Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

MICROLOCALIZATION OF SUBANALYTIC SHEAVES

Numéro 135
Nouvelle série

Luca PRELLI

2 0 1 3

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre National de la Recherche Scientifique

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 30 € ($ 45)
Abonnement Europe : 262 € hors Europe : 296 € ($ 444)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0249-633-X

ISBN 978-2-85629-768-1

Directeur de la publication : Marc PEIGNÉ

MÉMOIRES DE LA SMF 135

MICROLOCALIZATION OF SUBANALYTIC SHEAVES

Luca Prelli

Société Mathématique de France 2013
Publié avec le concours du Centre National de la Recherche Scientifique

L. Prelli
Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste 63,
35121 Padova, Italia..
E-mail : `lprelli@math.unipd.it`

2000 *Mathematics Subject Classification*. — 32C38, 35A27, 18F20, 32B20.

***Key words and phrases*. —** Algebraic analysis, specialization, normal deformation, microlocalization, subanalytic sheaves.

MICROLOCALIZATION OF SUBANALYTIC SHEAVES

Luca Prelli

***Abstract.* —** We define the specialization and microlocalization functors for subanalytic sheaves. Applying these tools to the sheaves of tempered and Whitney holomorphic functions, we generalize some classical constructions. We also prove that the microlocalizations of tempered and Whitney holomorphic functions have a natural structure of module over the ring of microdifferential operators, and are locally invariant under contact transformations.

***Résumé* (Microlocalisation des faisceaux sous-analytiques). —** On définit la spécialisation et la microlocalisation pour les faisceaux sous-analytiques. En appliquant ces outils aux faisceaux des fonctions holomorphes tempérées et de Whitney, on généralise des constructions classiques. On démontre aussi que les microlocalisations des fonctions holomorphes tempérées et de Whitney ont une structure naturelle de module sur l'anneau des opérateurs microdifférentiels, et sont localement invariants par transformations de contact.

CONTENTS

INTRODUCTION

After the fundamental works of Sato on hyperfunctions and microfunctions and the development of algebraic analysis, the methods of cohomological theory of sheaves became very useful for studying systems of PDE on real or complex analytic manifolds. Motivated by the study of solutions with growth conditions of a system of PDE (Riemann-Hilbert correspondence, Laplace transform, etc.), Kashiwara and Schapira in [**16**] introduced the notion of ind-sheaf, and defined the formalism of six Grothendieck operations in this framework. They defined the subanalytic site (a site whose open sets are subanalytic and the coverings are locally finite) and obtained the ind-sheaves of tempered and Whitney holomorphic functions (which are objects of the derived category of sheaves on this site) by including subanalytic sheaves into the category of ind-sheaves. Then, in [**28**], a direct, self-contained and elementary construction of the six Grothendieck operations for subanalytic sheaves was established. Important examples of applications of subanalytic sheaves to $\mathcal{D}$-modules can be found in [**24**] and [**25**].

The microlocalization functor for sheaves on a real analytic manifold was originally introduced by Sato to perform a microlocal analysis of the singularities of hyperfunction solutions of systems of linear PDE on complex manifolds. It was generalized to the framework of ind-sheaves in [**19**]. It is natural to ask if it is possible to develop microlocalization on the subanalytic site avoiding the heavy theory of ind-sheaves. The aim of this work is to extend some classical constructions for sheaves, as the functors of specialization and microlocalization, to the framework of subanalytic sheaves.

We introduce first the category of conic subanalytic sheaves on an analytic manifold endowed with an action of $\mathbb{R}^+$. In order to do that we have to choose a suitable definition: indeed there are several definitions, which are equivalent in the classical case but not in the framework of subanalytic sheaves. We choose the one which satisfies some desirable properties, as the equivalence with sheaves on the conic topology associated to the action. Thanks to this equivalence we can also represent conic sheaves as limits of conic $\mathbb{R}$-constructible sheaves. Then we extend the Fourier-Sato transform to the category of conic subanalytic sheaves on a vector bundle. This construction was also motivated by the sheaf theoretical interpretation given in [**31**] of the Laplace isomorphisms of Kashiwara and Schapira. At this point we can start studying subanalytic sheaves from a microlocal point of view by introducing the functors of specialization and microlocalization along a submanifold of a real analytic manifold. As an interesting application, the specialization is the key tool used in order to give a functorial construction of asymptotically developable functions (see also the recent developments in [**11**]). We give an estimate of the support of microlocalization using the subanalytic analogue of the notion of ind-microsupport of [**17**] and its functorial properties developed in [**23**]. We also show that the functor of microlocalization is related with the functor of ind-microlocalization defined in [**19**]. Then, applying specialization (resp. microlocalization) to the subanalytic sheaves of tempered and Whitney holomorphic functions, we generalize tempered and formal specialization (resp. microlocalization). In this way we get a unifying description of Andronikof's [**1**] and Colin's [**6**] "ad hoc" constructions.

As an application, we prove that the microlocalizations of $\mathcal{O}^t$ and $\mathcal{O}^{\mathrm{w}}$ have (in cohomology) a natural structure of $\mathcal{E}$-module and that locally they are invariant under contact transformations. Only in the case of $\mathcal{O}^t$ these results were proven in [**1**]. Furthermore, using DG-methods and ind-microlocalization, in [**10**] the author proved that the microlocalization of tempered holomorphic functions is an object of the derived category of $\mathcal{E}$-modules. The $\mathcal{E}$-module structure, combined with the estimate for the support of microlocalization, was essential for the proof of a Cauchy-Kowalevskaya-Kashiwara theorem with growth conditions given in [**29**].

In more details the contents of this work are as follows.

In Chapter 1 we recall the results on subanalytic sheaves of [**16**] and [**28**].

In Chapter 2 we construct the category of conic sheaves on a subanalytic site endowed with an action of $\mathbb{R}^+$.

In Chapter 3 we consider a vector bundle E over a real analytic manifold and its dual E^* endowed with the natural action of $\mathbb{R}^+$. We define the Fourier-Sato transform which gives an equivalence between conic subanalytic sheaves on E and conic subanalytic sheaves on E^*.

Then we define the functor ν_M^{sa} of specialization along a submanifold M of a real analytic manifold X (Chapter 4) and its Fourier-Sato transform, the functor μ_M^{sa} of microlocalization (Chapter 5). We introduce the functor μhom^{sa} for subanalytic sheaves and we give an estimate of its support using the notion of microsupport of [**17**]. Then we study its relation with the functor of ind-microlocalization of [**19**].

We apply these results in Chapter 6. We study the connection between specialization and microlocalization for subanalytic sheaves and the classical ones. Specialization of subanalytic sheaves generalizes tempered and formal specialization of [**1**] and [**6**], in particular when we specialize Whitney holomorphic functions we obtain the sheaves of functions asymptotically developable of [**22**] and [**36**]. Moreover, thanks to the functor of microlocalization, we are able to generalize tempered and formal microlocalization introduced by Andronikof in [**1**] and Colin in [**5**] respectively.

Chapter 7 is dedicated to the study of the microlocalization of tempered and Whitney holomorphic functions. We prove that the microlocalization of $\mathcal{O}^t$ and $\mathcal{O}^{\mathrm{w}}$ have (in cohomology) a natural structure of $\mathcal{E}$-module and that locally they are invariant under contact transformations.

We end this work with a short Appendix in which we recall the definitions and we collect some properties of subanalytic subsets and ind-sheaves, then we study the inverse image of the subanalytic sheaves of tempered and Whitney holomorphic functions.

Acknowledgments

We would like to thank Pierre Schapira who encouraged us to develop specialization and microlocalizaiton of subanalytic sheaves and A. D'Agnolo, S. Guillermou, S. Yamazaki for their many useful remarks. We thank E. Bierstone, M. Coste and M. Edmundo for their expertise in o-minimal and subanalytic geometry.

CHAPTER 1

REVIEW ON SHEAVES ON SUBANALYTIC SITES

In the following X will be a real analytic manifold and k a field. References are made to [**18**] and [**37**] for a complete exposition on sheaves on Grothendieck topologies, to [**16**] and [**28**] for an introduction to sheaves on subanalytic sites. We refer to [**3**] for the theory of subanalytic sets.

1.1. Sheaves on subanalytic sites

Let us recall some results of [**16**] and [**28**].

Denote by $\mathrm{Op}(X_{\mathrm{sa}})$ (resp. $\mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$) the category of open (resp. open relatively compact) subanalytic subsets of X. One endows $\mathrm{Op}(X_{\mathrm{sa}})$ with the following topology: $S \subset \mathrm{Op}(X_{\mathrm{sa}})$ is a covering of $U \in \mathrm{Op}(X_{\mathrm{sa}})$ if for any compact K of X there exists a finite subset $S_0 \subset S$ such that

$$K \cap \bigcup_{V \in S_0} V = K \cap U.$$

We will call X_{sa} the *subanalytic site*, and for $U \in \mathrm{Op}(X_{\mathrm{sa}})$ we denote by $U_{X_{\mathrm{sa}}}$ the category $\mathrm{Op}(X_{\mathrm{sa}}) \cap U$ with the topology induced by X_{sa}.

Let $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ denote the category of sheaves on X_{sa}.

Then $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ is a Grothendieck category, i.e. it admits a generator and small inductive limits, and small filtrant inductive limits are exact. In particular as a Grothendieck category, $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ has enough injective objects.

Let $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ be the abelian category of $\mathbb{R}$-constructible sheaves on X, and consider its subcategory $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_X)$ consisting of sheaves whose support is compact.

We denote by $\rho : X \to X_{\mathrm{sa}}$ the natural morphism of sites. We have functors

$$\mathrm{Mod}(k_X) \underset{\rho_!}{\overset{\rho_*}{\underset{\longleftarrow\rho^{-1}\longrightarrow}{\longrightarrow}}} \mathrm{Mod}(k_{X_{\mathrm{sa}}}).$$

The functors ρ^{-1} and ρ_* are the functors of inverse image and direct image respectively. The sheaf $\rho_! F$ is the sheaf associated with the presheaf $\mathrm{Op}(X_{\mathrm{sa}}) \ni U \mapsto F(\overline{U})$. In particular, for $U \in \mathrm{Op}(X)$ let k_U be the sheaf associated to the presheaf whose sections on $V \in \mathrm{Op}(X)$ are $= k$ if $V \subseteq U$ and $= 0$ otherwise. One has

$$\rho_! k_U \simeq \varinjlim_{V \Subset U} \rho_* k_V,$$

where $V \in \mathrm{Op}(X_{\mathrm{sa}})$. Let us summarize the properties of these functors:

▷ the functor ρ_* is fully faithful and left exact, the restriction of ρ_* to $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is exact;

▷ the functor ρ^{-1} is exact;

▷ the functor $\rho_!$ is fully faithful and exact;

▷ (ρ^{-1}, ρ_*) and $(\rho_!, \rho^{-1})$ are pairs of adjoint functors.

Notations 1.1.1. — Since the functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, we can identify $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ with its image in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$. When there is no risk of confusion we will write F instead of $\rho_* F$, for $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.

Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$. There exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_X)$ such that $F \simeq \varinjlim_i \rho_* F_i$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho"] & Y \arrow[d, "\rho"] \\ X_{\mathrm{sa}} \arrow[r, "f"] & Y_{\mathrm{sa}} \end{tikzcd} \tag{1.1.1}$$

We get external operations f^{-1} and f_*, which are always defined for sheaves on Grothendieck topologies. For subanalytic sheaves we can also define the functor of proper direct image

$$f_{!!} : \mathrm{Mod}(k_{X_{\mathrm{sa}}}) \longrightarrow \mathrm{Mod}(k_{Y_{\mathrm{sa}}}), \quad F \longmapsto \varinjlim_U f_* F_U \simeq \varinjlim_K f_* \Gamma_K F$$

where U ranges trough the family of relatively compact open subanalytic subsets of X and K ranges trough the family of subanalytic compact subsets of X. The notation $f_{!!}$ follows from the fact that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. If f is proper on $\mathrm{supp}(F)$ then $f_* F \simeq f_{!!} F$, in this case $f_{!!}$ commutes with ρ_*. While functors f^{-1} and $\otimes$ are exact, the functors $\mathcal{H}om$, f_* and $f_{!!}$ are left exact and admit right derived functors.

To derive these functors we use the category of quasi-injective objects. An object $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ is quasi-injective if for $U, V \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ with $V \subset U$ the restriction morphism $\Gamma(U; F) \to \Gamma(V; F)$ is surjective or, equivalently, if the functor

$\mathrm{Hom}_{k_{X_{\rm sa}}}(\,.\,,F)$ is exact on $\mathrm{Mod}^{\rm c}_{\mathbb{R}\text{-c}}(k_X)$. Quasi-injective objects are injective with respect to the functors $f_*, f_{!!}$ and, if $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, with respect to the functors $\mathrm{Hom}_{k_{X_{\rm sa}}}(G,\,.), \mathcal{H}om(G,\,.)$.

The functor $Rf_{!!}$ admits a right adjoint, denoted by $f^!$, and we get the usual isomorphisms between Grothendieck operations (projection formula, base change formula, Künneth formula, etc.) in the framework of subanalytic sites.

Let Z be a subanalytic locally closed subset of X. As in classical sheaf theory we define

$$\begin{aligned} \Gamma_Z : \mathrm{Mod}(k_{X_{\rm sa}}) &\longrightarrow \mathrm{Mod}(k_{X_{\rm sa}}), & F &\longmapsto \mathcal{H}om(\rho_* k_Z, F);\\ (.)_Z : \mathrm{Mod}(k_{X_{\rm sa}}) &\longrightarrow \mathrm{Mod}(k_{X_{\rm sa}}), & F &\longmapsto F \otimes \rho_* k_Z. \end{aligned}$$

Finally we recall the properties of the six Grothendieck operations and their relations with the functors ρ^{-1}, $R\rho_*$ and $\rho_!$. We refer to [**28**] for a detailed exposition.

- ▷ The functor $R^k\,\mathcal{H}om(F,\,.)$ commutes with filtrant $\varinjlim$ if $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.
- ▷ The functors $R^k f_{!!}$ and $H^k f^!$ commute with filtrant $\varinjlim$.
- ▷ The functor ρ^{-1} commutes with $\otimes$, f^{-1} and $Rf_{!!}$.
- ▷ The functor $R\rho_*$ commutes with $R\mathcal{H}om$, Rf_* and $f^!$.
- ▷ The functor $\rho_!$ commutes with $\otimes$ and f^{-1}.
- ▷ The restrictions of $\otimes$ and f^{-1} to the category of $\mathbb{R}$-constructible sheaves commute with ρ_*.
- ▷ If f is a topological submersion (i.e. it is locally isomorphic to a projection $Y \times \mathbb{R}^n \to Y$), then $f^! \simeq f^{-1} \otimes f^! k_Y$ commutes with ρ^{-1} and $Rf_{!!}$ commutes with $\rho_!$.

Moreover the functors Rf_*, $Rf_{!!}$ and $R\mathcal{H}om(F,\,.)$ with $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ have finite cohomological dimension.

1.2. Modules over a $k_{X_{\rm sa}}$-algebra

A sheaf of $k_{X_{\rm sa}}$-algebras (or a $k_{X_{\rm sa}}$-algebra, for short) is an object $\mathcal{R} \in \mathrm{Mod}(k_{X_{\rm sa}})$ such that $\Gamma(U;\mathcal{R})$ is a k-algebra for each $U \in \mathrm{Op}(X_{\rm sa})$ and the restriction maps are algebra morphisms.

A sheaf of (left) $\mathcal{R}$-modules is a sheaf F such that $\Gamma(U;F)$ has a structure of (left) $\Gamma(U;\mathcal{R})$-module for each $U \in \mathrm{Op}(X_{\rm sa})$.

Let us denote by $\mathrm{Mod}(\mathcal{R})$ the category of sheaves of (left) $\mathcal{R}$-modules.

The category $\mathrm{Mod}(\mathcal{R})$ is a Grothendieck category and the forgetful functor

$$\mathrm{for} : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(k_{X_{\rm sa}})$$

is exact.

The functors

$$\mathcal{H}om_{\mathcal{R}} : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(k_{X_{\mathrm{sa}}}),$$

$$\otimes_{\mathcal{R}} : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(k_{X_{\mathrm{sa}}})$$

are well defined. Remark that in the case of $\mathcal{R}$-modules the functor $\otimes_{\mathcal{R}}$ is only right exact and commutes with $\varinjlim$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a morphism of real analytic manifolds. Let $\mathcal{R}$ be a $k_{Y_{\mathrm{sa}}}$-algebra. The functors f^{-1}, f_* and $Rf_{!!}$ induce functors

$$f^{-1} : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{R}),$$

$$f_* : \mathrm{Mod}(f^{-1}\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}),$$

$$f_{!!} : \mathrm{Mod}(f^{-1}\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}).$$

Now we consider the derived category of sheaves of $\mathcal{R}$-modules. Thanks to flat objects we can find a left derived functor $\otimes^{L}_{\mathcal{R}}$ of the tensor product $\otimes_{\mathcal{R}}$.

Definition 1.2.1. — An object $F \in \mathrm{Mod}(\mathcal{R})$ is *quasi-injective* if its image via the forgetful functor is quasi-injective in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. Let $\mathcal{R}$ be a $k_{Y_{\mathrm{sa}}}$-algebra. One can prove that quasi-injective objects are injective with respect to the functors f_* and $f_{!!}$. The functors Rf_* and $Rf_{!!}$ are well defined and projection formula, base change formula remain valid for $\mathcal{R}$-modules. Moreover we have:

Theorem 1.2.2. — *The functor $Rf_{!!} : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$ admits a right adjoint. We denote the adjoint functor by*

$$f^{!} : D^+(\mathcal{R}) \longrightarrow D^+(f^{-1}\mathcal{R}).$$

CHAPTER 2

CONIC SHEAVES ON SUBANALYTIC SITES

We study here the category of conic sheaves on a subanalytic site. References are made to [**14**] for the classical theory of conic sheaves and to [**31**] for applications of conic subanalytic sheaves to the Laplace transform.

2.1. Conic sheaves on topological spaces

For the statements not proved here we refer to [**14**] and [**28**]. Let k be a field and X be a real analytic manifold endowed with an analytic action μ of $\mathbb{R}^+$. In other words we have an analytic map

$$\mu : X \times \mathbb{R}^+ \longrightarrow X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\mu(x, t_1 t_2) = \mu\big(\mu(x, t_1), t_2\big) \quad \text{and} \quad \mu(x, 1) = x.$$

Note that μ is open. Indeed let $U \in \mathrm{Op}(X)$ and $W \in \mathrm{Op}(\mathbb{R}^+)$. Then

$$\mu(U, W) = \bigcup_{t \in W} \mu(U, t),$$

and $\mu(\,.\,, t) : X \to X$ is a homeomorphism (with inverse $\mu(\,.\,, t^{-1})$). We have a diagram

$$X \xrightarrow{\ j\ } X \times \mathbb{R}^+ \overset{\mu}{\underset{p}{\rightrightarrows}} X,$$

where $j(x) = (x, 1)$ and p denotes the projection. We have $\mu \circ j = p \circ j = \mathrm{id}$.

Definition 2.1.1. — (i) Let S be a subset of X. We set

$$\mathbb{R}^+ S = \mu(S, \mathbb{R}^+).$$

If U belongs to $\mathrm{Op}(X)$, then $\mathbb{R}^+ U \in \mathrm{Op}(X)$ since μ is open.

(ii) Let S be a subset of X. We say that S is *conic* if $S = \mathbb{R}^+ S$. In other words, S is invariant by the action of μ.

(iii) An *orbit* of μ is the set $\mathbb{R}^+x$ with $x \in X$.

Let $S_1, S_2 \subset X$ and suppose that S_2 is conic. Then it is easy to check that

$$\mathbb{R}^+(S_1 \cap S_2) = \mathbb{R}^+S_1 \cap S_2.$$

Definition 2.1.2. — We say that a subset S of X is $\mathbb{R}^+$-*connected* if $S \cap \mathbb{R}^+x$ is connected for each $x \in S$.

Definition 2.1.3. — A sheaf $F \in \mathrm{Mod}(k_X)$ is *conic* if $\mu^{-1}F \simeq p^{-1}F$.

(i) We denote by $\mathrm{Mod}_{\mathbb{R}^+}(k_X)$ the subcategory of $\mathrm{Mod}(k_X)$ consisting of conic sheaves.

(ii) We denote by $D^{\mathrm{b}}_{\mathbb{R}^+}(k_X)$ the subcategory of $D^{\mathrm{b}}(k_X)$ consisting of objects F such that $H^j(F)$ belongs to $\mathrm{Mod}_{\mathbb{R}^+}(k_X)$ for all $j \in \mathbb{Z}$.

Let us assume the following hypothesis:

$$\text{(2.1.1)} \qquad \begin{cases} \text{(i) every point } x \in X \text{ has a fundamental neighborhood} \\ \quad\;\; \text{system consisting of } \mathbb{R}^+\text{-connected open subsets;} \\ \text{(ii) for any } x \in X \text{ the set } \mathbb{R}^+x \text{ is contractible.} \end{cases}$$

In this situation (see [**2**]) either $\mathbb{R}^+x \simeq \mathbb{R}$ or $\mathbb{R}^+x = x$.

Proposition 2.1.4. — *Let $U \in \mathrm{Op}(X)$ be $\mathbb{R}^+$-connected and let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_X)$. Then*

$$\mathrm{R}\Gamma(\mathbb{R}^+U; F) \xrightarrow{\sim} \mathrm{R}\Gamma(U; F).$$

Denote by $X_{\mathbb{R}^+}$ the topological space X endowed with the conic topology, i.e. $U \in \mathrm{Op}(X_{\mathbb{R}^+})$ if it is open for the topology of X and invariant by the action of $\mathbb{R}^+$.

Let us consider the natural map $\eta : X \to X_{\mathbb{R}^+}$. The restriction of η_* induces an exact functor denoted by $\widetilde{\eta}_*$ and we obtain a diagram

$$\text{(2.1.2)} \qquad \begin{tikzcd} \mathrm{Mod}_{\mathbb{R}^+}(k_X) \arrow[r, shift left, "\widetilde{\eta}_*"] \arrow[d] & \mathrm{Mod}(k_{X_{\mathbb{R}^+}}) \arrow[l, shift left, "\eta^{-1}"] \\ \mathrm{Mod}(k_X) \arrow[ur, "\eta_*"'] & \end{tikzcd}$$

Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_X)$. Let φ be the natural map from $\mathrm{R}\Gamma(\mathbb{R}^+U; F)$ to $\mathrm{R}\Gamma(U; \eta^{-1}F)$ defined by

$$\text{(2.1.3)} \qquad \begin{aligned} \mathrm{R}\Gamma(\mathbb{R}^+U; F) &\longrightarrow \mathrm{R}\Gamma(\mathbb{R}^+U; R\eta_*\eta^{-1}F) \\ &\simeq \mathrm{R}\Gamma(\mathbb{R}^+U; \eta^{-1}F) \longrightarrow \mathrm{R}\Gamma(U; \eta^{-1}F). \end{aligned}$$

Proposition 2.1.5. — *Let F be a sheaf over $X_{\mathbb{R}^+}$. Let U be an open set of X and assume that U is $\mathbb{R}^+$-connected. Then the morphism φ defined by* (2.1.3) *is an isomorphism.*

THEOREM 2.1.6. — *The functors $R\eta_*$ and η^{-1} induce equivalences of derived categories*

$$D^{\mathrm{b}}_{\mathbb{R}^+}(k_X) \underset{\eta^{-1}}{\overset{R\eta_*}{\rightleftarrows}} D^{\mathrm{b}}(k_{X_{\mathbb{R}^+}})$$

inverse to each others.

We need to introduce the subcategory of coherent conic sheaves.

DEFINITION 2.1.7. — Let $U \in \mathrm{Op}(X_{\mathbb{R}^+})$.

▹ U is said to be *relatively quasi-compact* if, for any covering $\{U_i\}_{i\in I}$ of $X_{\mathbb{R}^+}$, there exists $J \subset I$ finite such that $U \subset \bigcup_{i\in J} U_i$. We write

$$U \Subset X_{\mathbb{R}^+}.$$

▹ We will denote by $\mathrm{Op}^{\mathrm{c}}(X_{\mathbb{R}^+})$ the subcategory of $\mathrm{Op}(X_{\mathbb{R}^+})$ consisting of relatively quasi-compact open subsets.

One can check easily that if $U \in \mathrm{Op}^{\mathrm{c}}(X)$, then $\mathbb{R}^+U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathbb{R}^+})$.

DEFINITION 2.1.8. — Let $F \in \mathrm{Mod}(k_{X_{\mathbb{R}^+}})$.

(i) F is $X_{\mathrm{sa},\mathbb{R}^+}$*-finite* if there exists an epimorphism $G \twoheadrightarrow F$, with $G \simeq \bigoplus_{i\in I} k_{U_i}$, I finite and $U_i \in \mathrm{Op}^{\mathrm{c}}(X_{\mathbb{R}^+})$ subanalytic.

(ii) F is $X_{\mathrm{sa},\mathbb{R}^+}$*-pseudo-coherent* if for any morphism $\psi : G \to F$, where G is $X_{\mathrm{sa},\mathbb{R}^+}$-finite, $\ker\psi$ is $X_{\mathrm{sa},\mathbb{R}^+}$-finite.

(iii) F is $X_{\mathrm{sa},\mathbb{R}^+}$*-coherent* if it is both $X_{\mathrm{sa},\mathbb{R}^+}$-finite and $X_{\mathrm{sa},\mathbb{R}^+}$-pseudo-coherent.

We will denote by $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ the subcategory of $\mathrm{Mod}(k_{X_{\mathbb{R}^+}})$ consisting of $X_{\mathrm{sa},\mathbb{R}^+}$-coherent objects.

2.2. Conic sheaves on subanalytic sites

DEFINITION 2.2.1. — A sheaf of k-modules F on X_{sa} is *conic* if the restriction morphism $\Gamma(\mathbb{R}^+U; F) \to \Gamma(U; F)$ is an isomorphism for each $\mathbb{R}^+$-connected $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ with $\mathbb{R}^+U \in \mathrm{Op}(X_{\mathrm{sa}})$.

(i) We denote by $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ the subcategory of $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ consisting of conic sheaves.

(ii) We denote by $D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ the subcategory of $D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ consisting of objects F such that $H^j(F)$ belongs to $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ for all $j \in \mathbb{Z}$.

REMARK 2.2.2. — Let X be a real analytic manifold endowed with a subanalytic action μ of $\mathbb{R}^+$ and consider the diagram

$$X \times \mathbb{R}^+ \underset{p}{\overset{\mu}{\rightrightarrows}} X,$$

where p denotes the projection.

As in classical sheaf theory one can define the subcategory $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ of $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ consisting of sheaves satisfying

$$\mu^{-1}F \simeq p^{-1}F.$$

The categories $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ and $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ are not equivalent in general.

Indeed, let $X = \mathbb{R}$, set $X^+ = \{x \in \mathbb{R};\, x > 0\}$ and let μ be the natural action of $\mathbb{R}^+$ (i.e. $\mu(x,t) = tx$). Let us consider the sheaf $\rho_! k_{X^+} \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Then

$$\mu^{-1}\rho_! k_{X^+} \simeq \rho_! \mu^{-1} k_{X^+} \simeq \rho_! p^{-1} k_{X^+} \simeq p^{-1}\rho_! k_{X^+}.$$

Let

$$V = \big\{x \in \mathbb{R};\, 1 < x < 2\big\} \quad \text{and} \quad W_m = \big\{x \in \mathbb{R};\, \tfrac{1}{m} < x < m\big\},$$

where $m \in \mathbb{N} \setminus \{0\}$. Recall that $\rho_! k_{X^+} \simeq \varinjlim_{U \Subset X^+} \rho_* k_U \simeq \varinjlim_m \rho_* k_{W_m}$. We have

$$\Gamma(V; \rho_! k_{X^+}) \simeq \varinjlim_m \Gamma(V; k_{W_m}) \simeq k,$$

since $V \subset W_m$ for $m \geq 2$. On the other hand, let

$$V_n^+ = \big\{x \in \mathbb{R};\, 0 < x < n\big\},$$

where $n \in \mathbb{N}$. Since $\mathbb{R}^+ V = X^+$ we have

$$\Gamma(X^+; \rho_! k_{X^+}) \simeq \varprojlim_n \Gamma(V_n^+; \rho_! k_{X^+}) \simeq \varprojlim_n \varinjlim_m \Gamma(V_n^+; k_{W_m}),$$

(in the second isomorphism we used the fact that $V_n^+ \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ for each n) and $\Gamma(V_n^+; k_{W_m}) = 0$ for each $n, m \in \mathbb{N}$. Hence $\Gamma(V; \rho_! k_{X^+}) \not\simeq \Gamma(\mathbb{R}^+ V; \rho_! k_{X^+})$.

Definition 2.2.3. — We denote by :

▷ $\mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$ the full subcategory of $\mathrm{Op}(X_{\mathrm{sa}})$ consisting of conic subanalytic subsets, i.e. $U \in \mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$ if $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and it is invariant by the action of $\mathbb{R}^+$;

▷ $X_{\mathrm{sa},\mathbb{R}^+}$ the category $\mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$ endowed with the topology induced by X_{sa};

▷ $\rho_{\mathbb{R}^+} : X_{\mathbb{R}^+} \to X_{\mathrm{sa},\mathbb{R}^+}$ the natural morphism of sites.

Replacing $\mathcal{T}$ with $\mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$ in [**9**] we get the following results:

Theorem 2.2.4. — (i) *Let* $G \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ *and let* $\{F_i\}$ *be a filtrant inductive system in* $\mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. *Then we have an isomorphism*

$$\varinjlim_i \mathrm{Hom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(\rho_{\mathbb{R}^+ *} G, F_i) \xrightarrow{\sim} \mathrm{Hom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(\rho_{\mathbb{R}^+ *} G, \varinjlim_i F_i).$$

Moreover the functor of direct image $\rho_{\mathbb{R}^+ *}$ *associated with the morphism* $\rho_{\mathbb{R}^+}$ *in* (2.2.1) *is fully faithful and exact on* $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$.

(ii) *Let* $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. *There exists a small filtrant inductive system* $\{F_i\}_{i \in I}$ *in* $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ *such that* $F \simeq \varinjlim_i \rho_{\mathbb{R}^+ *} F_i$.

NOTATIONS 2.2.5. — Since $\rho_{\mathbb{R}^+ *}$ is fully faithful and exact on $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$, we can identify $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ with its image in $\mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. When there is no risk of confusion we will write F instead of $\rho_{\mathbb{R}^+ *}F$, for $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$.

We can also find a left adjoint to the functor $\rho_{\mathbb{R}^+}^{-1}$.

PROPOSITION 2.2.6. — *The functor $\rho_{\mathbb{R}^+}^{-1}$ admits a left adjoint, denoted by $\rho_{\mathbb{R}^+ !}$. It satisfies:*

(i) *the functor $\rho_{\mathbb{R}^+ !}$ is exact and commutes with $\otimes$;*

(ii) *for $F \in \mathrm{Mod}(k_{X_{\mathbb{R}^+}})$ and $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$, $\rho_{\mathbb{R}^+ !}F$ is the sheaf associated with the presheaf $U \mapsto \varinjlim_{V \Supset U} \Gamma(V;F)$;*

(iii) *for $U \in \mathrm{Op}(X_{\mathbb{R}^+})$ one has*

$$\rho_{\mathbb{R}^+ !}k_U \simeq \varinjlim_{\substack{V \Subset U \\ V \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})}} k_V.$$

DEFINITION 2.2.7. — An object $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ is *quasi-injective* if the functor $\mathrm{Hom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(\,\cdot\,,F)$ is exact in $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ or, equivalently (see Theorem 8.7.2 of [**18**]) if for each $U, V \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$ with $V \subset U$ the restriction morphism $\Gamma(U;F) \to \Gamma(V;F)$ is surjective.

The category of quasi-injective objects is cogenerating since it contains injective objects. Moreover it is stable by filtrant $\varinjlim$ and $\prod$. We have the following result

THEOREM 2.2.8. — *The family of quasi-injective sheaves is injective with respect to the functor $\mathrm{Hom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(G,\,.)$ for each $G \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$.*

In particular:

PROPOSITION 2.2.9. — *The family of quasi-injective sheaves is injective with respect to the functor $\Gamma(U;\,.\,)$ for any $U \in \mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$.*

Let $\eta : X \to X_{\mathbb{R}^+}$ and $\eta_{\mathrm{sa}} : X_{\mathrm{sa}} \to X_{\mathrm{sa},\mathbb{R}^+}$ be the natural morphisms of sites. We have a commutative diagram of sites

(2.2.1)
$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\eta"] & X_{\mathrm{sa}} \arrow[d, "\eta_{\mathrm{sa}}"] \\ X_{\mathbb{R}^+} \arrow[r, "\rho_{\mathbb{R}^+}"] & X_{\mathrm{sa},\mathbb{R}^+}. \end{tikzcd}$$

LEMMA 2.2.10. — *Let $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. Then $\eta_{\mathrm{sa}}^{-1}\rho_{\mathbb{R}^+ *}F \simeq \rho_*\eta^{-1}F$.*

Proof. — Since all these functors are exact on $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$, we may reduce to the case $F = k_U$ with $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$. Then we have

$$\eta_{\mathrm{sa}}^{-1}\rho_{\mathbb{R}^+*}k_U \simeq \eta_{\mathrm{sa}}^{-1}k_U \simeq k_U,$$

on the other hand we have $\rho_*\eta^{-1}k_U \simeq \rho_*k_U \simeq k_U$ and the result follows. □

Remark 2.2.11. — Remark that $\rho_! \circ \eta^{-1} \not\simeq \eta_{\mathrm{sa}}^{-1} \circ \rho_{\mathbb{R}^+!}$. In fact with the notations of Remark 2.2.2 we have

$$\rho_!\eta^{-1}k_{X^+} \simeq \varinjlim_m \rho_* k_{W_m}.$$

On the other hand, since $X^+ \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$ we have $\eta_{\mathrm{sa}}^{-1}\rho_{\mathbb{R}^+!}k_{X^+} \simeq k_{X^+}$.

2.3. An equivalence of categories

Let X be a real analytic manifold endowed with an action μ of $\mathbb{R}^+$. In the following we shall assume the hypothesis below:

$$(2.3.1)\qquad \begin{cases} \text{(i)} & \text{every } U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}}) \text{ has a finite covering consisting} \\ & \text{of } \mathbb{R}^+\text{-connected subanalytic open subsets;} \\ \text{(ii)} & \text{for any } U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}}) \text{ we have } \mathbb{R}^+U \in \mathrm{Op}(X_{\mathrm{sa}}); \\ \text{(iii)} & \text{for any } x \in X \text{ the set } \mathbb{R}^+x \text{ is contractible;} \\ \text{(iv)} & \text{there exists a covering } \{V_n\}_{n\in\mathbb{N}} \text{ of } X_{\mathrm{sa}} \text{ such that} \\ & V_n \text{ is } \mathbb{R}^+\text{-connected and } V_n \Subset V_{n+1} \text{ for each } n. \end{cases}$$

Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ such that $\mathbb{R}^+U$ is still subanalytic. Let φ be the natural map from $\Gamma(\mathbb{R}^+U;F)$ to $\Gamma(U;\eta_{\mathrm{sa}}^{-1}F)$ defined by

$$(2.3.2)\qquad \Gamma(\mathbb{R}^+U;F) \longrightarrow \Gamma(\mathbb{R}^+U;\eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1}F) \simeq \Gamma(\mathbb{R}^+U;\eta_{\mathrm{sa}}^{-1}F) \longrightarrow \Gamma(U;\eta_{\mathrm{sa}}^{-1}F).$$

Proposition 2.3.1. — *Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$, assume that U is $\mathbb{R}^+$-connected and $\mathbb{R}^+U \in \mathrm{Op}(X_{\mathrm{sa}})$. Then the morphism φ defined by (2.3.2) is an isomorphism.*

Proof. — (i) Assume that $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ is $\mathbb{R}^+$-connected. Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$, then $F = \varinjlim_i \rho_{\mathbb{R}^+*}F_i$, with $F_i \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. We have the chain of isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{k_{X_{\mathrm{sa}}}}(k_U, \eta_{\mathrm{sa}}^{-1}\varinjlim_i \rho_{\mathbb{R}^+*}F_i) \\ &\qquad \simeq \mathrm{Hom}_{k_{X_{\mathrm{sa}}}}(k_U, \varinjlim_i \rho_*\eta^{-1}F_i) \simeq \varinjlim_i \mathrm{Hom}_{k_X}(k_U, \eta^{-1}F_i) \\ &\qquad\quad \simeq \varinjlim_i \mathrm{Hom}_{k_{X_{\mathbb{R}^+}}}(k_{\mathbb{R}^+U}, F_i) \simeq \mathrm{Hom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(k_{\mathbb{R}^+U}, \varinjlim_i \rho_{\mathbb{R}^+*}F_i), \end{aligned}$$

where the first isomorphism follows since $\eta_{\mathrm{sa}}^{-1} \circ \rho_{\mathbb{R}^+ *} \simeq \rho_* \circ \eta^{-1}$ by Lemma 2.2.10 and the third one follows from the equivalence between conic sheaves on X and sheaves on $X_{\mathbb{R}^+}$. In the fourth isomorphism we used the fact that $\mathbb{R}^+ U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$.

(ii) Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. Let $\{V_n\}_{n\in\mathbb{N}} \in \mathrm{Cov}(X_{\mathrm{sa}})$ be a covering of X as in (2.3.1) (iv) and set $U_n = U \cap V_n$. We have:

$$\Gamma(U; \eta_{\mathrm{sa}}^{-1} F) \simeq \varprojlim_n \Gamma(U_n; \eta_{\mathrm{sa}}^{-1} F) \simeq \varprojlim_n \Gamma(\mathbb{R}^+ U_n; F) \simeq \Gamma(\mathbb{R}^+ U; F). \tag{2.3.3}$$ □

Corollary 2.3.2. — *Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ and let $U \in \mathrm{Op}(X_{\mathrm{sa}})$. Assume that U is $\mathbb{R}^+$-connected and $\mathbb{R}^+ U \in \mathrm{Op}(X_{\mathrm{sa}})$. There is an isomorphism*

$$\eta_{\mathrm{sa}*} \Gamma_U \eta_{\mathrm{sa}}^{-1} F \simeq \Gamma_{\mathbb{R}^+ U} F.$$

Proof. — Let $V \in \mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$. Then $\mathbb{R}^+(V \cap U) = V \cap \mathbb{R}^+ U$. We have the chain of isomorphisms

$$\begin{aligned}\Gamma(V; \eta_{\mathrm{sa}*} \Gamma_U \eta_{\mathrm{sa}}^{-1} F) &\simeq \Gamma(V \cap U; \eta_{\mathrm{sa}}^{-1} F) \simeq \Gamma(\mathbb{R}^+(V \cap U); F) \\ &\simeq \Gamma(V \cap \mathbb{R}^+ U; F) \simeq \Gamma(V; \Gamma_{\mathbb{R}^+ U} F),\end{aligned}$$

where the second isomorphism follows from Proposition 2.3.1. □

We can extend Lemma 2.2.10 to $\mathrm{Mod}(k_{X_{\mathbb{R}^+}})$.

Lemma 2.3.3. — *Let $F \in \mathrm{Mod}(k_{X_{\mathbb{R}^+}})$. Then $\eta_{\mathrm{sa}}^{-1} \rho_{\mathbb{R}^+ *} F \simeq \rho_* \eta^{-1} F$.*

Proof. — Let $F \in \mathrm{Mod}(k_{X_{\mathbb{R}^+}})$ and let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. Then

$$\Gamma(U; \rho_* \eta^{-1} F) \simeq \Gamma(U; \eta^{-1} F) \simeq \Gamma(\mathbb{R}^+ U; F),$$

where the second isomorphism follows from Proposition 2.1.5. On the other hand

$$\Gamma(U; \eta_{\mathrm{sa}}^{-1} \rho_{\mathbb{R}^+ *} F) \simeq \Gamma(\mathbb{R}^+ U; \rho_{\mathbb{R}^+ *} F) \simeq \Gamma(\mathbb{R}^+ U; F),$$

where the first isomorphism follows from Proposition 2.3.1. Hence by (2.3.1) (i) $\eta_{\mathrm{sa}}^{-1} \rho_{\mathbb{R}^+ *} F \simeq \rho_* \eta^{-1} F$. □

Let us consider the category $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ of conic sheaves on X_{sa}. The restriction of $\eta_{\mathrm{sa}*}$ induces a functor denoted by $\widetilde{\eta}_{\mathrm{sa}*}$ and we obtain a diagram

$$\begin{tikzcd} \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}}) \arrow[r, shift left, "\widetilde{\eta}_{\mathrm{sa}*}"] \arrow[d] & \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}}) \arrow[l, shift left, "\eta_{\mathrm{sa}}^{-1}"] \\ \mathrm{Mod}(k_{X_{\mathrm{sa}}}) \arrow[ur, "\eta_{\mathrm{sa}*}"'] & \end{tikzcd} \tag{2.3.4}$$

Theorem 2.3.4. — *The functors $\widetilde{\eta}_{\mathrm{sa}*}$ and η_{sa}^{-1} in (2.3.4) are equivalences of categories inverse to each others.*

Proof. — (i) Let $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$, and let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. We have

$$\Gamma(U;F) \simeq \Gamma(\mathbb{R}^+U;F) \simeq \Gamma(\mathbb{R}^+U;\widetilde{\eta}_{\mathrm{sa}*}F) \simeq \Gamma(U;\eta_{\mathrm{sa}}^{-1}\widetilde{\eta}_{\mathrm{sa}*}F).$$

The third isomorphism follows from Prop. 2.3.1. Then (2.3.1) (i) implies $\eta_{\mathrm{sa}}^{-1}\widetilde{\eta}_{\mathrm{sa}*} \simeq \mathrm{id}$.

(ii) For any $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa},\mathbb{R}^+})$ we have:

$$\Gamma(U;\eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1}F) \simeq \Gamma(U;\eta_{\mathrm{sa}}^{-1}F) \simeq \Gamma(U;F)$$

where the second isomorphisms follows from Prop. 2.3.1. This implies $\eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1} \simeq \mathrm{id}$. □

Notations 2.3.5. — Since η_{sa}^{-1} is fully faithful and exact we will often identify $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ with its image in $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. Hence, for $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ we shall often write F instead of $\eta_{\mathrm{sa}}^{-1}F$.

Thanks to Theorem 2.2.4 we can give another description of the category of conic sheaves.

Theorem 2.3.6. — *Let $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. Then there exists a small filtrant system $\{F_i\}$ in $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ such that $F \simeq \varinjlim_i \rho_*\eta^{-1}F_i$.*

This implies that each $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ can be seen as the inductive limit (in $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$) of a small filtrant system $\{F_i\}$ with $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_X)$ such that $F_i \simeq (F_i)_{U_i}$ for some $U_i \Subset X_{\mathbb{R}^+}$.

Remark 2.3.7. — Let $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. The functor of inverse image commutes with $\varinjlim$ and

$$\mu^{-1}\rho_*\eta^{-1}F \simeq \rho_*\mu^{-1}\eta^{-1}F \simeq \rho_*p^{-1}\eta^{-1}F \simeq p^{-1}\rho_*\eta^{-1}F.$$

Hence $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ implies $F \in \mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$, where $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ is the category introduced in Remark 2.2.2. Let

$$G = \varinjlim_i \rho_*G_i \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$$

with $G_i \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. Since $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ is stable by filtrant $\varinjlim$ we have that G belongs to $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$. Hence $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ is a full subcategory of $\mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ but $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}}) \not\simeq \mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}})$ in general. We have the chain of fully faithful functors

$$\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+}) \hookrightarrow \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}}) \hookrightarrow \mathrm{Mod}^{\mu}(k_{X_{\mathrm{sa}}}).$$

2.4. Derived category

Assume (2.3.1). Injective and quasi-injective objects of $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ are not contained in $\mathrm{Mod}_{\mathbb{R}^+}(k_{\mathrm{sa}})$. For this reason we are going to introduce a subcategory which is useful when we try to find acyclic resolutions.

LEMMA 2.4.1. — *Assume that X satisfies* (2.3.1)*. Then the following property is satisfied:*

$$(2.4.1)\qquad \begin{cases} \textit{Each finite covering of an } \mathbb{R}^+\textit{-connected } U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}}) \\ \textit{has a finite refinement } \{V_i\}_{i=1}^n \textit{ such that each ordered} \\ \textit{union } \bigcup_{i=1}^j V_i \textit{ is } \mathbb{R}^+\textit{-connected for each } j \in \{1,\dots,n\}. \end{cases}$$

Proof. — Let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. Then each finite covering of U admits a finite refinement consisting of $\mathbb{R}^+$-connected open subanalytic subsets. Let $\{U_\alpha\}_{\alpha=1}^n$ be a finite covering of U, $U_\alpha \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ $\mathbb{R}^+$-connected for each α. We will construct a refinement satisfying (2.4.1).

For $k = 1,\dots,n$ and $i = 2,\dots,n$ set $\sigma_k(i) = i-1$ if $i \le k$, $\sigma_k(i) = i$ if $i > k$ and

$$V_{k11} := U_k \quad \text{and} \quad V_{k1i} := U_{\sigma_k(i)} \cap \mathbb{R}^+\big(U_k \cap U_{\sigma_k(i)}\big) \text{ for } i \ge 2.$$

For $j = 2,\dots,n$ define recursively

$$V_{kj1} = \bigcup_{\ell=1}^{j-1}\bigcup_{i=1}^{n} V_{k\ell i} \quad \text{and} \quad V_{kji} = U_{\sigma_k(i)} \cap \mathbb{R}^+\big(V_{kj1} \cap U_{\sigma_k(i)}\big) \text{ for } i \ge 2.$$

Remark that $\bigcup_{p=1}^{j}\bigcup_{\ell=1}^{n}\bigcup_{i=1}^{n} V_{p\ell i} = \bigcup_{p=1}^{j} \mathbb{R}^+U_p \cap U$. By Lemma 2.4.2 below all the sets V_{kji} are $\mathbb{R}^+$-connected and $\{V_{kji}\}_{k,j,i}$ is a refinement of $\{U_\alpha\}_\alpha$ satisfying (2.4.1) (with the lexicographic order). □

LEMMA 2.4.2. — *Assume that X satisfies* (2.3.1) (iii)*. Let U, V, W be open and $\mathbb{R}^+$-connected. Then $U \cup (V \cap \mathbb{R}^+(U \cap V)) \cup (W \cap \mathbb{R}^+(U \cap W))$ is $\mathbb{R}^+$-connected.*

Proof. — In what follows, when we write $\mathbb{R}^+x$ we suppose that $\mathbb{R}^+x \simeq \mathbb{R}$. If $\mathbb{R}^+x = x$ everything becomes obvious.

(i) First remark that $U \cap V$ (resp. $U \cap W$, $V \cap W$) is $\mathbb{R}^+$-connected. Indeed, let $x_1, x_2 \in U \cap V \cap \mathbb{R}^+x$ for some $x \in X$. Then $x_1 = \mu(x,a)$, $x_2 = \mu(x,b)$. Every path in $\mathbb{R}^+x$ connecting x_1 and x_2 contains $\mu(x,[a,b])$. Since U and V are $\mathbb{R}^+$-connected then $U \cap V \supset \mu(x,[a,b])$. Remark that here and below in the notation $[a,b]$ we do not necessarily have $a \le b$.

(ii) Now let us prove that $U \cup (V \cap \mathbb{R}^+(U \cap V))$ is $\mathbb{R}^+$-connected. Let x_1, x_2 belong to $U \cup (V \cap \mathbb{R}^+(U \cap V)) \cap \mathbb{R}^+x$ for some $x \in X$. Then $x_1 = \mu(x,a)$, $x_2 = \mu(x,b)$. We want to prove that

$$\mu\big(x,[a,b]\big) \subset U \cup \big(V \cap \mathbb{R}^+(U \cap V)\big).$$

If $x_1, x_2 \in U$ it follows since U is $\mathbb{R}^+$-connected and if $x_1, x_2 \in V \cap \mathbb{R}^+(U \cap V)$ it follows from (i). So we may assume that $x_1 \in U$ and $x_2 \in V \cap \mathbb{R}^+(U \cap V)$. Since U is $\mathbb{R}^+$-connected and $x_2 \in \mathbb{R}^+ x_1$, there exists $y = \mu(x, c) \in U \cap V$. Then $\mu(x, [a, c]) \subset U$. In the same way $\mu(x, [b, c]) \subset V \cap \mathbb{R}^+(U \cap V)$ and hence

$$\mu\big(x, [a, c] \cup [b, c]\big) \subset U \cup \big(V \cap \mathbb{R}^+(U \cap V)\big).$$

(iii) Let us show that $U \cup (V \cap \mathbb{R}^+(U \cap V)) \cup (W \cap \mathbb{R}^+(U \cap W))$ is $\mathbb{R}^+$-connected. Let $x_1, x_2 \in U \cup (V \cap \mathbb{R}^+(U \cap V)) \cup (W \cap \mathbb{R}^+(U \cap W)) \cap \mathbb{R}^+ x$ for some $x \in X$. Then $x_1 = \mu(x, a)$, $x_2 = \mu(x, b)$. We want to prove that

$$\mu\big(x, [a, b]\big) \subset U \cup \big(V \cap \mathbb{R}^+(U \cap V)\big) \cup \big(W \cap \mathbb{R}^+(U \cap W)\big).$$

By (i) and (ii) we may reduce to the case $x_1 \in V$, $x_2 \in W$. As in (ii), there exist $y_1 = \mu(x, c) \in U \cap V$ and $y_2 = \mu(x, d) \in U \cap W$. Then $\mu(x, [c, d]) \in U$, $\mu(x, [a, c]) \subset V \cap \mathbb{R}^+(U \cap V)$ and $\mu(x, [b, d]) \subset W \cap \mathbb{R}^+(U \cap W)$. Hence $\mu(x, [c, d] \cup [a, c] \cup [b, d])$ is in $U \cup (V \cap \mathbb{R}^+(U \cap V)) \cup (W \cap \mathbb{R}^+(U \cap W))$ and the result follows. $\square$

Definition 2.4.3. — A sheaf $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ is $\mathbb{R}^+$-*quasi-injective* if for each $\mathbb{R}^+$-connected $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ the restriction morphism $\Gamma(X; F) \to \Gamma(U; F)$ is surjective.

Remark that the functor η_{sa}^{-1} sends quasi-injective objects of $\mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ to $\mathbb{R}^+$-quasi-injective objects since $\Gamma(U; \eta_{\mathrm{sa}}^{-1} F) \simeq \Gamma(\mathbb{R}^+ U; F)$ if $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ is $\mathbb{R}^+$-connected. Moreover the category of $\mathbb{R}^+$-quasi-injective objects is cogenerating since injective objects are cogenerating in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$.

Proposition 2.4.4. — *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence in* $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ *and assume that* F' *is* $\mathbb{R}^+$*-quasi-injective. Let* $U \in \mathrm{Op}(X_{\mathrm{sa}})$ *be* $\mathbb{R}^+$*-connected. Then the following sequence is exact:*

$$0 \to \Gamma(U; F') \longrightarrow \Gamma(U; F) \longrightarrow \Gamma(U; F'') \to 0.$$

Proof. — (i) Let us consider a $\mathbb{R}^+$-connected $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$. Let $s'' \in \Gamma(U; F'')$, and let $\{V_i\}_{i=1}^n$ be a finite covering of U satisfying (2.4.1) and such that there exists $s_i \in \Gamma(V_i; F)$ whose image is $s''|_{V_i}$. For $n \geq 2$ on $V_1 \cap V_2$ $s_1 - s_2$ defines a section of $\Gamma(V_1 \cap V_2; F')$ which extends to $s' \in \Gamma(X; F')$. Replace s_1 with $s_1 - s'$. We may suppose that $s_1 = s_2$ on $V_1 \cap V_2$. Then there exists $t \in \Gamma(V_1 \cup V_2; F)$ such that $t|_{V_i} = s_i$, for $i = 1, 2$. Thus the induction proceeds.

(ii) Let us consider a $\mathbb{R}^+$-connected $U \in \mathrm{Op}(X_{\mathrm{sa}})$. By (2.3.1) (iv) there exists a covering $\{V_n\}_{n \in \mathbb{N}}$ of X_{sa} such that V_n is $\mathbb{R}^+$-connected and $V_n \Subset V_{n+1}$ for each n. It follows from (i) that for each n the sequence

$$0 \to \Gamma(U \cap V_n; F') \longrightarrow \Gamma(U \cap V_n; F) \longrightarrow \Gamma(U \cap V_n; F'') \to 0$$

is exact. Moreover the morphism $\Gamma(U \cap V_{n+1}; F') \to \Gamma(U \cap V_n; F')$ is surjective for each n since F' is $\mathbb{R}^+$-quasi-injective. Then by the Mittag-Leffler property (see

Proposition 1.12.3 of [**14**]) the sequence

$$0 \to \varprojlim_n \Gamma(U \cap V_n; F') \longrightarrow \varprojlim_n \Gamma(U \cap V_n; F) \longrightarrow \varprojlim_n \Gamma(U \cap V_n; F'') \to 0$$

is exact. Since $\varprojlim_n \Gamma(U \cap V_n; G) \simeq \Gamma(U; G)$ for each $G \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ the result follows. □

Proposition 2.4.5. — *Let F', F be $\mathbb{R}^+$-quasi-injective and consider the exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Then F'' is $\mathbb{R}^+$-quasi-injective.*

Proof. — Let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected and let us consider the diagram

$$\begin{tikzcd} \Gamma(X; F) \arrow[r] \arrow[d, "\alpha"] & \Gamma(X; F'') \arrow[d, "\gamma"] \\ \Gamma(U; F) \arrow[r, "\beta"] & \Gamma(U; F''). \end{tikzcd}$$

The morphism α is surjective since F is $\mathbb{R}^+$-quasi-injective and β is surjective by Proposition 2.4.4. Then γ is surjective. □

It follows from the preceding results that

Proposition 2.4.6. — *$\mathbb{R}^+$-quasi-injective objects are injective with respect to the functor $\Gamma(U; .)$, with $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $\mathbb{R}^+$-connected.*

Corollary 2.4.7. — *$\mathbb{R}^+$-quasi-injective objects are injective with respect to the functor Γ_U, with $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $\mathbb{R}^+$-connected.*

Proof. — Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ and assume that F' is $\mathbb{R}^+$-quasi-injective. By Proposition 2.4.6 the sequence

$$0 \to \Gamma(U \cap V; F') \longrightarrow \Gamma(U \cap V; F) \longrightarrow \Gamma(U \cap V; F'') \to 0$$

is exact for any $V \in \mathrm{Op}(X_{\mathrm{sa}})$ and $\mathbb{R}^+$-connected. This implies that the sequence

$$0 \to \Gamma_U F' \longrightarrow \Gamma_U F \longrightarrow \Gamma_U F'' \to 0$$

is exact. □

Corollary 2.4.8. — *$\mathbb{R}^+$-quasi-injective objects are $\eta_{\mathrm{sa}*}$-injective.*

Proof. — Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ and assume that F' is $\mathbb{R}^+$-quasi-injective. By Proposition 2.4.6 the sequence

$$0 \to \Gamma(U; F') \longrightarrow \Gamma(U; F) \longrightarrow \Gamma(U; F'') \to 0$$

is exact for any $U \in \mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$. This implies that the following sequence is exact:

$$0 \to \eta_{\mathrm{sa}*} F' \longrightarrow \eta_{\mathrm{sa}*} F \longrightarrow \eta_{\mathrm{sa}*} F'' \to 0.$$ □

Theorem 2.4.9. — *The categories $D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ and $D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ are equivalent.*

Proof. — In order to prove this statement, it is enough to show that η_{sa}^{-1} is fully faithful. Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ and let $I^\bullet$ be an injective complex quasi-isomorphic to F. Since η_{sa}^{-1} sends injective objects to $\mathbb{R}^+$-quasi-injective objects which are $\eta_{\mathrm{sa}*}$-injective we have

$$R\eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1}F \simeq \eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1}I^\bullet \simeq I^\bullet \simeq F.$$

This implies $R\eta_{\mathrm{sa}*}\eta_{\mathrm{sa}}^{-1} \simeq \mathrm{id}$, hence η_{sa}^{-1} is fully faithful. □

Hence for each $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ we have $F \simeq \eta_{\mathrm{sa}}^{-1}F'$ with $F' \in D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. Remark that Theorem 2.4.9 also implies that conic sheaves are $\eta_{\mathrm{sa}*}$-acyclic.

PROPOSITION 2.4.10. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$. Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected and such that $\mathbb{R}^+U \in \mathrm{Op}(X_{\mathrm{sa}})$. There is an isomorphism*

$$\mathrm{R}\Gamma(\mathbb{R}^+U; F) \xrightarrow{\sim} \mathrm{R}\Gamma(U; \eta_{\mathrm{sa}}^{-1}F).$$

Proof. — Let $I^\bullet$ be a complex of injective objects quasi-isomorphic to F. Since η_{sa}^{-1} sends injective objects to $\mathbb{R}^+$-quasi-injective objects we have

$$\mathrm{R}\Gamma(\mathbb{R}^+U; F) \simeq \Gamma(\mathbb{R}^+U; I^\bullet) \xrightarrow{\sim} \Gamma(U; \eta_{\mathrm{sa}}^{-1}I^\bullet) \simeq \mathrm{R}\Gamma(U; \eta_{\mathrm{sa}}^{-1}F),$$

where the second isomorphism follows from Proposition 2.3.1. □

COROLLARY 2.4.11. — *Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ and let $U \in \mathrm{Op}(X_{\mathrm{sa}})$. Assume that U is $\mathbb{R}^+$-connected and $\mathbb{R}^+U \in \mathrm{Op}(X_{\mathrm{sa}})$. There is an isomorphism*

$$R\eta_{\mathrm{sa}*}\mathrm{R}\Gamma_U\eta_{\mathrm{sa}}^{-1}F \simeq \mathrm{R}\Gamma_{\mathbb{R}^+U}F.$$

Proof. — Let $V \in \mathrm{Op}(X_{\mathrm{sa},\mathbb{R}^+})$. As in Corollary 2.3.2,

$$\begin{aligned}\mathrm{R}\Gamma(V; R\eta_{\mathrm{sa}*}\mathrm{R}\Gamma_U\eta_{\mathrm{sa}}^{-1}F) &\simeq \mathrm{R}\Gamma(V\cap U; \eta_{\mathrm{sa}}^{-1}F) \simeq \mathrm{R}\Gamma(\mathbb{R}^+(V\cap U); F)\\ &\simeq \mathrm{R}\Gamma(V\cap \mathbb{R}^+U; F) \simeq \mathrm{R}\Gamma(V; \mathrm{R}\Gamma_{\mathbb{R}^+U}F),\end{aligned}$$

where the second isomorphism follows from Proposition 2.4.10. □

We extend Lemma 2.3.3 to $D^{\mathrm{b}}(k_{X_{\mathbb{R}^+}})$.

LEMMA 2.4.12. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathbb{R}^+}})$. Then*

$$\eta_{\mathrm{sa}}^{-1}R\rho_{\mathbb{R}^+*}F \simeq R\rho_*\eta^{-1}F.$$

Proof. — (i) Let $F \in \mathrm{Mod}(k_{X_{\mathbb{R}^+}})$ be injective. Then for each $\mathbb{R}^+$-connected U in $\mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$, one sees that $\mathrm{R}\Gamma(U; R\rho_*\eta^{-1}F) \simeq \mathrm{R}\Gamma(U; \eta^{-1}F) \simeq \mathrm{R}\Gamma(\mathbb{R}^+U; F)$ is concentrated in degree zero. Hence $\eta^{-1}F$ is $R\rho_*$-acyclic by (2.3.1) (i).

(ii) Let $F \in D^{\mathrm{b}}(k_{X_{\mathbb{R}^+}})$ and let $I^\bullet$ be a complex of injective objects quasi-isomorphic to F. Then $\eta_{\mathrm{sa}}^{-1}R\rho_{\mathbb{R}^+*}F \simeq \eta_{\mathrm{sa}}^{-1}\rho_{\mathbb{R}^+*}I^\bullet \simeq \rho_*\eta^{-1}I^\bullet \simeq R\rho_*\eta^{-1}F$, where the second isomorphism follows from Lemma 2.3.3 and the third one follows from (i). □

2.5. Operations

Let X be a real analytic manifold endowed with an analytic action of $\mathbb{R}^+$. We study the operations in the category of conic sheaves on X_{sa}.

PROPOSITION 2.5.1. — *The category* $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ *is stable under* $\varinjlim$ *and* $\varprojlim$.

Proof. — This is a consequence of the equivalence with $\mathrm{Mod}(k_{\mathrm{sa},\mathbb{R}^+})$. □

PROPOSITION 2.5.2. — *Let* $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_X)$ *and* $G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. *Then we have:*

(i) $\rho_* F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$;

(ii) $\rho^{-1} G \in \mathrm{Mod}_{\mathbb{R}^+}(k_X)$.

Proof. — (i) Let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. We have the chain of isomorphisms

$$\Gamma(U;\rho_*F) \simeq \Gamma(U;F) \simeq \Gamma(\mathbb{R}^+U;F) \simeq \Gamma(\mathbb{R}^+U;\rho_*F).$$

(ii) We have $G = \varinjlim\limits_j \rho_* G_j$, with $G_j \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. Then

$$\rho^{-1} \varinjlim_j \rho_* G_j \simeq \varinjlim_j \rho^{-1}\rho_* G_j \simeq \varinjlim_j G_j$$

and $\varinjlim\limits_j G_j$ belongs to $\mathrm{Mod}_{\mathbb{R}^+}(k_X)$. □

PROPOSITION 2.5.3. — *Let* $F, G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. *Then we have:*

(i) $F \otimes G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$,

(ii) $\mathcal{H}om(F,G) \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$.

Proof. — We have $F = \varinjlim\limits_i \rho_* F_i$ and $G = \varinjlim\limits_j \rho_* G_j$, with $F_i, G_j \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$, and

(i) $F \otimes G \simeq \varinjlim\limits_{i,j} \rho_*(F_i \otimes G_j)$ and $F_i \otimes G_j$ belongs to $\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ for each i, j;

(ii) $\mathcal{H}om(F,G) \simeq \varprojlim\limits_i \varinjlim\limits_j \rho_* \mathcal{H}om(F_i, G_j)$ and $\mathcal{H}om(F_i, G_j)$ is conic for each i, j. □

Let $f : X \to Y$ be a conic morphism of real analytic manifolds. We have a commutative diagram

$$\begin{tikzcd} X_{\mathrm{sa}} \arrow[r, "f"] \arrow[d, "\eta_{\mathrm{sa}}"] & Y_{\mathrm{sa}} \arrow[d, "\eta_{\mathrm{sa}}"] \\ X_{\mathrm{sa},\mathbb{R}^+} \arrow[r, "f"] & Y_{\mathrm{sa},\mathbb{R}^+}. \end{tikzcd} \tag{2.5.1}$$

PROPOSITION 2.5.4. — *Let* $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ *and* $G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$. *We have:*

(i) $f_* F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$;

(ii) $f^{-1} G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$.

Proof. — (i) Let $U \in \mathrm{Op}^{\mathrm{c}}(Y_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. Since f commutes with the action of $\mathbb{R}^+$, the set $f^{-1}(U)$ is $\mathbb{R}^+$ connected. We have the chain of isomorphisms

$$\Gamma\big(f^{-1}(U); F\big) \simeq \Gamma\big(\mathbb{R}^+ f^{-1}(U); F\big) \simeq \Gamma\big(f^{-1}(\mathbb{R}^+ U); F\big).$$

Hence $\Gamma(U; f_* F) \simeq \Gamma(\mathbb{R}^+ U; f_* F)$.

(ii) We have $G = \varinjlim\limits_{j} \rho_* G_j$, with $G_j \in \mathrm{Coh}(Y_{\mathrm{sa},\mathbb{R}^+})$. Then

$$f^{-1} \varinjlim_{j} \rho_* G_j \simeq \varinjlim_{j} f^{-1} \rho_* G_j \simeq \varinjlim_{j} \rho_* f^{-1} G_j$$

and $f^{-1} G_j$ is conic for each j. □

Proposition 2.5.5. — *Let $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$ and let $G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$. We have:*

(i) $\eta_{\mathrm{sa}*} f_* F \simeq f_* \eta_{\mathrm{sa}*} F$;

(ii) $\eta_{\mathrm{sa}*} f^{-1} G \simeq f^{-1} \eta_{\mathrm{sa}*} G$.

Proof. — Part (i) follows immediately from the commutativity of the diagram (2.5.1). Let us prove (ii). We have

$$f^{-1} G \simeq f^{-1} \eta_{\mathrm{sa}}^{-1} \eta_{\mathrm{sa}*} G \simeq \eta_{\mathrm{sa}}^{-1} f^{-1} \eta_{\mathrm{sa}*} G,$$

where the first isomorphism follows from Theorem 2.3.4 and the second one from the commutativity of the diagram (2.5.1). Composing with $\eta_{\mathrm{sa}*}$ and using Theorem 2.3.4 once again, we obtain the required isomorphism. □

Remark 2.5.6. — While (i) is true in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$, the isomorphism (ii) works only for conic sheaves. For example, let $X = \{0\}$, $Y = \mathbb{R}$ and let $f : \{0\} \to \mathbb{R}$ be the inclusion. Endow $\mathbb{R}$ with the action μ induced by the multiplication. Let B_ε be the open ball of radius $\varepsilon > 0$ centered at $\{0\}$. In this case

$$f^{-1} \eta_{\mathrm{sa}*} k_{B_\varepsilon} \simeq \Gamma(\mathbb{R}; k_{B_\varepsilon}) = 0 \quad \text{and} \quad \eta_{\mathrm{sa}*} f^{-1} k_{B_\varepsilon} \simeq f^{-1} k_{B_\varepsilon} \simeq k.$$

Proposition 2.5.7. — *Let $F, G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. Then we have:*

(i) $\eta_{\mathrm{sa}*}(F \otimes G) \simeq \eta_{\mathrm{sa}*} F \otimes \eta_{\mathrm{sa}*} G$;

(ii) $\eta_{\mathrm{sa}*} \mathcal{H}om(F, G) \simeq \mathcal{H}om(\eta_{\mathrm{sa}*} F, \eta_{\mathrm{sa}*} G)$.

Proof. — (i) We have the chain of isomorphisms

$$\begin{aligned}\eta_{\mathrm{sa}*} F \otimes \eta_{\mathrm{sa}*} G &\simeq \eta_{\mathrm{sa}*} \eta_{\mathrm{sa}}^{-1} (\eta_{\mathrm{sa}*} F \otimes \eta_{\mathrm{sa}*} G) \\ &\simeq \eta_{\mathrm{sa}*} (\eta_{\mathrm{sa}}^{-1} \eta_{\mathrm{sa}*} F \otimes \eta_{\mathrm{sa}}^{-1} \eta_{\mathrm{sa}*} G) \simeq \eta_{\mathrm{sa}*}(F \otimes G),\end{aligned}$$

where the first and the third isomorphisms follow from Theorem 2.3.4.

(ii) We have the chain of isomorphisms

$$\eta_{\mathrm{sa}*} \mathcal{H}om(F, G) \simeq \eta_{\mathrm{sa}*} \mathcal{H}om(\eta_{\mathrm{sa}}^{-1} \eta_{\mathrm{sa}*} F, G) \simeq \mathcal{H}om(\eta_{\mathrm{sa}*} F, \eta_{\mathrm{sa}*} G),$$

where the first isomorphism follows from Theorem 2.3.4. □

Now let us consider the operations in the derived category of conic subanalytic sheaves.

Proposition 2.5.8. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_X)$. Then $R\rho_* F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$.*

Proof. — There exists $F' \in D^{\mathrm{b}}(k_{X_{\mathbb{R}^+}})$ such that $F \simeq \eta^{-1} F'$. Then the result follows from Lemma 2.4.12. □

Proposition 2.5.9. — *Let $F, G \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. Then $R\mathcal{H}om(F,G) \in D^{+}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. In particular, if $H^k F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ for each $k \in \mathbb{Z}$, then $R\mathcal{H}om(F,G) \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$.*

Proof. — (i) Let us prove that if $F', G' \in \mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ and G' is injective, then $R\mathcal{H}om(\eta_{\mathrm{sa}}^{-1} F', \eta_{\mathrm{sa}}^{-1} G')$ is concentrated in degree zero. Let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ be $\mathbb{R}^+$-connected. Then

$$\begin{aligned}
\mathrm{R}\Gamma(U; R\mathcal{H}om(\eta_{\mathrm{sa}}^{-1} F', \eta_{\mathrm{sa}}^{-1} G')) &\simeq \mathrm{RHom}_{k_{X_{\mathrm{sa}}}}(\eta_{\mathrm{sa}}^{-1} F', \mathrm{R}\Gamma_U \eta_{\mathrm{sa}}^{-1} G') \\
&\simeq \mathrm{RHom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(F', R\eta_{\mathrm{sa}*} \mathrm{R}\Gamma_U \eta_{\mathrm{sa}}^{-1} G') \\
&\simeq \mathrm{RHom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(F', \mathrm{R}\Gamma_{\mathbb{R}^+ U} G') \\
&\simeq \mathrm{RHom}_{k_{X_{\mathrm{sa},\mathbb{R}^+}}}(F'_{\mathbb{R}^+ U}, G'),
\end{aligned}$$

which is concentrated in degree zero since G' is injective. The third isomorphism follows from Corollary 2.4.11. By (2.3.1) (i) this implies that $R\mathcal{H}om(\eta_{\mathrm{sa}}^{-1} F', \eta_{\mathrm{sa}}^{-1} G')$ is concentrated in degree zero.

(ii) Let $I^\bullet$ be a complex of injective objects of $\mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ such that $\eta_{\mathrm{sa}}^{-1} I^\bullet \simeq G$ and let $F' \in D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ such that $\eta_{\mathrm{sa}}^{-1} F' \simeq F$. By (i) we have

$$R\mathcal{H}om(F,G) \simeq \mathcal{H}om(\eta_{\mathrm{sa}}^{-1} F', \eta_{\mathrm{sa}}^{-1} I^\bullet)$$

and Proposition 2.5.3 (ii) implies that $\mathcal{H}om(\eta_{\mathrm{sa}}^{-1} F', \eta_{\mathrm{sa}}^{-1} I^\bullet)$ is a complex of conic sheaves. If $F \in D^{\mathrm{b}}(\mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+}))$, then $R\mathcal{H}om(F,G)$ has bounded cohomology: it is a consequence of the fact that $\mathcal{H}om(K, .)$ has finite cohomological dimension for each $K \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ (Corollary 2.3.3 of [**28**]). □

Proposition 2.5.10. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. Then $Rf_* F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$.*

Proof. — Remark that the functor η_{sa}^{-1} sends injective sheaves to f_*-acycic sheaves. This is a consequence of the fact that η_{sa}^{-1} sends injective sheaves to $\Gamma(U;\cdot)$-acyclic sheaves for each $\mathbb{R}^+$-connected $U \in \mathrm{Op}(X_{\mathrm{sa}})$. There exists $F' \in D^{\mathrm{b}}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ such that $F \simeq \eta_{\mathrm{sa}}^{-1} F'$. Let $I^\bullet$ be a bounded injective resolution of F'. Then $\eta_{\mathrm{sa}}^{-1} I^j$ is conic and f_*-acyclic for each j. We have $Rf_* F \simeq f_* \eta_{\mathrm{sa}}^{-1} I^\bullet$ and $f_* \eta_{\mathrm{sa}}^{-1} I^j$ is conic for each j. □

Proposition 2.5.11. — *Let $F, G \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$. We have:*

(i) $\eta_{\mathrm{sa}*} Rf_* F \simeq Rf_* \eta_{\mathrm{sa}*} F$;

(ii) $\eta_{\mathrm{sa}*}\, R\mathcal{H}om(F,G) \simeq R\mathcal{H}om(\eta_{\mathrm{sa}*} F, \eta_{\mathrm{sa}*} G)$.

Proof. — Part (i) follows immediately from the commutativity of the diagram (2.5.1) and the fact that conic sheaves are $\eta_{\mathrm{sa}*}$-acyclic. The proof of (ii) goes as Proposition 2.5.7 (ii) since $R\mathcal{H}om(F,G)$ is conic and conic sheaves are $\eta_{\mathrm{sa}*}$-acyclic. □

REMARK 2.5.12. — The commutation in the derived category between $\eta_{\mathrm{sa}*}$ and the functors f^{-1} and $\otimes$ follows immediately from Propositions 2.5.5 and 2.5.7 and the fact that conic sheaves are $\eta_{\mathrm{sa}*}$-acyclic.

PROPOSITION 2.5.13. — *Let $G \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$. Then $f^!G \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}})$.*

Proof. — We may reduce to the case $G \in \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$. Then

$$G = \varinjlim_j \rho_* G_j,$$

with $G_j \in \mathrm{Coh}(Y_{\mathrm{sa},\mathbb{R}^+})$. By Proposition 2.4.5 of [**28**] we have

$$H^k f^! G \simeq \varinjlim_i \rho_* H^k f^! G_j$$

for each $k \in \mathbb{Z}$ and the result follows since $H^k f^! G_j$ is conic for each $k \in \mathbb{Z}$ and for each j. □

REMARK 2.5.14. — The functor $f_{!!} : \mathrm{Mod}(k_{X_{\mathrm{sa}}}) \to \mathrm{Mod}(k_{Y_{\mathrm{sa}}})$ does not send conic sheaves to conic sheaves in general. In fact, let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection. It is a conic map with respect to the natural action of $\mathbb{R}^+$ on $\mathbb{R}^3$ and $\mathbb{R}^2$. Set

$$U = \big\{(x,y) \in \mathbb{R}^2;\ (x-1)^2 + y^2 < 1\big\}, \quad B_n = \big\{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 < n\big\},$$
$$B_n^+ = B_n \cap (\mathbb{R}^+ \times \mathbb{R}), \quad S = \mathbb{R}^+(\partial U \times \{1\}).$$

Let us consider the conic sheaf k_S. By definition of proper direct image we have

$$\Gamma(U; p_{!!}F) = \varinjlim_K \Gamma(p^{-1}(U); \Gamma_K F),$$

where K ranges through the family of subanalytic compact subsets of $\mathbb{R}^3$. Since U is bounded we have

$$\Gamma(U; p_{!!}k_S) \simeq \varinjlim_K \Gamma(U \times \mathbb{R}; \Gamma_K k_S) \simeq \varinjlim_m \Gamma(U \times \mathbb{R}; \Gamma_{\mathbb{R}^2 \times [-m,m]} k_S) \simeq k,$$

where $m \in \mathbb{N}$. On the other hand we have

$$\Gamma(\mathbb{R}^+U; p_{!!}k_S) \simeq \varprojlim_n \Gamma(B_n^+; p_{!!}k_S) \simeq \varprojlim_n \varinjlim_m \Gamma(B_n^+ \times \mathbb{R}; \Gamma_{\mathbb{R}^2 \times [-m,m]} k_S) = 0,$$

where $m, n \in \mathbb{N}$, since $\Gamma(B_n^+ \times \mathbb{R}; \Gamma_{\mathbb{R}^2 \times [-m,m]} k_S) = 0$ for each m, n.

Hence we shall need a new definition of proper direct image for conic sheaves.

Definition 2.5.15. — We define functor $f_{\mathbb{R}^+!!}$ of *proper direct image* for conic sheaves in the following way, where $F_i \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$:

$$f_{\mathbb{R}^+!!} : \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}}) \longrightarrow \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}}), \quad \varinjlim_i \rho_* F_i \longmapsto \varinjlim_i \rho_* f_! F_i$$

Let us see an explicit formula for the sections of $f_{\mathbb{R}^+!!}$. Let $U \in \mathrm{Op}^{\mathrm{c}}(Y_{\mathrm{sa},\mathbb{R}^+})$ and let $F = \varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$. We have the chain of isomorphisms

$$\begin{aligned}\Gamma(U; \varinjlim_i \rho_* f_! F_i) \simeq \varinjlim_i \Gamma(U; f_! F_i) &\simeq \varinjlim_{i,Z,K} \Gamma(f^{-1}(U); \Gamma_{Z\cap K} F_i)\\ &\simeq \varinjlim_{i,Z',K} \Gamma(f^{-1}(U); \Gamma_{Z'\cap K} F_i) \simeq \varinjlim_{Z',K} \Gamma(f^{-1}(U); \Gamma_{Z'\cap K} \varinjlim_i \rho_* F_i).\end{aligned}$$

Here Z ranges into the family of closed subanalytic subsets of $f^{-1}(U)$ such that $f : Z \to U$ is proper, Z' ranges through the family of closed conic subanalytic subsets of $f^{-1}(U)$ such that $f^{-1}(y) \cap \mathbb{R}^+ x = \{\mathrm{point}\}$ for any $y \in Y$, $x \in X$, and $K \Subset X_{\mathbb{R}^+}$ are conic and closed. The first isomorphism follows since $U \in \mathrm{Op}^{\mathrm{c}}(Y_{\mathrm{sa},\mathbb{R}^+})$, the third since F_i is conic for each i and the last one since $f^{-1}(U) \cap K \Subset X_{\mathbb{R}^+}$. This formula also explain why $f_{\mathbb{R}^+!!}$ does not depend on the choice of the family $\{F_i\}$.

Note that if $F \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$ then $f_{\mathbb{R}^+!!} \rho_* F \simeq \rho_* f_! F \not\simeq f_{!!} \rho_* F$. Moreover this definition is compatible with the classical one. In fact $f_{\mathbb{R}^+!!}$ commutes with ρ^{-1} and we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_{\mathbb{R}^+}(k_X) \arrow[r, "f_!"] \arrow[d, "\rho_*"] & \mathrm{Mod}_{\mathbb{R}^+}(k_Y) \\ \mathrm{Mod}_{\mathbb{R}^+}(k_{X_{\mathrm{sa}}}) \arrow[r, "f_{\mathbb{R}^+!!}"] & \mathrm{Mod}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}}). \arrow[u, "\rho^{-1}"'] \end{tikzcd}$$

Remark 2.5.16. — With the notation of Remark 2.5.14, we have

$$\Gamma(\mathbb{R}^+U; p_{\mathbb{R}^+!!} k_S) \simeq \Gamma(U; p_{\mathbb{R}^+!!} k_S) \simeq k.$$

In fact the restriction of p to $S \cap \{(x,y,z) \in \mathbb{R}^3;\ x > 0\}$ is proper.

It is easy to prove that projection formula and base change formula for conic sheaves are satisfied. Moreover, $\mathbb{R}^+$-quasi-injective objects are acyclic with respect to the functor $f_{\mathbb{R}^+!!}$, since they are $\mathcal{H}om(G, .)$-injective for each $G \in \mathrm{Coh}(X_{\mathrm{sa},\mathbb{R}^+})$.

In order to find a right adjoint to $Rf_{\mathbb{R}^+!!}$ we follow the method used to find a right adjoint to the functor proper direct image for subanalytic sheaves. We shall skip the details of the proof (which are an adaptation of the results of [**28**]). The subcategory $\mathcal{J}_{X_{\mathrm{sa},\mathbb{R}^+}}$ of $\mathbb{R}^+$-quasi-injective objects and the functor $f_{\mathbb{R}^+!!}$ have the following properties:

(i) $\mathcal{J}_{X_{\mathrm{sa},\mathbb{R}^+}}$ is cogenerating;

(ii) $\mathrm{Mod}(k_{X_{\mathrm{sa},\mathbb{R}^+}})$ has finite quasi-injective dimension;

(iii) $\mathcal{J}_{X_{\mathrm{sa},\mathbb{R}^+}}$ is $f_{\mathbb{R}^+!!}$-injective;

(iv) $\mathcal{J}_{X_{\mathrm{sa},\mathbb{R}^+}}$ is closed by small $\oplus$;

(v) $f_{\mathbb{R}^+!!}$ commutes with small $\oplus$.

As a consequence of the Brown representability theorem (see [**18**], Corollary 14.3.7 for details) we find a right adjoint to the functor $Rf_{\mathbb{R}^+!!}$, denoted by $f^!_{\mathbb{R}^+}$.

By adjunction $f^!_{\mathbb{R}^+}$ commutes with $R\rho_*$ and as in [**28**] one can prove that $H^k f^!_{\mathbb{R}^+}$ commutes with filtrant $\varinjlim$.

Hence $f^!_{\mathbb{R}^+}$ coincides with the restriction of $f^!$ to $D^{\mathrm{b}}_{\mathbb{R}^+}(k_{Y_{\mathrm{sa}}})$.

CHAPTER 3

FOURIER-SATO TRANSFORM FOR SUBANALYTIC SHEAVES

We construct here the Fourier-Sato transform for subanalytic sheaves. References are made to [**14**] for the classical Fourier-Sato transform.

3.1. Conic sheaves on vector bundles

Let $E \xrightarrow{\tau} Z$ be a real vector bundle, with dimension n over a real analytic manifold Z. Then $\mathbb{R}^+$ acts naturally on E by multiplication on the fibers. We identify Z with the zero-section of E and denote by $i : Z \hookrightarrow E$ the embedding. We set

$$\dot{E} = E \setminus Z$$

and $\dot{\tau}$: $\dot{E} \to Z$ denotes the projection.

LEMMA 3.1.1. — *The category* $\mathrm{Op}(E_{\mathrm{sa}})$ *satisfies* (2.3.1).

Proof. — Let us prove (2.3.1) (i). Let $U \in \mathrm{Op}^{\mathrm{c}}(X_{\mathrm{sa}})$ Let $\{V_i\}_{i\in\mathbb{N}}$ be a locally finite covering of Z with $V_i \in \mathrm{Op}^{\mathrm{c}}(Z_{\mathrm{sa}})$ such that $\dot{\tau}^{-1}(V_i) \simeq \mathbb{R}^m \times \mathbb{R}^n$ and let $\{U_i\}$ be a refinement of $\{V_i\}$ with $U_i \in \mathrm{Op}^{\mathrm{c}}(Z_{\mathrm{sa}})$ and $\overline{U}_i \subset V_i$ for each i. Then U is covered by a finite number of $\tau^{-1}(U_i)$ and $U \cap \tau^{-1}(U_i)$ is relatively compact in $\tau^{-1}(V_i)$ for each i. We may reduce to the case $E \simeq \mathbb{R}^m \times \mathbb{R}^n$. Let us consider the morphism of manifolds

$$\varphi : \mathbb{R}^m \times \mathbb{S}^{n-1} \times \mathbb{R} \longrightarrow \mathbb{R}^m \times \mathbb{R}^n, \quad (z, \vartheta, r) \longmapsto \big(z, ri(\vartheta)\big),$$

where $i : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ denotes the embedding. Then φ is proper and subanalytic. The subset $\varphi^{-1}(U)$ is subanalytic and relatively compact in $\mathbb{R}^m \times \mathbb{S}^{n-1} \times \mathbb{R}$.

(a) By Lemma A.1.11 $\varphi^{-1}(U \setminus Z)$ admits a finite cover $\{W_j\}_{j\in J}$ such that the intersections of each W_j with the fibers of $\pi : \mathbb{R}^m \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{S}^{n-1}$ are contractible or empty. Then $\varphi(W_j)$ is an open subanalytic relatively compact $\mathbb{R}^+$-connected subset of $\mathbb{R}^m \times \mathbb{R}^n$ for each j. In this way we obtain a finite covering of $U \setminus Z$ consisting of $\mathbb{R}^+$-connected subanalytic open subsets.

(b) Let $p \in \pi(\varphi^{-1}(U \cap Z))$. Then $\pi^{-1}(p) \cap \varphi^{-1}(U)$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R}$ containing 0. Set

$$W_Z = \{(p,r) \in \varphi^{-1}(U);\ m(p) < r < M(p)\}.$$

The set W_Z is open subanalytic (it is a consequence of Proposition 1.2, Chap. 6 of [**38**]), contains $\varphi^{-1}(U \cap Z)$ and its intersections with the fibers of π are contractible. Then $\varphi(W_Z)$ is an open $\mathbb{R}^+$-connected subanalytic neighborhood of $U \cap Z$ and it is contained in U.

By (a) there exists a finite covering $\{\varphi(W_j)\}_{j \in J}$ of $U \setminus Z$ consisting of $\mathbb{R}^+$-connected subanalytic open subsets, and $\varphi(W_Z) \cup \bigcup_{j \in J} \varphi(W_j) = U$.

By Proposition 8.3.8 of [**14**] the category $\mathrm{Op}(E_{\mathrm{sa}})$ also satisfies (2.3.1) (ii).

Moreover (2.3.1) (iii) and (iv) are clearly satisfied. □

Now let us consider E endowed with the conic topology. In this situation, an object $U \in \mathrm{Op}(E_{\mathbb{R}^+})$ is the union of $\dot{U} \in \mathrm{Op}(\dot{E}_{\mathbb{R}^+})$ and $U_Z \in \mathrm{Op}(Z)$ such that $\dot{\tau}^{-1}\ (U_Z) \subset \dot{U}$. If $U, V \in \mathrm{Op}(E_{\mathbb{R}^+})$, then $U \Subset V$ if $U_Z \Subset V_Z$ in Z and $\dot{U} \Subset \dot{V}$ in $\dot{E}_{\mathbb{R}^+}$ (this means that $\pi(\dot{U}) \Subset \pi(\dot{V})$ in $\dot{E}/\mathbb{R}^+$, where $\pi : \dot{E} \to \dot{E}/\mathbb{R}^+$ denotes the projection).

Applying Theorem 2.4.9 we have the following:

THEOREM 3.1.2. — *The categories $D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$ and $D^{\mathrm{b}}(k_{E_{\mathrm{sa},\mathbb{R}^+}})$ are equivalent.*

Consider the subcategory $\mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$ of $\mathrm{Mod}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$ consisting of sheaves whose support is compact on the base (i.e. $\tau(\mathrm{supp}(F))$ is compact in Z). Let us consider the natural map $\eta : E \to E_{\mathbb{R}^+}$. The restriction of η^{-1} to $\mathrm{Coh}(E_{\mathrm{sa},\mathbb{R}^+})$ gives rise to a functor

$$\bar{\eta}^{-1} : \mathrm{Coh}(E_{\mathrm{sa},\mathbb{R}^+}) \longrightarrow \mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E) \tag{3.1.1}$$

Since the functor η^{-1} is fully faithful and exact, we identify $\mathrm{Coh}(E_{\mathrm{sa},\mathbb{R}^+})$ as a subcategory of $\mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$.

THEOREM 3.1.3. — *The functor $\bar{\eta}^{-1}$ in* (3.1.1) *is an equivalence of categories.*

Proof. — (i) Let $F \in \mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$. Let us show that F is $E_{\mathrm{sa},\mathbb{R}^+}$-finite. We may reduce to the case $E \simeq \mathbb{R}^m \times \mathbb{R}^n$ and $Z \simeq \mathbb{R}^m \times \{0\}$. It is well known that if X is a real analytic manifold and $G \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_X)$, then G is quasi-isomorphic to a bounded complex of finite sums $\bigoplus_W k_W$, where $W \in \mathrm{Op}^{\mathrm{c}}_{\mathrm{sa}}(X)$.

Let us consider the diagram $Z \xhookrightarrow{i} E \xrightarrow{\tau} Z$, where i is the embedding. We have $\tau_* F \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_Z)$. Since F is conic $\tau_* F \simeq i^{-1} F$. We have an exact sequence

$$\bigoplus_{i \in I} k_{\tau^{-1}(V_i)} \longrightarrow \tau^{-1}\tau_* F \to 0, \tag{3.1.2}$$

where I is finite and $V_i \in \mathrm{Op}^{\mathrm{c}}_{\mathrm{sa}}(Z)$.

Now let us consider the diagram $S \xhookrightarrow{j} \dot{E} \xrightarrow{\pi} S$, where $S = \dot{E}/\mathbb{R}^+ \simeq \mathbb{R}^m \times \mathbb{S}^{n-1}$ and π is the projection. We have $j^{-1}F|_{\dot{E}} \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_S)$. Since $F|_{\dot{E}}$ is conic, one has $\pi^{-1}j^{-1}F|_{\dot{E}} \simeq F|_{\dot{E}}$. We have an exact sequence

$$\bigoplus_j k_{\pi^{-1}(U_j)} \longrightarrow F_{\dot{E}} \to 0. \tag{3.1.3}$$

where J is finite and $U_j \in \mathrm{Op}^{\mathrm{c}}_{\mathrm{sa}}(S)$.

It is easy to check that the morphism $\tau^{-1}\tau_* F \oplus F_{\dot{E}} \to F$ is an epimorphism and we obtain the result by (3.1.2) and (3.1.3).

(ii) Let us show that F is $E_{\mathrm{sa},\mathbb{R}^+}$-pseudo-coherent. Let $G = \bigoplus_{i\in I} k_{W_i}$, with I finite and $W_i \in \mathrm{Op}^{\mathrm{c}}(E_{\mathrm{sa},\mathbb{R}^+})$, and consider a morphism $\psi : G \to F$. Since F and G are $\mathbb{R}$-constructible and conic, then $\ker\psi$ belongs to $\mathrm{Mod}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$, and its support is still compact on the base. □

As a consequence of Theorems 2.3.6 and 3.1.3 one has the following:

Theorem 3.1.4. — *Let $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. Then there exists a small filtrant system $\{F_i\}$ in $\mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$ such that $F \simeq \varinjlim_i \rho_* F_i$.*

We end this section with the following result, which will be useful in § 3.2.

Lemma 3.1.5. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. Then:*
(i) $R\tau_* F \simeq i^{-1}F$;
(ii) $R\tau_{!!} F \simeq i^! F$.

Proof. — (i) The adjunction morphism defines

$$R\tau_* F \simeq i^{-1}\tau^{-1}R\tau_* F \longrightarrow i^{-1}F.$$

Let $V \in \mathrm{Op}^{\mathrm{c}}(Z_{\mathrm{sa}})$. Then

$$\varinjlim_{U\supset V} R^k\Gamma(U;F) \simeq \varinjlim_{\substack{U\supset V\\ \tau(U)=V}} R^k\Gamma(U;F) \simeq R^k\Gamma\big(\tau^{-1}(V);F\big) \simeq R^k\Gamma(V;R\tau_* F),$$

where $U \in \mathrm{Op}(E_{\mathrm{sa}})$ and $\mathbb{R}^+$-connected. The second isomorphism follows from Proposition 2.4.10.

(ii) The adjunction morphism defines

$$i^! F \longrightarrow i^!\tau^! R\tau_{!!} F \simeq R\tau_{!!} F.$$

Let $V \in \mathrm{Op}^{\mathrm{c}}(Z_{\mathrm{sa}})$, and let K be a compact subanalytic $\mathbb{R}^+$-connected neighborhood of V in E. Then $\tau^{-1}(V) \setminus K$ is $\mathbb{R}^+$-connected and subanalytic, and

$$\mathbb{R}^+\big(\tau^{-1}(V)\setminus K\big) = \tau^{-1}(V)\setminus Z.$$

By Proposition 2.4.10 we have the isomorphism

$$\mathrm{R}\Gamma\big(\tau^{-1}(V);\mathrm{R}\Gamma_Z F\big)\simeq \mathrm{R}\Gamma\big(\tau^{-1}(V);\mathrm{R}\Gamma_K F\big).$$

It follows from the definition of $R\tau_{!!}$ that for any $k\in\mathbb{Z}$ and $V\in\mathrm{Op}^{\mathrm{c}}(Z_{\mathrm{sa}})$ we have

$$R^k\Gamma(V;R\tau_{!!}F)\simeq \varinjlim_K R^k\Gamma\big(\tau^{-1}(V);\mathrm{R}\Gamma_K F\big),$$

where K ranges through the family of compact subanalytic $\mathbb{R}^+$-connected neighborhoods of V in E. On the other hand for any $k\in\mathbb{Z}$ we have

$$R^k\Gamma(V;i^!F)\simeq R^k\operatorname{Hom}(i_*k_V,F)\simeq R^k\operatorname{Hom}(i_*i^{-1}\tau^{-1}k_V,F)\simeq R^k\Gamma\big(\tau^{-1}(V);\mathrm{R}\Gamma_Z F\big)$$

and the result follows. □

3.2. Fourier-Sato transformation

Let $E\xrightarrow{\tau}Z$ be a real vector bundle, with dimension n over a real analytic manifold Z and $E^*\xrightarrow{\pi}Z$ its dual. We identify Z as the zero-section of E and denote by $i:Z\hookrightarrow E$ the embedding, we define similarly $i:Z\hookrightarrow E^*$. We denote by p_1 and p_2 the projections from $E\times_Z E^*$:

$$\begin{tikzcd}
& E\underset{Z}{\times}E^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
E \arrow[dr, "\tau"'] & & E^* \arrow[dl, "\pi"] \\
& Z &
\end{tikzcd}$$

We set

$$P:=\big\{(x,y)\in E\underset{Z}{\times}E^*;\ \langle x,y\rangle\geq 0\big\},\quad P':=\big\{(x,y)\in E\underset{Z}{\times}E^*;\ \langle x,y\rangle\leq 0\big\}$$

and we define the functors

$$\begin{aligned}
\Psi_{P'}&=Rp_{1*}\circ\mathrm{R}\Gamma_{P'}\circ p_2^!:D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}})\longrightarrow D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}),\\
\Phi_{P'}&=Rp_{2!!}\circ(.)_{P'}\circ p_1^{-1}:D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})\longrightarrow D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}),\\
\Psi_P&=Rp_{2*}\circ\mathrm{R}\Gamma_P\circ p_1^{-1}:D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})\longrightarrow D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}),\\
\Phi_P&=Rp_{1!!}\circ(.)_P\circ p_2^!:D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}})\longrightarrow D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}).
\end{aligned}$$

Remark 3.2.1. — These functors are well defined, more generally they send subanalytic sheaves to conic subanalytic sheaves.

Lemma 3.2.2. — *Let $F\in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. Then* $\mathrm{supp}((\mathrm{R}\Gamma_P(p_1^{-1}F))_{P'})$ *is contained in $Z\times_Z E^*$.*

Proof. — We may reduce to the case $F \in \mathrm{Mod}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. Then $F = \varinjlim_i \rho_* F_i$, with $F_i \in \mathrm{Mod}^{cb}_{\mathbb{R}\text{-c},\mathbb{R}^+}(k_E)$. We have

$$\begin{aligned} H^k\big(\mathrm{R}\Gamma_P(p_1^{-1} \varinjlim_i \rho_* F_i)_{P'}\big) &\simeq \varinjlim_i H^k\big(\mathrm{R}\Gamma_P(p_1^{-1}\rho_* F_i)_{P'}\big) \\ &\simeq \varinjlim_i \rho_* H^k\big(\mathrm{R}\Gamma_P(p_1^{-1}F_i)_{P'}\big) \\ &\simeq \varinjlim_i \rho_*\big(H^k(\mathrm{R}\Gamma_P(p_1^{-1}F_i)_{P'})\big)_{Z\times_Z E^*}, \end{aligned}$$

where the last isomorphism follows from Lemma 3.7.6 of [**14**]. □

LEMMA 3.2.3. — *Let A and B be two closed subanalytic subsets of E such that $A \cup B = E$, and let $F \in D^{\mathrm{b}}(k_{E_{\mathrm{sa}}})$. Then $\mathrm{R}\Gamma_A(F_B) \simeq (\mathrm{R}\Gamma_A F)_B$.*

Proof. — We have a natural arrow $(\Gamma_A F)_B \to \Gamma_A(F_B)$, and $R(\Gamma_A F)_B \simeq (\mathrm{R}\Gamma_A F)_B$ since $(\,.\,)_B$ is exact. Then we obtain a morphism $(\mathrm{R}\Gamma_A F)_B \to \mathrm{R}\Gamma_A(F_B)$. It is enough to prove that for any $k \in \mathbb{Z}$ and for any $F \in \mathrm{Mod}(k_{E_{\mathrm{sa}}})$ we have

$$(R^k\Gamma_A F)_B \xrightarrow{\sim} R^k\Gamma_A(F_B).$$

Since both sides commute with filtrant $\varinjlim$, we may assume $F \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_E)$. Then the result follows from the corresponding one for classical sheaves. □

PROPOSITION 3.2.4. — *The two functors $\Phi_{P'}, \Psi_P : D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}) \to D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}})$ are isomorphic.*

Proof. — We have the chain of isomorphisms:

$$\begin{aligned} \Phi_{P'}F = Rp_{2!!}(p_1^{-1}F)_{P'} &\simeq Rp_{2!!}\mathrm{R}\Gamma_P\big((p_1^{-1}F)_{P'}\big) \simeq Rp_{2!!}\big(\mathrm{R}\Gamma_P(p_1^{-1}F)\big)_{P'} \\ &\simeq Rp_{2*}\big(\mathrm{R}\Gamma_P(p_1^{-1}F)\big)_{P'} \simeq Rp_{2*}\mathrm{R}\Gamma_P(p_1^{-1}F). \end{aligned}$$

The first isomorphism follows from Lemma 3.1.5 (ii), the second from Lemma 3.2.3, the third one from Lemma 3.2.2 and the last one from Lemma 3.1.5 (i). □

DEFINITION 3.2.5. — Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$.

(i) The *Fourier-Sato transform* is the functor

$$(.)^\wedge : D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}) \longrightarrow D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}), \quad F^\wedge = \Phi_{P'}F \simeq \Psi_P F.$$

(ii) The *inverse Fourier-Sato transform* is the functor

$$(.)^\vee : D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}) \to D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}), \quad F^\vee = \Psi_{P'}F \simeq \Phi_P F.$$

It follows from the definition that the functors $^\wedge$ and $^\vee$ commute with $R\rho_*$ and ρ^{-1}. We have quasi-commutative diagrams

$$\begin{tikzcd} D^{\mathrm{b}}_{\mathbb{R}^+}(k_E) \arrow[r, shift left, "\wedge"] \arrow[d, "R\rho_*"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*}) \arrow[l, shift left, "\vee"] \arrow[d, "R\rho_*"] \\ D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}) \arrow[r, shift left, "\wedge"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}) \arrow[l, shift left, "\vee"] \end{tikzcd} \qquad \begin{tikzcd} D^{\mathrm{b}}_{\mathbb{R}^+}(k_E) \arrow[r, shift left, "\wedge"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*}) \arrow[l, shift left, "\vee"] \\ D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}) \arrow[r, shift left, "\wedge"] \arrow[u, "\rho^{-1}"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}). \arrow[l, shift left, "\vee"] \arrow[u, "\rho^{-1}"'] \end{tikzcd}$$

This implies that these functors are the extension to conic subanalytic sheaves of the classical Fourier-Sato and inverse Fourier-Sato transforms.

Theorem 3.2.6. — *The functors $^\wedge$ and $^\vee$ are equivalence of categories, inverse to each others. In particular we have*

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})}(F,G) \simeq \mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}})}(F^\wedge,G^\wedge).$$

Proof. — Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. The functors $^\wedge$ and $^\vee$ are adjoint functors, then we have a morphism $F \to F^{\wedge\vee}$. To show that it induces an isomorphism it is enough to check that $\mathrm{R}\Gamma(U;F) \to \mathrm{R}\Gamma(U;F^{\wedge\vee})$ is an isomorphism on a basis for the topology of E_{sa}. Hence we may assume that U is $\mathbb{R}^+$-connected. By Proposition 2.4.10 we may suppose that U is an open subanalytic cone of E. We have the chain of isomorphisms:

$$\begin{aligned} \mathrm{RHom}(k_U,F^{\wedge\vee}) = \mathrm{RHom}(k_U,\Psi_{P'}\Phi_{P'}F) &\simeq \mathrm{RHom}(\Phi_{P'}k_U,\Phi_{P'}F) \\ &\simeq \mathrm{RHom}(\Phi_{P'}k_U,\Psi_P F) \simeq \mathrm{RHom}(\Phi_P\Phi_{P'}k_U,F) \\ &\simeq \mathrm{RHom}(k_U,F), \end{aligned}$$

where the last isomorphism follows from Theorem 3.7.9 of [**14**] and from the fact that the functors $^\wedge$ and $^\vee$ commute with $R\rho_*$. Similarly we can show that for $G \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}})$ we have an isomorphism $G^{\vee\wedge} \xrightarrow{\sim} G$. □

Remark 3.2.7. — We have seen that the functors $^\vee$ and $^\wedge$ commute with ρ_* and ρ^{-1}. They do not commute with $\rho_!$ in general since it does not send conic sheaves to conic sheaves. We have the following quasi-commutative diagram

$$\begin{tikzcd} D^{\mathrm{b}}_{\mathbb{R}^+}(k_E) \arrow[r, shift left, "\wedge"] \arrow[d, "\rho_{\mathbb{R}^+!}"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*}) \arrow[l, shift left, "\vee"] \arrow[d, "\rho_{\mathbb{R}^+!}"] \\ D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}}) \arrow[r, shift left, "\wedge"] & D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E^*_{\mathrm{sa}}}) \arrow[l, shift left, "\vee"] \end{tikzcd}$$

Proposition 3.2.8. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{E_{\mathrm{sa}}})$. Denote by a the antipodal map.*

(i) *$F^{\wedge\wedge} \simeq F^a \otimes \mathrm{or}_{E|Z}[-n]$, where F^a denotes the inverse image of F by the antipodal map.*

(ii) *Let U be a convex conic subanalytic open subset of E^*. Then*

$$\mathrm{R}\Gamma(U;F^\wedge) \simeq \mathrm{R}\Gamma_{U^\circ}(E;F),$$

where U° denotes the polar cone.

(iii) *Let γ be a closed convex proper cone of E^{*} containing the zero section. Then*

$$\mathrm{R}\Gamma_{\gamma}(E^{*};F^{\wedge}) \simeq \mathrm{R}\Gamma(\mathrm{Int}\gamma^{\circ a};F) \otimes \mathrm{or}_{E|Z}[-n].$$

(iv) *We have*

$$(D'F)^{\vee} \simeq D'(F^{\wedge}), \qquad (DF)^{\vee} \simeq D(F^{\wedge})$$

(As usual, for $X = E, E^{*}$, $D'(.) = R\mathcal{H}om(.\,,k_X)$ and $D(.) = R\mathcal{H}om(.\,,\omega_X)$).

Proof. — The result follows adapting Proposition 3.7.12 of [**14**]. □

Let us study some functorial properties of the Fourier-Sato transform. Let Z' be another real analytic manifold and let $f : Z' \to Z$ be a real analytic map. Set $E' = Z' \times_Z E$ and denote by f_{τ} (resp. f_{π}) the map from E' to E (resp. from E'^{*} to E^{*}).

Proposition 3.2.9. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^{+}}(k_{E'_{\mathrm{sa}}})$. Then:*

$$(Rf_{\tau *}F)^{\wedge} \simeq Rf_{\pi *}(F^{\wedge}) \quad \textit{and} \quad (Rf_{\tau\mathbb{R}^{+}!!}F)^{\wedge} \simeq Rf_{\pi\mathbb{R}^{+}!!}(F^{\wedge}).$$

Let $G \in D^{\mathrm{b}}_{\mathbb{R}^{+}}(k_{E_{\mathrm{sa}}})$. Then:

$$(f_{\tau}^{!}G)^{\wedge} \simeq f_{\pi}^{!}(G^{\wedge}) \quad \textit{and} \quad (f_{\tau}^{-1}G)^{\wedge} \simeq f_{\pi}^{-1}(G^{\wedge}).$$

Proof. — The result follows adapting Proposition 3.7.13 of [**14**]. □

Let E_i, $i = 1,2$ be two real vector bundles over Z, $f : E_1 \to E_2$ a morphism of vector bundles. Set ${}^{t}f : E_2^{*} \to E_1^{*}$ the dual morphism.

Proposition 3.2.10. — (i) *Let $F \in D^{\mathrm{b}}_{\mathbb{R}^{+}}(k_{E_{1sa}})$. Then:*

$${}^{t}f^{!}(F^{\vee}) \simeq (Rf_{*}F)^{\vee}, \qquad {}^{t}f^{!}(F^{\wedge}) \simeq (Rf_{*}F)^{\wedge} \otimes \omega_{E_2^{*}/E_1^{*}},$$

$${}^{t}f^{-1}(F^{\wedge}) \simeq (Rf_{\mathbb{R}^{+}!!}F)^{\wedge}, \qquad {}^{t}f^{-1}(F^{\vee}) \simeq (Rf_{\mathbb{R}^{+}!!}F)^{\vee} \otimes \omega^{\otimes -1}_{E_2^{*}/E_1^{*}}.$$

(ii) *Let $G \in D^{\mathrm{b}}_{\mathbb{R}^{+}}(k_{E_{2sa}})$. Then:*

$$(f^{!}G)^{\wedge} \simeq (R\,{}^{t}f_{*}G^{\wedge}), \qquad (\omega^{\otimes -1}_{E_1/E_2} \otimes f^{!}G)^{\vee} \simeq (R\,{}^{t}f_{*}G^{\vee}),$$

$$(f^{-1}G)^{\vee} \simeq R\,{}^{t}f_{\mathbb{R}^{+}!!}G^{\vee}, \qquad (\omega_{E_1/E_2} \otimes f^{-1}G)^{\wedge} \simeq R\,{}^{t}f_{\mathbb{R}^{+}!!}G^{\wedge}.$$

Proof. — The result follows adapting Proposition 3.7.14 of [**14**]. □

Let E_i, $i = 1,2$ be two vector bundles over a real analytic manifold Z. We set for short $E = E_1 \times_Z E_2$ and $E^{*} = E_1^{*} \times_Z E_2^{*}$. We denote by $^{\wedge}$ the Fourier-Sato transform on E_i, $i = 1,2$ and E.

Proposition 3.2.11. — *Let $F_i \in D^{\mathrm{b}}_{\mathbb{R}^{+}}(k_{E_{i,sa}})$, $i = 1,2$. There is an isomorphism*

$$F_1^{\wedge} \ {}_{Z} F_2^{\wedge} \simeq \left(F_1 \ {}_{Z} F_2\right)^{\wedge}.$$

Proof. — Let p_i^j and p_i the i-th projection defined on $E_j \times_Z E_j^*$, $j = 1, 2$ and $E \times E^*$ respectively. Let

$$\begin{aligned} P'_j &= \big\{\langle x_j, y_j\rangle \leq 0\big\} \subset E_j \times_Z E_j^* \qquad (j = 1, 2), \\ P' &= \big\{\langle (x_1, x_2), (y_1, y_2)\rangle \leq 0\big\} \subset E \times_Z E^*. \end{aligned}$$

The Künneth formula gives rise to the isomorphisms:

$$\begin{aligned} F_1^\wedge \;{}_Z\; F_2^\wedge &\simeq Rp_{2!!}\big(p_1^{1-1}F_1 \;{}_Z\; p_1^{2-1}F_2\big)_{P'_1 \times_Z P'_2}, \\ \big(F_1 \;{}_Z\; F_2\big)^\wedge &\simeq Rp_{2!!}\big(p_1^{1-1}F_1 \;{}_Z\; p_1^{2-1}F_2\big)_{P'}. \end{aligned}$$

It is enough to show that for any sheaf $F \in D^+(k_{(E\times_Z E^*)_{\mathrm{sa}}})$ conic with respect to the actions of $\mathbb{R}^+$ on E_j and E_j^*, $j = 1, 2$, the morphism $Rp_{2!!}F_{P'} \to Rp_{2!!}F_{P'_1\times_Z P'_2}$ induces an isomorphism $R^k p_{2!!}F_{P'} \xrightarrow{\sim} R^k p_{2!!}F_{P'_1\times_Z P'_2}$ for any $k \in \mathbb{Z}$. We may reduce to the case F concentrated in degree zero. Then as in § 2.3 one can show that $F = \varinjlim_i \rho_* F_i$, with F_i conic with respect to the actions of $\mathbb{R}^+$ on E_j and E_j^*, $j = 1, 2$, $\mathbb{R}$-constructible and with compact support on the base for each i. We have the chain of isomorphisms

$$\begin{aligned} R^k p_{2!!}(\varinjlim_i \rho_* F_i)_{P'} \simeq{} & \varinjlim_i \rho_* R^k p_{2!}(F_i)_{P'} \\ &\simeq \varinjlim_i \rho_* R^k p_{2!}(F_i)_{P'_1\times_Z P'_2} \\ &\simeq R^k p_{2!!}(\varinjlim_i \rho_* F_i)_{P'_1\times_Z P'_2}. \end{aligned}$$

$R^k p_{2!!}$ commutes with ρ_* by Lemma 3.1.5 and the second isomorphism follows from Proposition 3.7.15 of [**14**]. □

CHAPTER 4

SPECIALIZATION OF SUBANALYTIC SHEAVES

We define here specialization for subanalytic sheaves. We refer to [**14**] for the classical theory of specialization.

4.1. Review on normal deformation

Let X be a real n-dimensional analytic manifold and let M be a closed submanifold of codimension ℓ. As usual we denote by $T_M X \xrightarrow{\tau} M$ the normal bundle. We identify M as the zero-section of $T_M X$ and denote $i : M \hookrightarrow T_M X$ the embedding.

We follow the notations of [**14**]. We consider the normal deformation of X, i.e. an analytic manifold $\widetilde{X}_M$, an application $(p,t) : \widetilde{X}_M \to X \times \mathbb{R}$, and an action of $\mathbb{R} \setminus \{0\}$ on $\widetilde{X}_M$ $(\widetilde{x}, r) \mapsto \widetilde{x} \cdot r$ satisfying

$$\begin{cases} p^{-1}(X \setminus M) & \text{isomorphic to } (X \setminus M) \times (\mathbb{R} \setminus \{0\}), \\ t^{-1}(c) & \text{isomorphic to } X \text{ for each } c \neq 0, \\ t^{-1}(0) & \text{isomorphic to } T_M X. \end{cases}$$

Let $s : T_M X \hookrightarrow \widetilde{X}_M$ be the inclusion, Ω the open subset of $\widetilde{X}_M$ defined by $\{t > 0\}$, $i_\Omega : \Omega \hookrightarrow \widetilde{X}_M$ and $\widetilde{p} = p \circ i_\Omega$. We get a commutative diagram

$$\begin{tikzcd} T_M X \arrow[r, "s"] \arrow[d, "\tau"] & \widetilde{X}_M \arrow[d, "p"] & \Omega \arrow[l, "i_\Omega"'] \arrow[dl, "\widetilde{p}"] \\ M \arrow[r, "i_M"] & X. & \end{tikzcd}$$

The morphism $\widetilde{p}$ is smooth and Ω is homeomorphic to $X \times \mathbb{R}^+$ by the map $(\widetilde{p}, t)$.

Definition 4.1.1. — Let S be a subset of X. The *normal cone to S along M*, denoted by $C_M(S)$, is the closed conic subset of $T_M X$ defined by

$$C_M(S) = T_M X \cap \overline{\widetilde{p}^{-1}(S)}.$$

Let us recall the following result of [**14**].

Lemma 4.1.2. — *Let V be a conic open subset of $T_M X$.*

(i) *Let W be an open neighborhood of V in $\widetilde{X}_M$ and let $U = \widetilde{p}(W \cap \Omega)$. Then $V \cap C_M(X \setminus U) = \varnothing$.*

(ii) *Let U be an open subset of X such that $V \cap C_M(X \setminus U) = \varnothing$. Then $\widetilde{p}^{-1}(U) \cup V$ is an open neighborhood of V in $\Omega \cup T_M X$.*

Let V be a conic subanalytic subset in $T_M X$. We introduce the conditions (Va) and (Vb) for V:

$$\begin{cases} (\mathrm{Va}): & V \subset T_M X \setminus i(M), \\ (\mathrm{Vb}): & \tau(V) \subset \tau\big(V \cap i(M)\big). \end{cases} \tag{4.1.1}$$

Note that each conic $V \in \mathrm{Op}(T_M X_{\mathrm{sa}})$ has a finite subanalytic open cover satisfying conditions (Va) or (Vb), since $V = (V \setminus i(M)) \cup \tau^{-1}(\tau(V \cap i(M))$. In [**11**], the local version of the following lemma was shown, however, the global one can be proved by the same argument as that in [**11**] for the one divisor case.

Lemma 4.1.3. — *Let V be a conic open subanalytic subset of $T_M X$ satisfying conditions* (Va) *or* (Vb) *of* (4.1.1). *For any subanalytic open neighborhood W of V in $\widetilde{X}_M$, there exists a subanalytic open neighborhood $\widetilde{W} \subset W$ of V such that:*

$$\begin{cases} \text{(i) the fibers of the map } \widetilde{p} : \widetilde{W} \cap \Omega \to X \text{ are connected,} \\ \text{(ii) } \widetilde{p}(\widetilde{W} \cap \Omega) \text{ is subanalytic in } X. \end{cases} \tag{4.1.2}$$

4.2. Specialization of subanalytic sheaves

Definition 4.2.1. — The *specialization along M* is the functor

$$\nu_M^{\mathrm{sa}} : D^{\mathrm{b}}(k_{X_{\mathrm{sa}}}) \longrightarrow D^{\mathrm{b}}(k_{T_M X_{\mathrm{sa}}}), \quad F \longmapsto s^{-1} \mathrm{R}\Gamma_\Omega p^{-1} F.$$

Theorem 4.2.2. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$.*

(i) *One has $\nu_M^{\mathrm{sa}} F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{T_M X_{\mathrm{sa}}})$.*

(ii) *Let V be a conic subanalytic open subset of $T_M X$ satisfying the condition either* (Va) *or* (Vb) *of* (4.1.1). *Then:*

$$H^j(V; \nu_M^{\mathrm{sa}} F) \simeq \varinjlim_U H^j(U; F),$$

where U ranges through the family of $\mathrm{Op}(X_{\mathrm{sa}})$ *such that $C_M(X \setminus U) \cap V = \varnothing$.*

(iii) *One has the isomorphisms*

$$(\nu_M^{\mathrm{sa}} F)_{|M} \simeq R\tau_*(\nu_M^{\mathrm{sa}} F) \simeq F_{|M}, \quad (\mathrm{R}\Gamma_M \nu_M^{\mathrm{sa}} F)_{|M} \simeq R\tau_{!!} \nu_M^{\mathrm{sa}} F \simeq (\mathrm{R}\Gamma_M F)_{|M}.$$

Proof. — (i) We may reduce to the case $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Hence $F = \varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_{X_{\mathrm{sa}}})$ for each i. We have

$$p^{-1} \varinjlim_i \rho_* F_i \simeq \varinjlim_i \rho_* p^{-1} F_i$$

and $p^{-1}F_i$ is $\mathbb{R}$-constructible and conic for each i. Hence $p^{-1}F$ is conic. Since the functors $\mathrm{R}\Gamma_\Omega$ and s^{-1} send conic sheaves to conic sheaves we obtain

$$s^{-1}\mathrm{R}\Gamma_\Omega p^{-1} F = \nu_M^{\mathrm{sa}} F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{T_M X_{\mathrm{sa}}}).$$

(ii) Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ such that $V \cap C_M(X \setminus U) = \varnothing$. We have the chain of morphisms

$$\begin{aligned} \mathrm{R}\Gamma(U;F) &\longrightarrow \mathrm{R}\Gamma(p^{-1}(U);p^{-1}F) \longrightarrow \mathrm{R}\Gamma(p^{-1}(U)\cap\Omega;p^{-1}F) \\ &\longrightarrow \mathrm{R}\Gamma(\widetilde{p}^{-1}(U)\cup V;\mathrm{R}\Gamma_\Omega p^{-1}F) \longrightarrow \mathrm{R}\Gamma(V;\nu_M^{\mathrm{sa}}F) \end{aligned}$$

where the third arrow exists since $\widetilde{p}^{-1}(U) \cup V$ is a neighborhood of V in $\overline{\Omega}$ by Lemma 4.1.2 (ii). Let us show that it is an isomorphism. Let V be a conic open subanalytic subset of $T_M X$ satisfying the condition either (Va) or (Vb) of (4.1.1). We have

$$H^k(V;\nu_M^{\mathrm{sa}}F) \simeq \varinjlim_W H^k(W;\mathrm{R}\Gamma_\Omega p^{-1}F) \simeq \varinjlim_W H^k(W\cap\Omega;p^{-1}F),$$

where W ranges through the family of subanalytic open neighborhoods of V in $\widetilde{X}_M$. By Lemma 4.1.3 we may assume that W satisfies (4.1.2). Since $p^{-1}F$ is conic, we have

$$\begin{aligned} H^k(W\cap\Omega;p^{-1}F) &\simeq H^k\big(p^{-1}(p(W\cap\Omega));p^{-1}F\big) \\ &\simeq H^k\big(p(W\cap\Omega)\times\{1\};p^{-1}F\big) \simeq H^k\big(p(W\cap\Omega);F\big), \end{aligned}$$

where the second isomorphism follows since every subanalytic neighborhood of $p(W\cap\Omega)\times\{1\}$ contains an $\mathbb{R}^+$-connected subanalytic neighborhood (the proof is similar to that of Lemma 4.1.3). By Lemma 4.1.2 (i) we have that $p(W\cap\Omega)$ ranges through the family of subanalytic open subsets U of X such that $V\cap C_M(X\setminus U)=\varnothing$ and we obtain the result.

(iii) The result follows adapting Theorem 4.2.3 (iv) of [**14**]. □

Proposition 4.2.3. — *Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ be quasi-injective. Then $\nu_M^{\mathrm{sa}}F$ is concentrated in degree zero.*

Proof. — Since $\nu_M^{\mathrm{sa}}F$ is conic, it is enough to prove that $H^j(V;\nu_M^{\mathrm{sa}}F)=0$, $j\neq 0$, when V is a conic subanalytic open subset of $T_M X$ satisfying the condition either (Va) or (Vb) of (4.1.1). By Theorem 4.2.2 we have

$$H^j(V;\nu_M^{\mathrm{sa}}F) \simeq \varinjlim_U H^j(U;F),$$

where U ranges through the family of $\mathrm{Op}(X_{\mathrm{sa}})$ such that $C_M(X\setminus U)\cap V=\varnothing$, and $H^j(U;F)=0$ if $j\neq 0$ since F is quasi-injective. □

Let us study the relation with the classical functor of specialization ν_M.

Proposition 4.2.4. — *Let $F \in D^{\mathrm{b}}(k_X)$. Then $\rho^{-1}\nu_M^{\mathrm{sa}} R\rho_* F \simeq \nu_M F$.*

Proof. — We have to show that for each $x \in T_M X$ we have

$$H^k(\rho^{-1}\nu_M^{\mathrm{sa}} R\rho_* F)_x \simeq H^k(\nu_M F)_x.$$

Hence we have to prove the isomorphism

$$\varinjlim_{x \in V} H^k(V; \nu_M^{\mathrm{sa}} R\rho_* F) \simeq \varinjlim_{x \in V} H^k(V; \nu_M F),$$

where V ranges through the family of open $\mathbb{R}^+$-connected relatively compact subanalytic subsets of $T_M X$ (which is cofinal in the family of neighborhoods of x). This is a consequence of Theorem 4.2.2 and Theorem 4.2.3 of [**14**]. $\square$

Remark 4.2.5. — Remark that the functor of specialization does not commute with $R\rho_*$ and ρ^{-1} in general. In fact let $V \in \mathrm{Op}^{\mathrm{c}}_{\mathrm{sa}}(T_M X)$ be $\mathbb{R}^+$-connected and let $F \in \mathrm{Mod}(k_X)$. Then

$$H^k(V; \nu_M^{\mathrm{sa}} R\rho_* F) \simeq \varinjlim_{U} H^k(U; F),$$

where U ranges through the family of $\mathrm{Op}(X_{\mathrm{sa}})$ such that $C_M(X \setminus U) \cap V = \varnothing$, which is not cofinal to the family of $\mathrm{Op}(X)$ such that $C_M(X \setminus U) \cap V = \varnothing$.

Now let $V \in \mathrm{Op}^{\mathrm{c}}_{\mathrm{sa}}(T_M X)$ be $\mathbb{R}^+$-connected and let $G \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Then

$$H^k(V; \nu_M \rho^{-1} G) \simeq \varinjlim_{U} \varprojlim_{W \Subset U} H^k(W; G),$$

where U ranges through the family of $\mathrm{Op}(X_{\mathrm{sa}})$ such that $C_M(X \setminus U) \cap V = \varnothing$ and $W \in \mathrm{Op}(X_{\mathrm{sa}})$. Then $H^k(\rho^{-1}\nu_M^{\mathrm{sa}} G)_x \neq H^k(\nu_M \rho^{-1} G)_x$.

Some interesting examples of this fact will be given in §6.3.

Let $f : X \to Y$ be a morphism of manifolds, let N be a closed submanifold of Y of codimension k and assume $f(M) \subset N$. We denote by f' the map from TX to $X \times_Y TY$ associated with f and by $f_\tau : X \times_Y TY \to TY$ the base change. We denote by Tf the composite map. Similarly, replacing X, Y, TX, TY by $M, N, T_M X, T_N Y$ we get the morphisms $f'_M, f_{M\tau}, T_M f$.

We have a commutative diagram, where all the squares are cartesian

$$\begin{tikzcd}
T_M X \arrow[r, "s_X"] \arrow[d, "T_M f"] & \widetilde{X}_M \arrow[d, "\widetilde{f}'"] & \Omega_X \arrow[l, "i_{\Omega_X}"'] \arrow[r, "\widetilde{p}_X"] \arrow[d, "\widetilde{f}"] & X \arrow[d, "f"] \\
T_N Y \arrow[r, "s_Y"] & \widetilde{Y}_N & \Omega_Y \arrow[l, "i_{\Omega_Y}"'] \arrow[r, "\widetilde{p}_Y"] & Y.
\end{tikzcd}$$

Recall that the following diagram is not cartesian in general:

(4.2.1)
$$\begin{tikzcd} \widetilde{X}_M \arrow[r, "p_X"] \arrow[d, "\widetilde{f}'"] & X \arrow[d, "f"] \\ \widetilde{Y}_N \arrow[r, "p_Y"] & Y. \end{tikzcd}$$

Definition 4.2.6. — (i) One says that f is *clean with respect to* N if $f^{-1}(N)$ is a submanifold M of X and the map ${}^t f'_M : M \times_N T^*_N Y \to T^*_M X$ is surjective.

(ii) One says that f is *transversal to* N if the map ${}^t f'|_{X \times_Y T^*_N Y} : X \times_Y T^*_N Y \to T^* Y$ is injective.

If f is transversal to N and $f^{-1}(N) = M$, then the square (4.2.1) is cartesian.

We will not prove the following results, which can be easily recovered adapting §IV.4.2 of [**14**], using the construction we did for subanalytic sheaves.

Proposition 4.2.7. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$.*

(i) *There exists a commutative diagram of canonical morphisms*

$$\begin{tikzcd} R(T_M f)_{\mathbb{R}^+ !!} \nu^{\mathrm{sa}}_M F \arrow[r] \arrow[d] & \nu^{\mathrm{sa}}_N R f_{\mathbb{R}^+ !!} F \arrow[d] \\ R(T_M f)_* \nu^{\mathrm{sa}}_M F & \nu^{\mathrm{sa}}_N R f_* F. \arrow[l] \end{tikzcd}$$

(ii) *Moreover if* $\mathrm{supp} F \to Y$ *and* $C_M(\mathrm{supp} F) \to T_N Y$ *are proper, and if* $\mathrm{supp} F \cap f^{-1}(N) \subset M$, *then the above morphisms are isomorphisms.*

In particular if f is clean with respect to N, proper on $\mathrm{supp} F$ *and* $f^{-1}(N) = M$, *then the above morphisms are isomorphisms.*

Proposition 4.2.8. — *Let $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$.*

(i) *There exists a commutative diagram of canonical morphisms*

$$\begin{tikzcd} \omega_{T_M X / T_N Y} \otimes (T_M f)^{-1} \nu^{\mathrm{sa}}_N G \arrow[r] \arrow[d] & \nu^{\mathrm{sa}}_M (\omega_{X/Y} \otimes f^{-1} G) \arrow[d] \\ (T_M f)^! \nu^{\mathrm{sa}}_N G & \nu^{\mathrm{sa}}_M f^! G \arrow[l] \end{tikzcd}$$

(ii) *Moreover if $f : X \to Y$ and $f_{|M} : M \to N$ are smooth the above morphisms are isomorphisms.*

Let X and Y be two real analytic manifolds and let M, N be two closed submanifolds of X and Y respectively.

Proposition 4.2.9. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ and $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. There is a natural morphism*

$$\nu^{\mathrm{sa}}_M F \quad \nu^{\mathrm{sa}}_N G \longrightarrow \nu^{\mathrm{sa}}_{M \times N}(F \quad G).$$

Corollary 4.2.10. — *Let $F, G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. There is a natural morphism*

$$\nu^{\mathrm{sa}}_M F \otimes \nu^{\mathrm{sa}}_M G \longrightarrow \nu^{\mathrm{sa}}_M (F \otimes G).$$

CHAPTER 5

MICROLOCALIZATION OF SUBANALYTIC SHEAVES

With the construction of the Fourier-Sato transform and the specialization we have all the tools to define the functor of microlocalization in the framework of subanalytic sites. See [**14**] for the classical theory of microlocalization. Then we introduce the functor μhom^{sa} for subanalytic sheaves, we study the relations with the notion of microsupport of [**17**] and with the functor of ind-microlocalization of [**19**].

5.1. Microlocalization of subanalytic sheaves

Let us denote by $T^*_M X$ the conormal bundle to M in X, i.e. the kernel of the map $M \times_X T^*X \to T^*M$. We denote by π the projection $T^*_M X \to M$.

Definition 5.1.1. — Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. The *microlocalization of F along M* is the Fourier-Sato transform of the specialization, i.e.

$$\mu^{\mathrm{sa}}_M F = (\nu^{\mathrm{sa}}_M F)^{\wedge}.$$

Theorem 5.1.2. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$.*

(i) $\mu^{\mathrm{sa}}_M F \in D^{\mathrm{b}}_{\mathbb{R}^+}(k_{T^*_M X_{\mathrm{sa}}})$.

(ii) *Let V be an open convex subanalytic cone of $T^*_M X$ satisfying (in $T^*_M X$) the condition either* (Va) *or* (Vb) *of* (4.1.1). *Then:*

$$H^j(V; \mu^{\mathrm{sa}}_M F) \simeq \varinjlim_{U,Z} H^j_Z(U; F),$$

where U ranges through the family of $\mathrm{Op}(X_{\mathrm{sa}})$ *such that $U \cap M = \pi(V)$ and Z through the family of closed subanalytic subsets such that $C_M(Z) \subset V^\circ$, where V° denotes the polar cone.*

(iii) *One has the isomorphisms*

$$(\mu^{\mathrm{sa}}_M F)_{|M} \simeq R\pi_*(\mu^{\mathrm{sa}}_M F) \simeq i^!_M F, \quad (\mathrm{R}\Gamma_M \mu^{\mathrm{sa}}_M F)_{|M} \simeq R\pi_{!!}\mu^{\mathrm{sa}}_M F \simeq i^{-1}_M F \otimes i^!_M k_X.$$

Proof. — The result follows from the functorial properties of the Fourier-Sato transform and Theorem 4.2.2. □

As in classical sheaf theory, we get the Sato's triangle for subanalytic sheaves:

$$F_{|M} \otimes \omega_{M|X} \longrightarrow \mathrm{R}\Gamma_M F_{|M} \longrightarrow R\dot{\pi}_* \mu_M^{\mathrm{sa}} F \xrightarrow{+}$$

where $\dot{\pi}$ is the restriction of π to $T^*_M X \setminus M$.

Proposition 5.1.3. — *Let $F \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ be quasi-injective. Then $\rho^{-1}\mu_M^{\mathrm{sa}} F$ is concentrated in degree zero.*

Proof. — The result follows from Theorem 5.1.2 (ii). □

Remark 5.1.4. — Remark that the functor of microlocalization does not commute with $R\rho_*$ and ρ^{-1} since specialization does not. If $F \in D^{\mathrm{b}}(k_X)$ we have

$$\rho^{-1}\mu_M^{\mathrm{sa}} R\rho_* F \simeq \mu_M F$$

since the Fourier-Sato transform commutes with ρ^{-1} and $\rho^{-1} \circ \nu_M^{\mathrm{sa}} \circ R\rho_* \simeq \nu_M$.

Let $f : X \to Y$ be a morphism of manifolds, let N be a closed submanifold of Y of codimension k and assume $f(M) \subset N$. The map Tf defines the maps

$$T^*X \xleftarrow{\;{}^t f'\;} X \times_Y T^*Y \xrightarrow{\;f_\pi\;} T^*Y$$

and similarly one can define the maps ${}^t f'_M$ and $f_{M\pi}$.

Applying the Fourier-Sato transform to the morphisms of §4.2 we get the following results (see also [**14**] §IV.4.3 for the classical case)

Proposition 5.1.5. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$.*

(i) *There exists a commutative diagram of canonical morphisms*

$$\begin{array}{ccc} Rf_{M\pi\mathbb{R}^+!!}{}^t f_M'^{-1}\mu_M^{\mathrm{sa}} F & \longrightarrow & \mu_N^{\mathrm{sa}} Rf_{!!} F \\ \downarrow & & \downarrow \\ Rf_{M\pi *}({}^t f_M'^{!}\mu_M^{\mathrm{sa}} F \otimes \omega_{X/Y} \otimes \omega_{M/N}^{\otimes -1}) & \longleftarrow & \mu_N^{\mathrm{sa}} Rf_* F. \end{array}$$

(ii) *Moreover if* $\mathrm{supp}F \to Y$ *and* $C_M(\mathrm{supp}F) \to T_N Y$ *are proper, and if* $\mathrm{supp}F \cap f^{-1}(N) \subset M$*, then the above morphisms are isomorphisms. In particular if f is clean with respect to N, proper on* $\mathrm{supp}F$ *and $f^{-1}(N) = M$, then the above morphisms are isomorphisms.*

Proposition 5.1.6. — *Let $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$.*

(i) *There exists a commutative diagram of canonical morphisms*

$$\begin{tikzcd} Rf'_{M\mathbb{R}^+!!}(\omega_{M/N} \otimes f^{-1}_{M\pi}\mu^{\mathrm{sa}}_N G) \arrow[r] \arrow[d] & \mu^{\mathrm{sa}}_M(\omega_{X/Y} \otimes f^{-1}G) \arrow[d] \\ Rf'_{M*}f^!_{M\pi}\mu^{\mathrm{sa}}_N G & \mu^{\mathrm{sa}}_M f^! G \arrow[l] \end{tikzcd}$$

(ii) *Moreover, if $f : X \to Y$ and $f|_M : M \to N$ are smooth, then the above morphisms are isomorphisms.*

Let X and Y be two real analytic manifolds and let M, N be two closed submanifolds of X and Y respectively.

Proposition 5.1.7. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ and $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. There is a natural morphism*

$$\mu^{\mathrm{sa}}_M F \quad \mu^{\mathrm{sa}}_N G \longrightarrow \mu^{\mathrm{sa}}_{M\times N}(F \quad G).$$

Corollary 5.1.8. — *Let M be a closed submanifold of X and let $\gamma : T^*_M X \times_M T^*_M X \to T^*_M X$ be the morphism given by the addition. There is a natural morphism*

$$R\gamma_{\mathbb{R}^+!!}\big(\mu^{\mathrm{sa}}_M F \underset{M}{\quad} \mu^{\mathrm{sa}}_M G\big) \longrightarrow \mu^{\mathrm{sa}}_M\,(F \quad G) \otimes \omega_{M/X}.$$

5.2. The functor μhom^{sa}

We denote by Δ the diagonal of $X\times X$, and we denote by δ the diagonal embedding. The normal deformation of the diagonal in $X \times X$ can be visualized by the diagram

$$\tag{5.2.1} \begin{tikzcd} TX \arrow[r, "\sim"] & T_\Delta(X\times X) \arrow[r, "s"] \arrow[d, "\tau_X"] & \widetilde{X\times X} \arrow[d, "p"] & \Omega \arrow[l, "i_\Omega"'] \arrow[dl, "\widetilde{p}"] \\ & \Delta \arrow[r, "\delta"] \arrow[dr, "\sim"] & X\times X \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & \\ & & X. & \end{tikzcd}$$

Set $p_i = q_i \circ p$, $i = 1, 2$. While $\widetilde{p}$ and p_i, $i = 1, 2$, are smooth, p is not, and moreover the square is not cartesian.

Definition 5.2.1. — Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. We set

$$\mu hom^{\mathrm{sa}}(F, G) := \mu^{\mathrm{sa}}_\Delta\, R\mathcal{H}om(q_2^{-1}F, q_1^! G) = \big(\nu^{\mathrm{sa}}_\Delta\, R\mathcal{H}om(q_2^{-1}F, q_1^! G)\big)^\wedge.$$

As in the classical case, there is a useful description of the functor μhom^{sa}.

Lemma 5.2.2. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$\mu hom^{\mathrm{sa}}(F, G) \simeq \big(s^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}G) \otimes s^{-1}p^{-1}q_1^! k_X\big)^\wedge.$$

Proof. — (i) We first need the following result: let $f : X \to Y$ be a morphism of real analytic manifolds and let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ such that $f_{|U}$ is smooth. Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y)$ and $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. Then

$$\mathrm{R}\Gamma_U f^{-1}\, R\mathcal{H}om(F,G) \simeq \mathrm{R}\Gamma_U\, R\mathcal{H}om(f^{-1}F, f^{-1}G). \tag{5.2.2}$$

We first reduce to the case $G = \varinjlim\limits_i \rho_* G_i$ with $G_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_Y)$ for each i. Let $U_{X_{\mathrm{sa}}}$ be the site induced by X_{sa} on U. We have

$$\begin{aligned} H^k i^{-1}_{U_{X_{\mathrm{sa}}}} f^{-1}\, R\mathcal{H}om(F, \varinjlim_i \rho_* G_i) &\simeq \varinjlim_i \rho_* H^k i_U^{-1} f^{-1}\, R\mathcal{H}om(F, G_i) \\ &\simeq \varinjlim_i \rho_* H^k\, R\mathcal{H}om(i_U^{-1} f^{-1}F, i_U^{-1} f^{-1} G_i) \\ &\simeq \varinjlim_i \rho_* H^k i_U^{-1}\, R\mathcal{H}om(f^{-1}F, f^{-1}G_i) \\ &\simeq H^k i^{-1}_{U_{X_{\mathrm{sa}}}}\, R\mathcal{H}om(f^{-1}F, \varinjlim_i \rho_* f^{-1} G_i). \end{aligned}$$

In the first and the last isomorphisms we used the commutation between ρ_* and the restriction to U given, for $V \in \mathrm{Op}(U_{X_{\mathrm{sa}}})$, by $i^{-1}_{U_{X_{\mathrm{sa}}}} \rho_* F(V) = F(V) = \rho_* i_U^{-1} F(V)$. The second isomorphism follows because $f \circ i_U$ is smooth. Composing with $Ri_{U_{X_{\mathrm{sa}}}*}$ we obtain the result.

(ii) Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. We have

$$\begin{aligned} \mu hom^{\mathrm{sa}}(F,G) &\simeq (s^{-1}\mathrm{R}\Gamma_\Omega p^{-1}\, R\mathcal{H}om(q_2^{-1}F, q_1^! G))^\wedge \\ &\simeq (s^{-1}\mathrm{R}\Gamma_\Omega\, R\mathcal{H}om(p^{-1}q_2^{-1}F, p^{-1}q_1^! G))^\wedge \\ &\simeq (s^{-1}\, R\mathcal{H}om((p^{-1}q_2^{-1}F)_\Omega, p^{-1}q_1^! G))^\wedge \\ &\simeq (s^{-1}(R\mathcal{H}om((p^{-1}q_2^{-1}F)_\Omega, p^{-1}q_1^{-1}G) \otimes p^{-1}q_1^! k_X))^\wedge \\ &\simeq (s^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}G) \otimes s^{-1}p^{-1}q_1^! k_X)^\wedge, \end{aligned}$$

where the second isomorphism follows from (5.2.2) with (U, X, f) replaced by $(\Omega, \widetilde{X \times X}, p)$. □

Let π denote the projection from $T^*_\Delta(X \times X)$ to $\Delta \simeq X$.

Proposition 5.2.3. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. There is a canonical isomorphism $\pi_* \mu hom^{\mathrm{sa}}(F,G) \simeq R\mathcal{H}om(F,G)$.*

Proof. — The result follows adapting Proposition 4.4.2 of [**14**]. □

Remark 5.2.4. — The functor μhom^{sa} is well defined also if $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. In this case we do not know if $\mu hom^{\mathrm{sa}}(F,G)$ has bounded cohomology or not.

Remark 5.2.5. — Adapting the results of § 4.2 and § 5.1 one gets the functorial properties of μhom for subanalytic sheaves. Since the proofs are essentially the same as the classical ones we will skip them and refer to [**14**].

Let $\pi : T^*X \to X$ be the projection and consider the canonical 1-form ω, the restriction to the diagonal of the map ${}^t\pi' : T^*X \times_X T^*X \to T^*T^*X$. We have a diagram

$$\begin{tikzcd}
T^*T^*X & T^*X \times_X T^*X \arrow[l, "{}^t\pi'"'] \arrow[r, "\pi_\pi"] & T^*X \\
& \Delta_{T^*X} \arrow[ul, "\omega"] \arrow[u, "\delta_{T^*X}"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Lemma 5.2.6. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ We have*

$$\omega^{-1}\mu hom^{\mathrm{sa}}(\pi^{-1}F, \pi^{-1}G) \simeq \mu hom^{\mathrm{sa}}(F, G).$$

Proof. — We have the isomorphism ${}^t\pi'_{!!}\pi_\pi^{-1}\mu hom^{\mathrm{sa}}(F, G) \xrightarrow{\sim} \mu hom^{\mathrm{sa}}(\pi^{-1}F, \pi^{-1}G)$. Hence we get the isomorphisms

$$\begin{aligned}
\omega^{-1}\mu hom^{\mathrm{sa}}(\pi^{-1}F, \pi^{-1}G) &\simeq \omega^{-1}{}^t\pi'_{!!}\pi_\pi^{-1}\mu hom^{\mathrm{sa}}(F, G) \\
&\simeq \delta^{-1}_{T^*X}\pi_\pi^{-1}\mu hom^{\mathrm{sa}}(F, G) \simeq \mu hom^{\mathrm{sa}}(F, G).
\end{aligned}$$ □

5.3. Microlocalization and microsupport

In [**14**] the authors prove that the support of $\mu hom(F, G)$ is contained in the intersection of the microsupports of F and G. We extend this result to the functor μhom^{sa}. Let X be a real analytic manifold and let $T^*X \xrightarrow{\pi} X$ be the cotangent bundle. We recall the following two equivalent definitions of microsupport of a subanalytic sheaf of [**17**]. For the notion of microsupport for classical sheaves we refer to [**14**]. For the functorial properties of the microsupport of subanalytic sheaves we refer to [**23**].

Definition 5.3.1. — The microsupport of $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$, denoted by $SS(F)$ is the subset of T^*X defined as follows. Let $p \in T^*X$, then $p \notin SS(F)$ if one of the following equivalent conditions is satisfied:

(i) There exist a conic neighborhood U of p and a small filtrant system $\{F_i\}$ in $C^{[a,b]}(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X))$ with $SS(F_i) \cap U = \varnothing$ such that F is quasi-isomorphic to $\varinjlim_i \rho_* F_i$ in a neighborhood of $\pi(p)$.

(ii) There exists a conic neighborhood U of p such that for any $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ with $\mathrm{supp}(G) \Subset \pi(U)$ and such that $SS(G) \subset U \cup T^*_X X$, one has

$$\mathrm{Hom}_{D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})}(G, F) = 0.$$

Remark 5.3.2. — In [**17**] microsupport was defined for ind-sheaves. The above definition follows from the equivalence between subanalytic sheaves and ind-$\mathbb{R}$-constructible sheaves (see [**23**] for details).

We need the following result of [**23**].

LEMMA 5.3.3. — *Let X, Y be two real analytic manifolds and let q_1, q_2 be the projections from $X \times Y$ to X and Y respectively. Let $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y)$ and $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$SS\big(R\mathcal{H}om(q_1^{-1}G, q_2^!F)\big) \subseteq SS(F) \times SS(G)^a. \tag{5.3.1}$$

Let M be a real closed submanifold of X. Let F be a conic subanalytic sheaf on T^*_MX, let S be a conic subset of T^*_MX and set ${}_SF = F \otimes \rho_{\mathbb{R}^+!}k_S$.

PROPOSITION 5.3.4. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$\mu^{\mathrm{sa}}_M F \simeq {}_{SS(F)\cap T^*_MX}(\mu^{\mathrm{sa}}_M F).$$

Proof. — Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ and let $p \notin SS(F)$. There exist a conic subanalytic neighborhood U of p and a small filtrant system $\{F_i\}$ in $C^{[a,b]}(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X))$ with $SS(F_i) \cap \overline{U} = \varnothing$ such that there exists $W \in \mathrm{Op}(X_{\mathrm{sa}})$ with $U \subseteq \pi^{-1}(W)$ and $F_W \simeq \varinjlim_i \rho_* F_i$. We have

$$H^k\mu^{\mathrm{sa}}_M F_W \simeq \varinjlim_i \rho_* H^k \mu_M F_{iW},$$

hence $(\mu^{\mathrm{sa}}_M F)|_U = 0$ since $\mathrm{supp}(\mu_M F_i) \subseteq SS(F_i)$. This implies $(\mu^{\mathrm{sa}}_M F)_V = 0$ for each $V \in \mathrm{Op}^{\mathrm{c}}((T^*_MX \setminus SS(F))_{\mathrm{sa},\mathbb{R}^+})$, hence $\mu^{\mathrm{sa}}_M F \simeq {}_{SS(F)}(\mu^{\mathrm{sa}}_M F)$. □

COROLLARY 5.3.5. — *Let $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$, $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$\mu hom^{\mathrm{sa}}(F, G) \simeq {}_{SS(F)\cap SS(G)}\big(\mu hom^{\mathrm{sa}}(F, G)\big). \tag{5.3.2}$$

The result follows from Proposition 5.3.4 and (5.3.1). □

COROLLARY 5.3.6. — *Let $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$, $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$\mathrm{supp}(\rho^{-1}\mu hom^{\mathrm{sa}}(F, G)) \subseteq SS(F) \cap SS(G).$$

Proof. — Applying ρ^{-1} to (5.3.2) we obtain the result. □

Let $f : X \to Y$ be a morphism of real analytic manifolds and denote the base change map by

$$f_\pi : X \times_Y T^*Y \longrightarrow T^*Y.$$

DEFINITION 5.3.7. — Let $f : X \to Y$ be a morphism of real analytic manifolds and let $F \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. One says that f is *non characteristic for $SS(F)$* if

$$f_\pi^{-1}\big(SS(F)\big) \cap T^*_XY \subseteq X \times_Y T^*_YY.$$

If f is a closed embedding X is said to be non characteristic.

Proposition 5.3.8. — *Let $f : X \to Y$ be a morphism of real analytic manifolds and let $F \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. Assume that f is non characteristic for $SS(F)$. Then the natural morphism*

$$f^{-1}F \otimes \omega_{X|Y} \longrightarrow f^!F$$

is an isomorphism.

Proof. — We may reduce to the case f closed embedding, hence we have to prove the isomorphism $F_{|X} \otimes \omega_{X|Y} \simeq \mathrm{R}\Gamma_X F_{|X}$ when $SS(F) \cap T^*_X Y \subseteq T^*_Y Y$. Consider the Sato's triangle

$$F_{|X} \otimes \omega_{X_{|_Y}} \longrightarrow \mathrm{R}\Gamma_X F_{|X} \longrightarrow R\dot{\pi}_* \mu^{\mathrm{sa}}_X F \xrightarrow{+} .$$

Since $SS(F) \cap T^*_X Y \subseteq T^*_Y Y$ we have $R\dot{\pi}_* \mu^{\mathrm{sa}}_X F = 0$ by Proposition 5.3.4 and the result follows. □

As usual, for $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ we define

$$D'F = R\mathcal{H}om(F, k_X).$$

Lemma 5.3.9. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and let $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Then*

$$D'F \quad G \xrightarrow{\sim} R\mathcal{H}om(q_1^{-1}F, q_2^{-1}G).$$

Proof. — We may reduce to the case $F = k_U$ with $D'k_U \simeq k_{\overline{U}}$ and $G \in \mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Set $G = \varinjlim_i \rho_* G_i$ with $G_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, we have the chain of isomorphisms

$$\begin{aligned} H^k(q_2^{-1}G)_{q_1^{-1}(\overline{U})} &\simeq \varinjlim_i \rho_* H^k(q_2^{-1}G_i)_{q_1^{-1}(\overline{U})} \\ &\simeq \varinjlim_i \rho_* R^k\Gamma_{q_1^{-1}(U)}(q_2^{-1}G_i) \simeq R^k\Gamma_{q_1^{-1}(U)}(q_2^{-1}G) \end{aligned}$$

where the second isomorphism follows from Proposition 3.4.4 of [**14**]. □

Proposition 5.3.10. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and let $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. Suppose that $SS(F) \cap SS(G) \subseteq T^*_X X$. Then*

$$D'F \otimes G \xrightarrow{\sim} R\mathcal{H}om(F, G).$$

Proof. — Let $\delta : \Delta \to X \times X$ be the embedding and let us consider the Sato's triangle

$$\begin{aligned} \delta^{-1}\, R\mathcal{H}om(q_1^{-1}F, q_2^! G) \otimes \omega_{\Delta_{|_X} \times X} &\longrightarrow \delta^!\, R\mathcal{H}om(q_1^{-1}F, q_2^! G) \\ &\longrightarrow R\dot{\pi}_* \mu hom^{\mathrm{sa}}(F, G) \xrightarrow{+} . \end{aligned}$$

We have $\delta^!\, R\mathcal{H}om(q_1^{-1}F, q_2^! G) \simeq R\mathcal{H}om(F, G)$. Moreover

$$\begin{aligned} \delta^{-1}\, R\mathcal{H}om(q_1^{-1}F, q_2^! G) \otimes \omega_{\Delta_{|_X} \times X} &\simeq \delta^{-1}\, R\mathcal{H}om(q_1^{-1}F, q_2^{-1}G) \\ &\simeq \delta^{-1}(D'F \quad G) \simeq D'F \otimes G, \end{aligned}$$

where the second isomorphism follows from Lemma 5.3.9. Then we obtain a distinguished triangle

$$D'F \otimes G \longrightarrow R\mathcal{H}om(F,G) \longrightarrow R\dot{\pi}_*\mu hom^{\mathrm{sa}}(F,G) \xrightarrow{+}$$

and the result follows since $R\dot{\pi}_*\mu hom^{\mathrm{sa}}(F,G) = 0$ by Corollary 5.3.5. □

5.4. The link with the functor μ of microlocalization

We will study the relation between microlocalization for subanalytic sheaves and the functor μ of [**19**]. Let X be a real analytic manifold and consider the normal deformation of Δ in $X \times X$ visualized by the diagram (5.2.1).

Lemma 5.4.1. — *Let $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$, then for $i = 1, 2$*

$$k_{\overline{\Omega}} \otimes p_i^{-1}G \simeq R\mathcal{H}om(k_{\Omega}, p_i^{-1}G), \tag{5.4.1}$$

$$k_{\Omega} \otimes p_i^{-1}G \simeq R\mathcal{H}om(k_{\overline{\Omega}}, p_i^{-1}G). \tag{5.4.2}$$

Proof. — Let us prove (5.4.1). Since for $i = 1, 2$ p_i is smooth, Proposition 3.16 of [**23**] implies that $SS(p_i^{-1}G) \cap SS(k_{\Omega})$ is contained on the zero section of $T^*(\widetilde{X \times X})$. Then the result is a consequence of Proposition 5.3.10, and the fact that $D'k_{\Omega} \simeq k_{\overline{\Omega}}$. The proof of (5.4.2) is similar. □

Let σ be a section of $T^*X \to X$ and consider the following commutative diagram with cartesian square

(5.4.3)
$$\begin{tikzcd}
TX & \\
T^*X \times_X TX \arrow[u, "\pi_2"] \arrow[d, "\pi_1"'] & TX \arrow[l, "\sigma'"] \arrow[ul, "\mathrm{id}"'] \arrow[d, "\tau_X"] \\
T^*X & X. \arrow[l, "\sigma"]
\end{tikzcd}$$

We set

$$P := \big\{((x,\xi),(x,v)) \in T^*X \times_X TX;\ \langle \xi, v\rangle \geq 0\big\},$$
$$P' := \big\{((x,\xi),(x,v)) \in T^*X \times_X TX;\ \langle \xi, v\rangle \leq 0\big\},$$
$$P_\sigma := \big\{(x,v) \in TX;\ \langle \sigma(x), v\rangle \geq 0\big\} = \sigma'^{-1}(P),$$
$$P'_\sigma := \big\{(x,v) \in TX;\ \langle \sigma(x), v\rangle \leq 0\big\} = \sigma'^{-1}(P').$$

The kernel K_σ is defined as follows

$$K_\sigma := Rp_{!!}(k_{\overline{\Omega}} \otimes \rho_! k_{P_\sigma}) \otimes \rho_! \delta_* \omega^{\otimes -1}_{\Delta|X\times X}. \tag{5.4.4}$$

PROPOSITION 5.4.2. — (i) *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. There is a natural arrow*

$$\varphi : R\mathcal{H}om(F, K_\sigma \circ G) \longrightarrow \sigma^{-1}\mu hom^{\mathrm{sa}}(F,G),$$

where $K_\sigma \circ G$ means $Rq_{1!!}(q_2^{-1}G \otimes K_\sigma)$.

(ii) *Let $\rho : X \to X_{\mathrm{sa}}$ be the natural functor of sites. Then $\rho^{-1}(\varphi)$ is an isomorphism.*

Proof. — $(\mathrm{i})_a$ Let $H \in D^{\mathrm{b}}(k_{TX_{\mathrm{sa}}})$. We have the chain of isomorphisms

$$\sigma^{-1}(H^\wedge) \simeq \sigma^{-1}R\pi_{1!!}(\pi_2^{-1}H \otimes k_{P'}) \simeq R\tau_{X!!}\sigma'^{-1}(\pi_2^{-1}H \otimes k_{P'}) \simeq R\tau_{X!!}(H \otimes k_{P'_\sigma}).$$

Consider the normal deformation of Δ in $X \times X$ visualized by the diagram (5.2.1). We have

$$\begin{aligned}\mu hom^{\mathrm{sa}}(F,G) &\simeq \big(s^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p^{-1}q_1^{-1}G) \otimes s^{-1}p^{-1}q_1^! k_X\big)^\wedge \\ &\simeq \big(s^{-1}\, R\mathcal{H}om(p_2^{-1}F, \mathrm{R}\Gamma_\Omega p_1^{-1}G) \otimes s^{-1}p^{-1}q_1^! k_X\big)^\wedge \\ &\simeq \big(s^{-1}\, R\mathcal{H}om(p_2^{-1}F, \mathrm{R}\Gamma_\Omega p_1^{-1}G) \otimes \tau_X^{-1}\omega^{\otimes -1}_{\Delta|X\times X}\big)^\wedge \\ &\simeq \big(s^{-1}\, R\mathcal{H}om(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}}) \otimes \tau_X^{-1}\omega^{\otimes -1}_{\Delta|X\times X}\big)^\wedge,\end{aligned}$$

where the second isomorphism follows from Lemma 5.2.2 and the last one follows from Lemma 5.4.1. Hence we get

$$\begin{aligned}\sigma^{-1}\mu hom^{\mathrm{sa}}(F,G) &\simeq R\tau_{X!!}\big(s^{-1}\, R\mathcal{H}om(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}}) \otimes \tau_X^{-1}\omega^{\otimes -1}_{\Delta|X\times X} \otimes k_{P'_\sigma}\big) \\ &\simeq Rp_{2!!}s_{!!}\big(s^{-1}\, R\mathcal{H}om(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}}) \otimes \tau_X^{-1}\omega^{\otimes -1}_{\Delta|X\times X} \otimes k_{P'_\sigma}\big) \\ &\simeq Rp_{2!!}\big(\, R\mathcal{H}om(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}}) \otimes p^{-1}\delta_*\omega^{\otimes -1}_{\Delta|X\times X} \otimes k_{P'_\sigma}\big).\end{aligned}$$

$(\mathrm{i})_b$ On the other hand we have the chain of isomorphisms

$$\begin{aligned}K_\sigma \circ G &\simeq Rq_{1!!}\big(q_2^{-1}G \otimes Rp_{!!}(k_{\overline{\Omega}} \otimes \rho_! k_{P_\sigma}) \otimes \rho_!\delta_*\omega^{\otimes -1}_{\Delta|X\times X}\big) \\ &\simeq Rp_{1!!}\big(p_2^{-1}G \otimes k_{\overline{\Omega}} \otimes \rho_! k_{P_\sigma} \otimes p^{-1}\rho_!\delta_*\omega^{\otimes -1}_{\Delta|X\times X}\big) \\ &\simeq Rp_{2!!}\big(p_1^{-1}G \otimes k_{\overline{\Omega}} \otimes \rho_! k_{P'_\sigma} \otimes p^{-1}\rho_!\delta_*\omega^{\otimes -1}_{\Delta|X\times X}\big) \\ &\simeq Rp_{2!!}\big(p_1^{-1}G \otimes k_{\overline{\Omega}} \otimes \rho_!(k_{P'_\sigma} \otimes p^{-1}\delta_*\omega^{\otimes -1}_{\Delta|X\times X})\big).\end{aligned}$$

Hence we get

$$\begin{aligned}R\mathcal{H}om(F, K_\sigma \circ G) &\simeq R\mathcal{H}om(F, Rp_{2!!}\big(p_1^{-1}G \otimes k_{\overline{\Omega}} \otimes \rho_!(k_{P'_\sigma} \otimes p^{-1}\delta_*\omega^{\otimes -1}_{\Delta|X\times X}))\big) \\ &\simeq Rp_{2!!}\, R\mathcal{H}om\,\big(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}} \otimes \rho_!(k_{P'_\sigma} \otimes p^{-1}\delta_*\omega^{\otimes -1}_{\Delta|X\times X})\big) \\ &\simeq Rp_{2!!}\big(\, R\mathcal{H}om(p_2^{-1}F, p_1^{-1}G \otimes k_{\overline{\Omega}}) \otimes \rho_!(k_{P'_\sigma} \otimes p^{-1}\delta_*\omega^{\otimes -1}_{\Delta|X\times X})\big).\end{aligned}$$

$(\mathrm{i})_c$ The adjunction morphism defines a morphism $\rho_! \to \rho_*$. It induces the morphism

$$\varphi : R\mathcal{H}om(F, K_\sigma \circ G) \longrightarrow \sigma^{-1}\mu hom^{\mathrm{sa}}(F,G).$$

(ii) Composing with ρ^{-1} we get $\rho^{-1}\circ\rho_! \xrightarrow{\sim} \rho^{-1}\circ\rho_* \simeq \mathrm{id}$. Hence we get

$$\rho^{-1}(\varphi):\rho^{-1}\,R\mathcal{H}om(F,K_\sigma\circ G)\xrightarrow{\sim}\rho^{-1}\sigma^{-1}\mu hom^{\mathrm{sa}}(F,G).\qquad\square$$

Let $\pi : T^*X \to X$ be the projection and consider the canonical 1-form ω, the restriction to the diagonal of the map ${}^t\pi'$. Replace X with T^*X and σ with ω in (5.4.4) and consider the microlocal kernel

$$K_\omega = Rp_{!!}(k_{\overline{\Omega}}\otimes\rho_! k_{P_\omega})\otimes\rho_!\delta_*\omega^{\otimes -1}_{\Delta_{T^*X}|T^*X\times T^*X}$$

Definition 5.4.3. — The *functor of microlocalization of* [**19**] is defined as

$$\mu : D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})\longrightarrow D^{\mathrm{b}}(k_{T^*X_{\mathrm{sa}}}),\quad F\longmapsto \mu F = K_\omega\circ\pi^{-1}F.$$

Remark 5.4.4. — The functor μ of [**19**] was defined for ind-sheaves. The above definition for subanalytic sheaves corresponds to the original one thanks to the compatibility conditions of §A.2.

Theorem 5.4.5. — (i) *Let $F\in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $G\in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$. There is a natural arrow*

$$\varphi : R\mathcal{H}om(\pi^{-1}F,\mu G)\longrightarrow \mu hom^{\mathrm{sa}}(F,G). \tag{5.4.5}$$

(ii) *Let $\rho : T^*X\to T^*X_{\mathrm{sa}}$ be the natural functor of sites. Then $\rho^{-1}(\varphi)$ is an isomorphism.*

Proof. — (i) By Lemma 5.2.6 and Proposition 5.4.2 (i) we get the morphisms

$$\mu hom^{\mathrm{sa}}(F,G)\simeq\omega^{-1}\mu hom^{\mathrm{sa}}(\pi^{-1}F,\pi^{-1}G)\longleftarrow R\mathcal{H}om(\pi^{-1}F,\mu G).$$

(ii) The result follows from Proposition 5.4.2 (ii). $\square$

Example 5.4.6. — The morphism (5.4.5) is not an isomorphism in general. For example let $F\in\mathrm{Mod}(k_{X_{\mathrm{sa}}})$. Then

$$R\mathcal{H}om(\pi^{-1}k_X,\mu F)\simeq R\mathcal{H}om(k_{T^*X},\mu F)\simeq\mu F;$$

on the other hand we have

$$\mu hom^{\mathrm{sa}}(k_X,F)\simeq j_*\mu^{\mathrm{sa}}_X F\simeq j_*F,$$

where $j : T^*_XX\hookrightarrow T^*X$ denotes the embedding of the zero section.

CHAPTER 6

HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS

We show how the functors we defined before generalize classical constructions. In § 6.3 we show the relation between specialization of Whitney holomorphic functions with the functor of formal specialization of [**6**], and the sheaf of asymptotically developable functions of [**22**] and [**36**]. In § 6.4 we study the microlocalization of tempered and Whitney holomorphic functions. We establish a relation with the functors of tempered and formal microlocalization introduced by Andronikof in [**1**] and Colin in [**5**].

6.1. Review on temperate and formal cohomology

From now on, the base field is $\mathbb{C}$. Let M be a real analytic manifold. One denotes

▷ $\mathcal{D}b_M$ the sheaf of Schwartz's distributions,
▷ $\mathcal{C}^\infty_M$ the sheaf of $\mathcal{C}^\infty$-functions,
▷ $\mathcal{A}_M$ the sheaf of analytic functions,
▷ $\mathcal{D}_M$ the sheaf of finite order differential operators with analytic coefficients.

Given a morphism $f : M \to N$ of real analytic manifolds, let $\mathcal{D}_{M\to N}$ and $\mathcal{D}_{N\leftarrow M}$ be the transfer bimodules. They are defined by

$$\mathcal{D}_{M\to N} = \mathcal{A}_M \otimes_{f^{-1}\mathcal{A}_N} f^{-1}\mathcal{A}_N,$$
$$\mathcal{D}_{N\leftarrow M} = \mathcal{A}_M^{\vee\otimes -1} \otimes_{\mathcal{A}_M} \mathcal{D}_{M\to N} \otimes_{f^{-1}\mathcal{A}_N} f^{-1}\mathcal{A}_N^{\vee\otimes -1},$$

where $\mathcal{A}_M^\vee$ (resp. $\mathcal{A}_N^\vee$) denotes the sheaf of real analytic densities (i.e. the tensor product in M (resp. N) between the sheaf of real analytic differential forms of maximal degree and the orientation sheaf).

In [**12**] the author defined the functor

$$T\mathcal{H}om(\,.\,, \mathcal{D}b_M) : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \longrightarrow \mathrm{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic subset of M and $Z = M \setminus U$. Then the sheaf $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)$ is defined by the exact sequence

$$0 \to \Gamma_Z \mathcal{D}b_M \longrightarrow \mathcal{D}b_M \longrightarrow T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M) \to 0.$$

This functor is exact and extends as a functor in the derived category, from $D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ to $D^{\mathrm{b}}(\mathcal{D}_M)$. Moreover the sheaf $T\mathcal{H}om(F, \mathcal{D}b_M)$ is soft for any $\mathbb{R}$-constructible sheaf F.

Definition 6.1.1. — Let Z be a closed subset of M. We denote by $\mathcal{I}^{\infty}_{M,Z}$ the sheaf of $\mathcal{C}^{\infty}$-functions on M vanishing up to infinite order on Z.

Definition 6.1.2. — A *Whitney function on a closed subset Z of M* is an indexed family $F = (F^k)_{k \in \mathbb{N}^n}$ consisting of continuous functions on Z such that for all $m \in \mathbb{N}$, for all $k \in \mathbb{N}^n$, $|k| \leq m$, for all $x \in Z$, for all $\varepsilon > 0$, there exists a neighborhood U of x such that for all $y, z \in U \cap Z$

$$\Big| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \Big| \leq \varepsilon d(y,z)^{m-|k|}.$$

We denote:

▷ $W^{\infty}_{M,Z}$ the space of Whitney $\mathcal{C}^{\infty}$-functions on Z,

▷ $\mathcal{W}^{\infty}_{M,Z}$ the sheaf $U \mapsto W^{\infty}_{U,U\cap Z}$.

In [**15**] the authors defined the functor

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \longrightarrow \mathrm{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M = \mathcal{I}^{\infty}_{M,Z} \quad \text{and} \quad \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M = \mathcal{W}^{\infty}_{M,Z}.$$

This functor is exact and extends as a functor in the derived category, from $D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ to $D^{\mathrm{b}}(\mathcal{D}_M)$. Moreover the sheaf $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M$ is soft for any $\mathbb{R}$-constructible sheaf F.

Now let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. The product $X \times \overline{X}$ is a complexification of $X_{\mathbb{R}}$ by the diagonal embedding $X_{\mathbb{R}} \hookrightarrow X \times \overline{X}$. One denotes by $\mathcal{O}_X$ the sheaf of holomorphic functions and by $\mathcal{D}_X$ the sheaf of finite order differential operators with holomorphic coefficients. For $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ one sets

$$T\mathcal{H}om(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{D}b_{X_{\mathbb{R}}})),$$
$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X_{\mathbb{R}}}),$$

and these functors are called the functors of temperate and formal cohomology respectively.

6.2. Tempered and Whitney holomorphic functions

Definition 6.2.1. — One denotes by $\mathcal{D}b^t_M$ the presheaf of tempered distributions on M_{sa} defined as

$$U \longmapsto \Gamma(M;\mathcal{D}b_M)/\Gamma_{M\setminus U}(M;\mathcal{D}b_M).$$

As a consequence of the Łojasievicz's inequalities [**20**], for $U,V \in \mathrm{Op}(M_{\mathrm{sa}})$ the sequence

$$0 \to \mathcal{D}b^t_M(U\cup V) \longrightarrow \mathcal{D}b^t_M(U)\oplus\mathcal{D}b^t_M(V) \longrightarrow \mathcal{D}b^t_M(U\cap V)\to 0$$

is exact. Then $\mathcal{D}b^t_M$ is a sheaf on M_{sa}. Moreover, by definition $\mathcal{D}b^t_M$ is quasi-injective.

Definition 6.2.2. — One denotes by $\mathcal{C}^{\infty,\mathrm{w}}_M$ the presheaf of Whitney $\mathcal{C}^\infty$-functions on M_{sa} defined as follows:

$$U \longmapsto \Gamma(M;H^0D'\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M).$$

As a consequence of a result of [**21**], for $U,V\in\mathrm{Op}(M_{\mathrm{sa}})$ the sequence

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_M(U\cup V) \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_M(U)\oplus\mathcal{C}^{\infty,\mathrm{w}}_M(V) \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_M(U\cap V)$$

is exact. Then $\mathcal{C}^{\infty,\mathrm{w}}_M$ is a sheaf on M_{sa}. Moreover if $U\in\mathrm{Op}(M_{\mathrm{sa}})$ is locally cohomologically trivial (l.c.t. for short), i.e. if $D'\mathbb{C}_U\simeq\mathbb{C}_{\overline{U}}$, the morphism

$$\Gamma(M;\mathcal{C}^{\infty,\mathrm{w}}_M) \longrightarrow \Gamma(U;\mathcal{C}^{\infty,\mathrm{w}}_M)$$

is surjective and $\mathrm{R}\Gamma(U;\mathcal{C}^{\infty,\mathrm{w}}_M)$ is concentrated in degree zero.

We have the following result (see [**16**], [**28**]).

Proposition 6.2.3. — *For each $F\in\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ one has the isomorphisms*

$$\rho^{-1}\,R\mathcal{H}om(F,\mathcal{D}b^t_M)\simeq T\mathcal{H}om(F,\mathcal{D}b_M),\quad \rho^{-1}\,R\mathcal{H}om(F,\mathcal{C}^{\infty,\mathrm{w}}_M)\simeq D'F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_M.$$

Now let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. One denotes by $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ the sheaves of tempered and Whitney holomorphic functions defined as follows:

$$\mathcal{O}^t_X := R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}},\mathcal{D}b^t_{X_{\mathbb{R}}}),\quad \mathcal{O}^{\mathrm{w}}_X := R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}},\mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathbb{R}}}).$$

The relation with the functors of temperate and formal cohomology are given by the following result (see [**16**], [**28**])

Proposition 6.2.4. — *For each $F\in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ one has the isomorphisms*

$$T\mathcal{H}om(F,\mathcal{O}_X)\simeq\rho^{-1}\,R\mathcal{H}om(F,\mathcal{O}^t_X),\quad D'F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X\simeq\rho^{-1}\,R\mathcal{H}om(F,\mathcal{O}^{\mathrm{w}}_X).$$

6.3. Asymptotic expansions

Let M be a real analytic manifold. We consider a slight generalization of the sheaf of Whitney $\mathcal{C}^\infty$-functions of [**16**].

Definition 6.3.1. — Let $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We define the presheaf $\mathcal{C}^{\infty,\mathrm{w}}_{M|F}$ on M_{sa} as follows:

$$U \longmapsto \Gamma\big(M; (H^0D'\mathbb{C}_U \otimes F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M\big).$$

Let $U, V \in \mathrm{Op}(M_{\mathrm{sa}})$, and consider the exact sequence

$$0 \to \mathbb{C}_{U\cap V} \longrightarrow \mathbb{C}_U \oplus \mathbb{C}_V \longrightarrow \mathbb{C}_{U\cup V} \to 0,$$

applying the functor $\mathcal{H}om(\,.\,, \mathbb{C}_M) = H^0D'(.)$ we obtain

$$0 \to H^0D'\mathbb{C}_{U\cup V} \longrightarrow H^0D'\mathbb{C}_U \oplus H^0D'\mathbb{C}_V \longrightarrow H^0D'\mathbb{C}_{U\cap V},$$

applying the exact functors $\cdot \otimes F$, $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M$ and taking global sections we obtain

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{M|F}(U\cup V) \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_{M|F}(U) \oplus \mathcal{C}^{\infty,\mathrm{w}}_{M|F}(V) \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_{M|F}(U\cap V).$$

This implies that $\mathcal{C}^{\infty,\mathrm{w}}_{M|F}$ is a sheaf on M_{sa}. Moreover if $U \in \mathrm{Op}(M_{\mathrm{sa}})$ is l.c.t., the morphism $\Gamma(M; \mathcal{C}^{\infty,\mathrm{w}}_{M|F}) \to \Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_{M|F})$ is surjective and $\mathrm{R}\Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_{M|F})$ is concentrated in degree zero. Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, we obtain an exact sequence in $\mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{M|F} \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_{M|G} \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_{M|H} \to 0. \tag{6.3.1}$$

We can easily extend the sheaf $\mathcal{C}^{\infty,\mathrm{w}}_{M|F}$ to the case of $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, taking a finite resolution of F consisting of locally finite sums $\oplus \mathbb{C}_V$, with V l.c.t. in $\mathrm{Op}^{\mathrm{c}}(M_{\mathrm{sa}})$. The sheaves $\mathcal{C}^{\infty,\mathrm{w}}_{M|\oplus\mathbb{C}_V}$ form a complex quasi-isomorphic to $\mathcal{C}^{\infty,\mathrm{w}}_{M|F}$ consisting of acyclic objects with respect to $\Gamma(U;\cdot)$, where U is l.c.t. in $\mathrm{Op}^{\mathrm{c}}(M_{\mathrm{sa}})$.

As in the case of Whitney $\mathcal{C}^\infty$-functions one can prove that, if $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$,

$$\rho^{-1}\, R\mathcal{H}om(G, \mathcal{C}^{\infty,\mathrm{w}}_{M|F}) \simeq (D'G \otimes F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M.$$

Example 6.3.2. — Setting $F = \mathbb{C}_M$ we obtain the sheaf of Whitney $\mathcal{C}^\infty$-functions. Let N be a closed analytic submanifold of M. Then $\mathcal{C}^{\infty,\mathrm{w}}_{M|\mathbb{C}_{M\setminus N}}$ is the sheaf of Whitney $\mathcal{C}^\infty$-functions vanishing on N with all their derivatives.

Notations 6.3.3. — Let Z be a locally closed subanalytic subset of M. We set for short $\mathcal{C}^{\infty,\mathrm{w}}_{M|Z}$ instead of $\mathcal{C}^{\infty,\mathrm{w}}_{M|\mathbb{C}_Z}$.

Let N be a closed analytic submanifold of M, let $T_NM \overset{\tau}{\to} N$ be the normal vector bundle and consider the normal deformation $\widetilde{M}_N$ as in §4.1.

Set $F = \mathbb{C}_{M\setminus N}$, $G = \mathbb{C}_M$, $H = \mathbb{C}_N$ in (6.3.1). The exact sequence

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{M|M\setminus N} \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_M \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_{M|N} \to 0$$

induces an exact sequence

$$0 \to \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|M\setminus N} \longrightarrow \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M} \longrightarrow \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|N} \to 0,$$

in fact let V be a l.c.t. conic subanalytic open subset of $T_N M$ and $U \in \mathrm{Op}(M_{\mathrm{sa}})$ such that $C_N(M \setminus U) \cap V = \varnothing$, then we can find a l.c.t. $U' \subset U$ satisfying the same property. Moreover it is easy to see that $\nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|N} \simeq \tau^{-1} \mathcal{C}^{\infty,\mathrm{w}}_{M|N}$, hence we get the exact sequence

$$0 \to \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|M\setminus N} \longrightarrow \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M} \longrightarrow \tau^{-1} \mathcal{C}^{\infty,\mathrm{w}}_{M|N} \to 0. \tag{6.3.2}$$

Remark 6.3.4. — Let $G \in D^{\mathrm{b}}(\rho_! \mathcal{D}_M)$. Then

$$\nu_N^{\mathrm{sa}} G \in D^{\mathrm{b}}(\rho_! \tau^{-1} i^{-1} \mathcal{D}_M).$$

Now let us study the relation with the constructions of [**6**]. In that work the author defined the functor of Whitney specialization as follows: let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, then

$$w\nu_N(F, \mathcal{C}^\infty_M) = s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M}_N}}\big(\mathcal{D}_{\widetilde{M}_N \to M}, (p^{-1}F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{\widetilde{M}_N}\big).$$

It is an object of $D^{\mathrm{b}}(\tau^{-1} i^{-1} \mathcal{D}_M)$. The stalks are given by the following formula: let $v \in T_N M$. Then

$$H^k\big(w\nu_N(F, \mathcal{C}^\infty_M)\big)_v \simeq \varinjlim_U H^k(M; F_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M), \tag{6.3.3}$$

where $U \in \mathrm{Op}(M_{\mathrm{sa}})$ l.c.t. such that $v \notin C_N(M \setminus U)$.

Proposition 6.3.5. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, there is a natural isomorphism in $D^{\mathrm{b}}(\tau^{-1} i^{-1} \mathcal{D}_M)$*

$$w\nu_N(F, \mathcal{C}^\infty_M) \simeq \rho^{-1} \nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|F}.$$

This means that Whitney specialization is obtained by specializing the sheaf $\mathcal{C}^{\infty,\mathrm{w}}_{M|F}$.

Proof. — We have the chain of morphisms in $D^{\mathrm{b}}(p^{-1}\mathcal{D}_M)$

$$\begin{aligned}
\rho^{-1}\mathrm{R}\Gamma_\Omega p^{-1} \mathcal{C}^{\infty,\mathrm{w}}_{M|F} \leftarrow \rho^{-1}\mathrm{R}\Gamma_\Omega p^! \mathcal{C}^{\infty,\mathrm{w}}_{M|F}[-1]& \\
\simeq \rho^{-1}\mathrm{R}\Gamma_\Omega\, R\mathcal{H}om_{\rho_!\mathcal{D}_{\widetilde{M}_N}}(\rho_! \mathcal{D}_{\widetilde{M}_N \to M}, \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{M}_N|p^{-1}F})& \\
\simeq R\mathcal{H}om_{\mathcal{D}_{\widetilde{M}_N}}(\mathcal{D}_{\widetilde{M}_N \to M}, \rho^{-1}\mathrm{R}\Gamma_\Omega \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{M}_N|p^{-1}F})& \\
\simeq R\mathcal{H}om_{\mathcal{D}_{\widetilde{M}_N}}(\mathcal{D}_{\widetilde{M}_N \to M}, (p^{-1}F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{\widetilde{M}_N}\big)&.
\end{aligned}$$

The first isomorphism follows from (A.4.3) and the last one follows since $D'\mathbb{C}_\Omega \simeq \mathbb{C}_{\overline{\Omega}}$. Applying the functor s^{-1} we obtain $w\nu_N(F, \mathcal{C}^\infty_M) \to \rho^{-1}\nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|F}$. Let $v \in T_N M$. By (6.3.3) and Theorem 4.2.2 (ii) it turns out that

$$H^k\big(w\nu_N(F, \mathcal{C}^\infty_M)\big)_v \simeq \varinjlim_U H^k(M; F_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M) \simeq \varinjlim_U H^k(U; \mathcal{C}^{\infty,\mathrm{w}}_{M|F}) \simeq H^k(\rho^{-1}\nu_N^{\mathrm{sa}} \mathcal{C}^{\infty,\mathrm{w}}_{M|F})_v,$$

where $U \in \mathrm{Op}(M_{\mathrm{sa}})$ l.c.t. such that $v \notin C_N(M \setminus U)$. This completes the proof. □

Assume that $M \simeq \{(x,y) \in \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}\}$ and $N \simeq \{0\} \times \mathbb{R}^{n-\ell}$.

▷ A *sector* S of M is a subanalytic open subset $S = U \times V$ with $U \in \mathrm{Op}(\mathbb{R}^{n-\ell}_{\mathrm{sa}})$ and $V = W \cap B(0,\varepsilon)$, where $W \in \mathrm{Op}(\mathbb{R}^\ell_{\mathrm{sa},\mathbb{R}^+})$ and $B(0,\varepsilon)$ is the open ball of center 0 and radius ε.

▷ We say that S' is a *subsector* of S if $\overline{S}' \setminus N \subset S$. We write for short $S' < S$.

Definition 6.3.6. — Let S be an open sector of M and let $f \in \mathcal{C}^\infty_M$. One says that *$f$ is asymptotically developable on S along M*, if there exists a formal series

$$\sum_{k \in \mathbb{N}^\ell} a_k(x) y^k$$

with $\mathcal{C}^\infty$ coefficients a_k such that, for all $S' < S$, $m \in \mathbb{N}$, there exists $C > 0$ such that

$$\forall (x,y) \in S', \quad \Big| f(x,y) - \sum_{|k| \leq m} a_k(x) y^k \Big| \leq C \|y\|^{m+1}.$$

One denotes by

▷ $\sigma_N(S) \subset \mathcal{C}^\infty_M$ the space of functions asymptotically developable along M,

▷ $\sigma^\infty_M = \{f \in \mathcal{C}^\infty_M(S),\ \forall k \in \mathbb{N}^\ell,\ D^k f \in \sigma_M(S)\}$.

Recall (see [**14**]) that locally we may assume

$$M \simeq \big\{(x,y) \in \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}\big\}, \quad N \simeq \{0\} \times \mathbb{R}^{n-\ell}$$

and we may identify $M \simeq T_N M$. A sector $S \subset M$ means a sector in the local model. We have the following result (see Proposition 2.10 of [**6**]).

Proposition 6.3.7. — *Let S be a sector of M. Then*

$$\Gamma(\mathbb{R}^+ S; \rho^{-1} \nu^{\mathrm{sa}}_N \mathcal{C}^{\infty,\mathrm{w}}_M) \simeq \sigma^\infty_N(S)$$

and $\Gamma(\mathbb{R}^+ S; \rho^{-1} \nu^{\mathrm{sa}}_N \mathcal{C}^{\infty,\mathrm{w}}_{M|M\setminus N})$ is the subspace of functions asymptotically developable to the identically zero series.

Applying the functor ρ^{-1} to the exact sequence (6.3.2) we obtain the exact sequence

$$0 \to \rho^{-1} \nu^{\mathrm{sa}}_N \mathcal{C}^{\infty,\mathrm{w}}_{M|M\setminus N} \longrightarrow \rho^{-1} \nu^{\mathrm{sa}}_N \mathcal{C}^{\infty,\mathrm{w}}_M \longrightarrow \rho^{-1} \tau^{-1} \mathcal{C}^{\infty,\mathrm{w}}_{M|N} \to 0,$$

where the surjective arrow is the map which associates with a function its asymptotic expansion.

Let X be a complex manifold and let Z be a complex submanifold of X. Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We denote by $\mathcal{O}^{\mathrm{w}}_{X|F}$ the sheaf defined as follows:

$$\mathcal{O}^{\mathrm{w}}_{X|F} := R\mathcal{H}om_{\rho_! \mathcal{D}_{\overline{X}}}(\rho_! \mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}|F}).$$

Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Then the exact sequence (6.3.1) gives rise to the distinguished triangle

$$\mathcal{O}^{\mathrm{w}}_{X|F} \longrightarrow \mathcal{O}^{\mathrm{w}}_{X|G} \longrightarrow \mathcal{O}^{\mathrm{w}}_{X|H} \xrightarrow{+} . \tag{6.3.4}$$

If we consider the functor of specialization of formal cohomology of [**6**]

$$w\nu_Z(F,\mathcal{O}_X) = R\mathcal{H}om_{\tau^{-1}\mathcal{D}_{\overline{X}}}\left(\tau^{-1}\mathcal{O}_{\overline{X}}, w\nu_Z(F,\mathcal{C}^\infty_{X_\mathbb{R}})\right),$$

we have the isomorphism

$$w\nu_Z(F,\mathcal{O}_X) \simeq \rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|F}.$$

Setting $F=\mathbb{C}_{X\setminus Z}$, $G=\mathbb{C}_X$, $H=\mathbb{C}_Z$ in (6.3.4) and applying the functor of specialization, we have the distinguished triangle

$$\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z} \longrightarrow \rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_X \longrightarrow \rho^{-1}\tau^{-1}\mathcal{O}^{\mathrm{w}}_{X|Z} \xrightarrow{+}. \tag{6.3.5}$$

The sheaves $\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_X$ and $\rho^{-1}\tau^{-1}\mathcal{O}^{\mathrm{w}}_{X|Z}$ are concentrated in degree zero. This follows from the following result of [**8**]: in the local model if $U\in \mathrm{Op}(X_{\mathrm{sa}})$ is convex, then $\mathrm{R}\Gamma(X;\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ is concentrated in degree zero. Moreover (see [**15**]) the sheaf $\rho^{-1}\mathcal{O}^{\mathrm{w}}_{X|Z}$ is isomorphic to the sheaf $\mathcal{O}_X\widehat{|}_Z$, the formal completion of $\mathcal{O}_X$ along Z. We have an exact sequence

$$0\to\rho^{-1}H^0\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}\longrightarrow\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}_X\to\tau^{-1}\mathcal{O}_X\widehat{|}_Z\longrightarrow\rho^{-1}H^1\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}\to 0.$$

Let $\sigma_Z^{\mathrm{hol}}(S)$ be the space of holomorphic functions asymptotically developable in S, having an asymptotic expansion with holomorphic coefficients. We have the following results of [**6**].

Proposition 6.3.8. — *Let S be a sector of X. Then*

$$\Gamma(\mathbb{R}^+S;\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_X)\simeq\sigma_Z^{\mathrm{hol}}(S)$$

and $\Gamma(\mathbb{R}^+S;\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z})$ is the subspace of functions asymptotically developable to the identically zero series.

Proposition 6.3.9. — *The distinguished triangle* (6.3.5) *induces an exact sequence outside the zero section*

$$0\to\rho^{-1}H^0\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|Z}|_{\dot{T}_ZX}\longrightarrow\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_X|_{\dot{T}_ZX}\longrightarrow\dot{\tau}^{-1}\,\mathcal{O}_X\widehat{|}_Z\to 0. \tag{6.3.6}$$

On the zero section we have the exact sequence

$$0\to\mathcal{O}_X|_Z\longrightarrow\mathcal{O}_X\widehat{|}_Z\longrightarrow\rho^{-1}H^1\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}|_Z\to 0. \tag{6.3.7}$$

Remark that on the exact sequence (6.3.7) we used Theorem 4.2.2 (iii) and the fact that $\rho^{-1}\mathcal{O}^{\mathrm{w}}_X\simeq\mathcal{O}_X$.

Example 6.3.10. — Set $X=\mathbb{C}$ and $Z=\{0\}$. Outside the zero section the sheaves $\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}$ and $\rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}^{\mathrm{w}}_X$ are the well-known sheaves $\mathcal{A}_0$ and $\mathcal{A}$ of Malgrange [**22**] and Sibuya [**36**]. These sheaves were defined in the real blow-up of the origin of $\mathbb{C}$ identified with $\mathbb{S}^1\times(\mathbb{R}^+\cup\{0\})$. Let π be the projection on $\mathbb{C}$. The sequence (6.3.6) is a generalization of the exact sequence in $\mathrm{Mod}(\mathbb{C}_{\mathbb{S}^1})$

$$0\to\mathcal{A}_0\longrightarrow\mathcal{A}\longrightarrow\pi^{-1}\mathbb{C}[[z]]\to 0,$$

and the sequence (6.3.7) is a generalization of the exact sequence

$$0 \to \mathbb{C}\{z\} \longrightarrow \mathbb{C}[[z]] \longrightarrow H^1(\mathbb{S}^1; \mathcal{A}_0) \to 0.$$

Example 6.3.11. — Let $X = \mathbb{C}$ and $Z = \{0\}$. The sheaf $\mathcal{A}$ is an example of the fact that specialization does not commute with ρ^{-1}. Indeed $\rho^{-1}\mathcal{O}_X^{\mathrm{w}} \simeq \mathcal{O}_X$ and $\mathcal{A} \simeq \rho^{-1}\nu_Z^{\mathrm{sa}}\mathcal{O}_X^{\mathrm{w}} \not\simeq \nu_Z\mathcal{O}_X$ (outside the zero section).

6.4. Microlocalization of $\mathcal{O}_X^t$ and $\mathcal{O}_X^{\mathrm{w}}$

Let $f : M \to N$ be a smooth morphism of real analytic manifolds. We have the following results (see the Appendix):

$$R\mathcal{H}om_{\rho_!\mathcal{D}_M}(\rho_!\mathcal{D}_{M\to N}, \mathcal{D}b_M^t) \simeq f^{-1}\mathcal{D}b_N^t,$$
$$R\mathcal{H}om_{\rho_!\mathcal{D}_M}(\rho_!\mathcal{D}_{M\to N}, \mathcal{C}_M^{\infty,\mathrm{w}}) \simeq f^{-1}\mathcal{C}_N^{\infty,\mathrm{w}}.$$

Let us consider the normal deformation of the diagonal in $M\times M$ of diagram (5.2.1). Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We recall the definitions of the Andronikof's functor of microlocalization of tempered distributions

$$t\mu hom(F, \mathcal{D}b_M) := \big(s^{-1}\big(\mathcal{D}_{M \xleftarrow{p_1} \widetilde{M\times M}} \otimes_{\mathcal{D}_{\widetilde{M\times M}}} T\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{D}b_{\widetilde{M\times M}})[-1]\big)\big)^\wedge$$

and Colin's microlocalization of the Whitney tensor product

$$F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{C}_M^\infty := \big(s^{-1}\, R\mathcal{H}om\,\big(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1} M}, (p_2^{-1}F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{\widetilde{M\times M}}\big)\big)^\vee.$$

They are objects of $D^{\mathrm{b}}(\mathcal{D}_M)$.

Remark 6.4.1. — Let N be a closed submanifold of M and consider the projection $\pi : T_N^*M \to N$ as in § 5.1. Let $H \in D^{\mathrm{b}}(\rho_!\mathcal{D}_M)$. As in Remark 6.3.4, $\mu_M^{\mathrm{sa}}H$ is an element of $D^{\mathrm{b}}(\rho_!\pi^{-1}i^{-1}\mathcal{D}_M)$. In particular in the case of the diagonal $\delta : \Delta \hookrightarrow M\times M$, if $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ and $G \in D^{\mathrm{b}}(\rho_!\mathcal{D}_M)$ we have $\mu hom^{\mathrm{sa}}(F, G) \in D^{\mathrm{b}}(\rho_!\mathcal{D}_M)$ (we do not write π^{-1} to lighten notations).

Theorem 6.4.2. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We have the isomorphisms in $D^{\mathrm{b}}(\mathcal{D}_M)$*

$$\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{D}b_M^t) \simeq t\mu hom(F, \mathcal{D}b_M), \tag{6.4.1}$$

$$\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{C}_M^{\infty,\mathrm{w}}) \simeq (D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{C}_M^\infty)^a, \tag{6.4.2}$$

where $(.)^a$ denotes the direct image of the antipodal map.

Proof. — Let $G \in D^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sa}}})$. By Lemma 5.2.2 we have

$$\mu hom^{\mathrm{sa}}(F, G) \simeq (s^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}G) \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M)^\wedge.$$

(i) Let us prove (6.4.1). Setting $G = \mathcal{D}b_M^t$ and composing with ρ^{-1} we have

$$\begin{aligned}
&\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{D}b_M^t)\\
&\quad\simeq \big(s^{-1}\rho^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}\mathcal{D}b_M^t) \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M\big)^\wedge\\
&\quad\simeq \big(s^{-1}\rho^{-1}\, R\mathcal{H}om_{\rho_!\mathcal{D}_{\widetilde{M\times M}}}(\rho_!\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, R\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{D}b^t_{\widetilde{M\times M}}))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M\big)^\wedge\\
&\quad\simeq \big(s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, \rho^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{D}b^t_{\widetilde{M\times M}}))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M\big)^\wedge\\
&\quad\simeq \big(s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, T\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{D}b_{\widetilde{M\times M}}))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M\big)^\wedge\\
&\quad\simeq \big(s^{-1}(\mathcal{D}_{M\xleftarrow{p_1}\widetilde{M\times M}} \otimes_{\mathcal{D}_{\widetilde{M\times M}}} T\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{D}b_{\widetilde{M\times M}})[-1])\big)^\wedge\\
&\quad\simeq t\mu hom(F, \mathcal{D}b_M).
\end{aligned}$$

(ii) Let us prove (6.4.2). Setting $G = \mathcal{C}_M^{\infty,\mathrm{w}}$ and composing with ρ^{-1} we have

$$\begin{aligned}
&\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{C}_M^{\infty,\mathrm{w}})\\
&\quad\simeq \big(s^{-1}\rho^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}\mathcal{C}_M^{\infty,\mathrm{w}}) \otimes s^{-1}p^{-1}q_1^!\mathbb{C}_M\big)^\wedge\\
&\quad\simeq \big(s^{-1}\rho^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, p_1^{-1}\mathcal{C}_M^{\infty,\mathrm{w}})\big)^{\vee a}\\
&\quad\simeq \big(s^{-1}\rho^{-1}\, R\mathcal{H}om_{\rho_!\mathcal{D}_{\widetilde{M\times M}}}(\rho_!\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, R\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{M\times M}}))\big)^{\vee a}\\
&\quad\simeq \big(s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, \rho^{-1}\, R\mathcal{H}om((p_2^{-1}F)_\Omega, \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{M\times M}}))\big)^{\vee a}\\
&\quad\simeq \big(s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, D'(p_2^{-1}F)_\Omega \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{\widetilde{M\times M}})\big)^{\vee a}\\
&\quad\simeq \big(s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, (p_2^{-1}D'F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{\widetilde{M\times M}})\big)^{\vee a}
\end{aligned}$$

where the last isomorphism follows since

$$D'((p_2^{-1}F)_\Omega) \simeq \mathrm{R}\Gamma_\Omega D'(p_2^{-1}F) \simeq \mathrm{R}\Gamma_\Omega p_2^{-1}D'F \simeq (p_2^{-1}D'F)_{\overline{\Omega}}.$$

Here we used Lemma 5.4.1 and the fact that p_2 is smooth. We have

$$s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{M\times M}}}\big(\mathcal{D}_{\widetilde{M\times M}\xrightarrow{p_1}M}, (p_2^{-1}D'F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{\widetilde{M\times M}})\big)^{\vee a} = (D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{C}^\infty_M)^a$$

and the result follows. □

Let X be a complex manifold and let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. In [**1**] and [**5**] the authors constructed the functors $t\mu hom(F, \mathcal{O}_X)$ of tempered microlocalization and $F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X$ of formal microlocalization taking the Dolbeaut resolutions of the real ones.

THEOREM 6.4.3. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We have the isomorphisms*

$$\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X) \simeq t\mu hom(F, \mathcal{O}_X), \tag{6.4.3}$$

$$\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^{\mathrm{w}}_X) \simeq (D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X)^a, \tag{6.4.4}$$

where $(.)^a$ denotes the direct image of the antipodal map.

Proof. — The result follows by taking Dolbeaut resolutions on the left and the right-hand sides of (6.4.1) and (6.4.2). Let us see the proof of (6.4.3). Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. As pointed out in Remark 6.4.1, the $\rho_!\mathcal{D}_{\overline{X}}$-module structure of $\mathcal{D}b^t_{X_{\mathbb{R}}}$ implies that $\mu hom^{\mathrm{sa}}(F, \mathcal{D}b^t_{X_{\mathbb{R}}}) \in D^{\mathrm{b}}(\rho !\mathcal{D}_{\overline{X}})$. The coherence of $\mathcal{O}_{\overline{X}}$ implies that

$$R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}\big(\rho_!\mathcal{O}_{\overline{X}}, \mu hom^{\mathrm{sa}}(F, \mathcal{D}b^t_{X_{\mathbb{R}}})\big) \simeq \mu hom^{\mathrm{sa}}\big(F, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_{\mathbb{R}}})\big).$$

Applying the functor ρ^{-1} we have

$$\begin{aligned}
\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X) &\simeq \rho^{-1}\, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}\big(\rho_!\mathcal{O}_{\overline{X}}, \mu hom^{\mathrm{sa}}(F, \mathcal{D}b^t_{X_{\mathbb{R}}})\big)\\
&\simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\big(\mathcal{O}_{\overline{X}}, \rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{D}b^t_{X_{\mathbb{R}}})\big)\\
&\simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\big(\mathcal{O}_{\overline{X}}, t\mu hom(F, \mathcal{D}b_{X_{\mathbb{R}}})\big)\\
&\simeq t\mu hom(F, \mathcal{O}_X).
\end{aligned}$$

The proof of (6.4.4) is similar. $\square$

CHAPTER 7

INTEGRAL TRANSFORMS

We give some applications related to the microlocalization of subanalytic sheaves. We show the existence of a natural action of tempered microdifferential operators on tempered and formal microlocalization. We show also the invariance under contact transformations of tempered and formal microlocalization.

7.1. $\mathcal{E}_X$-modules

Let X be a complex manifold of complex dimension d_X. Following the notations of [**14**] one sets

$$\mathcal{E}_X^{\mathbb{R}} = H^{d_X}(\mu_\Delta \mathcal{O}_{X\times X}^{(0,d_X)}).$$

It is a sheaf of rings over T^*X and for each $F \in D^{\mathrm{b}}(\mathbb{C}_X)$, $j \in \mathbb{Z}$ the sheaf $H^j\mu hom(F, \mathcal{O}_X)$ is naturally endowed with a structure of left $\mathcal{E}_X^{\mathbb{R}}$-module.

▷ The sheaf $\mathcal{E}_X^{\mathbb{R}}$ is called the ring of *microlocal operators* on X.

▷ It contains a subring, denoted by $\mathcal{E}_X$ and called the ring of (finite-order) *microdifferential operators*. We will not recall all the properties of this sheaf and refer to [**32**] for a detailed study.

In [**1**] the author introduced the sheaf $\mathcal{E}_X^{\mathbb{R},f}$ of *tempered microdifferential operators*

$$\mathcal{E}_X^{\mathbb{R},f} := H^{d_X}\big(t\mu hom(\mathbb{C}_\Delta, \mathcal{O}_{X\times X}) \overset{L}{\underset{\mathcal{O}_{X\times X}}{\otimes}} \mathcal{O}_{X\times X}^{(0,d_X)}\big).$$

It follows from Theorem 6.4.3 that

$$\mathcal{E}_X^{\mathbb{R},f} \simeq \rho^{-1} H^{d_X} \mu_\Delta^{\mathrm{sa}} \mathcal{O}_{X\times X}^{t(0,d_X)}.$$

Let us recall the following results:

▷ the sheaf $t\mu hom(\mathbb{C}_\Delta, \mathcal{O}_{X\times X})$ is concentrated in degree d_X;

▷ one has the ring inclusions $\mathcal{E}_X \subset \mathcal{E}_X^{\mathbb{R},f} \subset \mathcal{E}_X^{\mathbb{R}}$.

7.2. Integral transforms

Let X, Y, Z be three manifolds. Let q_{ij} be the (i,j)-th projection defined on $X \times Y \times Z$ and let p_{ij} be the (i,j)-th projection defined on $T^*X \times T^*Y \times T^*Z$. Let p_{ij}^a be the composition of p_{ij} with the antipodal map a and let $\delta : X \times Y \times Z \to X \times Y \times Y \times Z$ be the diagonal embedding. We denote by $p_2 : T^*Y \xrightarrow{\sim} T^*_{\Delta_Y}(Y \times Y)$ the isomorphism induced by the second projection. Consider the diagram

$$
\begin{tikzcd}
T^*(X \times Y) \times T^*(Y \times Z) & T^*X \times T^*Y \times T^*Z \arrow[l, "p_{12}^a \times p_{23}^a"'] \arrow[d, "\wr"', "\mathrm{id} \times p_2 \times a"] \arrow[ddd, bend left=60, "p_{13}^a"] \\
T^*(X \times Y) \times_Y T^*(Y \times Z)\delta_\pi \arrow[u] \arrow[d, "{}^t\delta'"'] & T^*X \times T^*_{\Delta_Y}(Y \times Y) \times T^*Z \arrow[l] \arrow[d] \\
T^*(X \times Y \times Z) & T^*X \times Y \times T^*Z \arrow[l, "{}^tq'_{13}"] \arrow[d, "q_{13\pi}"] \\
 & T^*X \times T^*Z
\end{tikzcd} \tag{7.2.1}
$$

For $F_1 \in D^{\mathrm{b}}(k_{(X \times Y)_{\mathrm{sa}}})$ and $F_2 \in D^{\mathrm{b}}(k_{(Y \times Z)_{\mathrm{sa}}})$ set

$$F_1 \circ F_2 = Rq_{13!!}(q_{12}^{-1}F_1 \otimes q_{23}^{-1}F_2)$$

and for $G_1 \in D^{\mathrm{b}}(k_{(T^*X \times T^*Y)_{\mathrm{sa}}})$ and $G_2 \in D^{\mathrm{b}}(k_{(T^*Y \times T^*Z)_{\mathrm{sa}}})$ set

$$G_1 \overset{a}{\circ} G_2 = Rp^a_{13\mathbb{R}^+!!}(p_{12}^{a-1}G_1 \otimes p_{23}^{a-1}G_2).$$

We need this proposition which follows from the functorial properties of μhom^{sa} (it is an adaptation of Proposition 4.4.11 of [**14**]).

PROPOSITION 7.2.1. — *Let us consider the sheaves*

$$K_1 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_{X \times Y}),\ \ F_1 \in D^{\mathrm{b}}(k_{(X \times Y)_{\mathrm{sa}}}),\ \ K_2 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_{Y \times Z}),\ \ F_2 \in D^{\mathrm{b}}(k_{(Y \times Z)_{\mathrm{sa}}}).$$

Suppose that q_{13} is proper on $\mathrm{supp}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$. There is a morphism

$$\mu hom^{\mathrm{sa}}(K_1, F_1) \overset{a}{\circ} \mu hom^{\mathrm{sa}}(K_2, F_2) \longrightarrow \mu hom^{\mathrm{sa}}(K_1 \circ K_2, F_1 \circ F_2). \tag{7.2.2}$$

PROPOSITION 7.2.2. — *Let $\lambda = \varnothing, t$. Let $K_1 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{X \times Y})$ and $K_2 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{Y \times Z})$. Suppose that q_{13} is proper on $\mathrm{supp}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$. Morphism (7.2.2) defines a morphism*

$$\begin{aligned} &\mu hom^{\mathrm{sa}}(K_1, \mathcal{O}^{\lambda(0,d_Y)}_{X \times Y}) \overset{a}{\circ} \mu hom^{\mathrm{sa}}(K_2, \mathcal{O}^{\lambda(0,d_Z)}_{Y \times Z}) \\ &\qquad\qquad \longrightarrow \mu hom^{\mathrm{sa}}(K_1 \circ K_2, \mathcal{O}^{\lambda(0,d_Z)}_{X \times Z})[-d_Y]. \end{aligned} \tag{7.2.3}$$

Proof. — It follows from (7.2.2) setting $F_1 = \mathcal{O}^{\lambda(0,d_Y)}_{X \times Y}$, $F_2 = \mathcal{O}^{\lambda(0,d_Z)}_{Y \times Z}$ and using the integration morphism $\mathcal{O}^{\lambda(0,d_Y)}_{X \times Y} \circ \mathcal{O}^{\lambda(0,d_Z)}_{Y \times Z} \to \mathcal{O}^{\lambda(0,d_Z)}_{X \times Z}[-d_Y]$. ☐

COROLLARY 7.2.3. — *Morphism (7.2.2) induces the ring structures on $\mathcal{E}^{\mathbb{R}}_X$ and $\mathcal{E}^{\mathbb{R},f}_X$.*

Proof. — Apply ρ^{-1} to (7.2.3) with $X = Y = Z$ and $K_1 = K_2 = \mathbb{C}_\Delta[-d_X]$. □

PROPOSITION 7.2.4. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Morphism* (7.2.2) *defines a morphism*

$$\mu^{\mathrm{sa}}_\Delta \mathcal{O}^{\lambda(0,d_X)}_{X\times X}[d_X] \otimes \mu hom^{\mathrm{sa}}(F, \mathcal{O}^\lambda_X) \longrightarrow \mu hom^{\mathrm{sa}}(F, \mathcal{O}^\lambda_X). \tag{7.2.4}$$

Proof. — We apply Proposition 7.2.1 with $X = Y$ and $Z = \{\text{point}\}$. We set

$$(K_1, K_2, F_1, F_2) = \big(\mathbb{C}_\Delta[-d_X], F, \mathcal{O}^{\lambda(0,d_X)}_{X\times X}, \mathcal{O}^\lambda_X\big).$$

In this case we have $\mathbb{C}_\Delta \circ F \simeq F$. We obtain the desired morphism using the integration morphism $\mathcal{O}^{\lambda(0,d_X)}_{X\times X} \circ \mathcal{O}^\lambda_X \to \mathcal{O}^\lambda_X[-d_X]$. □

Applying the functor ρ^{-1} to (7.2.4), we find the morphisms of [**1**] and [**14**] (recall that $\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X) \simeq t\mu hom(F, \mathcal{O}_X)$).

COROLLARY 7.2.5. — *Morphism* (7.2.4) *induces morphisms*

$$\mathcal{E}^{\mathbb{R},f}_X \otimes \rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X) \longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X), \tag{7.2.5}$$

$$\mathcal{E}^{\mathbb{R}}_X \otimes \mu hom(F, \mathcal{O}_X) \longrightarrow \mu hom(F, \mathcal{O}_X), \tag{7.2.6}$$

which, for each $k \in \mathbb{Z}$, induce a structure of $\mathcal{E}^{\mathbb{R},f}_X$-module (resp. $\mathcal{E}^{\mathbb{R}}_X$-module) on the sheaves $H^k\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}^t_X)$ (resp. $H^k\mu hom(F, \mathcal{O}_X)$).

Now we will study the action of $\mathcal{E}^{\mathbb{R},f}_X$ on formal microlocalization. We first recall the definition of the sheaf of tempered $\mathcal{C}^\infty$-functions.

DEFINITION 7.2.6. — Let X be a real analytic manifold and let $U \in \mathrm{Op}(X)$. Let $f \in \Gamma(U; \mathcal{C}^\infty_X)$. One says that:

▷ f has *poynomial growth at* $p \in X$ if for a local coordinate system $(x_1, \dots, x_n)$ around p, there exists a compact neighborhood K of p and $N \in \mathbb{N}$ such that

$$\sup_{x \in K\cap U} \big(d(x, K \setminus U)\big)^N \big|f(x)\big| < \infty;$$

▷ f is *tempered at* p if all its derivatives have polynomial growth at p;

▷ f is *tempered* if it is tempered at any point.

DEFINITION 7.2.7. — One denotes by $\mathcal{C}^{\infty,t}_X$ the presheaf of tempered $\mathcal{C}^\infty$-functions on X_{sa} defined as follows:

$$U \longmapsto \big\{f \in \Gamma(U; \mathcal{C}^\infty_X),\ f \text{ is tempered}\big\}.$$

As a consequence of a result of [**15**], for $U, V \in \mathrm{Op}(X_{\mathrm{sa}})$ the sequence

$$0 \to \mathcal{C}^{\infty,t}_X(U \cup V) \to \mathcal{C}^{\infty,t}_X(U) \oplus \mathcal{C}^{\infty,t}_X(V) \to \mathcal{C}^{\infty,t}_X(U \cap V)$$

is exact. Then $\mathcal{C}^{\infty,t}_X$ is a sheaf on X_{sa}. Moreover $\mathrm{R}\Gamma(U; \mathcal{C}^{\infty,t}_X)$ is concentrated in degree zero for any $U \in \mathrm{Op}(X_{\mathrm{sa}})$.

Let $T\mathcal{H}om(F, \mathcal{C}^\infty_X)$ be the sheaf of [**15**].

When $F = \mathbb{C}_U$, $U \in \mathrm{Op}(X_{\mathrm{sa}})$ it is defined by $V \mapsto \mathcal{C}^{\infty,t}_V(U \cap V)$.

We have the following results (see [**16**]).

PROPOSITION 7.2.8. — *For each $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ one has the isomorphism*

$$\rho^{-1} R\mathcal{H}om(F, \mathcal{C}^{\infty,t}_X) \simeq T\mathcal{H}om(F, \mathcal{C}^{\infty}_X).$$

PROPOSITION 7.2.9. — *Let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and $\overline{X}$ the conjugate manifold. Then*

$$\mathcal{O}^t_X \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X_{\mathbb{R}}}).$$

We prove the following result.

LEMMA 7.2.10. — *Let $f : X \to Y$ be a smooth morphism of real analytic manifolds. Then we have the isomorphism*

$$f^{-1}\mathcal{C}^{\infty,t}_Y \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{C}^{\infty,t}_X).$$

Proof. — First of all remark that there is a natural morphism

$$f^{-1}\mathcal{C}^{\infty,t}_Y \longrightarrow R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{C}^{\infty,t}_X).$$

In order to prove that it is an isomorphism we may reduce to the case of a projection $\pi : Y \times \mathbb{R} \to Y$. We shall prove that the morphism

$$\partial_t : \mathcal{C}^{\infty,t}_{Y\times\mathbb{R}} \longrightarrow \mathcal{C}^{\infty,t}_{Y\times\mathbb{R}},$$

where t denotes the variable in $\mathbb{R}$, is surjective. Let $U \in \mathrm{Op}^{\mathrm{c}}((Y \times \mathbb{R})_{\mathrm{sa}})$, then by Lemma A.1.11 it admits a finite covering $\{U_i\}_{i=1}^N$ such that each U_i is simply connected and the intersections of each U_i with the fibers of π are contractible (or empty). Hence we may reduce to the case that the intersections of U with the fibers of π are contractible (or empty). Moreover we can assume that

$$U = \big\{(x,t) \in Y \times \mathbb{R};\ f(x) < t < g(x)\big\},$$

where $f, g : \pi(U) \to \mathbb{R}$ are continuous subanalytic maps and $\pi(U)$ is simply connected. Let us consider $h, k : \pi(U) \to \mathbb{R}$ continuous subanalytic and $\varphi \in \Gamma(\pi(U); \mathcal{C}^{\infty}_Y)$ such that $f < h < \varphi < k < g$. Let $s \in \Gamma(U; \mathcal{C}^{\infty,t}_{Y\times\mathbb{R}})$ and define

$$\widetilde{s}(x,t) = \int_{(x,\varphi(x))}^{(x,t)} s(x,\tau)\mathrm{d}\tau.$$

Then $\widetilde{s} \in \Gamma(U; \mathcal{C}^{\infty}_{Y\times\mathbb{R}})$ and $\partial_t\widetilde{s} = s$. Moreover

$$\big|\widetilde{s}(x,t)\big| \leq \big|\varphi(x) - t\big| \sup_{(x,\tau)\in\{x\}\times[\varphi(x),t]} \big|s(x,\tau)\big|.$$

Since U is bounded, there exists $M > 0$ such that $|\varphi(x) - t| < M$ for each $(x,t) \in U$. Since s is tempered, for each $x \in \pi(U)$ and each $\tau \in [\varphi(x), t]$ there exist $c_1, r_1 > 0$

such that

$$
\begin{aligned}
|s(x,\tau)| &\leq c_1 \frac{1}{d((x,\tau),\partial U)^{r_1}} \\
&\leq c_1 \frac{1}{\min\{d((x,t),\partial U), d((x,h(x)),\partial U), d((x,k(x)),\partial U)\}^{r_1}}.
\end{aligned}
$$

As a consequence of Łojaciewicz's inequality (see Theorem 6.4 of [**3**]) there exist $c_2 > 0$ and $r_2 > 0$ such that

$$
d\big((x,h(x)),\partial U), d((x,k(x)),\partial U) \geq c_2 d\big(x,\partial(\pi(U))\big)^{r_2} \geq c_2 d\big((x,t),\partial U\big)^{r_2}.
$$

Hence there exist $c, r > 0$ such that

$$
\widetilde{s}(x,t) \leq c \frac{1}{d((x,t),\partial U)^r}
$$

and the result follows. □

LEMMA 7.2.11. — *Let $f : X \to Y$ be a smooth morphism of real analytic manifolds. Let $\mathcal{M},\mathcal{N} \in \mathcal{D}^{\mathrm{b}}(\mathcal{D}_X)$. There is a natural morphism*

$$
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{M}) \overset{L}{\underset{f^{-1}\mathcal{A}_Y}{\otimes}} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{N}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\mathcal{M}\overset{L}{\underset{\mathcal{A}_X}{\otimes}}\mathcal{N}\Big).
$$

Proof. — By Lemma 4.9 of [**13**] we have

$$
\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{D}_{X\to Y} \simeq \mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{f^{-1}\mathcal{A}_Y}{\otimes}} f^{-1}\mathcal{D}_Y. \tag{7.2.7}
$$

Then if $\mathcal{M}$ is a $\mathcal{D}_X$-module

$$
\begin{aligned}
\Big(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}\Big) \overset{L}{\underset{f^{-1}\mathcal{A}_Y}{\otimes}} f^{-1}\mathcal{D}_Y &\simeq \Big(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{f^{-1}\mathcal{A}_Y}{\otimes}} f^{-1}\mathcal{D}_Y\Big) \overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M} \\
&\simeq \Big(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{D}_{X\to Y}\Big) \overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M} \simeq \mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \Big(\mathcal{M}\overset{L}{\underset{\mathcal{A}_X}{\otimes}}\mathcal{D}_{X\to Y}\Big).
\end{aligned}
$$

Now when f is smooth

$$
\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \cdot \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\,.)[d_X - d_Y].
$$

Then if $\mathcal{N}$ is another $\mathcal{D}_X$-module

$$
\begin{aligned}
&R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{M}) \overset{L}{\underset{f^{-1}\mathcal{A}_Y}{\otimes}} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{N}) \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\mathcal{M} \overset{L}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{D}_{X\to Y} \overset{L}{\underset{f^{-1}\mathcal{D}_Y}{\otimes}} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{N})\Big) \\
&\qquad\qquad \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\mathcal{M}\overset{L}{\underset{\mathcal{A}_X}{\otimes}}\mathcal{N}\Big).
\end{aligned}
$$

□

Lemma 7.2.12. — *Let X be a real analytic manifold. Let $F,G\in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and let S be a closed subanalytic subset of X. There is a morphism*

$$\rho^{-1}\, R\mathcal{H}om\big(F,(\mathcal{C}_X^{\infty,t})_S\big)\otimes_{\mathcal{A}_X}\rho^{-1}\, R\mathcal{H}om\big(D'\big((F\otimes G)_S\big),\mathcal{C}_X^{\infty,\mathrm{w}}\big)\\ \longrightarrow \rho^{-1}\, R\mathcal{H}om\big(D'(G_S),\mathcal{C}_X^{\infty,\mathrm{w}}\big).$$

Proof. — (i) Let $V_1,V_2\in\mathrm{Op}(X_{\mathrm{sa}})$. The sheaf $\rho^{-1}\Gamma_{V_1}(\mathcal{C}_X^{\infty,t})_S$ is concentrated in degree zero since $\mathcal{C}_X^{\infty,t}$ is $\Gamma(U;\cdot)$-acyclic for each $U\in\mathrm{Op}(X_{\mathrm{sa}})$. Moreover the sheaves

$$\rho^{-1}\, R\mathcal{H}om(D'\mathbb{C}_{V_1\cap V_2\cap S},\mathcal{C}_X^{\infty,\mathrm{w}})\simeq\mathbb{C}_{V_1\cap V_2\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty},$$
$$\rho^{-1}\, R\mathcal{H}om(D'\mathbb{C}_{V_2\cap S},\mathcal{C}_X^{\infty,\mathrm{w}})\simeq\mathbb{C}_{V_2\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}$$

are also concentrated in degree zero. There is a morphism

$$\rho^{-1}\Gamma_{V_1}(\mathcal{C}_X^{\infty,t})_S\otimes_{\mathcal{A}_X}\mathbb{C}_{V_1\cap V_2\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}\to\mathbb{C}_{V_2\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}. \tag{7.2.8}$$

This follows since the multiplication of a function tempered on V_1 by a function vanishing with all its derivatives outside V_1 is a function vanishing with all its derivatives outside V_1.

(ii) By Theorem 1.1 of [**15**] the morphism (7.2.8) extends to a morphism

$$\rho^{-1}\Gamma_{V_1}(\mathcal{C}_X^{\infty,t})_S\otimes_{\mathcal{A}_X}G_{V_1\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}\longrightarrow G_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}, \tag{7.2.9}$$

functorial in $G\in\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. By adjuction this gives a morphism

$$\rho^{-1}\Gamma_{V_1}(\mathcal{C}_X^{\infty,t})_S\longrightarrow\mathcal{H}om_{\mathcal{A}_X}(G_{V_1\cap S}\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty},G_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}). \tag{7.2.10}$$

By Theorem 1.1 of [**15**] the morphism (7.2.10) extends to a morphism

$$\rho^{-1}\,\mathcal{H}om\big(F,(\mathcal{C}_X^{\infty,t})_S\big)\longrightarrow\mathcal{H}om_{\mathcal{A}_X}\big((F\otimes G)_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty},G_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}\big). \tag{7.2.11}$$

functorial in $F\in\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.

(iii) Let $F,G\in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We have the following chain of morphisms

$$\begin{aligned}\rho^{-1}\, R\mathcal{H}om(F,(\mathcal{C}_X^{\infty,t})_S)&\xrightarrow{\sim}R\big(\rho^{-1}\,\mathcal{H}om\big(F,(\mathcal{C}_X^{\infty,t})_S\big)\big)\\&\longrightarrow R\big(\mathcal{H}om_{\mathcal{A}_X}\big((F\otimes G)_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty},G_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}\big)\big)\\&\longrightarrow R\mathcal{H}om_{\mathcal{A}_X}\big((F\otimes G)_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty},G_S\overset{\mathrm{w}}{\otimes}\mathcal{C}_X^{\infty}\big),\end{aligned}$$

where the first isomorphism follows since ρ^{-1} is exact and $(.)_S$ sends quasi-injective objects to quasi-injective objects, the second arrow follows from (7.2.11) and the third one is a canonical morphism of derived functors (see [**18**], Proposition 13.3.13).

By adjunction we obtain the desired morphism. □

LEMMA 7.2.13. — *Let us consider the normal deformation of the diagonal in $X \times X$ of diagram (5.2.1). Let $F, G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. There is a morphism*

$$\rho^{-1}\nu^{\mathrm{sa}}_{\Delta}\, R\mathcal{H}om(q_1^{-1}F, q_2^{-1}\mathcal{C}^{\infty,t}_X) \otimes_{\mathcal{A}_X} \rho^{-1}\nu^{\mathrm{sa}}_{\Delta}\, R\mathcal{H}om\left(q_1^{-1}D'(F\otimes G), q_2^{-1}\mathcal{C}^{\infty,\mathrm{w}}_X\right) \longrightarrow \rho^{-1}\nu^{\mathrm{sa}}_{\Delta}\, R\mathcal{H}om(q_1^{-1}D'G, q_2^{-1}\mathcal{C}^{\infty,\mathrm{w}}_X).$$

Proof. — (i) As in the proof of Theorem 6.4.2, if X is a real analytic manifold, $K \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, $\lambda = t, \mathrm{w}$, we have

$$\begin{aligned}
&\rho^{-1}\nu^{\mathrm{sa}}_{\Delta}\, R\mathcal{H}om(q_1^{-1}K, q_2^{-1}\mathcal{C}^{\infty,\lambda}_X)\\
&\quad\simeq \rho^{-1}s^{-1}\, R\mathcal{H}om\left((p_1^{-1}K)_{\Omega}, p_2^{-1}\mathcal{C}^{\infty,\lambda}_X\right)\\
&\quad\simeq \rho^{-1}s^{-1}\, R\mathcal{H}om\left((p_1^{-1}K)_{\Omega}, R\mathcal{H}om_{\rho_!\mathcal{D}_{\widetilde{X\times X}}}(\rho_!\mathcal{D}_{\widetilde{X\times X}\to X}, \mathcal{C}^{\infty,\lambda}_{\widetilde{X\times X}})\right)\\
&\quad\simeq s^{-1}\, R\mathcal{H}om_{\mathcal{D}_{\widetilde{X\times X}}}\left(\mathcal{D}_{\widetilde{X\times X}\to X}, \rho^{-1}\, R\mathcal{H}om((p_1^{-1}K)_{\Omega}, \mathcal{C}^{\infty,\lambda}_{\widetilde{X\times X}})\right),
\end{aligned}$$

where the second isomorphism follows from Lemma 7.2.10.

(ii) By Lemma 5.4.1 for $H \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ we have

$$(p_1^{-1}D'H)_{\Omega} \simeq D'\left((p_1^{-1}H)_{\overline{\Omega}}\right) \quad\text{and}\quad \mathrm{R}\Gamma_{\Omega}p_2^{-1}\mathcal{C}^{\infty,t}_X \simeq (p_2^{-1}\mathcal{C}^{\infty,t}_X)_{\overline{\Omega}}.$$

(iii) By Lemma 7.2.11 with $(X,Y) = (\widetilde{X\times X}, X)$, $\mathcal{M} = R\mathcal{H}om(p_1^{-1}F, (\mathcal{C}^{\infty,t}_{\widetilde{X\times X}})_{\overline{\Omega}})$, $\mathcal{N} = R\mathcal{H}om(D'((p_1^{-1}(F\otimes G))_{\overline{\Omega}}), \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{X\times X}})$, we are reduced to find a morphism

$$\rho^{-1}\, R\mathcal{H}om\left(p_1^{-1}F, (\mathcal{C}^{\infty,t}_{\widetilde{X\times X}})_{\overline{\Omega}}\right) \otimes_{\mathcal{A}_{\widetilde{X\times X}}} \rho^{-1}\, R\mathcal{H}om\left(D'\left((p_1^{-1}(F\otimes G))_{\overline{\Omega}}\right), \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{X\times X}}\right) \longrightarrow \rho^{-1}\, R\mathcal{H}om\left(D'\left((p_1^{-1}G)_{\overline{\Omega}}\right), \mathcal{C}^{\infty,\mathrm{w}}_{\widetilde{X\times X}}\right)$$

which follows replacing (X,S,F,G) with $(\widetilde{X\times X}, \overline{\Omega}, p_1^{-1}F, p_1^{-1}G)$ in Lemma 7.2.12. □

Let us consider the complex case. Let X be a complex manifold.

LEMMA 7.2.14. — *Let $\mathcal{L}, \mathcal{H} \in D^{\mathrm{b}}(\mathcal{D}_{X_{\mathbb{R}}})$. There is a natural morphism*

$$R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{L}) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{H}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\Big(\mathcal{O}_{\overline{X}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_{\mathbb{R}}}}{\otimes}} \mathcal{H}\Big).$$

Proof. — By definition we have

$$\mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_{\mathbb{R}}}}{\otimes}} \mathcal{H} = \mathcal{D}_{X_{\mathbb{R}}\to X_{\mathbb{R}}\times X_{\mathbb{R}}} \overset{L}{\underset{\mathcal{A}_{X_{\mathbb{R}}\times X_{\mathbb{R}}}}{\otimes}} (\mathcal{L} \overset{\mathcal{D}}{\boxtimes} \mathcal{M}).$$

Hence we get

$$\mathcal{L} \overset{\mathcal{D}}{\boxtimes} \mathcal{H} \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X_{\mathbb{R}}}}\Big(\mathcal{D}_{X_{\mathbb{R}}\to X_{\mathbb{R}}\times X_{\mathbb{R}}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_{\mathbb{R}}}}{\otimes}} \mathcal{H}\Big).$$

There is a chain of morphisms

$$
\begin{aligned}
& R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{L}) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{H}) \\
& \simeq \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} \Big(R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{L}) \overset{\mathcal{D}}{\quad} R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{H}) \Big) \\
& \quad \longrightarrow \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}} \quad \mathcal{D}_{\overline{X}}} \Big(\mathcal{O}_{\overline{X}} \quad \mathcal{O}_{\overline{X}}, \mathcal{D}_{X\times X} \overset{L}{\underset{\mathcal{D}_X \quad \mathcal{D}_X}{\otimes}} (\mathcal{L} \quad \mathcal{H}) \Big) \\
& \quad \longrightarrow \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}\times\overline{X}}} \Big(\mathcal{O}_{\overline{X}\times\overline{X}}, \mathcal{D}_{\overline{X}\times\overline{X}} \overset{L}{\underset{\mathcal{D}_{\overline{X}} \quad \mathcal{D}_{\overline{X}}}{\otimes}} \mathcal{D}_{X\times X} \overset{L}{\underset{\mathcal{D}_X \quad \mathcal{D}_X}{\otimes}} (\mathcal{L} \quad \mathcal{H}) \Big) \\
& \quad \longrightarrow \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}\times\overline{X}}} \Big(\mathcal{O}_{\overline{X}\times\overline{X}}, \mathcal{L} \overset{\mathcal{D}}{\quad} \mathcal{H} \Big) \\
& \quad \longrightarrow \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}\times\overline{X}}} \Big(\mathcal{O}_{\overline{X}\times\overline{X}}, R\mathcal{H}om_{\mathcal{D}_{X_\mathbb{R}}}(\mathcal{D}_{X_\mathbb{R}\to X_\mathbb{R}\times X_\mathbb{R}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_\mathbb{R}}}{\otimes}} \mathcal{H}) \Big) \\
& \quad \simeq \mathcal{D}_{X\to X\times X} \overset{L}{\underset{\mathcal{D}_{X\times X}}{\otimes}} R\mathcal{H}om_{\mathcal{D}_X} \Big(\mathcal{D}_{X\to X\times X}, R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_\mathbb{R}}}{\otimes}} \mathcal{H}) \Big) \\
& \quad \longrightarrow R\mathcal{H}om_{\mathcal{D}_{\overline{X}}} \Big(\mathcal{O}_{\overline{X}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_\mathbb{R}}}{\otimes}} \mathcal{H} \Big). \qquad \square
\end{aligned}
$$

Lemma 7.2.15. — *Let us consider the normal deformation of the diagonal in $X \times X$ of diagram (5.2.1). Let $F, G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. There is a morphism*

$$
\begin{aligned}
\rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om(q_1^{-1}F, q_2^{-1}\mathcal{O}^t_X) \otimes_{\mathcal{O}_X} \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om \left(q_1^{-1}D'(F \otimes G), q_2^{-1}\mathcal{O}^{\mathrm{w}}_X \right) \\
\longrightarrow \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om(q_1^{-1}D'G, q_2^{-1}\mathcal{O}^{\mathrm{w}}_X).
\end{aligned}
$$

Proof. — If X is a complex manifold, $K \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, $\lambda = t, \mathrm{w}$, we have

$$
\begin{aligned}
& \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om(q_1^{-1}K, q_2^{-1}\mathcal{O}^{\lambda}_X) \\
& \qquad \simeq \rho^{-1}s^{-1} R\mathcal{H}om \left((p_1^{-1}K)_\Omega, p_2^{-1}\mathcal{O}^{\lambda}_X \right) \\
& \qquad \simeq \rho^{-1}s^{-1} R\mathcal{H}om \left((p_1^{-1}K)_\Omega, p_2^{-1} R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\lambda}_{X_\mathbb{R}}) \right) \\
& \qquad \simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \rho^{-1}s^{-1} R\mathcal{H}om \left((p_1^{-1}K)_\Omega, p_2^{-1}\mathcal{C}^{\infty,\lambda}_{X_\mathbb{R}}) \right) \\
& \qquad \simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om \left(q_1^{-1}K, q_2^{-1}\mathcal{C}^{\infty,\lambda}_{X_\mathbb{R}}) \right).
\end{aligned}
$$

Set

$$
\mathcal{L} = \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om(q_1^{-1}F, q_2^{-1}\mathcal{C}^{\infty,t}_{X_\mathbb{R}}), \;\; \mathcal{H} = \rho^{-1}\nu^{\mathrm{sa}}_{\Delta} R\mathcal{H}om \left(q_1^{-1}D'(F \otimes G), q_2^{-1}\mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}}) \right).
$$

By Lemma 7.2.14 there is a natural morphism

$$
R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{L}) \overset{L}{\underset{\mathcal{O}_X}{\otimes}} R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{H}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{\overline{X}}} \Big(\mathcal{O}_{\overline{X}}, \mathcal{L} \overset{L}{\underset{\mathcal{A}_{X_\mathbb{R}}}{\otimes}} \mathcal{H} \Big).
$$

Then the result follows from Lemma 7.2.13. $\square$

LEMMA 7.2.16. — *Let $f : X \to Y$ be a smooth morphism of complex manifolds. Then there is a natural morphism*

$$Rf_{!!}\Omega_X^{\mathrm{w}}[d_X] \longrightarrow \Omega_Y^{\mathrm{w}}[d_Y].$$

Proof. — By Theorem A.4.7 we have the isomorphism

$$f^! \mathcal{O}_Y^{\mathrm{w}}[2d_Y] \xrightarrow{\sim} R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{D}_{X\to Y}, \mathcal{O}_X^{\mathrm{w}})[2d_X].$$

We have $R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}_X^{\mathrm{w}}) \simeq \rho_!\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \mathcal{O}_X^{\mathrm{w}}[d_Y - d_X]$. Hence we get

$$\rho_!\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \mathcal{O}_X^{\mathrm{w}} \simeq f^!\mathcal{O}_Y^{\mathrm{w}}[d_Y - d_X].$$

By adjunction we get

$$Rf_{!!}\Big(\rho_!\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \mathcal{O}_X^{\mathrm{w}}\Big) \longrightarrow \mathcal{O}_Y^{\mathrm{w}}[d_Y - d_X].$$

From this we can deduce

$$Rf_{!!}\Omega_X^{\mathrm{w}} \longrightarrow Rf_{!!}\Big(\Omega_X^{\mathrm{w}} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \rho_!\mathcal{D}_{X\to Y}\Big) \longrightarrow \Omega_Y^{\mathrm{w}}[d_Y - d_X]. \qquad \square$$

Let us consider the diagram (7.2.1) with $Z = \{\text{point}\}$. Set

$$p_X : T^*X \times T^*Y \longrightarrow T^*X, \qquad p_Y : T^*X \times T^*Y \longrightarrow T^*Y,$$
$$q_X : X \times Y \longrightarrow X, \qquad q_Y : X \times Y \longrightarrow Y.$$

PROPOSITION 7.2.17. — *Let $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $K \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{X\times Y})$ such that q_Y is proper on* $\mathrm{supp}(q_X^{-1}G) \cap \mathrm{supp}(K)$. *Then we have a morphism*

$$(7.2.12) \qquad \rho^{-1}\mu hom^{\mathrm{sa}}(K, \mathcal{O}_{X\times Y}^{t(0,d_Y)})[d_Y] \overset{a}{\circ} \rho^{-1}\mu hom^{\mathrm{sa}}\left(D'(K\circ G), \mathcal{O}_Y^{\mathrm{w}}\right) \longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}(D'G, \mathcal{O}_X^{\mathrm{w}}).$$

Proof. — We will prove the assertion in several steps. Set

$$H_1 = \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om(q_1^{-1}K, q_2^!\mathcal{O}_{X\times Y}^{t(0,d_Y)}) \simeq \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om(q_1^{-1}K, q_2^{-1}\mathcal{O}_{X\times Y}^{t(0,d_Y)})[2d_{X\times Y}],$$
$$H_2 = \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(K\circ G), q_2^!\mathcal{O}_Y^{\mathrm{w}}\right)$$
$$\simeq \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(K\circ G), q_2^{-1}\mathcal{O}_Y^{\mathrm{w}}\right)[2d_Y].$$

Since the Fourier-Sato transform commutes with ρ^{-1} we have

$$H_1^\wedge \simeq \rho^{-1}\mu hom^{\mathrm{sa}}(K, \mathcal{O}_{X\times Y}^{t(0,d_Y)}), \quad H_2^\wedge \simeq \rho^{-1}\mu hom^{\mathrm{sa}}\left(D'(K\circ G), \mathcal{O}_Y^{\mathrm{w}}\right).$$

(i) By the commutativity of the diagram (7.2.1) we have an isomorphism

$$Rp^a_{X!}\big((H_1^\wedge)^a \otimes p_Y^{a-1}H_2^\wedge\big) \simeq Rq_{X\pi!}{}^t q_X'^{-1} R^t\delta_!'\delta_\pi^{-1}(H_1^\wedge \quad H_2^\wedge).$$

(ii) By Proposition 3.7.15 of [**14**] we have an isomorphism

$$(H_1)^\wedge \quad (H_2)^\wedge \xrightarrow{\sim} (H_1 \quad H_2)^\wedge,$$

(iii) Denote by $Tq_Y : T(X \times Y) \to TY$ the tangent map. By Propositions 3.7.13 and 3.7.14 of [**14**] we have the isomorphism

$$R^t\delta'_!\delta_\pi^{-1}(H_1 \quad H_2)^\wedge \simeq (H_1 \otimes Tq_Y^{-1}H_2)^\wedge[-2d_Y].$$

(iv) We have the chain of morphisms

$$\begin{aligned}
&Tq_Y^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om(q_1^{-1}D'(K \circ G), q_2^{-1}\mathcal{O}_Y^{\mathrm{w}})\\
&\quad\simeq \nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}q_Y^{-1}D'(K \circ G), q_2^{-1}q_Y^{-1}\mathcal{O}_Y^{\mathrm{w}}\right)\\
&\qquad\simeq \nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(q_Y^{-1}q_{Y_*}(K \otimes q_X^{-1}G)), q_2^{-1}q_Y^{-1}\mathcal{O}_Y^{\mathrm{w}}\right)\\
&\qquad\quad\longrightarrow \nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(K \otimes q_X^{-1}G), q_2^{-1}q_Y^{-1}\mathcal{O}_Y^{\mathrm{w}}\right)\\
&\qquad\qquad\longrightarrow \nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(K \otimes q_X^{-1}G), q_2^{-1}\mathcal{O}_{X\times Y}^{\mathrm{w}}\right),
\end{aligned}$$

where the first isomorphism follows since q_Y is smooth, the second one since $\mathrm{supp}(q_X^{-1}G) \cap \mathrm{supp}(K)$ is proper over Y.

(v) We have a morphism

$$(H_1 \otimes Tq_Y^{-1}H_2)^\wedge[-2d_Y] \longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}\left(q_X^{-1}D'G, \mathcal{O}_{X\times Y}^{\mathrm{w}(0,d_Y)}\right).$$

To prove the existence of this morphism we shall prove the morphism

$$H_1 \otimes Tq_Y^{-1}H_2[-2d_Y] \longrightarrow \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}q_X^{-1}D'G, q_2^{-1}\mathcal{O}_{X\times Y}^{\mathrm{w}(0,d_Y)}\right)[2d_{X\times Y}].$$

Hence by (iv) we may reduce to the case of the morphism

$$\begin{aligned}
\rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om(q_1^{-1}K, q_2^{-1}\mathcal{O}_{X\times Y}^t) \otimes \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(K \otimes q_X^{-1}G), q_2^{-1}\mathcal{O}_{X\times Y}^{\mathrm{w}}\right)&\\
\longrightarrow \rho^{-1}\nu_\Delta^{\mathrm{sa}}\, R\mathcal{H}om\left(q_1^{-1}D'(q_X^{-1}G), q_2^{-1}\mathcal{O}_{X\times Y}^{\mathrm{w}}\right).&
\end{aligned}$$

This is a consequence of Lemma 7.2.15 with (X, F, G) replaced by $(X \times Y, K, q_X^{-1}G)$.

(vi) We have the chain of morphisms

$$\begin{aligned}
&Rq_{X\pi!}{}^tq_X'^{-1}\rho^{-1}\mu hom^{\mathrm{sa}}(q_X^{-1}D'G, \mathcal{O}_{X\times Y}^{\mathrm{w}(0,d_Y)})\\
&\quad\longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}(Rq_{X*}q_X^{-1}D'G, Rq_{X!!}\mathcal{O}_{X\times Y}^{\mathrm{w}(0,d_Y)})\\
&\qquad\longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}(D'G, \mathcal{O}_X^{\mathrm{w}})[-d_Y],
\end{aligned}$$

where the second morphism is a consequence of the integration morphism

$$Rq_{X!!}\mathcal{O}_{X\times Y}^{\mathrm{w}(0,d_Y)} \longrightarrow \mathcal{O}_X^{\mathrm{w}}[-d_Y]$$

defined in Lemma 7.2.16 (see also Remark 3.4 of [**15**]) and the fact that $Rq_{X*}q_X^{-1} \simeq \mathrm{id}$. Composing morphisms (i)–(vi) we get the desired morphism. □

Corollary 7.2.18. — *Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Morphism* (7.2.12) *defines a morphism*

$$\mathcal{E}_X^{\mathbb{R},f} \otimes \rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}_X^{\mathrm{w}}) \longrightarrow \rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}_X^{\mathrm{w}}) \tag{7.2.13}$$

which induces a structure of $\mathcal{E}_X^{\mathbb{R},f}$-module on $H^k\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}_X^{\mathrm{w}})$ for each $k \in \mathbb{Z}$.

Proof. — We apply Proposition 7.2.17 setting $X = Y$ and $(G, K) = (D'F, \mathbb{C}_\Delta)$. In this case we have $D'(\mathbb{C}_\Delta \circ D'F) \simeq D'D'F \simeq F$. □

In this way we find the morphism of [**5**]

$$(\mathcal{E}_X^{\mathbb{R},f})^a \otimes F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X \longrightarrow F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X$$

(recall that $\rho^{-1}\mu hom^{\mathrm{sa}}(F, \mathcal{O}_X^{\mathrm{w}}) \simeq (D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X)^a$).

Remark 7.2.19. — The integration morphism in Proposition 7.2.17 (vi) can be directly constructed starting from the integration for Whitney $\mathcal{C}^\infty$-functions. Let $f : X \to Y$ be a smooth morphism. Given a l.c.t. $U \in \mathrm{Op}^{\mathrm{c}}(Y_{\mathrm{sa}})$ we have

$$\begin{aligned}\Gamma(U; f_{!!}\mathcal{C}_X^{\infty,\mathrm{w}\vee}) &\simeq \Gamma_c(Y; f_! \rho^{-1} R\mathcal{H}om(\mathbb{C}_{f^{-1}(U)}, \mathcal{C}_X^{\infty,\mathrm{w}\vee}) \\ &\simeq \Gamma_c(X; \mathbb{C}_{f^{-1}(\overline{U})} \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^{\infty\vee}) \xrightarrow{\int} \Gamma(Y; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_Y^{\infty\vee}) \simeq \Gamma(U; \mathcal{C}_Y^{\infty,\mathrm{w}\vee}).\end{aligned}$$

Remark 7.2.20. — Let us consider the compatibility between this morphism and the one of Andronikof ([**1**], Proposition 3.3.10). Steps (i) to (iii) of Proposition 7.2.17 are the same. We need the compatibility between the multiplications. We will see the compatibility between

$$\rho^{-1}\mathrm{R}\Gamma_Z \mathcal{O}_X^t \otimes \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^t) \longrightarrow \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^t)$$

and

$$\rho^{-1}\mathrm{R}\Gamma_Z \mathcal{O}_X^t \otimes \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^{\mathrm{w}}) \longrightarrow \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^{\mathrm{w}})$$

when $Z \subset X$ is closed subanalytic and $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.

We reduce to the case of a real analytic manifold and we use the fact that

$$\rho^{-1} R\mathcal{H}om(G, \mathcal{C}_X^{\infty,t}) \simeq T\mathcal{H}om(G, \mathcal{C}_X^\infty) \quad \text{and} \quad \rho^{-1} R\mathcal{H}om(G, \mathcal{C}_X^{\infty,\mathrm{w}}) \simeq D'G \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty$$

for $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Define

$$F \overset{\mathrm{w}}{\otimes} T\mathcal{H}om(G, \mathcal{C}_X^\infty) = T\mathcal{H}om(G, F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty)$$

saying that, if U, V are open subanalytic

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} T\mathcal{H}om(\mathbb{C}_V, \mathcal{C}_X^\infty) = T\mathcal{H}om(\mathbb{C}_V, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty)$$

are $\mathcal{C}^\infty$-functions tempered on V and vanishing up to infinity outside U. Then we have

$$\begin{aligned}T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}_X^\infty) \otimes T\mathcal{H}om(F, \mathcal{C}_X^\infty) &\longrightarrow T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}_X^\infty) \otimes T\mathcal{H}om(F, \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \\ &\longrightarrow T\mathcal{H}om(F_Z, \mathcal{C}_X^\infty) \longrightarrow T\mathcal{H}om(F, \mathcal{C}_X^\infty)\end{aligned}$$

and

$$\begin{aligned}T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}_X^\infty) \otimes D'F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty &\longrightarrow T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}_X^\infty) \otimes (D'F)_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty \\ &\longrightarrow T\mathcal{H}om(\mathbb{C}_Z, D'F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \longrightarrow D'F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty.\end{aligned}$$

The first and the third arrows of the two diagrams are clearly compatible. Let us see the compatibility between the second arrows. Note that $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ plays no role in these arrows (it denotes a growth conditions which is preserved after the multiplication), so in order to better understand how they are constructed we set $F = \mathbb{C}_X$. Let $U = X \setminus Z$. Then $T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}^\infty_X)$ and $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ are represented by the complexes

$$0 \to \mathcal{C}^\infty_X \to T\mathcal{H}om(\mathbb{C}_U, \mathcal{C}^\infty_X),$$
$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \to \mathcal{C}^\infty_X \to 0$$

where in both cases $\mathcal{C}^\infty_X$ is the degree zero of the complex. The morphism is induced by the following diagram, where the vertical arrows are given by multiplication

$$\begin{tikzcd}
\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \otimes \mathcal{C}^\infty_X \arrow[r] \arrow[d] & \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \otimes T\mathcal{H}om(\mathbb{C}_U, \mathcal{C}^\infty_X) \oplus \mathcal{C}^\infty_X \otimes \mathcal{C}^\infty_X \arrow[r] \arrow[d] & \mathcal{C}^\infty_X \otimes T\mathcal{H}om(\mathbb{C}_U, \mathcal{C}^\infty_X) \arrow[d] \\
\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \arrow[r] & \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \oplus \mathcal{C}^\infty_X \arrow[r] & T\mathcal{H}om(\mathbb{C}_U, \mathcal{C}^\infty_X).
\end{tikzcd}$$

In the complex in the second line the first arrow is given by $s \mapsto (s, s)$ and the second one by $(u, v) \mapsto u - v$. Computing the cohomology, it is quasi-isomorphic to $T\mathcal{H}om(\mathbb{C}_Z, \mathcal{C}^\infty_X)$.

7.3. Microlocal integral transformations

In the case of a contact transformation the hypothesis of properness of the previous section are not satisfied. Hence we are going to define microlocal operations on $\mu hom^{\mathrm{sa}}(\,.\,, \mathcal{O}^\lambda_X)$ extending those of [**14**] and [**1**]. Let $\Omega \subset T^*X$. Denote by

- ▷ $D^{\mathrm{b}}(X_{\mathrm{sa}}, \Omega)$ the category $D^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})/\mathcal{N}_\Omega$;
- ▷ $D^{\mathrm{b}}(X, \Omega)$ the category $D^{\mathrm{b}}(\mathbb{C}_X)/\mathcal{N}_\Omega$;
- ▷ $D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(X, \Omega)$ the category $D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)/\mathcal{N}_\Omega$;

where $\mathcal{N}_\Omega = \{F \in D^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}); SS(F) \cap \Omega = \varnothing\}$ (resp. $F \in D^{\mathrm{b}}(\mathbb{C}_X)$, resp. $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$).

It follows from Corollary 5.3.5 that the functor

$$\rho^{-1}\mu hom^{\mathrm{sa}} : D^{\mathrm{b}}(X_{\mathrm{sa}}, \Omega)^{\mathrm{op}} \times D^{\mathrm{b}}(X_{\mathrm{sa}}, \Omega) \longrightarrow D^{\mathrm{b}}(\Omega)$$

is well defined.

Notations 7.3.1. — If there is no risk of confusion we will write for short $\mu hom(\,.\,, \mathcal{O}^\lambda_X)$ instead of $\rho^{-1}\mu hom^{\mathrm{sa}}(\,.\,, \mathcal{O}^\lambda_X)$.

Denote by $\underset{\mu}{\circ}$ the microlocal composition of kernels of [**14**] (and [**1**] for $\mathbb{R}$-constructible sheaves). As usual, given $K \in D^{\mathrm{b}}(\mathbb{C}_{(X\times Y)_{\mathrm{sa}}})$ and $F \in D^{\mathrm{b}}(\mathbb{C}_{Y_{\mathrm{sa}}})$ we set

$$\Phi^\mu_K F = K \underset{\mu}{\circ} F.$$

Proposition 7.3.2. — (i) *Let X, Y be two complex analytic manifolds, let $K \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{X\times Y})$, $p_X \in T^*X$, $p_Y \in T^*Y$ such that $SS(K) \cap (\{p_X\} \times T^*Y) \subseteq (p_X, p_Y^a)$ in a neighborhood of this point. Then for each $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_Y)$ and $G \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ there are morphisms*

$$\mu hom(K, \mathcal{O}^{t(0,d_Y)}_{X\times Y})_{(p_X,p_Y^a)}[d_Y] \otimes \mu hom(F, \mathcal{O}^t_Y)_{p_Y} \longrightarrow \mu hom(\Phi^{\mu}_K F, \mathcal{O}^t_X)_{p_X}, \tag{7.3.1}$$

$$\mu hom(K, \mathcal{O}^{t(0,d_Y)}_{X\times Y})_{(p_X,p_Y^a)}[d_Y] \otimes \mu hom(D'(\Phi^{\mu}_K G), \mathcal{O}^{\mathrm{w}}_Y)_{p_Y} \longrightarrow \mu hom(D'G, \mathcal{O}^{\mathrm{w}}_X)_{p_X}. \tag{7.3.2}$$

(ii) *Let Z be another complex analytic manifold, let $K_1 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{X\times Y})$ and $K_2 \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{Y\times Z})$ be microlocally composable at $(p_X, p_Y, p_Z) \in T^*X \times T^*Y \times T^*Z$, i.e.*

$$\big(SS(K_1) \times_{T^*Y} SS(K_2)\big) \cap p_{13}^{a-1}(p_X, p_Z^a) \subseteq \big\{((p_X, p_Y^a), (p_Y, p_Z^a))\big\}$$

in a neighborhood of $((p_X, p_Y^a), (p_Y, p_Z^a))$. Then there is a morphism

$$\mu hom(K_1, \mathcal{O}^{t(0,d_Y)}_{X\times Y})_{(p_X,p_Y^a)} \otimes \mu hom(K_2, \mathcal{O}^{t(0,d_Z)}_{Y\times Z})_{(p_Y,p_Z^a)} \longrightarrow \mu hom(K_1 \underset{\mu}{\circ} K_2, \mathcal{O}^{t(0,d_Z)}_{X\times Z})_{(p_X,p_Z^a)}[-d_Y]$$

Proof. — The result follows thanks to the morphisms defined in the previous section and adapting the proof of Proposition 3.3.12 of [**1**]. □

7.4. Contact transformations

Let X, Y be two complex analytic manifolds of the same complex dimension n and let $\Omega_X \subset T^*X$, $\Omega_Y \subset T^*Y$ be two open subanalytic subsets. Let χ be a contact transformation from Ω_X to Ω_Y. Let $\Lambda \subset \Omega_X \times \Omega_Y^a$ be the Lagrangian manifold associated with the graph of χ (i.e. $(p_X, p_Y^a) \in \Lambda$ if $p_Y = \chi(p_X)$). We denote by p_1 and p_2^a the projections from Λ to Ω_X and Ω_Y respectively.

Let $(p_X, p_Y) \in \Omega_X \times \Omega_Y$ and consider $K \in D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(X \times Y, (p_X, p_Y^a))$ satisfying the following properties (for the definition of simple sheaf we refer to [**14**]):

$$SS(K) \subset \Lambda \quad \text{and} \quad K \text{ is simple with shift } 0 \text{ along } \Lambda. \tag{7.4.1}$$

In this situation we have the following results of [**17**] and [**1**].

Proposition 7.4.1. — *Let $K \in D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(X \times Y, (p_X, p_Y^a))$ satisfying (7.4.1). Set*

$$K^* = r_*\, R\mathcal{H}om(K, \omega_{X\times Y|Y}),$$

where $r : X \times Y \to Y \times X$ is the canonical map. Then the functors

$$\Phi^{\mu}_K : D^{\mathrm{b}}(X_{\mathrm{sa}}, p_X) \longrightarrow D^{\mathrm{b}}(Y_{\mathrm{sa}}, p_Y),$$
$$\Phi^{\mu}_{K^*} : D^{\mathrm{b}}(Y_{\mathrm{sa}}, p_Y) \longrightarrow D^{\mathrm{b}}(X_{\mathrm{sa}}, p_X)$$

are equivalences of categories inverse to each other.

Lemma 7.4.2. — *Let* $K \in D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(X \times Y, (p_X, p_Y^a))$ *satisfying* (7.4.1). *Then* $\mu hom(K, \mathcal{O}^t_{X\times Y})$ *is concentrated in degree zero.*

Proposition 7.4.3. — *Let* $K \in D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(X \times Y, (p_X, p_Y^a))$ *satisfying* (7.4.1) *and let* $s \in \mu hom(K, \mathcal{O}^{t(0,n)}_{X\times Y})_{(p_X,p_Y^a)}$.

(i) *For each* $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(Y, p_Y)$ *there are morphisms induced by* s

$$\varphi_s : \mu hom(F, \mathcal{O}^t_Y)_{p_Y}[n] \longrightarrow \mu hom(\Phi^\mu_K F, \mathcal{O}^t_X)_{p_X},$$
$$\psi_s : \mu hom(D'(\Phi^\mu_K F), \mathcal{O}^{\mathrm{w}}_X)_{p_X}[n] \longrightarrow \mu hom(D'F, \mathcal{O}^{\mathrm{w}}_Y)_{p_Y}.$$

(ii) *Let* Z *be a* n*-dimensional complex analytic manifold,* $\Omega_Z \subset T^*Z$ *and let*

$$\chi' : \Omega_Y \longrightarrow \Omega_Z$$

be a contact transformation. Let Λ' *be the Lagrangian submanifold associated with the graph of* χ'. *Let* $K' \in D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(Y \times Z, (p_Y, p_Z^a))$ *satisfying* (7.4.1) *and* $s \in \mu hom(K', \mathcal{O}^{t(0,n)}_{Y\times Z})$. *Then*

$$\varphi_s \circ \varphi'_{s'} = (\varphi \circ \varphi')_{s\circ s'} \quad \text{and} \quad \psi_s \circ \psi'_{s'} = (\psi \circ \psi')_{s\circ s'},$$

where $s \circ s'$ *is the image of* $s \otimes s'$ *by the morphism*

$$\mu hom(K, \mathcal{O}^t_{X\times Y})_{(p_X,p_Y^a)} \otimes \mu hom(K', \mathcal{O}^t_{Y\times Z})_{(p_Y,p_Z^a)} \longrightarrow \mu hom(K \underset{\mu}{\circ} K'[n], \mathcal{O}^t_{X\times Z})_{(p_X,p_Z^a)}.$$

(iii) *Let* $P \in \mathcal{E}^{\mathbb{R},f}_{X,p_X}$ *and* $Q \in \mathcal{E}^{\mathbb{R},f}_{Y,p_Y}$ *such that* $Ps = sQ$. *Then:*

$$P \circ \varphi_s = \varphi_s \circ Q$$

(*and similarly for* ψ_s).

Proof. — (i) Similar to Proposition 5.2.1 (i) of [**1**]. There exists a neighborhood Ω of (p_X, p_Y^a) such that $s \in \Gamma(\Omega; \mu hom(K, \mathcal{O}^{t(0,n)}_{X\times Y})$ and we may suppose that Λ is closed in Ω. Set $\mathcal{K} = \mu hom(K, \mathcal{O}^{t(0,n)}_{X\times Y})$. Then

$$s \in \Gamma(\Omega, \mathcal{K}) \simeq \mathrm{Hom}(\mathbb{C}_\Lambda, \mathcal{K}). \tag{7.4.2}$$

Moreover we can find a relatively compact neighborhood V_Y of $\pi_Y(p_Y)$ such that

$$\Phi^\mu_K F = \Phi_{K_{X\times V_Y}} F = K_{X\times V_Y} \circ F.$$

Now set

$$\mathcal{F}_1 = \mu hom(\Phi_{K_{X\times V_Y}} F, \mathcal{O}^t_X), \qquad \mathcal{G}_1 = \mu hom(D'F, \mathcal{O}^{\mathrm{w}}_Y),$$
$$\mathcal{F}_2 = \mu hom(F, \mathcal{O}^t_Y)[n], \qquad \mathcal{G}_2 = \mu hom\big(D'(\Phi_{K_{X\times V_Y}}), \mathcal{O}^{\mathrm{w}}_X\big)[n].$$

Then the morphisms φ_s and ψ_s are given by the diagrams

$$\mathcal{F}_2|_{\Omega_Y} \xrightarrow{\sim} (\mathbb{C}^a_\Lambda \circ \mathcal{F}_2)|_{\Omega_X} \longrightarrow (\mathcal{K}^a \circ \mathcal{F}_2)|_{\Omega_X} \longrightarrow \mathcal{F}_1|_{\Omega_X},$$
$$\mathcal{G}_2|_{\Omega_X} \xrightarrow{\sim} (\mathbb{C}^a_\Lambda \circ \mathcal{G}_2)|_{\Omega_Y} \longrightarrow (\mathcal{K}^a \circ \mathcal{G}_2)|_{\Omega_Y} \longrightarrow \mathcal{G}_1|_{\Omega_Y}$$

where the first arrows are given by (7.4.2) and the second ones by (7.3.1) and (7.3.2).

(ii) The arrow follows from (i) and the associativity of the composition.

(iii) See [**1**], Proposition 5.2.1 (iii). □

THEOREM 7.4.4. — *Let χ be a contact transformation from Ω_X to Ω_Y and let Λ be the Lagrangian manifold associated with the graph of χ. Then there exist K in $D^{\mathrm{b}}_{\mathbb{C}\text{-c}}(X\times Y,(p_X,p_Y^a))$ satisfying* (7.4.1) *and $s\in\mu hom(K,\mathcal{O}^{t(0,n)}_{X\times Y})_{(p_X,p_Y^a)}$ such that:*

(i) *the correspondence $\mathcal{E}_{X,p_X}\ni P\mapsto Q\in\mathcal{E}_{Y,p_Y}$ such that $Ps=s\,Q$ is an isomorphism of rings,*

(ii) *for each $F\in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(Y,p_Y)$ the morphisms induced by s*

$$\varphi_s:\mu hom(F,\mathcal{O}^t_Y)_{p_Y}[n]\longrightarrow\mu hom(\Phi^\mu_K F,\mathcal{O}^t_X)_{p_X}$$

$$\psi_s:\mu hom(D'(\Phi^\mu_K F),\mathcal{O}^{\mathrm{w}}_X)_{p_X}[n]\longrightarrow\mu hom(D'F,\mathcal{O}^{\mathrm{w}}_Y)_{p_Y}.$$

are isomorphisms compatible with (i).

Proof. — The proof is similar to the proof of Proposition 5.2.2 of [**1**]. □

REMARK 7.4.5. — Set $F=\Phi^\mu_{K^*}G$ with $G\in D^b_{\mathbb{R}\text{-c}}(X,p_X)$ then $\Phi^\mu_K F\simeq G$ and $D'F\simeq\Psi^\mu_{K^*}D'G\simeq\Phi^\mu_K D'G$, where $\Psi^\mu_{K^*}=Rq_{Y*}\circ R\mathcal{H}om(K^*,\cdot)\circ q_X^!$ and the second isomorphism follows from Proposition 7.1.9 of [**14**]. Hence, replacing X with Y and $D'G$ with F we obtain the isomorphism

$$\mu hom(F,\mathcal{O}^{\mathrm{w}}_Y)_{p_Y}[n]\xrightarrow{\sim}\mu hom(\Phi^\mu_K F,\mathcal{O}^{\mathrm{w}}_X)_{p_X}.$$

APPENDIX A

REVIEW ON SUBANALYTIC SETS

A.1. Properties of subanalytic subsets

We recall briefly some properties of subanalytic subsets. Reference are made to [**3**] for the theory of subanalytic subsets and to [**7**] and [**38**] for the more general theory of o-minimal structures. Let X be a real analytic manifold.

DEFINITION A.1.1. — Let A be a subset of X.

(i) A is said to be *semi-analytic* if it is locally analytic, i.e. each $x \in A$ has a neighborhood U_x such that $X \cap U_x = \bigcup_{i\in I}\bigcap_{j\in J} X_{ij}$, where I, J are finite sets and either $X_{ij} = \{y \in U_x;\, f_{ij}(y) > 0\}$ or $X_{ij} = \{y \in U_x;\, f_{ij}(y) = 0\}$ for some analytic function f_{ij}.

(ii) A is said to be *subanalytic* if it is locally a projection of a relatively compact semi-analytic subset, i.e. each $x \in A$ has a neighborhood U_x such that there exists a real analytic manifold Y and a relatively compact semi-analytic subset $A' \subset X \times Y$ satisfying $X \cap U_x = \pi(A')$, where $\pi : X \times Y \to X$ denotes the projection.

(iii) Let Y be a real analytic manifold. A continuous map $f : X \to Y$ is *subanalytic* if its graph is subanalytic in $X \times Y$.

Let us recall some results on subanalytic subsets.

PROPOSITION A.1.2. — *Let A, B be subanalytic subsets of X. Then*

$$A \cup B, \quad A \cap B, \quad \overline{A}, \quad \partial A, \quad A \setminus B$$

are subanalytic.

PROPOSITION A.1.3. — *Let A be a subanalytic subset of X. Then the connected components of A are locally finite.*

PROPOSITION A.1.4. — *Let $f : X \to Y$ be a subanalytic map. Let A be a relatively compact subanalytic subset of X. Then $f(A)$ is subanalytic.*

DEFINITION A.1.5. — A simplicial complex (K, Δ) is the data consisting of a set K and a set Δ of subsets of K satisfying the following axioms:

(S1) any $\sigma \in \Delta$ is a finite and non-empty subset of K;

(S2) if τ is a non-empty subset of an element σ of Δ, then τ belongs to Δ;

(S3) for any $p \in K$, $\{p\}$ belongs to Δ;

(S4) for any $p \in K$, the set $\{\sigma \in \Delta; p \in \sigma\}$ is finite.

If (K, Δ) is a simplicial complex, an element of K is called a vertex. Let $\mathbb{R}^K$ be the set of maps from K to $\mathbb{R}$ equipped with the product topology. To $\sigma \in \Delta$ one associates $|\sigma| \subset \mathbb{R}^K$ as follows:

$$|\sigma| = \Big\{x \in \mathbb{R}^K;\ x(p) = 0 \text{ for } p \notin \sigma,\, x(p) > 0 \text{ for } p \in \sigma \text{ and } \sum_p x(p) = 1\Big\}.$$

As usual we set:

$$|K| = \bigcup_{\sigma\in\Delta} |\sigma|, \quad U(\sigma) = \bigcup_{\substack{\tau\in\Delta \\ \tau\supset\sigma}} |\tau|,$$

and for $x \in |K|$:

$$U(x) = U\big(\sigma(x)\big),$$

where $\sigma(x)$ is the unique simplex such that $x \in |\sigma|$.

THEOREM A.1.6. — *Let $X = \bigsqcup_{i\in I} X_i$ be a locally finite partition of X consisting of subanalytic subsets. Then there exists a simplicial complex (K, Δ) and a subanalytic homeomorphism $\psi : |K| \xrightarrow{\sim} X$ such that*

(i) *for any $\sigma \in \Delta$, $\psi(|\sigma|)$ is a subanalytic submanifold of X;*

(ii) *for any $\sigma \in \Delta$ there exists $i \in I$ such that $\psi(|\sigma|) \subset X_i$.*

Let us recall the definition of a subfamily of the subanalytic subsets of $\mathbb{R}^n$ which has some very good properties.

DEFINITION A.1.7. — A subanalytic subset A of $\mathbb{R}^n$ is said to be *globally subanalytic* if it is subanalytic in the projective space $\mathbb{P}^n(\mathbb{R})$. Here we identify $\mathbb{R}^n$ with a submanifold of $\mathbb{P}^n(\mathbb{R})$ via the map $(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$.

An equivalent way to define globally subanalytic subsets is by means of the map $\tau_n : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\tau_n(x_1, \dots, x_n) := \Big(\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}}\Big).$$

In particular relatively compact subanalytic subsets are globally subanalytic.

DEFINITION A.1.8. — A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *globally subanalytic* if its graph is globally subanalytic.

Proposition A.1.9. — *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a globally subanalytic map. Let A be a globally subanalytic subset of $\mathbb{R}^n$. Then $f(A)$ is globally subanalytic.*

Now we recall the notion of cylindrical cell decomposition, a useful tool to study the geometry of a subanalytic subset. We refer to [**7**] and [**38**] for a complete exposition.

A *cyindrical cell decomposition* (ccd for short) of $\mathbb{R}^n$ is a finite partition of $\mathbb{R}^n$ into subanalytic subsets, called the *cells of the ccd.* It is defined by induction on n:

$n = 1$. — A ccd of $\mathbb{R}$ is given by a finite subdivision $a_1 < \cdots < a_\ell$ of $\mathbb{R}$. The cells of $\mathbb{R}$ are the points $\{a_i\}$, $1 \leq i \leq \ell$, and the intervals (a_i, a_{i+1}), $0 \leq i \leq \ell$, where $a_0 = -\infty$ and $a_{\ell+1} = +\infty$.

$n > 1$. — A ccd of $\mathbb{R}^n$ is given by a ccd of $\mathbb{R}^{n-1}$ and, for each cell D of $\mathbb{R}^{n-1}$, continuous subanalytic functions $\zeta_{D,1} < \cdots < \zeta_{D,\ell_D} : D \to \mathbb{R}$. The cells of $\mathbb{R}^n$ are :

▷ the graphs $\bigl\{(x, \zeta_{D,i}(x));\ x \in D\bigr\}$, $1 \leq i \leq \ell_D$, and

▷ the bands $\bigl\{(x, y) \in D \times \mathbb{R};\ \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\bigr\}$ for $0 \leq i \leq \ell_D$,

where $\zeta_{D,0} = -\infty$ and $\zeta_{D,\ell_D+1} = +\infty$.

Theorem A.1.10. — *Let $A_1, \ldots, A_k$ be globally subanalytic subsets of $\mathbb{R}^n$. There exists a ccd of $\mathbb{R}^n$ such that each A_i is a union of cells.*

We end this section with the following useful result.

Lemma A.1.11. — *Let U be a globally subanalytic subset of $\mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection. Then U admits a finite open covering $\{U_i\}$ such that each U_i is simply connected and the intersection of each U_i with the fibers of π is contractible or empty.*

Proof. — Up to take the image of U by the homeomorphism

$$\varphi : \mathbb{R}^n \longrightarrow (-1,1)^n, \quad (x_1, \ldots, x_n) \longmapsto \Bigl(\frac{x_1}{\sqrt{1+x_1^2}}, \ldots, \frac{x_n}{\sqrt{1+x_n^2}}\Bigr)$$

we may assume that U is bounded. Then it follows from a result of [**40**] that U can be covered by finitely many open cells, and cells satisfy the desired properties. □

A.2. Ind-sheaves and subanalytic sites

Let us recall some results of [**16**]. One denotes by

▷ $\mathrm{I}(k_X)$ the category of ind-sheaves of k-vector spaces on X, that is

$$\mathrm{I}(k_X) = \mathrm{Ind}\bigl(\mathrm{Mod}^{\mathrm{c}}(k_X)\bigr),$$

where $\mathrm{Mod}^{\mathrm{c}}(k_X)$ denotes the full subcategory of $\mathrm{Mod}(k_X)$ consisting of sheaves with compact support on X:

▷ $D^{\mathrm{b}}(\mathrm{I}(k_X))$ the bounded derived category of $\mathrm{I}(k_X)$.

There are three functors relating ind-sheaves and classical sheaves:

$$\begin{aligned}
&\iota : \mathrm{Mod}(k_X) \longrightarrow \mathrm{I}(k_X), && F \longmapsto \text{“}\varinjlim_{U \Subset X}\text{”} F_U,\\
&\alpha : \mathrm{I}(k_X) \longrightarrow \mathrm{Mod}(k_X), && \text{“}\varinjlim_{i}\text{”} F_i \longmapsto \varinjlim_{i} F_i,\\
&\beta : \mathrm{Mod}(k_X) \longrightarrow \mathrm{I}(k_X), && \text{left adjoint to } \alpha.
\end{aligned}$$

These functors satisfy the following properties:

▷ the functor ι is fully faithful, exact and commutes with $\varprojlim$;

▷ the functor α is exact and commutes with $\varinjlim$ and $\varprojlim$;

▷ the functor β is fully faithful, exact and commutes with $\varinjlim$;

▷ (α, ι) and (β, α) are pairs of adjoint functors.

Since ι is fully faithful and exact we identify $\mathrm{Mod}(k_X)$ (resp. $D^{\mathrm{b}}(k_X)$) with a full abelian subcategory of $\mathrm{I}(k_X)$ (resp. $D^{\mathrm{b}}(\mathrm{I}(k_X))$).

The category $\mathrm{I}(k_X)$ admits an internal hom denoted by $\mathcal{I}hom$ and this functor admits a left adjoint, denoted by $\otimes$. One can also define an external

$$\mathcal{H}om : \mathrm{I}(k_X) \times \mathrm{I}(k_X) \longrightarrow \mathrm{Mod}(k_X)$$

and one has

$$\mathcal{H}om(F, G) = \alpha\, \mathcal{I}hom(F, G) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{I}(k_X)}(F, G) = \Gamma(X; \mathcal{H}om(F, G)).$$

The functor $\otimes$ is exact while $\mathcal{I}hom$ and $\mathcal{H}om$ are left exact and admit right derived functors $R\mathcal{I}hom$ and $R\mathcal{H}om$.

Consider a morphism of real analytic manifolds $f : X \to Y$. One defines the external operations

$$\begin{aligned}
f^{-1} &: \mathrm{I}(k_Y) \longrightarrow \mathrm{I}(k_X), && \text{“}\varinjlim_{i}\text{”} G_i \longmapsto \text{“}\varinjlim_{i, U \Subset X}\text{”} (f^{-1}G_i)_U,\\
f_* &: \mathrm{I}(k_X) \longrightarrow \mathrm{I}(k_Y), && \text{“}\varinjlim_{i}\text{”} F_i \longmapsto \varprojlim_{U \Subset X} \varinjlim_{i} f_* \Gamma_U F_i,\\
f_{!!} &: \mathrm{I}(k_X) \longrightarrow \mathrm{I}(k_Y), && \text{“}\varinjlim_{i}\text{”} F_i \longmapsto \text{“}\varinjlim_{i}\text{”} f_! F_i,
\end{aligned}$$

where the notation $f_{!!}$ is chosen to stress the fact that $f_{!!} \circ \iota \not\simeq \iota \circ f_!$ in general.

While f^{-1} is exact, the other functors admit right derived functors. One can show that the functor $Rf_{!!}$ admits a right adjoint denoted by $f^!$ and we get the usual formalism of the six Grothendieck operations. Almost all the formulas of the classic theory of sheaves remain valid for ind-sheaves.

There is a strict relation between ind-sheaves and sheaves on the subanalytic site associated with X. Set for short

$$\mathrm{I}_{\mathbb{R}\text{-c}}(k_X) = \mathrm{Ind}\big(\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(k_X)\big).$$

THEOREM A.2.1. — *One has an equivalence of categories*

$$\mathrm{I}_{\mathbb{R}\text{-c}}(k_X) \xrightarrow{\sim} \mathrm{Mod}(k_{X_{\mathrm{sa}}}), \quad \text{“}\varinjlim_i\text{”} F_i \longmapsto \varinjlim_i \rho_* F_i.$$

Let us recall the following functor defined in [**16**]:

$$I_{\mathcal{T}} : \mathrm{Mod}(k_{X_{\mathrm{sa}}}) \longrightarrow \mathrm{I}(k_X), \quad \varinjlim_i \rho_* F_i \longmapsto \text{“}\varinjlim_i\text{”} F_i.$$

It is fully faithful, exact and commutes with $\varinjlim$ and $\otimes$. It admits a right adjoint

$$J_{\mathcal{T}} : \mathrm{I}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{\mathrm{sa}}})$$

satisfying, for each $U \in \mathrm{Op}(X_{\mathrm{sa}})$, $\Gamma(U; J_{\mathcal{T}} F) = \mathrm{Hom}_{\mathrm{I}(k_X)}(k_U, F)$. This functor is right exact and commutes with filtrant inductive limits. Moreover we have $RJ_{\mathcal{T}} \circ I_{\mathcal{T}} \simeq \mathrm{id}$ and

$$RJ_{\mathcal{T}}\, R\mathcal{I}hom(I_{\mathcal{T}} F, G) \simeq R\mathcal{H}om(F, RJ_{\mathcal{T}} G).$$

We have the following relations:

$$RJ_{\mathcal{T}} \circ \iota \simeq R\rho_* \quad \text{and} \quad \alpha \simeq \rho^{-1} \circ J_{\mathcal{T}},$$
$$\alpha \circ I_{\mathcal{T}} \simeq \rho^{-1} \quad \text{and} \quad I_{\mathcal{T}} \circ \rho_! \simeq \beta.$$

Let $f : X \to Y$ be a morphism of real analytic manifolds and let U be an open subanalytic subset of X.

LEMMA A.2.2. — *Let $F \in D^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ and $G \in D^{\mathrm{b}}(k_{Y_{\mathrm{sa}}})$. We have*

(i) $I_{\mathcal{T}} \circ Rf_{!!} F \simeq Rf_{!!} \circ I_{\mathcal{T}} F$;
(ii) $I_{\mathcal{T}} \circ f^{-1} G \simeq f^{-1} \circ I_{\mathcal{T}} G$;
(iii) $I_{\mathcal{T}} \circ f^! G \simeq f^! \circ I_{\mathcal{T}} G$;
(iv) $I_{\mathcal{T}} F_U \simeq (I_{\mathcal{T}} F)_U$;
(v) $I_{\mathcal{T}} \circ \mathrm{R}\Gamma_U F \simeq \mathrm{RI}\Gamma_U \circ I_{\mathcal{T}} F$.

A.3. Inverse image for tempered holomorphic functions

The results of § A.3 have already been proved in [**16**] using ind-sheaves, for sake of completeness we reproduce here the proofs with slight modifications. Let $f : M \to N$ be a morphism of oriented real analytic manifolds of dimension d_M and d_N. Set

$$d = d_N - d_M.$$

LEMMA A.3.1. — *Let F be an $\mathcal{A}_M$-module locally free of finite rank. Then, for $k \neq 0$:*

$$R^k f_{!!}(\mathcal{D}b_M^t \otimes_{\rho_! \mathcal{A}_M} \rho_! F) = 0.$$

Proof. — It is a consequence of the fact that $\mathcal{D}b_M^t$ is quasi-injective and Proposition 1.6.5 of [**28**]. □

Lemma A.3.2. — *Let M and N be orientable real manifolds. There is a natural morphism of complexes*

$$f_{!!}(\mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega^\bullet_M)[d_M] \longrightarrow \mathcal{D}b^t_N \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\Omega^\bullet_N[d_N].$$

Proof. — Let $U \in \mathrm{Op}^{\mathrm{c}}(N_{\mathrm{sa}})$. We have the chain of morphisms

$$\begin{aligned}
\Gamma\Big(U; f_{!!}\Big(\mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega^{d_M-i}_M\Big)\Big) &\simeq \Gamma\Big(N; \rho^{-1}\,\mathcal{H}om\,\Big(\mathbb{C}_U, f_{!!}(\mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega^{d_M-i}_M)\Big)\Big)\\
&\simeq \Gamma\Big(N; f_!\rho^{-1}\,\mathcal{H}om\,\Big(f^{-1}\mathbb{C}_U, \mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega^{d_M-i}_M\Big)\Big)\\
&\simeq \Gamma\Big(N; f_!\; T\mathcal{H}om\,\Big(f^{-1}\mathbb{C}_U, \mathcal{D}b_M \underset{\mathcal{A}_M}{\otimes} \Omega^{d_M-i}_M\Big)\Big)\\
&\longrightarrow \Gamma_c\big(N; T\mathcal{H}om\,\Big(\mathbb{C}_U, \mathcal{D}b_N \underset{\mathcal{A}_N}{\otimes} \Omega^{d_N-i}_N\Big)\Big)\\
&\simeq \Gamma\Big(U; \mathcal{D}b^t_N \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\Omega^{d_N-i}_N\Big),
\end{aligned}$$

where the arrow is a consequence of Proposition 4.3 of [**15**]. □

Proposition A.3.3. — *There is a natural morphism in $D^{\mathrm{b}}(\rho_!\mathcal{D}^{\mathrm{op}}_M)$:*

$$Rf_{!!}\Big(\mathcal{D}b^{t\vee}_M \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N}\Big) \longrightarrow \mathcal{D}b^{t\vee}_N. \tag{A.3.1}$$

Proof. — The Spencer resolution of $\mathcal{D}_{M\to N}$ gives rise to the quasi-isomorphism

$$\mathcal{D}_{M\to N} \overset{\sim}{\leftarrow} \mathcal{D}_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{\mathcal{A}_M}{\otimes} \mathcal{D}_{M\to N} \simeq \mathcal{D}_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{f^{-1}\mathcal{A}_M}{\otimes} f^{-1}\mathcal{D}_N$$

from which we obtain the following quasi-isomorphism for $\mathcal{D}b^{t\vee}_M \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N}$ in $D^{\mathrm{b}}(\rho_! f^{-1}\mathcal{D}^{\mathrm{op}}_N)$:

$$\begin{aligned}
\mathcal{D}b^{t\vee}_M \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N} &\simeq \Big(\mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega_M\Big) \underset{\rho_!\mathcal{D}_M}{\otimes} \Big(\rho_!\Big(\mathcal{D}_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{f^{-1}\mathcal{A}_N}{\otimes} f^{-1}\mathcal{D}_N\Big)\Big)\\
&\simeq \mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Big(\Omega_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{f^{-1}\mathcal{A}_N}{\otimes} f^{-1}\mathcal{D}_N\Big)\\
&\simeq \mathcal{D}b^t_M \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Big(\Omega^\bullet_M \underset{f^{-1}\mathcal{A}_N}{\otimes} f^{-1}\mathcal{D}_N\Big)[d_M].
\end{aligned}$$

Applying $Rf_{!!}$ we obtain:

$$\begin{aligned}
Rf_{!!}\Big(\mathcal{D}b_M^{t\vee} \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N}\Big) &\simeq Rf_{!!}\Big(\mathcal{D}b_M^t \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Big(\Omega_M^\bullet \underset{f^{-1}\mathcal{A}_N}{\otimes} f^{-1}\mathcal{D}_N\Big)\Big)[d_M] \\
&\simeq Rf_{!!}\Big(\mathcal{D}b_M^t \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega_M^\bullet\Big) \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\mathcal{D}_N[d_M] \\
&\simeq f_{!!}\Big(\mathcal{D}b_M^t \underset{\rho_!\mathcal{A}_M}{\otimes} \rho_!\Omega_M^\bullet\Big) \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\mathcal{D}_N[d_M] \\
&\longrightarrow \mathcal{D}b_N^t \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\Omega_N^\bullet \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\mathcal{D}_N[d_N] \\
&\simeq \mathcal{D}b_N^t \underset{\rho_!\mathcal{A}_N}{\otimes} \rho_!\Omega_N = \mathcal{D}b_N^{t\vee},
\end{aligned}$$

where the third isomorphism follows from Lemma A.3.1 and the arrow from Lemma A.3.2. □

By adjunction we get a morphism

$$\mathcal{D}b_M^{t\vee} \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N} \longrightarrow f^!\mathcal{D}b_N^{t\vee}. \tag{A.3.2}$$

Theorem A.3.4. — *The morphism* (A.3.2) *is an isomorphism.*

Proof. — Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ with compact support. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{RHom}(F, f^!\mathcal{D}b_N^{t\vee}) &\simeq \mathrm{RHom}(Rf_{!!}F, \mathcal{D}b_N^{t\vee}) \simeq \mathrm{R}\Gamma\big(N, T\mathcal{H}om(Rf_!F, \mathcal{D}b_N^\vee)\big) \\
&\simeq \mathrm{R}\Gamma\Big(N, Rf_!\Big(T\mathcal{H}om(F, \mathcal{D}b_M^\vee) \overset{L}{\underset{\mathcal{D}_M}{\otimes}} \mathcal{D}_{M\to N}\Big)\Big) \\
&\simeq \mathrm{R}\Gamma\Big(M, T\mathcal{H}om(F, \mathcal{D}b_M^\vee) \overset{L}{\underset{\mathcal{D}_M}{\otimes}} \mathcal{D}_{M\to N}\Big) \simeq \mathrm{RHom}\Big(F, \mathcal{D}b_M^{t\vee} \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \rho_!\mathcal{D}_{M\to N}\Big),
\end{aligned}$$

where the third isomorphism follows from Theorem 4.4 of [**15**]. □

By the equivalence between left and right $\mathcal{D}$-modules, we have an isomorphism

$$\rho_!\mathcal{D}_{N\leftarrow M} \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \mathcal{D}b_M^t \xrightarrow{\sim} f^!\mathcal{D}b_N^t. \tag{A.3.3}$$

Corollary A.3.5. — *When f is smooth we have an isomorphism*

$$f^{-1}\mathcal{D}b_N^t \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_M}(\rho_!\mathcal{D}_{M\to N}, \mathcal{D}b_M^t).$$

Proof. — The result is obtained by the following isomorphisms

$$\begin{aligned}
R\mathcal{H}om_{\rho_!\mathcal{D}_M}(\rho_!\mathcal{D}_{M\to N}, \mathcal{D}b_M^t) &\simeq \rho_!\, R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{D}_{M\to N}, \mathcal{D}_M) \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \mathcal{D}b_M^t \\
&\simeq \rho_!\mathcal{D}_{N\leftarrow M} \overset{L}{\underset{\rho_!\mathcal{D}_M}{\otimes}} \mathcal{D}b_M^t[d] \simeq f^!\mathcal{D}b_N^t[d] \simeq f^{-1}\mathcal{D}b_N^t.
\end{aligned}$$

The first isomorphism is obtained by replacing $\mathcal{D}_{M\to N}$ with its Koszul complex. The second follows from the smoothness of f and the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{D}_{M\to N}, \mathcal{D}_M) \simeq \mathcal{D}_{N\leftarrow M}[d].$$

The last isomorphism follows since we have the isomorphism $f^!(.)[d] \simeq f^{-1}$ when f is smooth. □

From now on X will be a complex manifold of complex dimension d_X, with structure sheaf $\mathcal{O}_X$. We denote by $\overline{X}$ the complex conjugate manifold (with structure sheaf $\mathcal{O}_{\overline{X}}$), and $X_{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. Let $\mathcal{O}^t_X$ be the sheaf of tempered holomorphic functions on X. We also consider the sheaf $\Omega^t_X \in D^{\mathrm{b}}(\rho_! \mathcal{D}^{\mathrm{op}}_X)$:

$$\Omega^t_X := \mathcal{D}b^{t\vee}_{X_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_{\overline{X}}}{\otimes}} \rho_!\mathcal{O}_{\overline{X}}[-d_X].$$

Proposition A.3.6. — *Let $f : X \to Y$ be a holomorphic map between complex manifolds. Then*

$$\Omega^t_X \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \rho_!\mathcal{D}_{X\to Y}[d_X] \simeq f^!\Omega^t_Y[d_Y]. \tag{A.3.4}$$

Proof. — We have the chain of isomorphisms

$$\begin{aligned} f^!\Big(\mathcal{D}b^{t\vee}_{Y_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_{\overline{X}}}{\otimes}} \rho_!\mathcal{O}_{\overline{Y}}\Big) &\simeq f^!\mathcal{D}b^{t\vee}_{Y_{\mathbb{R}}} \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}{\otimes}} \rho_! f^{-1}\mathcal{O}_{\overline{Y}} \\ &\simeq \mathcal{D}b^{t\vee}_{X_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_{X_{\mathbb{R}}}}{\otimes}} \rho_!\mathcal{D}_{X_{\mathbb{R}}\to Y_{\mathbb{R}}} \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}{\otimes}} \rho_! f^{-1}\mathcal{O}_{\overline{Y}} \\ &\simeq \Big(\mathcal{D}b^{t\vee}_{X_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \rho_!\mathcal{D}_{X\to Y}\Big) \overset{L}{\underset{\rho_!\mathcal{D}_{\overline{X}}}{\otimes}} \rho_!\mathcal{D}_{\overline{X}\to\overline{Y}} \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}{\otimes}} \rho_! f^{-1}\mathcal{O}_{\overline{Y}} \\ &\simeq \Big(\mathcal{D}b^{t\vee}_{X_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \rho_!\mathcal{D}_{X\to Y}\Big) \overset{L}{\underset{\rho_!\mathcal{D}_{\overline{X}}}{\otimes}} \rho_!\mathcal{O}_{\overline{X}} \simeq \Big(\mathcal{D}b^{t\vee}_{X_{\mathbb{R}}} \overset{L}{\underset{\rho_!\mathcal{D}_{\overline{X}}}{\otimes}} \rho_!\mathcal{O}_{\overline{X}}\Big) \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \rho_!\mathcal{D}_{X\to Y}\ , \end{aligned}$$

where the second isomorphism follows from Proposition A.3.4. □

By the equivalence between left and right $\mathcal{D}$-modules, we have an isomorphism

$$\rho_!\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\rho_!\mathcal{D}_X}{\otimes}} \mathcal{O}^t_X[d_X] \xrightarrow{\sim} f^!\mathcal{O}^t_Y[d_Y]. \tag{A.3.5}$$

Corollary A.3.7. — *When f is smooth we have an isomorphism*

$$f^{-1}\mathcal{O}^t_Y \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}^t_X).$$

Proof. — The proof is similar to that of Corollary A.3.5. □

A.4. Inverse image for Whitney holomorphic functions

Let $f : M \to N$ be a morphism of oriented real analytic manifolds of dimensions d_M and d_N. Set

$$d = d_N - d_M.$$

Lemma A.4.1. — *The sheaf $f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d]$ is concentrated in degree zero.*

Proof. — If f is smooth, then $f^!(.)[d] \simeq f^{-1}$, and the result is clear. Let f be a closed embedding. Then $Rf_* \simeq Rf_! \simeq f_!$. Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{RHom}(D'F, f^!\mathcal{C}^{\infty,\mathrm{w}}_N)[d] &\simeq \mathrm{RHom}(f_!D'F[-d], \mathcal{C}^{\infty,\mathrm{w}}_N)\\ &\simeq \mathrm{RHom}\big(D'(f_!F); \mathcal{C}^{\infty,\mathrm{w}}_N\big) \simeq \mathrm{R}\Gamma(N, f_!F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_N).\end{aligned}$$

The second isomorphism follows since

$$Rf_*DF \simeq D(Rf_!F)$$

(where $D(.) = R\mathcal{H}om(\,.\,,\omega_M))$ if $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ and $Rf_* \simeq Rf_! \simeq f_!$ since f is a closed embedding. Let $U \in \mathrm{Op}^{\mathrm{c}}(M_{\mathrm{sa}})$ be locally cohomologically trivial. We have $D'\mathbb{C}_{\overline{U}} \simeq \mathbb{C}_U$, and if $k \neq 0$ we get

$$R^{k+d}\Gamma(U; f^!\mathcal{C}^{\infty,\mathrm{w}}_N) \simeq R^k\Gamma(X; f_!\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_N) = 0$$

since $f_!\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_N$ is soft. Hence $f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d]$ is concentrated in degree zero on a basis for the topology of M_{sa} and the result follows. □

LEMMA A.4.2. — *There is a natural morphism in* $\mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}})$

$$\rho_!\mathcal{A}_M \underset{\rho_!f^{-1}\mathcal{A}_N}{\otimes} f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d] \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_M.$$

Proof. — Let $U \in \mathrm{Op}^{\mathrm{c}}(M_{\mathrm{sa}})$ be locally cohomologically trivial. We have the chain of morphisms

$$\begin{aligned}\Gamma(U; f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d]) &\simeq \mathrm{R}\Gamma(N; Rf_!\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_N)\\ &\longrightarrow \mathrm{R}\Gamma(M; f^{-1}Rf_!\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M)\\ &\longrightarrow \mathrm{R}\Gamma(M; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M) \simeq \Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_M),\end{aligned}$$

where the first isomorphism has been proved in Lemma A.4.1 and the first arrow follows from Theorem 3.3 of [**15**]. In this way we construct a $\rho_!f^{-1}\mathcal{A}_N$-linear morphism $f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d] \to \mathcal{C}^{\infty,\mathrm{w}}_M$. The inclusion $\rho_!\mathcal{A}_M \to \mathcal{C}^{\infty,\mathrm{w}}_M$ and multiplication imply the desired morphism. □

PROPOSITION A.4.3. — *There is a natural morphism in* $D^{\mathrm{b}}(\rho_!\mathcal{D}_M)$*:*

$$\rho_!\mathcal{D}_{M\to N} \underset{\rho_!f^{-1}\mathcal{D}_N}{\overset{L}{\otimes}} f^!\mathcal{C}^{\infty,\mathrm{w}}_N[d] \longrightarrow \mathcal{C}^{\infty,\mathrm{w}}_M. \tag{A.4.1}$$

Proof. — The Spencer resolution of $\mathcal{D}_{M\to N}$ gives rise to the quasi-isomorphism

$$\mathcal{D}_{M\to N} \xleftarrow{\sim} \mathcal{D}_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{\mathcal{A}_M}{\otimes} \mathcal{D}_{M\to N} \simeq \mathcal{D}_M \underset{\mathcal{A}_M}{\otimes} \overset{\bullet}{\bigwedge}\Theta_M \underset{f^{-1}\mathcal{A}_N}{\otimes} f^{-1}\mathcal{D}_N$$

from which we obtain

$$\begin{aligned}
&\rho_! \mathcal{D}_{M\to N} \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_N}{\otimes}} f^! \mathcal{C}_N^{\infty,\mathrm{w}}[d] \\
&\quad\simeq \rho_! \mathcal{D}_M \underset{\rho_! \mathcal{A}_M}{\otimes} \rho_! \overset{\bullet}{\bigwedge} \Theta_M \underset{\rho_! f^{-1}\mathcal{A}_M}{\otimes} \rho_! f^{-1}\mathcal{D}_N \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_N}{\otimes}} f^! \mathcal{C}_N^{\infty,\mathrm{w}}[d] \\
&\quad\simeq \rho_! \mathcal{D}_M \underset{\rho_! \mathcal{A}_M}{\otimes} \rho_! \overset{\bullet}{\bigwedge} \Theta_M \underset{\rho_! f^{-1}\mathcal{A}_N}{\otimes} f^! \mathcal{C}_N^{\infty,\mathrm{w}}[d] \\
&\quad\longrightarrow \rho_! \mathcal{D}_M \underset{\rho_! \mathcal{A}_M}{\otimes} \rho_! \overset{\bullet}{\bigwedge} \Theta_M \underset{\rho_! \mathcal{A}_M}{\otimes} \mathcal{C}_M^{\infty,\mathrm{w}} \simeq \mathcal{C}_M^{\infty,\mathrm{w}},
\end{aligned}$$

where the arrow follows from Lemma A.4.2. □

By adjunction we get a morphism

$$f^! \mathcal{C}_N^{\infty,\mathrm{w}}[d] \longrightarrow R\mathcal{H}om_{\rho_! \mathcal{D}_M}(\rho_! \mathcal{D}_{M\to N}, \mathcal{C}_M^{\infty,\mathrm{w}}). \tag{A.4.2}$$

Theorem A.4.4. — *The morphism* (A.4.2) *is an isomorphism.*

Proof. — Let $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{RHom}(D'F, f^! \mathcal{C}_N^{\infty,\mathrm{w}})[d] &\simeq \mathrm{R}\Gamma(Y; Rf_! F \overset{\mathrm{w}}{\otimes} \mathcal{C}_N^{\infty}) \\
&\simeq \mathrm{RHom}_{\mathcal{D}_M}(\mathcal{D}_{M\to N}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^{\infty}) \\
&\simeq \mathrm{RHom}_{\mathcal{D}_M}\big(\mathcal{D}_{M\to N}, \rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty,\mathrm{w}})\big) \\
&\simeq \mathrm{RHom}_{\rho_! \mathcal{D}_M}\big(\rho_! \mathcal{D}_{M\to N}, R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty,\mathrm{w}})\big) \\
&\simeq \mathrm{RHom}\big(D'F, R\mathcal{H}om_{\rho_! \mathcal{D}_M}(\rho_! \mathcal{D}_{M\to N}, \mathcal{C}_M^{\infty,\mathrm{w}})\big),
\end{aligned}$$

where the second isomorphism follows from Theorem 3.5 of [**15**]. □

Corollary A.4.5. — *When f is smooth we have an isomorphism*

$$f^{-1} \mathcal{C}_N^{\infty,\mathrm{w}} \xrightarrow{\sim} R\mathcal{H}om_{\rho_! \mathcal{D}_M}(\rho_! \mathcal{D}_{M\to N}, \mathcal{C}_M^{\infty,\mathrm{w}}).$$

Proof. — It follows from the fact that $f^!(.)[d] \simeq f^{-1}$ when f is smooth. □

Remark A.4.6. — There is a similar isomorphism for $\mathcal{C}_{N|F}^{\infty,\mathrm{w}}$, $F \in D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_N)$, namely

$$f^! \mathcal{C}_{N|F}^{\infty,\mathrm{w}}[d] \simeq R\mathcal{H}om_{\rho_! \mathcal{D}_M}(\rho_! \mathcal{D}_{M\to N}, \mathcal{C}_{M|f^{-1}F}^{\infty,\mathrm{w}}). \tag{A.4.3}$$

The proof is the same as the one for $\mathcal{C}_N^{\infty,\mathrm{w}}$. We only considered the case $F = \mathbb{C}_X$ to lighten notations.

From now on X will be a complex manifold of complex dimension d_X, with structure sheaf $\mathcal{O}_X$. We denote by $\overline{X}$ the complex conjugate manifold (with structure sheaf $\mathcal{O}_{\overline{X}}$), and $X_{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. Let $\mathcal{O}_X^{\mathrm{w}}$ be the sheaf of Whitney holomorphic functions on X.

Theorem A.4.7. — *Let $f : X \to Y$ be a morphism of complex manifolds. Then*

$$f^! \mathcal{O}_Y^{\mathrm{w}}[2d_Y] \xrightarrow{\sim} R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{D}_{X\to Y}, \mathcal{O}_X^{\mathrm{w}})[2d_X]. \tag{A.4.4}$$

Proof. — Remark that, if $\mathcal{M} \in D^{\mathrm{b}}(\rho_!\mathcal{D}_{X_\mathbb{R}})$ we have

$$\begin{aligned}
&R\mathcal{H}om_{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}\big(\rho_! f^{-1}\mathcal{O}_{\overline{Y}}, R\mathcal{H}om_{\rho_!\mathcal{D}_{X_\mathbb{R}}}(\rho_!\mathcal{D}_{X_\mathbb{R}\to Y_\mathbb{R}}, \mathcal{M})\big)\\
&\quad\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}\big(\rho_!\mathcal{D}_{X\to Y}, R\mathcal{H}om_{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}\big(\rho_! f^{-1}\mathcal{O}_{\overline{Y}}, R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{D}_{\overline{X}\to\overline{Y}}, \mathcal{M})\big)\big)\\
&\quad\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}\big(\rho_!\mathcal{D}_{X\to Y}, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!(\mathcal{D}_{\overline{X}\to\overline{Y}} \overset{L}{\underset{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}{\otimes}} f^{-1}\mathcal{O}_{\overline{Y}}), \mathcal{M})\big)\\
&\quad\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}\big(\rho_!\mathcal{D}_{X\to Y}, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{M})\big).
\end{aligned}$$

We have the chain of isomorphisms

$$\begin{aligned}
f^!\mathcal{O}^{\mathrm{w}}_Y[2d_Y] &\simeq f^!\, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{Y}}}(\rho_!\mathcal{O}_{\overline{Y}}, \mathcal{C}^{\infty,\mathrm{w}}_{Y_\mathbb{R}})[2d_Y]\\
&\simeq R\mathcal{H}om_{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}(f^{-1}\rho_!\mathcal{O}_{\overline{Y}}, f^!\mathcal{C}^{\infty,\mathrm{w}}_{Y_\mathbb{R}})[2d_Y]\\
&\simeq R\mathcal{H}om_{\rho_! f^{-1}\mathcal{D}_{\overline{Y}}}\big(\rho_! f^{-1}\mathcal{O}_{\overline{Y}}, R\mathcal{H}om_{\rho_!\mathcal{D}_{X_\mathbb{R}}}(\rho_!\mathcal{D}_{X_\mathbb{R}\to Y_\mathbb{R}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}})\big)[2d_X]\\
&\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}\big(\rho_!\mathcal{D}_{X\to Y}, R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}})\big)[2d_X]\\
&\simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}^{\mathrm{w}}_X)[2d_X].
\end{aligned}$$

□

COROLLARY A.4.8. — *When f is smooth we have an isomorphism*

$$f^{-1}\mathcal{O}^{\mathrm{w}}_Y \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}^{\mathrm{w}}_X).$$

Proof. — The proof is similar to that of Corollary A.4.5. □

REMARK A.4.9. — As above, there is a similar isomorphism for $\mathcal{O}^{\mathrm{w}}_{Y|F}$, with F in $D^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_Y)$, namely

$$f^!\mathcal{O}^{\mathrm{w}}_{Y|F}[2d_Y] \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathcal{O}^{\mathrm{w}}_{X|f^{-1}F})[2d_X]. \tag{A.4.5}$$

BIBLIOGRAPHY

[1] ANDRONIKOF (E.) – *Microlocalisation tempérée*, Mémoires Soc. Math. France, vol. **57**, 1994.

[2] BECK (A.) – *Continuous flows in the plane*, Grundlehren der Math., vol. **201**, Springer-Verlag, New York-Heidelberg, 1974.

[3] BIERSTONE (E.) & MILMANN (D.) – *Semianalytic and subanalytic sets*, Publ. IHÉS, t. **67** (1988), pp. 5–42.

[4] BJÖRK (J. E.) – *Analytic $\mathcal{D}$-modules and applications*, Math. Appl., vol. **247**, Kluwer Academic Publishers Group, Dordrecht, 1993.

[5] COLIN (V.) – *Formal microlocalization*, C. R. Acad. Sci. Paris Math., t. **327** (1998), pp. 289–293.

[6] ______, *Specialization of formal cohomology and asymptotic expansions*, Publ. RIMS, Kyoto Univ., t. **37** (2001), pp. 37–69.

[7] COSTE (M.) – *An introduction to o-minimal geometry*, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica.

[8] D'AGNOLO (A.) & SCHAPIRA (P.) – *Leray's quantization of projective duality*, Duke Math. J., t. **84** (1996), pp. 453–496.

[9] EDMUNDO (M.) & PRELLI (L.) – *Sheaves on $\mathcal{T}$-topologies*, arXiv:1002.0690.

[10] GUILLERMOU (S.) – *DG-methods for microlocalization*, Publ. RIMS, Kyoto Univ., t. **47** (2011), pp. 99–140.

[11] HONDA (N.) & PRELLI (L.) – *Multi-specialization and multi-asymptotic expansions*, Advances in Math., t. **232** (2013), pp. 432–498.

[12] KASHIWARA (M.) – *The Riemann-Hilbert problem for holonomic systems*, Publ. RIMS, Kyoto Univ., t. **20** (1984), pp. 319–365.

[13] ______, *$\mathcal{D}$-modules and microlocal calculus*, in *Transl. Math. Monogr.*, Iwanami Series in Modern Math., vol. **217**, Amer. Math. Soc., Providence, 2003.

[14] Kashiwara (M.) & Schapira (P.) – *Sheaves on manifolds*, Grundlehren der Math., vol. **292**, Springer-Verlag, Berlin, 1990.

[15] ______, *Moderate and formal cohomology associated with constructible sheaves*, Mémoires Soc. Math. France, vol. **64**, 1996.

[16] ______, *Ind-sheaves*, Astérisque, t. **271** (2001).

[17] ______, *Microlocal study of ind-sheaves I: microsupport and regularity*, Astérisque, t. **284** (2003), pp. 143–164.

[18] ______, *Categories and sheaves*, Grundlehren der Math., vol. **332**, Springer-Verlag, Berlin, 2006.

[19] Kashiwara (M.), Schapira (P.), Ivorra (F.) & Waschkies (I.) – *Microlocalization of ind-sheaves*, in *Studies in Lie theory*, Progress in Math., vol. **243**, Birkhäuser, 2006, pp. 171–221.

[20] Łojaciewicz (S.) – *Sur le problème de la division*, Studia Mathematica, t. **8** (1959), pp. 87–136.

[21] Malgrange (B.) – *Ideals of differentiable functions*, Tata Institute, Oxford University Press, 1967.

[22] ______, *Équations différentielles à coefficients polynomiaux*, Progress in Math., vol. **96**, Birkhäuser, 1991.

[23] Martins (A. R.) – *Functorial properties of the microsupport and regularity for ind-sheaves*, Math. Zeitschrift, t. **260** (2008), pp. 541–556.

[24] Morando (G.) – *An existence theorem for tempered solutions of $\mathcal{D}$-modules on complex curves*, Publ. RIMS, Kyoto Univ., t. **43** (2007), pp. 625–659.

[25] ______, *Tempered solutions of $\mathcal{D}$-modules on complex curves and formal invariants*, Ann. Inst. Fourier, t. **59** (2009), pp. 1611–1639.

[26] Prelli (L.) – *Sheaves on subanalytic sites*, Ph.D. Thesis, Universities of Padova and Paris VI, 2006.

[27] ______, *Microlocalization of subanalytic sheaves*, C. R. Acad. Sci. Paris Math., t. **345** (2007), pp. 127–132.

[28] ______, *Sheaves on subanalytic sites*, Rend. Sem. Mat. Univ. Padova, t. **120** (2008), pp. 167–216.

[29] ______, *Cauchy-Kowaleskaya-Kashiwara theorem with growth conditions*, Math. Zeitschrift, t. **265** (2010), pp. 115–124.

[30] ______, *Microlocalization with growth conditions of holomorphic functions*, C. R. Acad. Sci. Paris Math., t. **348** (2010), pp. 1263–1266.

[31] ______, *Conic sheaves on subanalytic sites and Laplace transform*, Rend. Sem. Mat. Univ. Padova, t. **125** (2011), pp. 173–206.

[32] Schapira (P.) – *Microdifferential systems in the complex domain*, Grundlehren der Math., vol. **269**, Springer-Verlag, Berlin, 1985.

[33] SCHAPIRA (P.) & SCHNEIDERS (J. P.) – *Index theorem for elliptic pairs*, Astérisque, t. **224** (1994).

[34] SCHNEIDERS (J. P.) – *An introduction to $\mathcal{D}$-modules*, Bull. Soc. Roy. Sci. Liège, t. **63** (1994), pp. 223–295.

[35] SGA4 – *Séminaire Géom. Algébrique du Bois-Marie by M. Artin, A. Grothendieck, J.-L. Verdier*, Lecture Notes in Math., vol. **269**, Springer-Verlag, Berlin, 1972.

[36] SIBUYA (Y.) – *Linear ordinary differential equation in the complex domain: problems of analytic continuation*, Transl. Math. Monogr., vol. **82**, Amer. Math. Soc., Providence, 1990.

[37] TAMME (G.) – *Introduction to étale cohomology*, Universitext, Springer-Verlag, Berlin, 1994.

[38] VAN DEN DRIES (L.) – *Tame topology and o-minimal structures*, Lecture Notes Series, vol. **248**, Cambridge University Press, Cambridge, 1998.

[39] WASCHKIES (I.) – *Microlocal perverse sheaves*, Bull. Soc. Math. de France, t. **132** (2004), pp. 397–462.

[40] WILKIE (A. J.) – Covering definable open sets by open cells, in *O-minimal Structures, M. Edmundo, D. Richardson, and A. Wilkie, eds,* Proceedings of the RAAG Summer School Lisbon 2003, Lecture Notes in Real Algebraic and Analytic Geometry, Cuvillier Verlag, 2005.