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DIAGONAL OPERATORS ON SPACES OF MEASURABLE FUNCTIONS

by

M. ORHON and T. TERZIOGLU

1. Introduction.

We denote by L the set of equivalence classes of real-valued measurable functions on a fixed measure space (X, Σ, μ) . L is an algebra with unit and a vector lattice with respect to almost everywhere pointwise operations. The space of essentially bounded real-valued functions $L^\infty = L^\infty(\mu)$ is a normed subalgebra of L and L is a module over L^∞ with respect to almost everywhere pointwise multiplication. A subspace M of L is a solid sublattice of L if and only if M is an L^∞ -submodule of L [4]. We will call an L^∞ -submodule M of L a locally convex L^∞ -module if M is a locally convex vector space whose topology is given by a family of seminorms p satisfying

$$p(af) \leq \|a\|_\infty p(f) \quad a \in L^\infty, f \in M.$$

Such a seminorm is called a scalar L^∞ -seminorm [7]. Since a scalar L^∞ -seminorm defined on a solid sublattice of L is a lattice seminorm and vice versa, M is a locally convex L^∞ -module if and only if it is a locally convex vector lattice and solid in L [4]. The Banach spaces $L^p(\mu)$, $1 \leq p < \infty$, and Köthe spaces equipped with Köthe topologies [12] are examples of locally convex L^∞ -modules.

A linear operator T mapping a subspace M of L into another subspace of L will be called diagonal if there is a locally measurable real-valued function g on X such that $Tf = gf$ for every f in M . A linear operator T mapping an L^∞ -submodule M into another L^∞ -submodule of L will be called L^∞ -linear if $T(af) = aT(f)$ for every a in L^∞ and f in M .

From now on M and N will denote locally convex L^∞ -modules (or equivalently, locally convex solid sublattices of L). Further N is assumed to be order complete. If A is a subset of L , then A^+ denotes the set of positive elements of A .

We present our results without proofs; a full account will appear elsewhere. Finally, we wish to express our gratitude to the Scientific and Technical Research Council of Turkey for their support.

2. L^∞ -linear operators.

Let \mathcal{C} be the set of positive continuous linear operators from M into N . Then $\mathfrak{L}(M, N) = \mathcal{C} - \mathcal{C}$ is a solid sublattice of the space $L^b(M, N)$ of order bounded linear operators from M into N . By $\mathfrak{H}_\infty(M, N)$ we denote the space of continuous L^∞ -linear operators from M into N .

LEMMA. - $\mathfrak{H}_\infty(M, N)$ is a sublattice of $\mathfrak{L}(M, N)$.

A locally convex L^∞ -module M is said to have the dominated convergence property if for every sequence (f_n) in L with $|f_n| \leq g$ for some g in M and $\lim f_n(x) = f(x)$ on X , we have $\lim f_n = f$ in M .

PROPOSITION 1. - Let Λ be a Köthe space, T a Köthe topology on Λ and Λ^\times the α -dual of Λ . Consider the following conditions :

- a) T is compatible with the duality $(\Lambda, \Lambda^\times)$.
- b) If $f_n \in \Lambda$ and $f_n(x) \downarrow 0$ on X then $\lim f_n = 0$ in $\Lambda(T)$
- c) $\Lambda(T)$ has the dominated convergence property.
- d) If p is one of the scalar L^∞ -seminorms defining the topology T on Λ and $f \in \Lambda$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $p(\chi_E f) < \epsilon$.

We have (a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d).

We will also consider the following condition.

(A) for every $f \in M^+$ there is an increasing sequence (s_n) of positive simple functions of bounded support such that $s_n(x) \uparrow f_n(x)$ on X and $\lim s_n = f$ in M .

The Banach spaces $L^p(\mu)$, $1 \leq p < \infty$, satisfy this condition.

PROPOSITION 2. - If a Köthe space $\Lambda(T)$ has the dominated convergence property, it satisfies (A).

From now on we assume $L^1(\mu)' = L^\infty(\mu)$.

A diagonal operator is certainly L^∞ -linear. Under certain assumptions the converse is also true.

PROPOSITION 3. - a) Let M satisfy condition (A). If for every set of finite measure B , the characteristic function $\chi_B \in M$, then every element of $\mathfrak{H}_\infty(M, N)$ is a diagonal operator.

b) If M is a Köthe space which has the dominated convergence property, then every element of $\mathfrak{H}_\infty(M, N)$ is a diagonal operator.

Remark : The hypothesis of the proposition is satisfied by $L^p(\mu)$, $1 \leq p < \infty$. On the other hand, if $T : L^\infty \rightarrow N$ is L^∞ -linear, since $T(f) = T(1)f$ for every $f \in L^\infty$, it is also diagonal.

The set of idempotents in L^∞ is denoted by I_∞ and non-negative finite linear combinations of elements of I_∞ are dense in $(L^\infty)^+$. If $\chi \in I_\infty$, then $\chi' = 1 - \chi \in I_\infty$ also.

PROPOSITION 4. - There is a projection P of $\mathfrak{L}(M,N)$ onto $\mathfrak{H}_\infty(M,N)$ with $0 \leq P \leq I$.

The projection is constructed in successive steps. First, for $T \in \mathcal{C}$ and $f \in M^+$ we define an element of N by

$$P(T)(f) = \bigwedge_{I \in I_\infty} \{ \chi T(\chi f) + \chi' T(\chi' f) \}.$$

We prove that $P(T)$ is additive on M^+ and then extend it to a positive linear operator on M . In the next step P is proved to be additive on \mathcal{C} and then extended to $\mathfrak{L}(M,N)$.

Remark 1. If we define an L^∞ -module structure on $\mathfrak{L}(M,N)$ by letting $(a.T)(f) = T(af)$ for f in M and a in L^∞ , then P is also L^∞ -linear.

Remark 2. If we take μ to be the counting measure on the set of positive integers, a Köthe space becomes a solid sequence space [5]. Certain operators on sequence spaces can be represented by infinite-matrices [8 ; p. 20]. If (t_{ij}) is the matrix which represents the operators T , then $P(T)$ is the operator represented by the diagonal of the matrix (t_{ij}) .

Let M and N be Banach sublattices of L , and $\mathcal{N}(M,N)$ the space of nuclear operators from M into N with the nuclear norm $r(\cdot)$. Every nuclear operator can be written as the difference of two positive nuclear operators. If $u_i \in M'$ and $g_i \in N$, $i=1, \dots, n$, by $\sum_{i=1}^n u_i \otimes g_i$ we denote the nuclear operator which sends each $f \in M$ to $\sum_{i=1}^n u_i(f) g_i$. We consider the following conditions on a Banach L^∞ -module Q .

- (B) Given $f \in Q$ and $\epsilon > 0$, there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\|f \chi_E\| < \epsilon$.
- (C) The support of each $f \in Q$ is σ -finite.
- (D) Q has the dominated convergence property.

By $\mathcal{N}_\infty(M,N)$ we will denote the space of nuclear L^∞ -linear operators from M into N with the nuclear norm.

PROPOSITION 5. - Let M and N the Banach L^∞ -modules. If M satisfies (B), N satisfies (C) and (D) and further for every finite family of atoms $\{x_1, \dots, x_n\}$, $u \in M'$ and $g \in N$ we have

$$(**) \quad r\left(\sum x_k u \otimes x_k g\right) \leq \left\| \left(\sum x_k\right) u \right\| \left\| \left(\sum x_k\right) g \right\|$$

then the projection P maps $\mathcal{N}(M, N)$ onto $\mathcal{N}_\infty(M, N)$ such that $r(P(T)) \leq r(T)$ for each $T \in \mathcal{N}(M, N)$.

Remark : If M' has property (B) instead of M, M has property (C) instead of N or if M' has property (D) instead of N, the result still holds.

3. Diagonal and nuclear diagonal operators on L^p -spaces.

Let M and N be two normed L^∞ -modules and $M \otimes N$ the complete projective tensor product as defined by Grothendieck [3]. Let K be the smallest closed subspace of $M \otimes N$ containing all elements of the form $(af \otimes g) - (f \otimes ag)$ for every $a \in L^\infty$, $f \in M$ and $g \in N$. The quotient space $M \otimes N / K$ with the quotient norm is called the normed L^∞ -tensor product of M and N, and denoted by $M \otimes_\infty N$. If $f \otimes_\infty g$ denotes $f \otimes g \text{ mod } K$ for each $f \in M$, $g \in N$, then for $u \in M \otimes_\infty N$ the norm is given by [4 and 9]

$$\gamma_\infty(u) = \inf \left\{ \sum \|f_i\| \|g_i\| : u = \sum f_i \otimes_\infty g_i, f_i \in M, g_i \in N \right\}.$$

With a measure space (X, Σ, μ) we associate for every real number $s > 0$ a weighted counting measure space as follows : ψ is the set of equivalence classes of atoms of μ together with the equivalence class of sets of μ -measure zero. We let $\mu_\alpha = \mu(A)$ for any $A \in \alpha$, where $\alpha \in \psi$. For any subset S of ψ we define

$$\tilde{\mu}^s(S) = \sum_{\alpha \in S} \mu_\alpha^s.$$

PROPOSITION 6. - (Harte). Let $1/p + 1/q = 1/r \leq 1$ where $1 \leq p, q \leq \infty$. Then $L^p(\mu) \otimes_\infty L^q(\mu)$ is isometrically L^∞ -isomorphic with $L^r(\mu)$.

In the result complementary to this we have to use the weighted counting measure constructed above.

PROPOSITION 7. - Let $s = 1/p + 1/q > 1$ where $1 \leq p, q \leq \infty$. Then $L^p(\mu) \otimes_\infty L^q(\mu)$ is isometrically L^∞ -isomorphic with $L^1(\tilde{\mu}^s)$.

This result can be found in [6]. Next we give characterizations of diagonal operators between L^p -spaces as another L^p -space. Again we have two cases, the

first due to Harte [4] and the second to Orhon [6].

PROPOSITION 8. - Let $1/q - 1/p = 1/r$ where $1 \leq p, q \leq \infty$. Then $\mathfrak{H}_\infty(L^p(\mu), L^q(\mu))$ is isometrically L^∞ -isomorphic with $L^r(\mu)$.

In the result complementary to this we again need the weighted counting measure.

PROPOSITION 9. - Let $1 \leq p < q \leq \infty$. Then $\mathfrak{H}_\infty(L^p(\mu), L^q(\mu))$ is isometrically L^∞ -isomorphic with $L^\infty(\tilde{\mu})$.

Remark: Diagonal operators between l^p -spaces were characterized by A. Tong [11]. G. Crofts [1] has considered diagonal operators between sequence spaces.

Using the projection constructed in proposition 4 and its properties discussed in proposition 5, we can define a continuous linear operator from $L^p(\mu) \otimes_\infty L^q(\mu)$ onto the space $\mathfrak{H}_\infty(L^p, L^q)$ of diagonal nuclear operators. This enables us to characterize $\mathfrak{H}_\infty(L^p, L^q)$ by using propositions 6 and 7.

PROPOSITION 10. - $\mathfrak{H}_\infty(L^p(\mu), L^q(\mu))$ is isometrically isomorphic with

- (i) $L^1(\tilde{\mu}^{-1/r})$, if $1 \leq q < p < \infty$ and $1/r = 1/q - 1/p$.
- (ii) $L^1(\psi_0)$, if $1 \leq p = q < \infty$ where ψ_0 denotes the set of equivalence classes of atoms of μ .
- (iii) $L^s(\tilde{\mu}^{-1-s})$, if $1 \leq p < \infty$ and $s = pq/pq - q + p$.
- (iv) $L^{p'}(\tilde{\mu}^{-1-p'})$, if $1 < p < \infty$ and $q = \infty$.

Remark: In proposition 10 the cases $\mathfrak{H}_\infty(L^\infty, L^p)$, $1 \leq p < \infty$ and $\mathfrak{H}_\infty(L^1, L^\infty)$ are not covered. In the case $\mathfrak{H}_\infty(L^1, L^\infty)$ our method breaks down, since in this case the projection p (cap.) does not take nuclear operators to nuclear diagonal operators. Nuclear diagonal operators on l^p -spaces were characterized by A. Tong [11].

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