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REPRESENTATION THEORY OF LOCALLY m-CONVEX TOPOLOGICAL ALGEBRAS

by

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INTRODUCTION.

The purpose of this paper is to point out the intimate relation between representations of "generalized group algebras", as it has been considered by A. Hausner in [3], and the general theory of topological tensor product algebras, in particular, locally m-convex ones [12], [7]. More specifically, the present material may be considered as another application of the basic formula relating the spectrum of a topological tensor product algebra to the spectra of the factor algebras.

Proofs of the results reported herein, as well as further details along the lines of this paper will appear elsewhere.

1. Preliminaries.

All vector spaces and algebras considered in the following are over the field of complex numbers. All topological spaces are assumed to be Hausdorff. For the terminology applied concerning locally m-convex topological algebras, we refer to [12]. On the other hand, we shall also use the terminology and previous results of this author regarding topological tensor products of topological algebras without further discussion. (In this respect, cf. for instance [7] or [11]).

Now, if E is a locally m-convex (topological) algebra, then by definition there exists a local basis in E consisting of m-barrels (balanced, convex, closed, absorbing and idempotent subsets of E [9]). On the other hand, if every m-barrel is a neighborhood of the zero element in E, then E is called an m-barreled (locally m-convex) algebra (ibid.). Besides this class of topological algebras, we also consider in the following those locally m-convex algebras, the topology of which coincides with that of the uniform convergence on the (closed) equicontinuous subsets of their spectra (Michael algebras, cf. for instance, [10] p. 475). In this respect, we remark that a Michael algebra E is also an m-barreled one if, and only if, every (weakly) bounded subset of its spectrum $\mathfrak{M}(E)$ is also equicontinuous.

Finally, we shall also consider below on a locally m-convex algebra E the inverse image (locally m-convex algebra) topology $\tau^{\#}$ defined on E by the respective Gel'fand map $g : E \to \mathcal{C}_c(\mathfrak{M}(E))$ (: the algebra of continuous complex-valued maps on $\mathfrak{M}(E)$ in the (locally m-convex algebra) topology of compact convergence in $\mathfrak{M}(E)$). We shall denote the corresponding topological algebra by $E[\tau^{\#}]$.

In particular, one has the preceding situation by considering certain locally m-convex algebras equipped with an <u>involution</u>, i.e. a hermitian involutive (algebra) anti-homomorphism, $x \to x^*$, of E into itself (<u>$*$-algebras</u>).

Thus, let E be a locally m-convex (topological) algebra equipped with an identity element and an involution such that the following condition is satisfied.

(1.1) <u>There exists a family</u> $\Gamma = (p_\alpha)_{\alpha \in I}$ <u>of submultiplicative seminorms defining the topology of</u> E [12] <u>such that</u> $p_\alpha(x^*x) = (p_\alpha(x))^2$, <u>for every</u> $x \in E$ <u>and for every index</u> $\alpha \in I$.

The preceding condition implies, in particular, that the algebra E is (<u>functionally</u>) semi-simple, in the sense that the respective Gel'fand map is injective. Thus, we may consider E as a subalgebra of $\mathcal{C}(\mathfrak{M}(E))$, which is, in particular, commutative. (In this respect, cf. also [10], p. 474, Scholium).

Under the preceding circumstances, E carries now the Michael topology, so that in case it is also m-barreled, its topology coincides with that of the uniform convergence on the compact subsets of $\mathfrak{M}(E)$, i.e. one has in this case the relation

$$(1.2) \qquad E[\tau^{\#}] \subsetneq \mathcal{C}_c(\mathfrak{M}(E)) ,$$

within a topological algebraic (into) isomorphism.

In particular, one concludes that the involution in E is continuous and moreover a <u>functional</u> one, in the sense that one has, for every $x \in E$, the relation :

$$(1.3) \qquad g(x^*) = \overline{g(x)} ,$$

where g denotes the respective Gel'fand map of E and the "bar", complex conjugation.

Thus, one concludes, in particular, that <u>the algebra E is selfadjoint with a continuous involution</u>. Hence, by a direct application of the Stone-Weirstrass theorem, one has the relation :

$$(1.4) \qquad \overline{E[\tau^{\#}]} = \mathcal{C}_c(\mathfrak{M}(E)) ,$$

so that if E is, moreover, complete, one obtains :

$$(1.5) \qquad E[\tau^{\#}] = \mathcal{C}_c(\mathfrak{M}(E)) ,$$

within a topological algebraic (onto) isomorphism (defined by the corresponding Gel'fand map g).

2. Representation theory.

Let E be a topological algebra and let H be a topological vector space. By a continuous representation of E on H, we mean a continuous (algebra) homomorphism of E into $\mathcal{L}_\tau(H)$, the algebra of continuous linear endomorphisms of H equipped with a topology τ making it a topological algebra.

In particular, if H is a (complex) Hilbert space, a continuous representation of a topological algebra E in $\mathcal{L}_b(H)$ will be called a uniformly continuous representation of E on H, where now the topology b (: bounded convergence on H) coincides with the "uniform operator topology" u on the space $\mathcal{L}(H)$.

On the other hand, a $*$-representation of (a $*$-algebra) E on (the Hilbert space) H is one, which preserves the involution of the corresponding algebras E and $\mathcal{L}(H)$.

Now, suppose that T is a completely regular (Hausdorff) space and let $\mathcal{C}_c(T)$ be the algebra of complex-valued continuous functions on T equipped with the topology of compact convergence, so that $\mathcal{C}(T)$ thus topologized becomes a locally m-convex (topological) algebra.

On the other hand, the topological dual of the respective locally convex space $\mathcal{C}(T)$ can be realized as the space of regular Borel measures on T with compact support (cf. for instance [1] p. 203, theorem 1 (iii)). Based on this result and the relevant considerations in [6], one gets the following.

THEOREM 2.1. - *Let T be a completely regular space and let $\mathcal{C}_c(T)$ be the (locally m-convex topological) algebra of complex-valued continuous functions on T endowed with the topology of compact convergence in T. Moreover, let H be a reflexive locally convex (topological vector) space and let :*

$$(2.1) \qquad A : \mathcal{C}_c(T) \to \mathcal{L}_\tau(H) ,$$

with $\tau \geqslant s$ (: uniform topology of simple convergence in H), be a continuous representation of $\mathcal{C}_c(T)$ on H. Then, there exists a uniquely defined idempotent $\mathcal{L}(H)$-valued measure on T, in such a way that one has :

$$(2.2) \qquad \ell(A_x \Phi) = \int_T x(t)\, d\ell(P_t(\Phi)) ,$$

for any $\Phi \in H$ and $\ell \in H'$, with $x \in \mathcal{C}(T)$. In particular, if H is a Hilbert space and A denotes a uniformly continuous representation of $\mathcal{C}_c(T)$ on H, one obtains

(2.3) $$(A_x \Phi, \Psi) = \int_T x(t)\, d(P_t \Phi, \Psi) ,$$

for every pair (Φ, Ψ) of elements of H, with $x \in \mathcal{C}(T)$, where P_t now denotes a projection-valued measure on T.

Concerning the preceding result, we also remark that there is actually a bijection between continuous representations given by (2.1) above and idempotent $\mathcal{L}(H)$-valued measures on T, in such a way that the relation (2.2) holds true.

Now, applying the preceding theorem to the particular completely regular space defined by the spectrum $\mathfrak{M}(E)$ (: Gel'fand space [10]) of a locally m-convex algebra E, one obtains the following result.

THEOREM 2.2. - Let E be a commutative semi-simple locally m-convex (topological) algebra endowed with a continuous involution for which it is also self-adjoint, and moreover suppose that (1.4) above holds true. Then, to every continuous $*$-representation $A : E[\tau^{\#}] \to \mathcal{L}_u(H)$ of the algebra $E[\tau^{\#}]$ on a Hilbert space H, there corresponds a uniquely defined projection-valued measure P^A on the spectrum $\mathfrak{M}(E)$ of E in such a way that the following relation holds true :

(2.4) $$(A_x \Phi, \Psi) = \int_{\mathfrak{M}(E)} \hat{x}(f)\, d\,(P^A_f \Phi, \Psi) ,$$

with $x \in E$ ($\hat{x}$ denotes the Gel'fand transform of x [10]), and for every pair (Φ, Ψ) of elements of H, this correspondence being, moreover, a bijection between the respective classes of objects.

By what has been said in the preceding section, we conclude that the above theorem is valid if, in particular, E is a commutative semi-simple m-barreled locally m-convex (topological) algebra with an identity element and an involution satisfying the relation (1.1) above.

3. Tensor products.

We specialize in the sequel to the case of tensor product algebras, which are endowed with suitable "compatible" topologies [11], and seek out representations of such algebras in the sense of the preceding section. In this respect, one obtains the following general result, which we shall also use in the next section. That is, we have.

THEOREM 3.1. - Let E, F, G be topological algebras with (jointly) continuous multiplication in such a way that all of them have approximate identities and the algebra G is complete. Moreover, let $E \hat{\otimes}_\tau F$ be the completion of the tensor product algebra $E \otimes F$ under an "admissible" topology τ [8] making it a topological algebra with continuous multiplication. Then, for every element $h \in \mathcal{H}om(E \hat{\otimes}_{\tau} F, \mathcal{L}_s(G))$ such that :

$$\overline{\mathrm{Im}(h) \cdot G} = G \tag{3.1}$$

(the "bar" means topological closure), there exist, uniquely defined, elements $f \in \mathcal{H}om(E, \mathcal{L}_s(G))$ and $g \in \mathcal{H}om(F, \mathcal{L}_s(G))$ such that :

$$h = f \otimes g \text{ , with } h(x \otimes y) = (f \otimes g)(x \otimes y) = f(x)\, g(y) = g(y)\, f(x) \text{ ,} \tag{3.2}$$

for any elements $x \in E$ and $y \in F$.

The preceding theorem generalizes the situation one has in the case that the algebras considered have identity elements (cf. for instance [11] p. 80, theorem 3.1).

On the other hand, if $\mathcal{H}om_0(E \hat{\otimes}_{\tau} F, \mathcal{L}_s(G))$ denotes the subset of $\mathcal{H}om(E \hat{\otimes}_{\tau} F, \mathcal{L}_s(G))$ (: the set of continuous (algebra) homomorphisms between the respective topological algebras) determined by the relation (3.1) above, then one obtains a continuous injection

$$\Phi : \mathcal{H}om_0(E \hat{\otimes}_{\tau} F, \mathcal{L}_s(G)) \to \mathcal{H}om(E, \mathcal{L}_s(G)) \times \mathcal{H}om(F, \mathcal{L}_s(G)) \text{ ,} \tag{3.3}$$

whose range are those (f,g) as above, for which one has $f(x)\, g(y) = g(y)\, f(x)$, for every pair $(x,y) \in E \times F$, in such a way that the corresponding (continuous) homomorphism $h = f \otimes g$ satisfies the relation (3.1) above.

We finally remark that one can also conclude the bicontinuity of the preceding map Φ under suitable restrictions for the topological algebras involved, in analogy with the situation one has in the case the algebras considered have identity elements (cf. also [11], p. 79, § 3).

We conclude by citting a similar result, in this respect, of A. Guichardet, concerning a tensor product of C^*-algebras (cf. [2], p. 193, proposition 1), which has also contributed to the present setting.

4. Generalized group algebras.

As an application of the results referred to in the foregoing, we specialize below to the case of "generalized group algebras", which are a particular instance of topological tensor product algebras (cf. for instance [7]).

Thus, by applying to the case under consideration the results of the preceding section, we first obtain the following theorem.

THEOREM 4.1. - *Let E be a commutative complete locally m-convex (topological) algebra with an identity element (denoted by 1) and a continuous involution for which it is also self-adjoint. Moreover, let G be a locally compact abelian group and let $L^1(G, E)$ be the corresponding "generalized group algebra"* [7]. *Finally, let $A : L^1(G, E) \to \mathcal{L}_u(H)$ be a uniformly continuous $*$-representation of the algebra $L^1(G, E)$ on a (complex) Hilbert space H such that :*

$$\overline{A(L^1(G)\otimes 1)} = H \tag{4.1}$$

(the "bar" means topological closure). Then, there exists a uniformly continuous representation $T : E \to \mathcal{L}_u(H)$ of the algebra E on H and a weakly continuous unitary representation (U_a) of G in H commuting with (T_x), in such a way that one has the relation :

$$(A_f\ \Phi, \Psi) = \int_G (T_{f(a)}\ U_a\ \Phi, \Psi)\ da\ , \tag{4.2}$$

where da denotes the Haar measure on G, for every $f \in L^1(G, E)$ and for all elements Φ, Ψ in H.

On the other hand, one obtains the following converse to the preceding result. That is, we have :

THEOREM 4.2. - *Let E be a commutative complete locally m-convex algebra with an identity element and a continuous involution. Moreover, let G be a locally compact abelian group and let $L^1(G, E)$ be the respective generalized group algebra of G. Finally, let (T_x) be a uniformly continuous representation of E on a Hilbert space H, and (U_a) a weakly continuous unitary representation of G in H, which commutes with (T_x). Then, there exists a uniformly continuous representation (A_h) of the algebra $L^1(G, E)$ on H in such a way that, for every $\Phi \in H$, the element $A_h\Phi \in H$ is given by the Pettis integral*

$$A_h\ \Phi = \int_G T_{h(a)}\ U_a\ \Phi\ da\ . \tag{4.3}$$

Moreover, if the algebra E is self-adjoint and (T_x) is a $*$-representation of E in $\mathcal{L}_u(H)$, then the same holds true for the representations (A_h) as above.

By applying the preceding results, we are now in a position to state the following theorem, which also motivated the present study. It also constitutes an extension of a previous result of A. Hausner obtained into the context of the theory of Banach algebras by using different techniques (cf. [3], p. 4, theorem 2). Besides it is a form of a *generalized* SNAG (: Stone-Naimark-Ambrose-Godement) *Theorem* (cf. also [3], p. 5, in particular, the comments following the proof of theorem 2). Thus, we have :

THEOREM 4.3. - *Let E be a commutative complete semi-simple locally m-convex (topological) algebra with an identity element and a locally equicontinuous spectrum* [9], *equipped with a continuous involution for which it is also self-adjoint. Moreover, let H be a (complex) Hilbert space and (T_x) a uniformly continuous $*$-representation of $E[\tau^{\#}]$ in $\mathcal{L}_u(H)$. Finally, let G be a locally compact abelian group, and let (U_a) be a weakly continuous unitary representation of G in H commuting with (T_x). Then, there exists a projection-valued measure $P(h, \hat{\alpha})$ on $\mathfrak{M}(E)\times\hat{G}$, where $\mathfrak{M}(E)$ denotes the spectrum of E and $\hat{G}$ the dual group of G, in such a way that one has the following relation :*

$$(4.4) \qquad (T_x U_a \Phi, \Psi) = \int_{\mathfrak{M}(E)\times\hat{G}} \hat{x}(h)\,(a, \hat{\alpha})\, d(P(h, \hat{\alpha})\,\Phi, \Psi)\,,$$

for any $x\in E$, $a\in G$ and all elements Φ, Ψ of H.

Concerning the representation (T_x) in the preceding theorem, we remark that we simply hypothesize the respective map $T : E \to \mathcal{L}_u(H)$ to be continuous when E is equipped with the topology $\tau^{\#}$, i.e., the initial (locally m-convex algebra) topology defined on E by the Gel'fand map $g : E \to \mathcal{C}_c(\mathfrak{M}(E))$. This implies, in particular, that the map $T \circ g^{-1} : g(E) \to \mathcal{L}_u(H)$ is a continuous representation of the algebra $g(E)$ on H, where the algebra $g(E) = E^{\Lambda}$ (: *Gel'fand transform of the algebra* E) is a dense subalgebra of $\mathcal{C}_c(\mathfrak{M}(E))$, as it is concluded by the hypothesis on E (cf. also section 1 in the preceding), so that one may apply theorem 2.1 above. On the other hand, regarding the topological algebras considered in the preceding theorem, we also refer to the comments following theorem 2.2 in the foregoing.

By concluding the present discussion, we finally remark that an extension to the case of "non-unitary representations" concerning, in particular, the results in this section is possible. In this respect, some relevant recent considerations by R. A. Hirschfeld seems to provide a suitable framework thereof (cf. for instance [5] or [4]). Regarding this point of view however, we intend to be more specific in some other place.

Remark : The topological algebra E considered in theorem 4.3 above should moreover satisfy the condition : E is m-barrelled [9] in such a way that the respective Gel'fand transform algebra $E^{\Lambda} = g(E)$ is contained in the Banach algebra $\mathcal{C}_o(\mathfrak{M}(E))$ of complex-valued continuous functions on $\mathfrak{M}(E)$, which "vanish at infinity", the topology of the latter algebra being that of the uniform convergence in $\mathfrak{M}(E)$.

In this respect, we note that since $g(E) \subsetneq \mathcal{C}_o(\mathfrak{M}(E))$, as above, the algebra E is bounded [10 ; p. 470], so that, since it is also m-barrelled, its spectrum $\mathfrak{M}(E)$ is an equicontinuous subset of the respective weak dual space E'_s

[ibid. ; p. 471, corollary] and hence a locally compact (Hausdorff) space [9 ; p. 302, theorem 2.1]. Besides, the presence of the identity element in E is not essential.

A detailed proof of the preceding theorem, as well as an abstract version of it, in terms of general topological tensor algebras, which gives a better insight into the situation described by theorem 4.3, are given in a subsequent publication.

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