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A Poincaré lemma for Whitney–de Rham complex

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Abstract – Let M be a real analytic manifold, Z a closed subanalytic subset of M. We show that the Whitney–de Rham complex over Z is quasi-isomorphic to the constant sheaf $\mathbb{C}_Z$.

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1. Introduction

In [5], Kashiwara–Schapira introduced the Whitney functor (real case) and formal cohomology functor (complex case), then they introduced the notion of ind-sheaves and they also defined Grothendieck six operations in this framework in [6]. As applications, they defined the Whitney $\mathcal{C}^\infty$ functions and Whitney holomorphic functions on the subanalytic site as examples of ind-sheaves. A more elementary study for sheaves on the subanalytic site is performed in [7] and [8].

Let M be a real analytic manifold, by Poincaré lemma, it is well known that the de Rham complex over M is isomorphic to $\mathbb{C}_M$. The aim of this paper is to show that a theorem of [1] follows easily from a deep result of Kashiwara on regular holonomic $\mathcal{D}$-module in [3]. More precisely, we show that

Main theorem (Theorem 3.3). *Let M be a real analytic manifold of dimension n and Z a closed subanalytic subset of M. Then we have*

$$\mathbb{C}_Z \xrightarrow{\sim} \big(0 \longrightarrow \mathcal{W}^{\infty}_{M,Z} \xrightarrow{d} \mathcal{W}^{(\infty,1)}_{M,Z} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}^{(\infty,n)}_{M,Z} \longrightarrow 0\big),$$

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where $\mathcal{W}^{\infty}_{M,Z}$ denotes the sheaf of Whitney functions on Z and $\mathcal{W}^{(\infty,i)}_{M,Z}$ denotes the sheaf of differential forms of degree i with coefficients in $\mathcal{W}^{\infty}_{M,Z}$ for each i, i.e., the Whitney–de Rham complex is isomorhpic to $\mathbb{C}_Z$.

2. Review on Whitney and formal cohomology functors

In this section, we review some results on Whitney and formal cohomology functors. References are made to [5], [6], [7], [8], and [9].

Let M be a real analytic manifold, we denote by $\mathcal{A}_M$, $\mathcal{C}^{\infty}_M$ the sheaf of complex-valued real analytic functions and $\mathcal{C}^{\infty}$ functions on M. We denote by $\mathcal{D}_M$ the sheaf of rings on M of finite-order differential operators with coefficients in $\mathcal{A}_M$.

We denote by $\mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on M and $\mathrm{Mod}(\mathcal{D}_M)$ the abelian category of left $\mathcal{D}_M$-modules. We also denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_M)$ the bounded derived category consisting of objects whose cohomology groups belong to $\mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_M)$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_M)$ the derived category of $\mathrm{Mod}(\mathcal{D}_M)$ with bounded cohomologies.

Definition 2.1. Let Z be a closed subset of M. We denote by $\mathcal{I}^{\infty}_{M,Z}$ the sheaf of $\mathcal{C}^{\infty}$ functions on M vanishing up to infinite order on Z.

Definition 2.2. A *Whitney function* on a closed subset Z of M is an indexed family

$$F = (F^k)_{k\in\mathbb{N}^n}$$

consisting of continuous functions on Z such that for all $m \in \mathbb{N}$ and $k \in \mathbb{N}^n$ with $|k| \le m$, and all $x \in Z$ and $\varepsilon > 0$ there exists a neighborhood U of x such that

$$\left| F^k(z) - \sum_{|j+k|\le m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \le \varepsilon d(y,z)^{m-|k|}, \quad \text{for all} y, z \in U \cap Z.$$

We denote by $W^{\infty}_{M,Z}$ the space of Whitney C^{∞} functions on Z. We denote by $\mathcal{W}^{\infty}_{M,Z}$ the sheaf $U \mapsto W^{\infty}_{U,U\cap Z}$.

In [5], the authors defined the Whitney tensor product functor

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M \colon \mathrm{Mod}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_M) \longrightarrow \mathrm{Mod}(\mathcal{D}_M)$$

in the following way. Let U be an open subanalytic subset of M and $Z = M \setminus U$. Then $\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M = \mathcal{I}^{\infty}_{M,Z}$ and $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M = \mathcal{W}^{\infty}_{M,Z}$. This functor is exact and extends as a functor in the derived category, from $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_M)$ to $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_M)$. Moreover, the sheaf $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M$ is soft for any $\mathbb{R}$-constructible sheaf F.

Now let X be a complex manifold and we denote by $\mathcal{D}_X$ the sheaf of rings on X of finite-order differential operators. We still denote by X the real underlying manifold and we denote by $\overline{X}$ the complex manifold conjugate to X. One defines the functor of formal cohomology as follows:

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$, we set

$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X}),$$

where $\mathcal{D}_{\overline{X}}$ denotes the sheaf of rings on $\overline{X}$ of finite-order differential operators.

Let M be a real analytic manifold, X a complexification of M, $\imath\colon M \hookrightarrow X$ the embedding. We recall the following result.

Theorem 2.3 ([5], Theorem 5.10). *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$. Then we have*

$$\imath_* F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq \imath_*(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{M}).$$

In particular,

$$\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq \mathcal{C}^{\infty}_{M}.$$

The following proposition is the key point of this paper which follows from a deep result of [3].

Proposition 2.4 ([5], Corollary 6.2). *Let $\mathfrak{M}$ be a regular holonomic $\mathcal{D}_X$-module, and let F be an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$. Then, the natural morphism:*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F \otimes \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X). \tag{2.1}$$

is an isomorphism.

3. Main result

Let X be a complex manifold of dimension n. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order differential operators and Θ_X the sheaf of vector fields on X.

First we recall the following basic result in $\mathcal{D}_X$-module theory.

Proposition 3.1 ([4], Proposition 1.6). *The complex*

$$0 \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n} \Theta_X \longrightarrow \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{2} \Theta_X \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \longrightarrow \mathcal{D}_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

is exact.

LEMMA 3.2. *Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. Then we have*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \simeq \Big[\mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \bigwedge^n \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}\Big].$$

PROOF. By Proposition 3.1, we have

$$\begin{aligned} &R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \\ &\simeq \Big[\mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X, \mathcal{M}) \longrightarrow \cdots \\ &\qquad\qquad \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X, \mathcal{M})\Big] \\ &\simeq \Big[\mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{M}) \longrightarrow \cdots \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^n \Theta_X, \mathcal{M})\Big] \\ &\simeq \Big[\mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \bigwedge^n \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}\Big] \end{aligned}$$

where

$$\Omega^1_X := \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{O}_X). \qquad \square$$

Let M be a real analytic manifold, X a complexification of M and Z a closed subanalytic subset of M. We denote by Ω^1_X the sheaf of differential one-form on X and

$$\mathcal{A}^{(i)}_Z := \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \otimes \mathcal{O}_X), \tag{3.1}$$

$$\mathcal{W}^{(\infty,i)}_{M,Z} := \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \simeq \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} (\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M). \tag{3.2}$$

Now we are ready to prove the main theorem of this paper below.

THEOREM 3.3. *Let M be a real analytic manifold of dimension n and Z a closed subanalytic subset of M. Then we have*

$$\mathbb{C}_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{W}^\infty_{M,Z} \xrightarrow{d} \mathcal{W}^{(\infty,1)}_{M,Z} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}^{(\infty,n)}_{M,Z} \longrightarrow 0),$$

where $\mathcal{W}^\infty_{M,Z}$ denotes the sheaf of Whitney functions on Z and $\mathcal{W}^{(\infty,i)}_{M,Z}$ denotes the sheaf of differential forms of degree i with coefficients in $\mathcal{W}^\infty_{M,Z}$ for each i which are defined in (3.2), i.e., the Whitney–de Rham complex is isomorhpic to $\mathbb{C}_Z$.

Proof. Take $\mathfrak{M} = \mathcal{O}_X$ and $F = \mathbb{C}_Z$ in Proposition 2.4.

On the one hand, we show that the left hand side of (2.1) is $\mathbb{C}_Z$. By Theorem 2.3 and Lemma 3.2, we get the following complex

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{A}_M^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_M^{(n)} \longrightarrow 0$$

which is exact by Poincaré lemma where $\mathcal{A}_M^{(i)}$'s are defined in (3.1) by taking $Z = M$. Tensoring $\mathbb{C}_Z$, we obtain the following exact sequence

$$0 \longrightarrow \mathbb{C}_Z \longrightarrow \mathcal{A}_Z^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_Z^{(n)} \longrightarrow 0.$$

Therefore,

$$\mathbb{C}_Z \xrightarrow{\sim} \big(0 \longrightarrow \mathcal{A}_Z^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_Z^{(n)} \longrightarrow 0\big).$$

On the other hand, the right hand side of (2.1) is the Whitney–de Rham complex

$$0 \longrightarrow \mathcal{W}_{M,Z}^{\infty} \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,n)} \longrightarrow 0.$$

Now the result follows from the isomoprhism of (2.1). □

Remark 3.4. This theorem is in some sense dual to a theorem of Grothendieck in [2] which asserts that if U is subanalytic then the de Rham cohomology may be calculated with holomoprhic functions which are meromorphic on the complementary of U (U is the complementary of a closed hypersurface in the complex analytic space), that is, holomorphic functions with temperate growth.

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