\rho}_w)^{\mathrm{cris}}$, which is to say that $\bigcap \mathfrak{p} \subset \alpha^{-1}(\mathfrak{q})$, as wanted. □

Remark 3.8. In fact, one can show that the projection $R(\bar{\rho}_w) \to R(\bar{\rho}_w)^{\mathrm{cris}}$ is an isomorphism (that is, $\bigcap \mathfrak{p} = 0$ in the above notation). For instance, see Theorem A in [Che1], when $\bar{\rho}_w$ is irreducible. Consequently, so is $R(\bar{\pi}_w)^{det} \to R(\bar{\pi}_w)^{\mathrm{cris}}$, and

$$\mathbb{V}\colon R(\bar{\rho}_w) \xrightarrow{\sim} R(\bar{\pi}_w)^{det}$$

is a strengthening of Theorem 3.6. We note that an alternative approach to some of these results can be found in the important recent work of Paskunas [Pas].

3.7 – *Zariski density of crystalline points*

First, let us get the definition of "classical and crystalline" in place.

Definition 3.9. A prime ideal $\mathfrak{p} \subset \mathbb{T}(K^p)_{\mathcal{O}}$ is called *classical* if $\mathfrak{p} = \ker(\lambda)$, for some eigensystem $\lambda\colon \mathbb{T}(K^p)_{\mathcal{O}} \to \overline{\mathbb{Q}}_p$ associated with an automorphic representation π of $G(\mathbb{A})$, of tame level K^p (and possibly non-trivial weight). If moreover π_p is unramified, we say $\mathfrak{p}$ is classical and crystalline. We denote by $\mathcal{C}$ the set of all classical and crystalline points $\mathfrak{p}$ in $\operatorname{Spec} \mathbb{T}(K^p)_{\mathcal{O}}$.

Note that prime ideals $\mathfrak{p}$ in the localization $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$ correspond to prime ideals $\mathfrak{p} \subset \mathbb{T}(K^p)_{\mathcal{O}}$ such that $\mathfrak{p} \subset \mathfrak{m}$. We will pass between these points of view, with no mention. We will let $\mathcal{C}_{\mathfrak{m}}$ denote the set of $\mathfrak{p} \in \mathcal{C}$ contained in $\mathfrak{m}$.

Proposition 3.10. *The submodule* $\bigoplus_{\mathfrak{p}\in\mathcal{C}} \widehat{H}^0(K^p)_E[\mathfrak{p}]^{\mathrm{alg}}$ *is dense in* $\widehat{H}^0(K^p)_E$. *(Similarly,* $H = \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}} \otimes E$ *contains* $\bigoplus_{\mathfrak{p}\in\mathcal{C}_{\mathfrak{m}}} H[\mathfrak{p}]^{\mathrm{alg}}$ *as a dense submodule.)*

Proof. See Corollary 4 in section 7.5 of [So2]. □

We have the following corollary to the above result:

Corollary 3.11. *One has* $\bigcap_{\mathfrak{p}\in\mathcal{C}_{\mathfrak{m}}} \mathfrak{p} = 0$ *(that is,* $\mathcal{C}_{\mathfrak{m}}$ *is Zariski dense in* $\operatorname{Spec} \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$*).*

Proof. Any $t \in \bigcap_{\mathfrak{p}\in\mathcal{C}_{\mathfrak{m}}} \mathfrak{p}$ must act trivially on all of $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$, so $t = 0$. □

Remark 3.12. The Zariski-density of crystalline points have been proved by one of us (P.C.) in the generality of a PEL-type Shimura variety, see [Cho]. The argument is much similar to the zero-dimensional case, except for a few issues with cohomology in higher degree.

3.8 – *Reformulation of Theorem 1.1*

We are finally in possession of all the ingredients needed to define a certain module $\mathcal{X}$, in terms of which Theorem 1.1 gets a simple formulation. For each $w|p$, we have $\pi_{\mathfrak{m},w}$, a deformation of $\bar{\pi}_w \leftrightarrow \bar{\rho}_w$ over $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$, with central character $\det(\rho_{\mathfrak{m},w})\epsilon$, such that $\rho_{\mathfrak{m},w} = \mathbb{V}(\pi_{\mathfrak{m},w})$. First, we introduce

$$\pi_{\mathfrak{m}} = \bigotimes_{v|p} \pi_{\mathfrak{m},\tilde{v}}.$$

(Here $v|p$ varies over places of F, and $\tilde{v}|v$ is our choice of a place of $\mathcal{K}$ above v. The tensor product is over the ring $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$.) This $\pi_{\mathfrak{m}}$ is thus a $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$-module, with a linear action of the group $G(\mathbb{Q}_p)$, which we always identify with $\prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$, using our collection of places $\{\tilde{v}\}$. Furthermore, $\pi_{\mathfrak{m}}$ has a natural $\mathfrak{m}$-adic topology, from $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$.

Strongly inspired by Section 6.3 in [Em2], we introduce the (multiplicity) module:

$$\mathcal{X} = \mathcal{X}_{K^p} := \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}[G(\mathbb{Q}_p)]}(\pi_{\mathfrak{m}}, \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}).$$

(See also Section 4.1 in [Bre]). This parametrizes maps

$$\pi_{\mathfrak{m}} \longrightarrow \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}},$$

and as we will see below its specialization $\mathcal{X}[\mathfrak{p}_x]$ parametrizes the maps

$$\bigotimes_{v|p} B(\rho_{x,\tilde{v}}) \longrightarrow \widehat{H}^0(K^p)_E[\mathfrak{p}_x]$$

relevant for Theorem 1.1, for primes $\mathfrak{p} = \mathfrak{p}_x$. Here the big difference with [Em2] and [Bre], is the lack of a Galois-action on $\widehat{H}^0(K^p)$, so we look at continuous homomorphism out of $\pi_{\mathfrak{m}}$, as opposed to $\rho_{\mathfrak{m}} \otimes \pi_{\mathfrak{m}}$. Moreover, we find it simpler to work with a fixed tame level K^p throughout (and hence a fixed eigenvariety).

We find it useful to spell out the continuity assumption: $\mathcal{X}$ consists of $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$-linear, $G(\mathbb{Q}_p)$-equivariant, homomorphisms of the form

$$\eta\colon \pi_{\mathfrak{m}} \longrightarrow \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}},$$

such that for all $s \in \mathbb{Z}_{>0}$, there is a $t \in \mathbb{Z}_{>0}$, such that

$$\eta(\mathfrak{m}^t \pi_{\mathfrak{m}}) \subset \varpi^s \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}.$$

(The reader may want to compare this to the first paragraph of 4.4 in [Bre].)

THEOREM 3.13. $\mathcal{X}[\mathfrak{p}] \neq 0$, *for **all** prime ideals* $\mathfrak{p} \subset \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$.

This implies our main result.

LEMMA 3.14. *Theorem* 3.13 $\implies$ *Theorem* 1.1.

PROOF. If $\mathfrak{p}$ is a prime, the $\mathfrak{p}$-torsion $\mathcal{X}[\mathfrak{p}]$ consists of those η which factor through

$$\pi_{\mathfrak{m}}(\mathfrak{p}) = \pi_{\mathfrak{m}}/\mathfrak{p}\pi_{\mathfrak{m}} \simeq \bigotimes_{v|p}(\pi_{\mathfrak{m},\tilde{v}}/\mathfrak{p}\pi_{\mathfrak{m},\tilde{v}}),$$

this tensor product being over the field $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}/\mathfrak{p}$. Since $\mathbb{V}$ is an exact functor,

$$\rho_{\mathfrak{m},\tilde{v}}(\mathfrak{p}) = \rho_{\mathfrak{m},\tilde{v}}/\mathfrak{p}\rho_{\mathfrak{m},\tilde{v}} = \mathbb{V}(\pi_{\mathfrak{m},\tilde{v}}/\mathfrak{p}\pi_{\mathfrak{m},\tilde{v}}).$$

If we take $\mathfrak{p} = \mathfrak{p}_x$, as in Theorem 1.1, the left-hand side is $\rho_{x,\tilde{v}}$. Consequently,

$$\pi_{\mathfrak{m}}(\mathfrak{p}_x) = \bigotimes_{v|p} B(\rho_{x,\tilde{v}}),$$

since B and $\mathbb{V}$ are each others inverse (see Definition 3.3.15 on p. 26 in [EM2]). In conclusion, the non-vanishing $\mathcal{X}[\mathfrak{p}_x] \neq 0$ of Theorem 3.13 amounts to the existence of a nonzero, continuous, E-linear, $G(\mathbb{Q}_p)$-equivariant, homomorphism

$$\eta\colon \pi_{\mathfrak{m}}(\mathfrak{p}_x) = \bigotimes_{v|p} B(\rho_{x,\tilde{v}}) \longrightarrow \hat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}[\mathfrak{p}_x] \simeq \hat{H}^0(K^p)_E[\mathfrak{p}_x]$$

(Since $\pi_{\mathfrak{m}}(\mathfrak{p}_x)$ is annihilated by $\mathfrak{p}_x$, so is the image of η.) Finally, since the target is complete, and η is continuous, it extends uniquely to the completion (with respect to the tensor product norm, cf. (1) at the end of the next section). This shows Theorem 1.1. $\square$

3.9 – *Non-vanishing at classical crystalline points*

Recall that $\bar{\rho}\colon \Gamma_{\mathcal{K}} \to \mathrm{GL}_2(k)$ is an absolutely irreducible representation, associated with the maximal ideal $\mathfrak{m} \subset \mathbb{T}(K^p)_{\mathcal{O}}$. In this section we will make the rather bold assumption that all its restrictions $\bar{\rho}_w\colon \Gamma_{\mathcal{K}_w} \to \mathrm{GL}_2(k)$ remain absolutely irreducible, for $w|p$. This is to avoid having to deal with $B(\rho)$, for reducible crystalline ρ. We are hopeful that one can adapt the approach of [BE], and relax this condition. For now, we wish to keep things simple by imposing this hypothesis.

In the next section, by a formal Nakayama type argument, we will verify that it is enough to prove Theorem 3.13 for a Zariski dense subset of $\mathfrak{p} \subset \mathfrak{m}$. Here we will prove the non-vanishing for all classical and crystalline points $\mathfrak{p}$.

LEMMA 3.15. $\mathcal{X}[\mathfrak{p}] \neq 0$, *for all* $\mathfrak{p} \in \mathcal{C}_{\mathfrak{m}}$.

PROOF. Let $\mathfrak{p} \subset \mathfrak{m}$ be a classical crystalline point of $\operatorname{Spec} \mathbb{T}(K^p)_{\mathcal{O}}$. Say, $\mathfrak{p} = \ker(\lambda_\pi)$, for an automorphic π, which is unramified at p (and of tame level K^p, and some weight). By the observations made in the proof of Lemma 3.14, we seek a nonzero, continuous, E-linear, $G(\mathbb{Q}_p)$-equivariant map,

$$\eta: \bigotimes_{v|p} B(\rho_{\pi,\tilde{v}}) \simeq \pi_{\mathfrak{m}}/\mathfrak{p}\pi_{\mathfrak{m}} \longrightarrow \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}[\mathfrak{p}] \simeq \widehat{H}^0(K^p)_E[\mathfrak{p}],$$

after possibly passing to a finite extension of E (containing the residue field of $\mathfrak{p}$). Here $\rho_\pi = \rho_{\pi,\iota}$ is as in Theorem 2.1; we suppress ι. Note that $\bar{\rho}_\pi \simeq \bar{\rho}$, since $\mathfrak{p} \subset \mathfrak{m}$. In particular, by our "bold" assumption on $\bar{\rho}_{\tilde{v}}$, we infer that the crystalline representation $\rho_{\pi,\tilde{v}}$ is absolutely irreducible (in fact, residually).

Now, for absolutely irreducible crystalline ρ, the p-adic local Langlands correspondent $B(\rho)$ has a simple description, due to Berger and Breuil, [BB], which we briefly recall. Following the recipe of [BS], one first associates a locally algebraic representation $\mathrm{BS}(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ out of the p-adic Hodge theoretical data of ρ as follows:

$$\mathrm{BS}(\rho) := \xi(\rho) \otimes_E \pi(\rho) = \det{}^{\kappa_1} \mathrm{Sym}^{\kappa_2-\kappa_1-1}(E^2) \otimes_E \pi(\rho),$$

where $\kappa_1 < \kappa_2$ are the Hodge–Tate weights of ρ, and the smooth factor $\pi(\rho)$ is given by the generic local Langlands correspondence. Thus, $\pi(\rho)$ is a full unramified principal series, possibly reducible. When it is irreducible, one has

$$\mathrm{WD}(\rho)^{F-ss} \simeq \mathrm{rec}(\pi(\rho) \otimes |\det|^{-1/2}).$$

In the reducible case, $\mathrm{WD}(\rho)$ corresponds to the Langlands quotient of $\pi(\rho)$. By Theorem 2.3.2 in [BER], which summarizes some of the main results of [BB], one knows that $\mathrm{BS}(\rho)$ admits a separated $\mathrm{GL}_2(\mathbb{Q}_p)$-stable $\mathcal{O}$-lattice, which is *finitely generated* over $\mathrm{GL}_2(\mathbb{Q}_p)$. Clearly all such lattices are commensurable, and $B(\rho)$ is the completion of $\mathrm{BS}(\rho)$ with respect to any one of them. Thus $B(\rho)$ becomes a topologically irreducible, admissible, unitary Banach E-space representation.

Specializing this discussion to $\rho = \rho_{\pi,\tilde{v}}$, Theorem 2.1 (especially local-global compatibility at the places above p) allows us to compute $\mathrm{BS}(\rho)$ in terms of π,

$$\xi(\rho) = \xi_{\tilde{v}}, \quad \pi(\rho) = \mathrm{BC}_{\tilde{v}|v}(\pi_v),$$

where $\xi_{\tilde{v}}$ denotes the irreducible algebraic representation of $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$, over E, related to π_∞ as follows. A priori, π_∞ is a representation of $G(\mathbb{C})$, restricted to $G(\mathbb{R})$. Via ι, we view π_∞ as a representation of $G(\overline{\mathbb{Q}}_p)$, and restrict it to $G(\mathbb{Q}_p)$. The resulting representation is $\xi = \bigotimes_{v|p} \xi_{\tilde{v}}$. For more details, see 2.4 in [So2]. So,

$$\bigotimes\nolimits_{v|p} \mathrm{BS}(\rho_{\pi,\tilde{v}}) \simeq \xi \otimes \pi_p,$$

both viewed as representations of $G(\mathbb{Q}_p)$, identified with $\prod_{v|p} \mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$.

We now invoke Corollary 3.2 which shows the existence of embeddings

$$\bigotimes\nolimits_{v|p} \mathrm{BS}(\rho_{\pi,\tilde{v}}) \simeq \xi \otimes \pi_p \hookrightarrow \hat{H}^0(K^p)_E[\mathfrak{p}]^{\mathrm{alg}},$$

parametrized by $(\pi_f^p)^{K^p}$ (repeated m_π times).

If $\mathcal{L}$ is an arbitrary Banach E-space, with a unitary action of $GL_2(\mathcal{K}_{\tilde{v}})$, then any equivariant map $i: \mathrm{BS}(\rho_{\pi,\tilde{v}}) \to \mathcal{L}$ is automatically continuous; with respect to the topology given by a finite type lattice $\Lambda \subset \mathrm{BS}(\rho_{\pi,\tilde{v}})$. This is almost immediate. If Λ is generated by $\{\lambda_j\}$ as a $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$-module, then $i(\Lambda)$ is contained in the ball in $\mathcal{L}$ (centered at zero) with radius $\max \|i(\lambda_j)\|_{\mathcal{L}}$. Thus, for example, if there was only one place $v|p$ of F (that is, if $F = \mathbb{Q}$), any map

$$\mathrm{BS}(\rho_{\pi,\tilde{v}}) \hookrightarrow \hat{H}^0(K^p)_E[\mathfrak{p}]$$

automatically extends to the completion $B(\rho_{\pi,\tilde{v}})$, which yields the desired map η.

Suppose for simplicity we only have two places $\{v_1, v_2\}$ of F above p. The previous discussion gives rise to a $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}_1})$-equivariant embedding,

$$\mathrm{BS}(\rho_{\pi,\tilde{v}_1}) \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{K}_{\tilde{v}_2})}(\mathrm{BS}(\rho_{\pi,\tilde{v}_2}), \hat{H}^0(K^p)_E[\mathfrak{p}]).$$

All the homomorphisms on the right-hand side are automatically continuous. So, the target may be identified with the Banach space of bounded transformations,

$$\mathcal{L}_{\mathrm{GL}_2(\mathcal{K}_{\tilde{v}_2})}(B(\rho_{\pi,\tilde{v}_2}), \hat{H}^0(K^p)_E[\mathfrak{p}]),$$

which then must contain $B(\rho_{\pi,\tilde{v}_1})$. Composing with the evaluation map, yields

$$B(\rho_{\pi,\tilde{v}_1}) \otimes B(\rho_{\pi,\tilde{v}_2}) \hookrightarrow \hat{H}^0(K^p)_E[\mathfrak{p}],$$

which is necessarily continuous relative to the tensor-product norm (and therefore extends to the completed tensor product). Since all invariant norms on $B(\rho_{\pi,\tilde{v}_i})$ are equivalent, see Corollary 5.3.4 on p. 56 in [BB], a short argument (which we include below) shows that *any* norm on their tensor product is dominated by (a constant multiple of) the tensor product norm; as defined in paragraph 17 in [Sc].

We may continue this inductively in order to deal with three or more places $v|p$. $\square$

For completeness let us give the short argument mentioned in the above proof. If V and W are normed vector spaces over a non-archimedean field K, one defines the tensor product norm on $V \otimes_K W$ as on p. 103 of [Sc]:

$$(1) \qquad \|x\|_t := \inf\{\max_i \|v_i\|_V \cdot \|w_i\|_W\}, \qquad x = \sum_i v_i \otimes w_i \in V \otimes_K W.$$

Here the infimum is taken over all possible expansions $x = \sum_i v_i \otimes w_i$. By Proposition 17.4 on p. 105 of [Sc] this defines a norm (as opposed to just a seminorm) with the property that $\|v \otimes w\|_t = \|v\|_V \cdot \|w\|_W$.

Now suppose V carries a K-linear action of a group G, and that $\|\cdot\|_V$ is the only G-invariant norm on V up to equivalence. Similarly, W carries an H-action, and $\|\cdot\|_W$ is the only H-invariant norm on W (up to equivalence).

Lemma 3.16. *Let $\|\cdot\|$ be any $G \times H$-invariant norm on $V \otimes_K W$. Then there is a constant $C > 0$ such that $\|x\| \le C\|x\|_t$ for all $x \in V \otimes_K W$.*

Proof. For any fixed $v \neq 0$, the formula $\|w\|_v := \|v \otimes w\|$ defines an H-invariant norm on W, so by assumption there is a constant C_v such that $\|w\|_v \le C_v\|w\|_W$ for all $w \in W$. Thus it makes sense to define

$$\|v\|' := \sup_{w \neq 0} \frac{\|v \otimes w\|}{\|w\|_W} \le C_v < \infty.$$

This is easily seen to be a G-invariant norm on V, from which we deduce the existence of a constant C such that $\|v\|' \le C\|v\|_V$ for all $v \in V$. In other words,

$$\|v \otimes w\| \le C\|v\|_V\|w\|_W = C\|v \otimes w\|_t,$$

for all v, w. Consequently, for any $x = \sum_i v_i \otimes w_i$, we get the estimate

$$\|x\| \le \max_i \|v_i \otimes w_i\| \le C \cdot \max_i \|v_i \otimes w_i\|_t.$$

Taking the infimum over all expansions of x yields the result. □

In the proof of Lemma 3.15, we apply Lemma 3.16 with $V := \mathrm{BS}(\rho_{\pi,\tilde{v}_1})$ and $W := \mathrm{BS}(\rho_{\pi,\tilde{v}_2})$, viewed as representations of $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}_1})$ and $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}_2})$ respectively, and equipped with the unique invariant norms.

3.10 – *Non-vanishing at all* $\mathfrak{p} \subset \mathfrak{m}$

To finish the proof of Theorem 1.1, we have to deduce Theorem 3.13 from Lemma 3.15; again under the assumption that $\mathfrak{m}$ is associated with a $\bar{\rho}$, with irreducible restrictions $\bar{\rho}_w$, for $w|p$. Thus, knowing that $\mathcal{X}[\mathfrak{p}] \neq 0$ for all $\mathfrak{p} \in \mathcal{C}_{\mathfrak{m}}$, we will deduce this for all primes $\mathfrak{p} \subset \mathfrak{m}$ whatsoever. This proceeds exactly as the proof of Proposition 4.7 in [Bre], making use of the density of $\mathcal{C}_{\mathfrak{m}}$ (Corollary 3.11 above).

Below we will use Nakayama's lemma, which requires the following preliminary result.

Lemma 3.17. $\mathrm{Hom}_{\mathcal{O}}(\mathcal{X}, \mathcal{O})$ *is a finitely generated* $\mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$*-module.*

Proof. For simplicity, let $\mathbb{T} = \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$ throughout this proof (a complete local $\mathcal{O}$-algebra). By Proposition C.5 on p. 104 in [Em2], we need to show $\mathcal{X}$ is cofinitely generated over $\mathbb{T}$ (cf. Definition C.1 in loc. cit., $\mathcal{X}$ clearly satisfies the first three properties, since $\hat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$ is ϖ-adically complete, separated, and $\mathcal{O}$-torsion-free). By C.1 it remains to show $(\mathcal{X}/\varpi\mathcal{X})[\mathfrak{m}]$ is finite-dimensional over k (the fourth property). Note that there is a natural reduction map, $\eta \mapsto \bar{\eta}$,

$$\mathcal{X}/\varpi\mathcal{X} \longrightarrow \mathrm{Hom}_{\mathbb{T}_k[G(\mathbb{Q}_p)]}(\pi_{\mathfrak{m}}/\varpi\pi_{\mathfrak{m}}, H^0(K^p)_{k,\mathfrak{m}}),$$

which is injective. Since $\varpi \in \mathfrak{m}$, after taking the $\mathfrak{m}$-torsion, we get

$$(\mathcal{X}/\varpi\mathcal{X})[\mathfrak{m}] \hookrightarrow \mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\textstyle\bigotimes_{v|p} \bar{\pi}_{\mathfrak{m},\tilde{v}}, H^0(K^p)_{k,\mathfrak{m}}),$$

using that $\pi_{\mathfrak{m}} = \bigotimes_{v|p} \pi_{\mathfrak{m},\tilde{v}}$, where $\pi_{\mathfrak{m},\tilde{v}}$ is a deformation of $\bar{\pi}_{\mathfrak{m},\tilde{v}} \leftrightarrow \bar{\rho}_{\mathfrak{m},\tilde{v}}$ over $\mathbb{T}$ ($\bar{\pi}_{\mathfrak{m},\tilde{v}}$ is the residual representation obtained from the evaluation of $\pi_{\mathfrak{m},\tilde{v}}$ at the special point). We will show that the ambient space of Hom's is finite-dimensional. We use a trick from p. 78 of [Em2], from the proof of his Theorem 6.3.12. Since the representation $\bar{\pi}_{\mathfrak{m},\tilde{v}}$ has finite length, choose a finite-dimensional k-subspace $W_{\tilde{v}}$, which generates $\bar{\pi}_{\mathfrak{m},\tilde{v}}$ as a $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$-representation (cf. the definition of $\mathbb{V}$ in Section 3.5). Put $W = \bigotimes_{v|p} W_{\tilde{v}}$, a representation of $G(\mathbb{Q}_p)$ over k. Furthermore, since $W_{\tilde{v}}$ is smooth and finite-dimensional, we can choose a compact open subgroup $K_{\tilde{v}}$ fixing $W_{\tilde{v}}$ pointwise. Let $K_p = \prod_{v|p} K_{\tilde{v}}$. By restriction,

$$\mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\textstyle\bigotimes_{v|p} \bar{\pi}_{\mathfrak{m},\tilde{v}}, H^0(K^p)_{k,\mathfrak{m}}) \hookrightarrow \mathrm{Hom}_{k[K_p]}(W, H^0(K^p)_{k,\mathfrak{m}}).$$

Moreover, since K_p acts trivially on W, the latter space can be thought of as

$$\mathrm{Hom}_{k[K_p]}(W, H^0(K^p)_{k,\mathfrak{m}}) \simeq W^\vee \otimes_k H^0(K_pK^p)_{k,\mathfrak{m}},$$

which obviously has finite dimension over k. $\square$

We now have everything in place to finish the proof.

Lemma 3.15 $\Rightarrow$ *Theorem* 3.13: Let $\mathfrak{p} \subset \mathfrak{m}$ be a prime ideal of $\mathbb{T} = \mathbb{T}(K^p)_{\mathcal{O},\mathfrak{m}}$. By Lemma C. 14 on p. 108 of [EM2], the natural restriction map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{X}, \mathcal{O}) \otimes_{\mathbb{T}} (\mathbb{T}/\mathfrak{p}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}[\mathfrak{p}], \mathcal{O})$$

becomes an isomorphism after tensoring $-\bigotimes_{\mathcal{O}} E$. By the anti-equivalence of Proposition C.5 in loc. cit., it therefore suffices to show that

$$M/\mathfrak{p}M \neq 0, \quad M = \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}, \mathcal{O}).$$

Once we show $\mathbb{T}$ acts faithfully on M, we are done by Nakayama's lemma (which applies since M is finitely generated, as shown above). Suppose $M = \mathfrak{p}M$. Then there is a $t \equiv 1 \pmod{\mathfrak{p}}$ in $\mathbb{T}$ such that $tM = 0$. Clearly a contradiction.

To show faithfulness, first note that $M/\mathfrak{p}M$ is a vector space over $\mathbb{T}/\mathfrak{p}$, a finite field extension of E. Thus, $\mathbb{T}/\mathfrak{p}$ acts faithfully on $M/\mathfrak{p}M$, whenever the latter is nonzero. If $t \in \mathbb{T}$ acts trivially on M, it acts trivially on every $M/\mathfrak{p}M$, and therefore t belongs to every $\mathfrak{p}$ for which $M/\mathfrak{p}M$ is nonzero. If this holds for a Zariski dense set $\mathcal{S}$ of primes, take $\mathcal{S} = \mathcal{C}_{\mathfrak{m}}$ for instance, we infer that

$$t \in \bigcap_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p} = 0.$$

That is, $t = 0$; which is to say $\mathbb{T}$ acts faithfully on M.

4. Theorem 1.1 $\Longrightarrow$ Theorem 1.2

We deduce Theorem 1.2 in the introduction. This is in some sense the novelty of our paper. We show classicality of a point on an eigenvariety, which requires knowledge about how refinements on the "automorphic" side interfere with triangulations on the "Galois" side. For this we need a comparison of eigenvarieties. Those from [CHE2], and those from [KPX]. This is (at least in part) the subject matter of [HE], [HESC], and [BHS].

4.1 – *Classical weights*

First off, let us note that x at least has a classical weight $\chi_x = \psi_x \theta_x$, where θ_x is smooth, and ψ_x is a dominant algebraic character of $T(\mathbb{Q}_p)$. Indeed, for any x, the Hodge–Tate–Sen weights $\{\kappa_1, \kappa_2\}$ of $\rho_{x,\tilde{v}}$ are encoded in $\chi_x = \bigotimes_{v|p} \chi_{x,\tilde{v}}$ as follows. On a compact open subgroup of $T_{\mathrm{GL}(2)}(\mathcal{K}_{\tilde{v}})$, the character $\chi_{x,\tilde{v}}$ takes the form

$$\chi_{x,\tilde{v}}(t) = t_1^{\kappa_1} t_2^{\kappa_2 - 1}.$$

Since $\rho_{x,\tilde{v}}$ is Hodge–Tate, $\kappa_1 < \kappa_2$ are integers, and $\psi_{x,\tilde{v}}(t) = t_1^{\kappa_1} t_2^{\kappa_2 - 1}$ is an algebraic character on all of $T_{\mathrm{GL}(2)}(\mathcal{K}_{\tilde{v}})$, which is dominant (relative to the lower-triangular Borel), and agrees with $\chi_{x,\tilde{v}}$ in a neighborhood of the identity.

4.2 – *Modularity*

By assumption, $\rho_{x,\tilde{v}}$ is absolutely irreducible. By well-known properties of the p-adic local Langlands correspondence, we infer that $B(\rho_{x,\tilde{v}})$ is topologically irreducible. Thus the space of locally algebraic vectors $B(\rho_{x,\tilde{v}})^{\mathrm{alg}}$ is dense, since it is nonzero by the de Rham assumption in Theorem 1.2. Consequently, $\bigotimes_{v|p} B(\rho_{x,\tilde{v}})^{\mathrm{alg}}$ is dense in $\widehat{\bigotimes}_{v|p} B(\rho_{x,\tilde{v}})$, and by continuity of the map in Theorem 1.1 it restricts to a nonzero $G(\mathbb{Q}_p)$-equivariant map

$$\bigotimes_{v|p} B(\rho_{x,\tilde{v}})^{\mathrm{alg}} \longrightarrow \widehat{H}^0(K^p)_E[\mathfrak{p}_x]^{\mathrm{alg}}.$$

In particular, the target is non-trivial, which is to say there is an automorphic representation π of $G(\mathbb{A})$, with coefficients in E, such that π_∞ has highest weight ψ_x, and the spherical Hecke algebra $\mathcal{H}(K^p)^{\mathrm{sph}}$ acts on $\pi_f^{K^p} \neq 0$ via the eigensystem $\lambda_x \colon \mathcal{H}(K^p)^{\mathrm{sph}} \to E$, with kernel $\mathfrak{p}_x$. This shows that ρ_x is at least modular; it is associated with the automorphic representation π. To conclude that $x \in X_{\mathrm{cl}}$, we are left with verifying that there is an embedding $\pi_p \hookrightarrow i_B(\theta_x)$. Here B is the product of the upper-triangular $B_{\mathrm{GL}(2)}(\mathcal{K}_{\tilde{v}})$, and the induction is not unitarily normalized (no modulus factor $\delta_B^{\frac{1}{2}}$ involved). We say θ_x is an accessible refinement of $\pi_p = \bigotimes_{v|p} \pi_v$.

4.3 – *Refinements*

We will use the following result on refinements.

THEOREM 4.1. *When $\chi_{x,\tilde{v}}$ is regular, $\rho_{x,\tilde{v}}$ is trianguline. In fact, its étale (ϕ, Γ)-module $D_{\mathrm{rig}}(\rho_{x,\tilde{v}})$ admits a triangulation,*

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow D_{\mathrm{rig}}(\rho_{x,\tilde{v}}) \longrightarrow \mathcal{R}(\delta_2) \longrightarrow 0,$$

where $\mathcal{R}$ is the Robba ring over E, and the δ_i are E-valued (necessarily) continuous characters of $\mathbb{Q}_p^\times = \mathcal{K}_{\tilde{v}}^\times$, related to the weight $\chi_{x,\tilde{v}}$ via the formula

$$\chi_{x,\tilde{v}} = \delta_1 \otimes \delta_2 \epsilon^{-1},$$

where ϵ is the "cyclotomic" character of $\mathbb{Q}_p^\times$, which kills p (sending $x \mapsto x|x|$).

Proof. When $F = \mathbb{Q}$, this is Corollary 4.3 in [He], which in fact gives an analogue for $\mathrm{GL}(n)$. (This uses the residual irreducibility of ρ_x.) The result appears in general also as Theorem 6.3.13 in [KPX] or Theorem A in [Berg2], for the space of trianguline (ϕ, Γ)-modules. Theorem 3.5 on p. 17 of [HeSc] and Theorem 4.8 on p. 40 of [BHS] relate this space to the automorphically defined eigenvariety X we consider in Theorem 1.2. □

Remark 4.2. We will not be precise about what we mean by $\chi_{x,\tilde{v}}$ being regular (see [He] page 7 for the definition). It is automatically satisfied if $\rho_{x,\tilde{v}}$ is potentially semistable with distinct Hodge–Tate weights $\kappa_1 < \kappa_2$, which we assume.

Consequently, $B(\rho_{x,\tilde{v}})$ is a p-adic unitary principal series for $\mathrm{GL}_2(\mathbb{Q}_p)$, and Colmez has recently described its locally analytic vectors $B(\rho_{x,\tilde{v}})^{\mathrm{an}}$, thereby proving conjectures of Breuil and Emerton. (We refer to [Em3] for a precise definition of locally analytic vectors.) A different proof was given by Liu, Xie, and Zhang, [LXZ]. We recall Colmez's result below.

Theorem 4.3. *Suppose $\delta_1\delta_2^{-1} \neq x^k|x|$, for any $k \in \mathbb{Z}_+$, and $\rho_{x,\tilde{v}}$ is irreducible. Then $B(\rho_{x,\tilde{v}})^{\mathrm{an}}$ sits in an exact sequence of locally analytic representations,*

$$0 \longrightarrow i_B(\delta_2 \otimes \delta_1\epsilon^{-1})^{\mathrm{an}} \longrightarrow B(\rho_{x,\tilde{v}})^{\mathrm{an}} \longrightarrow i_B(\delta_1 \otimes \delta_2\epsilon^{-1})^{\mathrm{an}} \longrightarrow 0.$$

(*Here the induction is unnormalized. The notation i_B was introduced just before Theorem 1.1 in the introduction.*)

Proof. This is Theorem 0.7, part (i), on page 7 in [Co2]. See also the main Theorem 1.2 in [LXZ], which gives an alternative proof. □

Taking locally algebraic vectors, which is left exact, yields

$$0 \longrightarrow i_B(\delta_2 \otimes \delta_1\epsilon^{-1})^{\mathrm{alg}} \longrightarrow B(\rho_{x,\tilde{v}})^{\mathrm{alg}} \longrightarrow i_B(\delta_1 \otimes \delta_2\epsilon^{-1})^{\mathrm{alg}}.$$

Here the rightmost term is $i_B(\chi_{x,\tilde{v}})^{\mathrm{alg}}$, by Theorem 4.1, and the leftmost term is $i_B(\chi'_{x,\tilde{v}})^{\mathrm{alg}}$, where $\chi'_{x,\tilde{v}}$ is the character sending $t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T_{\mathrm{GL}(2)}(\mathcal{K}_{\tilde{v}})$ to

$$\chi'_{x,\tilde{v}}(t) = \chi_{x,\tilde{v}}\left(\begin{pmatrix} t_2 & \\ & t_1 \end{pmatrix}\right) \cdot \epsilon\left(\frac{t_1}{t_2}\right).$$

In particular, its algebraic part $\psi'_{x,\tilde{v}}(t) = t_1^{\kappa_2} t_2^{\kappa_1 - 1}$ is *not* dominant, relative to the lower triangular Borel, and consequently we have $i_B(\psi'_{x,\tilde{v}})^{\mathrm{alg}} = 0$.

LEMMA 4.4. *The two extreme terms of the exact sequence are*

(*a*) $i_B(\chi'_{x,\tilde{v}})^{\mathrm{alg}} = 0$,

(*b*) $i_B(\chi_{x,\tilde{v}})^{\mathrm{alg}} \simeq i_B(\psi_{x,\tilde{v}})^{\mathrm{alg}} \otimes i_B(\theta_{x,\tilde{v}})$,

where, in (*b*)*, the first induction is algebraic induction* (*thus* $\xi_{x,\tilde{v}} = i_B(\psi_{x,\tilde{v}})^{\mathrm{alg}}$ *is irreducible algebraic of highest weight* $\psi_{x,\tilde{v}}$)*, and the second is smooth induction.*

PROOF. For (b), note that there is a natural multiplication map,

$$i_B(\theta_{x,\tilde{v}}) \longrightarrow \mathrm{Hom}(\xi_{x,\tilde{v}}, i_B(\chi_{x,\tilde{v}}))^{sm},$$

which is $\mathrm{GL}_2(\mathcal{K}_{\tilde{v}})$-equivariant. It is injective. Indeed, a function in the kernel would annihilate the highest weight vector in $\xi_{x,\tilde{v}}$, and therefore vanish on $\overline{N}$, and hence on the dense open Bruhat cell $B\overline{N}$. To show surjectivity, it suffices to show

$$i_B(\theta_{x,\tilde{v}})^K \xrightarrow{\sim} \mathrm{Hom}_K(\xi_{x,\tilde{v}}, i_B(\chi_{x,\tilde{v}})),$$

for all sufficiently small compact open subgroups K. All we have to do is count dimensions. On the left-hand side, we get $|B\backslash G/K|$, at least for small enough K. To deal with the right-hand side, think of it as

$$(\xi^{\vee}_{x,\tilde{v}} \otimes i_B(\chi_{x,\tilde{v}}))^K = i_B(\xi^{\vee}_{x,\tilde{v}}|_B \otimes \chi_{x,\tilde{v}})^K.$$

If K is small enough, the latter is identified with $|B\backslash G/K|$ copies of $(\xi^{\vee}_{x,\tilde{v}}|_B \otimes \psi_{x,\tilde{v}})^B$, since $B \bigcap K$ is Zariski-dense, which is the line spanned by the highest weight vector in $\xi^{\vee}_{x,\tilde{v}}$. (Note here that $\xi^{\vee}_{x,\tilde{v}}$ has highest weight $\psi^{-1}_{x,\tilde{v}}$ relative to B.) As a result, the initial "multiplication map" is an isomorphism, and

$$\xi_{x,\tilde{v}} \otimes i_B(\theta_{x,\tilde{v}}) \xrightarrow{\sim} i_B(\chi_{x,\tilde{v}})^{\xi_{x,\tilde{v}}\text{-alg}} = i_B(\chi_{x,\tilde{v}})^{\mathrm{alg}},$$

where the last equality is deduced by looking at the highest weight vectors as in the above argument (cf. the paragraph preceding the Lemma). The same arguments works for (a). If $i_B(\chi'_{x,\tilde{v}})$ had W-algebraic vectors, for some W, one could deduce that $W^{\vee}$ has highest weight $\chi'^{-1}_{x,\tilde{v}}$ relative to B, but then $\chi'_{x,\tilde{v}}$ would be $\overline{B}$-dominant, which we already observed is not the case. $\square$

We conclude that there is an embedding

$$B(\rho_{x,\tilde{v}})^{\mathrm{alg}} \hookrightarrow \xi_{x,\tilde{v}} \otimes i_B(\theta_{x,\tilde{v}}).$$

On the other hand, since $\rho_{x,\tilde{v}}$ is assumed to be potentially semistable, with distinct Hodge–Tate weight, the algebraic vectors can be expressed as $\xi_{x,\tilde{v}} \otimes \pi_{x,\tilde{v}}$, where

$\pi_{x,\tilde{v}}$ arises from $\mathrm{WD}(\rho_{x,\tilde{v}})^{F-ss}$ via the generic local Langlands correspondence (suitably normalized). We infer that there is an inclusion of $\pi_{x,\tilde{v}}$,

$$\pi_{x,\tilde{v}} \hookrightarrow \mathrm{Hom}(\xi_{x,\tilde{v}}, \xi_{x,\tilde{v}} \otimes i_B(\theta_{x,\tilde{v}}))^{sm} = (\mathrm{End}(\xi_{x,\tilde{v}}) \otimes i_B(\theta_{x,\tilde{v}}))^{sm} = i_B(\theta_{x,\tilde{v}}),$$

since $\xi_{x,\tilde{v}}|_K$ remains irreducible for any K. This proves what we want. Using Theorem 1.1, we have already deduced $\rho_x \simeq \rho_\pi$ is modular. From local-global compatibility above p (due to Barnet-Lamb, Gee, Geraghty, Taylor, and Caraiani) we conclude that $\pi_{x,\tilde{v}} = \mathrm{BC}_{\tilde{v}|v}(\pi_v)$. (Indeed, since ρ_π is irreducible, $\mathrm{BC}_{\mathcal{K}/F}(\pi)$ must be cuspidal, hence globally generic, and therefore π_v is generic, so in this case "generic" local Langlands is just "classical" local Langlands). Finally, by taking the tensor product over all $v|p$, we get the desired embedding $\pi_p \hookrightarrow i_B(\theta_x)$, which shows x must be a classical point. This finishes the proof of Theorem 1.2.

References

[BLGGT1] T. Barnet-Lamb – T. Gee – D. Geraghty – R. Taylor, *Local–global compatibility for $l = p$,* I. Ann. Fac. Sci. Toulouse Math. (6) **21** (2012), no. 1, pp. 57–92.

[BLGGT2] T. Barnet-Lamb – T. Gee – D. Geraghty – R. Taylor, *Local–global compatibility for $l = p$,* II. Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, pp. 165–179.

[BC] J. Bellaiche – G. Chenevier, *Families of Galois representations and Selmer groups,* Asterisque 324, 2009.

[Ber] L. Berger, *La correspondance de Langlands locale p-adique pour* $\mathrm{GL}_2(\mathbb{Q}_p)$*,* Séminaire Bourbaki, Vol. 2009/2010, Exposés 1012–1026, Asterisque 339, 2011, Exp. no. 1017, viii, pp. 157–180.

[Berg] J. Bergdall, *Ordinary modular forms and companion points on the eigencurve,* J. Number Theory **134** (2014), pp. 226–239.

[Berg2] J. Bergdall, *Paraboline variation of p-adic families of (ϕ, Γ)-modules,* preprint 2014. arXiv:1410.3412 [math.NT]

[BB] L. Berger – C. Breuil, *Sur quelques representations potentiellement cristallines de* $\mathrm{GL}_2(\mathbb{Q}_p)$*,* Asterisque 330, 2010, pp. 155–211.

[BE] C. Breuil – M. Emerton, *Representations p-adiques ordinaires de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et compatibilite local-global,* Asterisque 331, 2010, pp. 255–315.

[BHS] C. Breuil – E. Hellmann – B. Schraen, *Une interprétation modulaire de la variété trianguline,* preprint 2015. arXiv:1411.7260 [math.NT]

[BHS2] C. Breuil – E. Hellmann – B. Schraen, *Smoothness and classicality on eigenvarieties,* preprint 2015, arXiv:1510.01222 [math.NT]

[BS] C. Breuil – P. Schneider, *First steps towards p-adic Langlands functoriality,* J. Reine Angew. Math. **610** (2007), pp. 149–180.

[Bre] C. Breuil, *Correspondance de Langlands p-adique, compatibilité local-global et applications [d'après Colmez, Emerton, Kisin, …],* Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. Astérisque 348, 2012, Exp. no. 1031, viii, pp. 119–147.

[Buz] K. Buzzard, *Eigenvarieties,* in *L-functions and Galois representations,* papers from the symposium held at the University of Durham, Durham, July 19–30, 2004, edited by D. Burns, K. Buzzard and J. Nekovář, London Mathematical Society Lecture Note Series, 320, Cambridge University Press, Cambridge, 2007. pp. 59-120.

[Car] A. Caraiani, *Monodromy and local-global compatibility for $l = p$,* Algebra Number Theory **8** (2014), no. 7, pp. 1597–1646.

[Che1] G. Chenevier, *Sur la densité des représentations cristallines de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*,* Math. Ann. **355** (2013), no. 4, pp. 1469–1525.

[Che2] G. Chenevier, *Une application des variétés de Hecke des groupes unitaires,* preprint 2009. http://gaetan.chenevier.perso.math.cnrs.fr/articles/famgal.pdf

[Cho] P. Chojecki, Density of crystalline points on unitary Shimura varieties, Int. J. Number Theory **9** (2013), no. 3, pp. 729–744.

[CS] P. Chojecki – C. Sorensen, *Strong local-global compatibility in the p-adic Langlands program for $U(2)$,* preprint 2014. arXiv:1406.1828 [math.NT]

[CM] R. Coleman – B. Mazur, *The eigencurve,* in *Galois representations in arithmetic algebraic geometry,* papers from the London Mathematical Society Symposium held in Durham, July 9–18, 1996, edited by A. J. Scholl and R. L. Taylor, London Mathematical Society Lecture Note Series, 254. Cambridge University Press, Cambridge, 1998, pp. 1–113.

[Co1] P. Colmez, *Representations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (ϕ, Γ)*-modules,* Asterisque 330, 2010, pp. 281–509.

[Co2] P. Colmez, *La série principale unitaire de* $\mathrm{GL}_2(\mathbb{Q}_p)$*: vecteurs localement analytiques,* in *Automorphic forms and Galois representations,* Vol. 1, Proceedings of the 94th London Mathematical Society LMS-EPSRC Symposium on Galois Representations and Automorphic Forms held at the University of Durham, Durham, July 18–28, 2011, edited by F. Diamond, P. L. Kassaei, and M. Kim, London Mathematical Society Lecture Note Series, 414, Cambridge University Press, Cambridge, 2014, pp. 286–358.

[CY] T. Barnet-Lamb – D. Geraghty – M. Harris – R. Taylor, *A family of Calabi-Yau varieties and potential automorphy* II, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, pp. 29–98.

[Din] Y. Ding, *$\mathcal{L}$-invariants and local-global compatibility for the group* GL_2/F*,* Forum Math. Sigma **4** (2016), e13, 49 pp.

[Em1] M. Emerton, *A local-global compatibility conjecture in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$, Pure Appl. Math. Q. **2** (2006), no. 2, special Issue in honor of J. H. Coates, Part 2, pp. 279–393.

[Em2] M. Emerton, *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$, preprint 2011.
http://www.math.uchicago.edu/~emerton/pdffiles/lg.pdf

[Em3] M. Emerton, *Locally analytic vectors in representations of locally p-adic analytic groups,* preprint 2011, to appear in Mem. Amer. Math. Soc.
http://www.math.uchicago.edu/~emerton/pdffiles/analytic.pdf

[Em4] M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms,* Invent. Math. **164** (2006), no. 1, pp. 1–84.

[EH] M. Emerton – D. Helm, *The local Langlands correspondence for* $\mathrm{GL}(n)$ *in families,* Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, pp. 655–722.

[HT] M. Harris – R. Taylor, *The geometry and cohomology of some simple Shimura varieties,* with an appendix by V. G. Berkovich, Annals of Mathematics Studies, 151, Princeton University Press, Princeton, N.J., 2001.

[He] E. Hellmann, *Families of trianguline representations and finite slope spaces,* preprint 2014. arXiv:1202.4408 [math.AG]

[HeSc] E. Hellmann – B. Schraen, *Density of potentially crystalline representations of fixed weight,* Compos. Math. **152** (2016), no. 8, pp. 1609–1647.

[Ki1] M. Kisin, *Deformations of* $G_{\mathbb{Q}_p}$ *and* $\mathrm{GL}_2(\mathbb{Q}_p)$ *representations,* Asterisque 330, 2010, pp. 529–542.

[Ki2] M. Kisin, *The Fontaine–Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, pp. 641–690.

[KPX] K. S. Kedlaya – J. Pottharst – L. Xiao, *Cohomology of arithmetic families of* (ϕ, Γ)*-modules,* J. Amer. Math. Soc. **27** (2014), no. 4, pp. 1043–1115.

[Loe] D. Loeffler, *Overconvergent algebraic automorphic forms,* Proc. Lond. Math. Soc. (3) **102** (2011), no. 2, pp. 193–228.

[LXZ] R. Liu – B. Xie – Y. Zhang, *Locally analytic vectors of unitary principal series of* $\mathrm{GL}_2(\mathbb{Q}_p)$, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), no. 1, pp. 167–190.

[Min] A. Mínguez, *Unramified representations of unitary groups,* in *On the stabilization of the trace formula,* edited by L. Clozel, M. Harris, J.-P. Labesse, and B.-C. Ngô, Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications, 1, International Press, Somerville, MA, 2011, pp. 389–410.

[Pas] V. Paskunas, *The image of Colmez's Montreal functor,* Publ. Math. Inst. Hautes Études Sci. **118** (2013), pp. 1–191.

[Rog] J. Rogawski, *Automorphic representations of unitary groups in three variables,* Annals of Mathematics Studies, 123. Princeton University Press, Princeton, N.J., 1990.

[Sc] P. Schneider, *Nonarchimedean functional analysis,* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[So1] C. Sorensen, *A proof of the Breuil–Schneider conjecture in the indecomposable case,* Ann. of Math. (2) **177** (2013), no. 1, pp. 367–382.

[So2] C. Sorensen, *Eigenvarieties and invariant norms,* Pacific J. Math. **275** (2015), no. 1, pp. 191–230.

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