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***Abstract*. –** There has been a great deal of work done in recent years on weighted Bergman spaces A_α^p on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, where $0 < p < \infty$ and $\alpha > -1$. We extend this study in a very natural way to the case where α is *any* real number and $0 < p \leq \infty$. This unified treatment covers all classical Bergman spaces, Besov spaces, Lipschitz spaces, the Bloch space, the Hardy space H^2, and the so-called Arveson space. Some of our results about integral representations, complex interpolation, coefficient multipliers, and Carleson measures are new even for the ordinary (unweighted) Bergman spaces of the unit disk.

***Résumé* (Théorie des espaces de Bergman dans la boule unité de $\mathbb{C}^n$)**

Ces dernières années il y a eu un grand nombre de travaux sur les espaces de Bergman pondérés A_α^p sur la boule unité $\mathbb{B}_n$ de $\mathbb{C}^n$, où $0 < p < \infty$ et $\alpha > -1$. Nous étendons cette étude, de manière très naturelle, au cas où α est un nombre réel *quelconque* et $0 < p \leq \infty$. Ce traitement unifié couvre tous les espaces de Bergman classiques, les espaces de Bésov, de Lipschitz, l'espace de Bloch, l'espace H^2 de Hardy, et celui appelé espace d'Arveson. Certains de nos résultats autour de la représentation entière, de l'interpolation complexe, des multiplicateurs de coefficients et des mesures de Carleson, sont nouveaux, y compris pour les espaces de Bergman ordinaires (non-pondérés) sur le disque unité.

CONTENTS

CHAPTER 1

INTRODUCTION

Throughout the paper we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n dimensional complex Euclidean space. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \big\{ z \in \mathbb{C}^n : |z| < 1 \big\}.$$

We use $H(\mathbb{B}_n)$ to denote the space of all holomorphic functions in $\mathbb{B}_n$.

For any $-\infty < \alpha < \infty$ we consider the positive measure

$$\mathrm{d}v_\alpha(z) = (1 - |z|^2)^\alpha \mathrm{d}v(z),$$

where $\mathrm{d}v$ is volume measure on $\mathbb{B}_n$. It is easy to see that $\mathrm{d}v_\alpha$ is finite if and only if $\alpha > -1$. When $\alpha > -1$, we normalize $\mathrm{d}v_\alpha$ so that it is a probability measure.

Bergman spaces with standard weights are defined as

$$A_\alpha^p = H(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, \mathrm{d}v_\alpha),$$

where $p > 0$ and $\alpha > -1$. Here the assumption that $\alpha > -1$ is essential, because the space $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$ does not contain any holomorphic function other than 0 when $\alpha \leq -1$. When $\alpha = 0$, we use A^p to denote the ordinary unweighted Bergman spaces. Bergman spaces with standard weights on the unit ball have been studied by numerous authors in recent years. See Aleksandrov [**2**], Beatrous-Burbea [**11**], Coifman-Rochberg [**21**], Rochberg [**46**], Rudin [**47**], Stoll [**57**], and Zhu [**71**] for results and references.

In this paper we are going to extend the definition of A_α^p to the case in which α is any real number and develop a theory for the extended family of spaces. More specifically, we study the following topics about the generalized spaces A_α^p: various

characterizations, integral representations, atomic decomposition, complex interpolation, optimal pointwise estimates, duality, reproducing kernels when $p = 2$, Carleson type measures, and various special cases. A few of these are straightforward consequences or generalizations of known results in the case $\alpha > -1$ (we included them here with full proofs for the sake of a complete and coherent theory), thanks to the isomorphism between A_α^p and A^p via fractional integral and differential operators, while most others require new techniques and reveal new properties. Several of our results are new even in the case of ordinary Bergman spaces of the unit disk.

Our starting point is the observation that, for $p > 0$ and $\alpha > -1$, a holomorphic function f in $\mathbb{B}_n$ belongs to A_α^p if and only if the function $(1 - |z|^2)Rf(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$, where

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$$

is the radial derivative of f. This result is well known to experts in the field and is sometimes referred to as a theorem of Hardy and Littlewood (especially in the one-dimensional case). See Beatrous [9], Pavlovic [42], or Theorem 2.16 of Zhu [71]. More generally, we can repeatedly apply this result and show that, for any positive integer k, a holomorphic function f is in A_α^p if and only if the function $(1 - |z|^2)^k R^k f(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

Now for $p > 0$ and $-\infty < \alpha < \infty$ we fix a nonnegative integer k with $pk + \alpha > -1$ and define A_α^p as the space of holomorphic functions f in $\mathbb{B}_n$ such that the function $(1 - |z|^2)^k R^k f(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$. As was mentioned in the previous paragraph, this definition of A_α^p is consistent with the traditional definition when $\alpha > -1$. Also, it is easy to show (see Section 4) that the definition of A_α^p is independent of the integer k.

We also study a companion family of spaces defined using the sup-norm of a combination of powers of $1 - |z|^2$ and partial derivatives of a holomorphic function f in $\mathbb{B}_n$. More specifically, for any real α we define Λ_α to be the space of holomorphic functions f in $\mathbb{B}_n$ such that the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded in $\mathbb{B}_n$, where k is any nonnegative integer with $k > \alpha$. We are going to call them holomorphic Lipschitz spaces. Once again, it can be shown that the definition of Λ_α is independent of the choice of the integer k.

The two families of spaces A_α^p and Λ_α, with $0 < p < \infty$ and α real, cover any space (except H^∞ and its equivalents) of holomorphic functions that is defined in terms of membership in $L^p(\mathbb{B}_n, \mathrm{d}v)$, $0 < p \leq \infty$, for any combination of partial derivatives of f and powers of $1 - |z|^2$. These spaces have appeared before in the literature under different names. For example, for any positive p and real s there is the classical diagonal Besov space B_p^s consisting of holomorphic functions f in $\mathbb{B}_n$ such that $(1 - |z|^2)^{k-s} R^k f(z)$ belongs to $L^p(\mathrm{d}v_{-1})$, where k is any positive integer greater

than s. It is clear that $B_p^s = A_\alpha^p$ with $\alpha = -(ps+1)$; and $A_\alpha^p = B_p^s$ with $s = -(\alpha+1)/p$. Thus our spaces A_α^p are exactly the diagonal Besov spaces. See Ahern-Cohen [1], Arazy-Fisher-Janson-Peetre [4], Arcozzi-Rochberg-Sawyer [7], Frazier-Jawerth [26], Hahn-Youssfi [30], [29], Kaptanoglu [33], [34], Nowark [40], Peloso [44], and Zhu [71] for some recent results on such Besov spaces and more references. In particular, our spaces A_α^p are the same as the spaces B_q^p (with $q = \alpha$) in Kaptanoglu [33], although an unnecessary condition $-qp + q > -1$ was imposed in [33].

On the other hand, if s is a positive integer, p is positive, and α is real, then there is the Sobolev space $W_{s,\alpha}^p$ consisting of holomorphic functions f in $\mathbb{B}_n$ such that the partial derivatives of f of order up to N all belong to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$. It is easy to see that our generalized Bergman spaces are exactly the holomorphic Sobolev spaces. See Ahern-Cohen [1], Aleksandrov [2], Beatrous-Burbea [11] for results and more references.

Therefore, for those who are more familiar or more comfortable with Besov or Sobolev spaces, our paper can be considered a unified theory for such spaces as well. However, we believe that most people nowadays are familiar and comfortable with the term "Bergman spaces", and our theory here is almost identical to the theory of ordinary Bergman spaces (as presented in Zhu [71] for example), so it is also reasonable for us to call A_α^p weighted Bergman spaces. We can stretch this a little further. More specifically, there has been a sizable amount of work in the literature about spaces of holomorphic functions satisfying the condition

$$\sup_{z \in \mathbb{B}_n} \left(1 - |z|^2\right)^t |f(z)| < \infty,$$

where $t > 0$. Such spaces are special cases of our Lipschitz spaces Λ_α and they have been called Bergman spaces as well by some authors; see Seip [50], [49] for example. Therefore, we do not feel guilty to use the term "Bergman spaces" to include all our Lipschitz spaces Λ_α.

It is apparent that this work is a natural extension of the recent book [71]. There is undoubtedly some overlapping between the two. In particular, the notation here is identical to that used in [71], and several techniques developed in [71] are used repeatedly in this paper. Since we are developing a more general theory here, complete proofs are included for all but the obvious.

As was mentioned a little earlier, the spaces we study here are not new. There are several papers in the literature that are very much related to this one, for example, [11] and [33]. In fact, almost every result here has its origin somewhere else. Therefore, in subsequent chapters, whenever a major theorem is proved, we will try to bring the reader's attention to these other sources where earlier versions or special cases of the particular result can be found. These repetitive references may be annoying to the

reader on the one hand, and on the other hand they may prove more offensive to authors whose papers have somehow been overlooked or inaccurately quoted. Whatever the case, we apologize in advance. It is not our intention to claim a known result ours. We also greatly appreciate the referee's complete understanding of this dilemma, as well as his/her suggestions on how to improve the presentation of the paper.

CHAPTER 2

VARIOUS SPECIAL CASES

In this chapter we spell out to the reader several special cases of the spaces A_α^p and Λ_α. In particular, this partially shows the scope of the paper and gives an orientation to those readers who are only concerned with certain special cases.

As was mentioned in the introduction, when $\alpha > -1$, the spaces A_α^p are traditionally called weighted Bergman spaces. In this case, a holomorphic function f in $\mathbb{B}_n$ belongs to A_α^p (see Zhu [**71**]) if and only if

$$\int_{\mathbb{B}_n} \left|f(z)\right|^p \left(1 - |z|^2\right)^\alpha \mathrm{d}v(z) < \infty.$$

When $\alpha = -(n+1)$, or $n + \alpha + 1 = 0$, we have mentioned several times earlier that the space A_α^p is traditionally denoted by B_p and is called a diagonal Besov space. Alternatively, a holomorphic function f in $\mathbb{B}_n$ belongs to the Besov space B_p if and only if

$$\int_{\mathbb{B}_n} \left|\left(1 - |z|^2\right)^k R^k f(z)\right|^p \mathrm{d}\tau(z) < \infty,$$

where k is any positive integer with $pk > n$ and

$$\mathrm{d}\tau(z) = \frac{\mathrm{d}v(z)}{(1 - |z|^2)^{n+1}}$$

is the Möbius invariant measure on $\mathbb{B}_n$. See Zhu [**71**].

When $\alpha = -1$ and $p = 2$, the space A_α^p coincides with the classical Hardy space H^2. See (20) later in the paper and (1.22) of Zhu [**71**]. Recall that H^p consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\sup_{0 < r < 1} \int_{\mathbb{S}_n} \left|f(r\zeta)\right|^p \mathrm{d}\sigma(\zeta) < \infty,$$

where $\mathrm{d}\sigma$ is the normalized surface measure on the unit sphere $\mathbb{S}_n$.

When $\alpha = -n$ and $p = 2$, the space A_α^p is the so-called Arveson space, which is usually defined as the Hilbert space of holomorphic functions in $\mathbb{B}_n$ whose reproducing kernel is given by (see Theorem 41)

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle}.$$

This space has attracted much attention lately in the study of multi-variable operator theory. We mention Arveson's [**8**] influential paper and the recent monograph [**17**] by Chen and Guo.

When $0 < \alpha < 1$, the space Λ_α is the classical Lipschitz space of holomorphic functions f in $\mathbb{B}_n$ satisfying the condition

$$\sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in \mathbb{B}_n, \, z \neq w \right\} < \infty.$$

See Section 6.4 of Rudin [**47**]. The space Λ_1 is also called the Zygmund class, especially in the case when $n = 1$.

When $\alpha = 0$, the space Λ_α is just the classical Bloch space, consisting of functions $f \in H(\mathbb{B}_n)$ such that

$$\sup_{z \in \mathbb{B}_n} \left(1 - |z|^2 \right) \left| Rf(z) \right| < \infty.$$

When $\alpha < 0$, the spaces Λ_α have appeared in the literature under the name of growth spaces. In this case, a holomorphic function f in $\mathbb{B}_n$ belongs to Λ_α if and only if

$$\sup_{z \in \mathbb{B}_n} \left(1 - |z|^2 \right)^{|\alpha|} \left| f(z) \right| < \infty.$$

The term "Bloch type spaces" or α-Bloch spaces can also be found in recent literature. More specifically, for any $\alpha > 0$ the α-Bloch space is denoted by $\mathscr{B}_\alpha$ and consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\sup_{z \in \mathbb{B}_n} \left(1 - |z|^2 \right)^\alpha \left| Rf(z) \right| < \infty.$$

It is then clear that the α-Bloch space $\mathscr{B}_\alpha$ is the same as our generalized Lipschitz space $\Lambda_{1-\alpha}$. See Zhu [**71**].

CHAPTER 3

PRELIMINARIES

In this chapter we present preliminary material on Bergman kernel functions and fractional differential and integral operators. This material will be heavily used in later chapters.

▷ Throughout the paper we use

$$m = (m_1, \cdots, m_n)$$

to denote an n-tuple of nonnegative integers. It is customary to write

$$|m| = m_1 + \cdots + m_n \quad \text{and} \quad m! = m_1! \ldots m_n!.$$

If $z = (z_1, \cdots, z_n)$ is a point in $\mathbb{C}^n$, we write

$$z^m = z_1^{m_1} \cdots z_n^{m_n}.$$

The following multi-nomial formula will be used (implicitly) several times later on:

$$\langle z, w \rangle^k = \sum_{|m|=k} \frac{k!}{m!} \, z^m \, \overline{w}^m. \tag{1}$$

▷ If f is a holomorphic function in $\mathbb{B}_n$, it has a unique Taylor series,

$$f(z) = \sum_m a_m z^m.$$

If we define

$$f_k(z) = \sum_{|m|=k} a_m z^m, \quad k = 0, 1, 2, \ldots,$$

then each f_k is a homogeneous polynomial of degree k, and we can rearrange the Taylor series of f as follows:

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

This is called the homogeneous expansion of f.

▷ Using homogeneous expansion of f we can write the radial derivative Rf as

$$Rf(z) = \sum_{k=1}^{\infty} k f_k(z).$$

More general, for any real number t, we can define the following fractional radial derivative for a holomorphic function f in $\mathbb{B}_n$:

$$R^t f(z) = \sum_{k=1}^{\infty} k^t f_k(z).$$

▷ When we work with partial derivatives, we will use the following notation, where m is any n-tuple of nonnegative integers:

$$\partial^m f = \frac{\partial^{|m|} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}.$$

▷ An important tool in the study of holomorphic function spaces is the notion of fractional differential and integral operators. There are numerous types of fractional differential and integral operators, we introduce one that is intimately related to and interacts well with the Bergman kernel functions. More specifically, for any complex parameters s and t with the property that neither $n + s$ nor $n + s + t$ is a negative integer, we define two operators $R^{s,t}$ and $R_{s,t}$ on $H(\mathbb{B}_n)$ as follows. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of a holomorphic function in $\mathbb{B}_n$, we define

$$R^{s,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s)\Gamma(n+1+k+s+t)}{\Gamma(n+1+s+t)\Gamma(n+1+k+s)} f_k(z).$$

If $H(\mathbb{B}_n)$ is equipped with the topology of "uniform convergence on compact sets", it is easy to see that each $R^{s,t}$ is a continuous invertible operator on $H(\mathbb{B}_n)$. We use $R_{s,t}$ to denote the inverse of $R^{s,t}$ on $H(\mathbb{B}_n)$. Thus

$$R_{s,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s+t)\Gamma(n+1+k+s)}{\Gamma(n+1+s)\Gamma(n+1+k+s+t)} f_k(z).$$

When s is real and $t > 0$, it follows from Stirling's formula that

$$\frac{\Gamma(n+1+s)\Gamma(n+1+k+s+t)}{\Gamma(n+1+s+t)\Gamma(n+1+k+s)} \sim k^t$$

as $k \to \infty$. In this case, $R^{s,t}$ is indeed a fractional radial differential operator of order t and $R_{s,t}$ is a fractional radial integral operator of order t.

The operators $R^{s,t}$ and $R_{s,t}$ seem to have first appeared in Peloso [**44**], and independently in Zhu [**68**], [**69**], [**70**], as a way to define and study holomorphic function spaces on the unit ball, and more generally, on bounded symmetric domains. This

type of fractional differential and integral operators also became an important tool in the books by Arcozzi-Rochberg-Sawyer [7] and Zhu [71].

Kaptanoglu [33], [34], [35] used these operators in a slightly more general way. More specifically, the technical conditions that $n + s$ and $n + s + t$ should not be negative integers can be removed if one is willing to make a separate definition for $R^{s,t}$ (and $R_{s,t}$) in this case. However, since these operators are meant to transform the kernel function $(1 - \langle z, w \rangle)^{-(n+1+s)}$ to $(1 - \langle z, w \rangle)^{-(n+1+s+t)}$, it is clear that the technical conditions mentioned above are natural. Otherwise, these functions would become polynomials and the corresponding reproducing Hilbert spaces would become finite dimensional. Besides, in all our applications, it always involves in choosing a sufficiently large parameter s, and with the technical conditions imposed on s and t, there is never a lack of s for such choices. Also, the use of complex parameters does not present any extra difficulty and will be more convenient for us on several occasions.

LEMMA 1. – *Suppose neither $n + s$ nor $n + s + t$ is a negative integer. Then*

$$R_{s,t} = R^{s+t,-t}.$$

Proof. – This follows directly from the definition of these operators. □

LEMMA 2. – *Suppose s, t, and λ are complex parameters such that none of $n + \lambda$, $n + \lambda + t$, and $n + \lambda + s + t$ is a negative integer. Then*

$$R^{\lambda,t} R^{\lambda+t,s} = R^{\lambda,s+t}.$$

Proof. – This also follows from the definition of these operators. □

As was mentioned earlier, the main advantage of the operators $R^{s,t}$ and $R_{s,t}$ is that they interact well with Bergman kernel functions. This is made precise by the following result.

PROPOSITION 3. – *Suppose neither $n + s$ nor $n + s + t$ is a negative integer. Then*

$$R^{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+s}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}},$$

$$R_{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+s}}.$$

Furthermore, these relations uniquely determine the operators $R^{s,t}$ and $R_{s,t}$ on $H(\mathbb{B}_n)$.

Proof. – See Proposition 1.14 of [**71**]. The proof there is for the case when s and t are real. But obviously the same proof works for complex parameters as well. $\square$

Most of the time we use the above proposition as follows. If a holomorphic function f in $\mathbb{B}_n$ has an integral representation

$$f(z) = \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu(w)}{(1 - \langle z, w \rangle)^{n+1+s}},$$

then

$$R^{s,t} f(z) = \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu(w)}{(1 - \langle z, w \rangle)^{n+1+s+t}}.$$

In particular, if $\alpha > -1$ and $n + \alpha + t$ is not a negative integer, then

$$R^{\alpha,t} f(z) = \lim_{r \to 1^-} \int_{\mathbb{B}_n} \frac{f(rw) \mathrm{d}v_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}}$$

for every function $f \in H(\mathbb{B}_n)$. See Corollary 2.3 of [**71**].

PROPOSITION 4. – *Suppose N is a positive integer and s is a complex number such that $n + s$ is not a negative integer. Then the operator $R^{s,N}$ is a linear partial differential operator on $H(\mathbb{B}_n)$ of order N with polynomial coefficients, that is,*

$$R^{s,N} f(z) = \sum_{|m| \leq N} p_m(z) \partial^m f(z),$$

where each p_m is a polynomial.

Proof. – The proof of Proposition 1.15 of [**71**] works for complex parameters as well.
$\square$

PROPOSITION 5. – *Suppose s and t are complex parameters such that neither $n + s$ nor $n + s + t$ is a negative integer. If $\alpha = s + N$ for some positive integer N, then*

$$R^{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} = \frac{h(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}},$$

where h is a certain one-variable polynomial of degree N. Similarly, there exists a one-variable polynomial q of degree N such that

$$R_{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} = \frac{q(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

Proof. – See the proof of Lemma 2.18 of [**71**] for the result concerning $R^{s,t}$. Combining this with Lemma 1, the result for $R_{s,t}$ follows as well.

Alternatively, we can use Proposition 3 to write

$$R^{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} = R^{s,t} R^{s,N} \frac{1}{(1 - \langle z, w \rangle)^{n+1+s}}.$$

Since $R^{s,N}$ and $R^{s,t}$ commute, another application of Proposition 3 gives

$$R^{s,t}\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} = R^{s,N}\frac{1}{(1-\langle z,w\rangle)^{n+1+s+t}}.$$

The desired result then follows from Proposition 4. □

We also include an easy but important fact concerning the radial derivative.

LEMMA 6. – *For any positive integer k the operator R^k is a kth order partial differential operator on $H(\mathbb{B}_n)$ with polynomial coefficients.*

Proof. – Obvious. □

We are going to need two integral estimates involving Bergman kernel functions.

PROPOSITION 7. – *Suppose s and t are real numbers with $s > -1$. Then the integral*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^s\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+s+t}}$$

has the following asymptotic behavior as $|z| \to 1^-$:

(a) *If $t < 0$, then $I(z)$ is continuous on $\overline{\mathbb{B}}_n$. In particular, $I(z)$ is bounded for $z \in \mathbb{B}_n$.*
(b) *If $t > 0$, then $I(z)$ is comparable to $(1-|z|^2)^{-t}$.*
(c) *If $t = 0$, then $I(z)$ is comparable to $-\log(1-|z|^2)$.*

Proof. – See Proposition 1.4.10 of Rudin [47]. □

PROPOSITION 8. – *Suppose a and b are complex parameters. If S and T are integral operators defined by*

$$Sf(z) = \left(1-|z|^2\right)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b f(w)\,\mathrm{d}v(w)}{(1-\langle z,w\rangle)^{n+1+a+b}},$$

$$Tf(z) = \left(1-|z|^2\right)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b f(w)\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+a+b}},$$

then for any $1 \le p < \infty$ and α real, the following conditions are equivalent:

(a) *The operator S is bounded on $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.*
(b) *The operator T is bounded on $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.*
(c) *The parameters satisfy $-p\,\mathrm{Re}\,a < \alpha + 1 < p(\mathrm{Re}\,b + 1)$.*

Proof. – See [**36**] or Theorem 2.10 of [**71**]. Once again, those proofs are given for real parameters, but the proof for the complex case is essentially the same. The only extra attention to pay is this: when $\lambda = u + iv$ is a complex constant, we have

$$\left(1 - \langle z, w \rangle\right)^{\lambda} = \left|1 - \langle z, w \rangle\right|^{\lambda} \exp(iu\theta - v\theta),$$

where θ is the argument of $1 - \langle z, w \rangle$, say $\theta \in [0, 2\pi)$. It follows that

$$\left|(1 - \langle z, w \rangle)^{\lambda}\right| = \left|1 - \langle z, w \rangle\right|^{u} \exp(-v\theta).$$

Since v is a constant and $\theta \in [0, 2\pi)$, we see that

$$\left|(1 - \langle z, w \rangle)^{\lambda}\right| \sim \left|1 - \langle z, w \rangle\right|^{u} = \left|1 - \langle z, w \rangle\right|^{\mathrm{Re}\,\lambda}. \qquad \square$$

Note that certain special cases of the above proposition can be found in Forelli-Rudin [**25**] and Rudin [**47**].

PROPOSITION 9. – *Suppose* $\mathrm{Re}\,\alpha > -1$. *Then there exists a constant* c_{α} *such that*

$$f(z) = c_{\alpha} \int_{\mathbb{B}_n} \frac{f(w)(1 - |w|^2)^{\alpha}\,\mathrm{d}v(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z \in \mathbb{B}_n,$$

where f *is any holomorphic function in* $\mathbb{B}_n$ *such that*

$$\int_{\mathbb{B}_n} \left|f(z)\right|\left(1 - |z|^2\right)^{\alpha}\mathrm{d}v(z) < \infty.$$

Proof. – See Theorem 7.1.4 of Rudin [**47**] or Theorem 2.2 of Zhu [**71**]. $\qquad \square$

CHAPTER 4

ISOMORPHISM OF BERGMAN SPACES

Our first main result shows that for fixed p, the spaces A_α^p are all isomorphic. A word of caution is necessary here: while the isomorphism among A_α^p reduces the topological structure of A_α^p to that of the ordinary Bergman space A^p, it does not help too much when the properties of individual functions are concerned. This is clear in the Hilbert space case: the Hardy space H^2, the Bergman space A^2, and the Dirichlet space B_2 are all isomorphic as Hilbert spaces, but their respective function theories behave much differently from one to another.

There is a good amount of overlap between the material in this and the next chapter with the results in Beatrous-Burbea [**10**], [**11**], Kaptanoglu [**34**], and Peloso [**44**]. We present independent proofs here in order to achieve a complete and coherent theory. As was mentioned in the introduction, the spaces B_q^p in Kaptanoglu [**34**] and our spaces A_α^p are actually the same (with the identification of α and q), while the family of spaces A_q^p in Beatrous-Burbea [**11**] covers ordinary Bergman spaces (our A_α^p with $\alpha > -1$) and Hardy spaces H^p.

THEOREM 10. – *Suppose $p > 0$ and α is real. If s is a complex parameter such that neither $n + s$ nor $n + s + (\alpha/p)$ is a negative integer, then a holomorphic function f in $\mathbb{B}_n$ is in A_α^p if and only if $R_{s,\alpha/p}f$ is in A^p. Equivalently, $R_{s,\alpha/p}$ is an invertible operator from A_α^p onto A^p.*

Proof. – Recall that a holomorphic function f in $\mathbb{B}_n$ is in A_α^p if and only if there exists a nonnegative integer k with $pk + \alpha > -1$ such that the function $(1 - |z|^2)^k R^k f(z)$ is in $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$. Obviously, this is equivalent to the condition that

$$(2) \qquad R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk+\alpha}).$$

By Theorem 2.16 of [**71**], the condition that $R_{s,\alpha/p}f \in A^p$ is equivalent to

$$\left(1 - |z|^2\right)^k R^k R_{s,\alpha/p}f(z) \in L^p(\mathbb{B}_n, \mathrm{d}v).$$

Since R^k commutes with $R_{s,\alpha/p}$, the above condition is equivalent to

$$R_{s,\alpha/p}R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk}).$$

If $\alpha > 0$, then by Theorem 2.19 of [71], the above condition is equivalent to

$$\left(1 - |z|^2\right)^{\alpha/p} R^{s,\alpha/p} R_{s,\alpha/p} R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk}).$$

Since $R^{s,\alpha/p}$ is the inverse of $R_{s,\alpha/p}$, the above condition is equivalent to

$$\left(1 - |z|^2\right)^{\alpha/p} R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk}),$$

which is the same as (2). This proves the theorem for $\alpha > 0$.

If $\alpha = 0$, the operator $R_{s,\alpha/p}$ becomes the identity operator, and the desired result is trivial.

If $\alpha < 0$, then by Lemma 1, we have $R_{s,\alpha/p}f \in A^p$ if and only if $R^{s+\alpha/p,-\alpha/p}f \in A^p$, which, according to Theorem 2.16 of [71], is equivalent to

$$\left(1 - |z|^2\right)^k R^k R^{s+\alpha/p,-\alpha/p} f \in L^p(\mathbb{B}_n, \mathrm{d}v).$$

Since R^k commutes with $R^{s+\alpha/p,-\alpha/p}$, the above condition is equivalent to $R^{s+\alpha/p,-\alpha/p}R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk})$, or

$$\left(1 - |z|^2\right)^{-\alpha/p} R^{s+\alpha/p,-\alpha/p} R^k f \in L^p(\mathbb{B}_n, \mathrm{d}v_{pk+\alpha}).$$

Since $\alpha < 0$, it follows from Theorem 2.19 of [71] that the above condition is equivalent to (2). This proves the desired result for $\alpha < 0$ and completes the proof of the theorem. $\square$

As a consequence, we obtain the following result which shows that the definition of A_α^p is actually independent of the integer k used. This is of course a phenomenon that has been well known to experts in the field.

COROLLARY 11. – *Suppose $p > 0$ and α is real. Then the following conditions are equivalent for holomorphic functions f in $\mathbb{B}_n$:*

(a) *$f \in A_\alpha^p$, that is, for some positive integer k with $kp + \alpha > -1$ the function $(1 - |z|^2)^k R^k f(z)$ is in $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.*

(b) *For every positive integer k with $kp + \alpha > -1$ the function $(1 - |z|^2)^k R^k f(z)$ is in $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.*

Proof. – This follows from the proof of Theorem 10. This also follows from the equivalence of (a) and (d) in Theorem 2.16 of [71]. $\square$

Since the polynomials are dense in A^p, and since the operators $R^{s,t}$ and $R_{s,t}$ map the set of polynomials onto the set of polynomials, we conclude from Theorem 10 that the polynomials are dense in each space A^p_α.

The following result is a generalization of Theorem 10.

THEOREM 12. – *Suppose α is real, β is real, and $p > 0$. Let $t = (\alpha - \beta)/p$ and let s be a complex parameter such that neither $n + s$ nor $n + s + t$ is a negative integer. Then the operator $R_{s,t}$ maps A^p_α boundedly onto A^p_β.*

Proof. – We can approximate s by a sequence $\{s_k\}$ of complex numbers such that each of the operators $R_{s_k,t}$, $R^{s_k+t,\beta/p}$, and $R_{s_k,\alpha/p}$ is well defined. According to Lemmas 1 and 2, we have

$$R_{s_k,t} = R^{s_k+t,\beta/p} R_{s_k,\alpha/p}.$$

Since $R^{s_k+t,\beta/p}$ is the inverse of $R_{s_k+t,\beta/p}$, it follows from Theorem 10 that each $R_{s_k,t}$ maps A^p_α boundedly onto A^p_β. Since $R_{s,t}$ is well defined, an easy limit argument then shows that $R_{s,t}$ maps A^p_α boundedly onto A^p_β. $\qquad\square$

For any positive p and real α we let N be the smallest nonnegative integer such that $pN + \alpha > -1$ and define

$$(3) \qquad \|f\|_{p,\alpha} = |f(0)| + \Big[\int_{\mathbb{B}_n} \big(1 - |z|^2\big)^{pN} \big|R^N f(z)\big|^p \mathrm{d}v_\alpha(z) \Big]^{1/p}$$

for $f \in A^p_\alpha$. Then A^p_α becomes a Banach space when $p \geq 1$. For $0 < p < 1$ the space A^p_α is a topological vector space with a complete metric

$$(4) \qquad d(f,g) = \|f - g\|^p_{p,\alpha}.$$

The metric d is invariant in the sense that

$$d(f,g) = d(f - g, 0).$$

In particular, A^p_α is an F-space. One of the properties of an F-space that we will use later is that the closed graph theorem is valid for it.

CHAPTER 5

SEVERAL CHARACTERIZATIONS OF A_α^p

In this chapter we obtain various characterizations of A_α^p in terms of fractional differential operators and in terms of higher order derivatives.

THEOREM 13. – *Suppose $p > 0$ and α is real. Then the following conditions are equivalent for holomorphic functions f in $\mathbb{B}_n$.*

(a) $f \in A_\alpha^p$.

(b) *For some nonnegative integer k with $kp + \alpha > -1$ the functions*

$$\left(1 - |z|^2\right)^{|m|} \partial^m f(z),$$

where $|m| = k$, all belong to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

(c) *For every nonnegative integer k with $kp + \alpha > -1$ the functions*

$$\left(1 - |z|^2\right)^{|m|} \partial^m f(z),$$

where $|m| = k$, all belong to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

Proof. – Fix a nonnegative integer k with $pk + \alpha > -1$ and assume that

$$\left(1 - |z|^2\right)^k \partial^m f(z) \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$$

for all $|m| = k$, then

$$\left(1 - |z|^2\right)^k \partial^m f(z) \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$$

for all $|m| \le k$; see Theorem 2.17 of [**71**]. Since R^k is a linear partial differential operator on $H(\mathbb{B}_n)$ with polynomial coefficients (see Lemma 6), we have

$$\left(1 - |z|^2\right)^k R^k f(z) \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha),$$

or $f \in A_\alpha^p$. This proves that condition (b) implies (a). That condition (c) implies (b) is obvious.

Next assume that $f \in A_\alpha^p$. Then by Theorem 10, the function $g = R_{\beta,\alpha/p}f$ is in A^p, where β is a sufficiently large (to be specified later) positive number. By Proposition 9, we have

$$R_{\beta,\alpha/p}f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}}.$$

Apply $R^{\beta,\alpha/p}$ to both sides and use Proposition 3. We obtain

$$(5) \qquad f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta+\alpha/p}}.$$

If $p \geq 1$ and k is any nonnegative integer such that $pk + \alpha > -1$, then we choose β large enough so that

$$(6) \qquad -pk < \alpha + 1 < p\left(\beta + \frac{\alpha}{p}\right).$$

Rewrite the reproducing formula (5) as

$$f(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\beta+\alpha/p}h(w)\mathrm{d}v(w)}{(1 - \langle z, w\rangle)^{n+1+\beta+\alpha/p}},$$

where

$$h(z) = \left(1 - |z|^2\right)^{-\alpha/p}g(z).$$

Differentiating under the integral sign, we obtain a positive constant C (depending on the parameters but not on f and z) such that

$$\left(1 - |z|^2\right)^k\left|\partial^m f(z)\right| \leq C\left(1 - |z|^2\right)^k \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\beta+\alpha/p}|h(w)|\mathrm{d}v(w)}{|1 - \langle z, w\rangle|^{n+k+1+\beta+\alpha/p}},$$

where $|m| = k$. Since $h \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$, it follows from (6) and Proposition 8 that the functions $(1 - |z|^2)^k\partial^m f(z)$, where $|m| = k$, all belong to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

The case $0 < p < 1$ calls for a different proof. In this case, we differentiate under the integral sign in (5) and obtain a constant $C > 0$ (depending on the parameters but not on f and z) such that

$$\left(1 - |z|^2\right)^k\left|\partial^m f(z)\right| \leq C\left(1 - |z|^2\right)^k \int_{\mathbb{B}_n} \frac{|g(w)|(1 - |w|^2)^\beta \mathrm{d}v(w)}{|1 - \langle z, w\rangle|^{n+k+1+\beta+\alpha/p}},$$

where $|m| = k$. We write

$$\beta = \frac{n + 1 + \beta'}{p} - (n + 1)$$

and assume that β is large enough so that $\beta' > 0$. Then we can apply Lemma 2.15 of [**71**] to show that the integral

$$\int_{\mathbb{B}_n} \frac{|g(w)|(1 - |w|^2)^\beta \mathrm{d}v(w)}{|1 - \langle z, w\rangle|^{n+k+1+\beta+\alpha/p}}$$

is less than or equal to a positive constant times

$$\left[\int_{\mathbb{B}_n} \left| \frac{g(w)}{(1 - \langle z, w \rangle)^{n+k+1+\beta+\alpha/p}} \right|^p (1 - |w|^2)^{\beta'} \, dv(w) \right]^{1/p}.$$

It follows that there exists a positive constant C' such that

$$\left(1 - |z|^2\right)^{kp} |\partial^m f(z)|^p \leq C' \left(1 - |z|^2\right)^{kp} \int_{\mathbb{B}_n} \frac{|g(w)|^p (1 - |w|^2)^{\beta'} \, dv(w)}{|1 - \langle z, w \rangle|^{p(n+k+1+\beta)+\alpha}},$$

where $|m| = k$. Integrate both sides against the measure dv_α and apply Fubini's theorem. We see that the integral

$$\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{kp} |\partial^m f(z)|^p dv_\alpha(z)$$

is less than or equal to C' times

$$\int_{\mathbb{B}_n} |g(w)|^p \left(1 - |w|^2\right)^{\beta'} dv(w) \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{kp+\alpha} dv(z)}{|1 - \langle z, w \rangle|^{p(n+k+1+\beta)+\alpha}}.$$

Estimating the inner integral above according to Proposition 7, we find another constant $C'' > 0$ such that

$$\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{kp} |\partial^m f(z)|^p dv_\alpha(z) \leq C'' \int_{\mathbb{B}_n} |g(w)|^p dv(w)$$

for all $|m| = k$. This proves that (a) implies (c), and completes the proof of the theorem. $\qquad \square$

Note that several special cases of the above theorem are well known. See Beatrous-Burbea [11] or Pavlovic [42] for example. In fact, any nontangential partial differential operator of order k with C^∞ coefficients may be used in place of R^k; see Peloso [44]. The proof above uses several techniques developed in Zhu [71].

THEOREM 14. – *Suppose $p > 0$, α is real, and f is holomorphic in $\mathbb{B}_n$. Then the following conditions are equivalent:*

(a) *$f \in A_\alpha^p$.*

(b) *There exists some real t with $pt + \alpha > -1$ such that the function*

$$\left(1 - |z|^2\right)^t R^{s,t} f(z)$$

is in $L^p(\mathbb{B}_n, dv_\alpha)$, where s is any real parameter such that neither $n + s$ nor $n + s + t$ is a negative integer.

(c) *For every real t with $pt + \alpha > -1$ the function*

$$\left(1 - |z|^2\right)^t R^{s,t} f(z)$$

is in $L^p(\mathbb{B}_n, dv_\alpha)$, where s is any real parameter such that neither $n + s$ nor $n + s + t$ is a negative integer.

Proof. – It is obvious that condition (c) implies (b). To show that condition (b) implies (a), we fix a sufficiently large positive number β and apply Proposition 9 to write

$$R^{s,t}f(z) = c_{t+\beta}\int_{\mathbb{B}_n}\frac{R^{s,t}f(w)(1-|w|^2)^{t+\beta}\,\mathrm{d}v(w)}{(1-\langle z,w\rangle)^{n+1+t+\beta}},$$

where $c_{t+\beta}$ is a positive constant such that $c_{t+\beta}\mathrm{d}v_\beta$ is a probability measure on $\mathbb{B}_n$. Apply R^k to both sides, where k is a nonnegative integer such that $kp + \alpha > -1$. Then there exists a polynomial h of degree k such that

$$R^k R^{s,t}f(z) = \int_{\mathbb{B}_n}\frac{h(\langle z,w\rangle)R^{s,t}f(w)\,\mathrm{d}v_{t+\beta}(w)}{(1-\langle z,w\rangle)^{n+1+k+t+\beta}}.$$

If β is chosen so that $\beta - s$ is a sufficiently large positive integer, we first write

$$h(\langle z,w\rangle) = \sum_{j=0}^{k}c_j(1-\langle z,w\rangle)^j,$$

then apply the operator $R_{s,t}$ to every term according to the second part of Proposition 5, and then combine the various terms. The result is that

$$R_{s,t}R^k R^{s,t}f(z) = \int_{\mathbb{B}_n}\frac{g(z,w)R^{s,t}f(w)\,\mathrm{d}v_{t+\beta}(w)}{(1-\langle z,w\rangle)^{n+1+k+\beta}},$$

where g is a polynomial. Since the operators $R_{s,t}$, R^k, and $R^{s,t}$ commute with each other, and since $R_{s,t}$ is the inverse of $R^{s,t}$, we obtain a constant $C > 0$ such that

$$\left(1-|z|^2\right)^k\left|R^k f(z)\right| \le C\left(1-|z|^2\right)^k\int_{\mathbb{B}_n}\frac{(1-|w|^2)^t|R^{s,t}f(w)|\,\mathrm{d}v_\beta(w)}{|1-\langle z,w\rangle|^{n+1+k+\beta}}.$$

We then follow the same arguments as in the proof of Theorem 13 to show that the condition

$$\left(1-|z|^2\right)^t\left|R^{s,t}f(z)\right| \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$$

implies

$$\left(1-|z|^2\right)^k R^k f(z) \in L^p(\mathbb{B}_n, \mathrm{d}v_\alpha).$$

This proves that condition (b) implies (a).

To show that condition (a) implies (c), we fix a function $f \in A_\alpha^p$ and choose a sufficiently large positive number β such that the function $g = R_{\beta,\alpha/p}f$ is in A^p. We then follow the same arguments as in the proof of Theorem 13 to finish the proof. The only adjustment to make here is this: instead of differentiating under the integral sign, we apply the operator $R^{s,t}$ inside the integral sign and take advantage of Proposition 5 (assuming that β is chosen so that $\beta - s$ is a positive integer). We leave the details to the interested reader. $\qquad\square$

Several special cases of the above theorem have appeared before. See Kaptanoglu [33], [34] and Peloso [44] for the case $\alpha = -(n+1)$, and Zhu [71] for the case $\alpha > -1$. The book [11] of Beatrous and Burbea also contains a version of the result for $\alpha > -1$ which is based on a different family of fractional radial differential operators.

CHAPTER 6

HOLOMORPHIC LIPSCHITZ SPACES

The classical Lipschitz space Λ_α, $0 < \alpha < 1$, consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\big|f(z) - f(w)\big| \leq C|z - w|^\alpha, \quad z, w \in \mathbb{B}_n,$$

where C is a positive constant depending on f. It is well known that a holomorphic function f is in Λ_α if and only if there exists a positive constant C such that

$$\big(1 - |z|^2\big)^{1-\alpha}\big|Rf(z)\big| \leq C, \quad z \in \mathbb{B}_n.$$

See Rudin [**47**] and Zhu [**71**].

In this chapter we extend the theory of Lipschitz spaces Λ_α to the full range $-\infty < \alpha < \infty$. More specifically, for any real number α we let Λ_α denote the space of holomorphic functions f in $\mathbb{B}_n$ such that for some nonnegative integer $k > \alpha$ the function $(1 - |z|^2)^{k-\alpha}R^k f(z)$ is bounded in $\mathbb{B}_n$. We first prove that the definition of Λ_α is independent of the integer k used.

LEMMA 15. – *Suppose f is holomorphic in $\mathbb{B}_n$. Then the following conditions are equivalent:*

(a) *There exists some nonnegative integer $k > \alpha$ such that the function*

$$\big(1 - |z|^2\big)^{k-\alpha}R^k f(z)$$

is bounded in $\mathbb{B}_n$.

(b) *For every nonnegative integer $k > \alpha$ the function*

$$\big(1 - |z|^2\big)^{k-\alpha}R^k f(z)$$

is bounded in $\mathbb{B}_n$.

Proof. – Suppose k is a nonnegative integer with $k > \alpha$. Let $N = k + 1$.

If the function $(1 - |z|^2)^{N-\alpha} R^N f(z)$ is bounded in $\mathbb{B}_n$, then an elementary integral estimate based on the identity

$$R^k f(z) - R^k f(0) = \int_0^1 \frac{R^N f(tz)}{t} \, \mathrm{d}t$$

shows that the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded in $\mathbb{B}_n$.

Conversely, if the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded, then there exists a constant $c > 0$ such that

$$R^k f(z) = c \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{k-\alpha} R^k f(w) \mathrm{d}v(w)}{(1 - \langle z, w \rangle)^{n+1+k-\alpha}};$$

see Proposition 9. Taking the radial derivative on both sides, we get

$$R^N f(z) = C \int_{\mathbb{B}_n} \frac{\langle z, w \rangle (1 - |w|^2)^{k-\alpha} R^k f(w) \mathrm{d}v(w)}{(1 - \langle z, w \rangle)^{n+1+N-\alpha}},$$

where $C = c(n+1+k-\alpha)$. This combined with Proposition 7 shows that the function $(1 - |z|^2)^{N-\alpha} R^N f(z)$ is bounded in $\mathbb{B}_n$.

Therefore, the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded if and only if the function $(1 - |z|^2)^{k+1-\alpha} R^{k+1} f(z)$ is bounded, where k is any nonnegative integer satisfying $k > \alpha$. This clearly proves the desired result. $\square$

The above lemma is most likely known to experts in the field, although we could not find a precise reference. In the case $\alpha > 0$, the above result as well as everything else in this chapter can be found in Zhu [**71**].

In what follows we let k be the smallest nonnegative integer greater than α and define a norm on Λ_α by

$$\|f\|_\alpha = \left| f(0) \right| + \sup_{z \in \mathbb{B}_n} \left(1 - |z|^2 \right)^{k-\alpha} \left| R^k f(z) \right|.$$

It is then easy to check that Λ_α becomes a nonseparable Banach space when equipped with this norm.

We write $\mathscr{B} = \Lambda_0$. This is called the Bloch space. It is clear that $f \in \mathscr{B}$ if and only if

$$\sup_{z \in \mathbb{B}_n} \left(1 - |z|^2 \right) \left| Rf(z) \right| < \infty.$$

See [**71**] for more information about $\mathscr{B}$. Our next result shows that all the spaces Λ_α are isomorphic to the Bloch space.

THEOREM 16. – *Suppose s is complex and α is real such that neither $n+s$ nor $n+s+\alpha$ is a negative integer. Then the operator $R^{s,\alpha}$ maps Λ_α onto $\mathscr{B}$.*

Proof. – Suppose f is holomorphic in $\mathbb{B}_n$. Then $R^{s,\alpha}f$ is in the Bloch space if and only if the function $(1 - |z|^2)^k R^k R^{s,\alpha}f(z)$ is bounded in $\mathbb{B}_n$, where k is any positive integer. See Lemma 15 above.

If $f \in \Lambda_\alpha$, then the function

$$g(z) = \left(1 - |z|^2\right)^{k-\alpha} R^k f(z)$$

is bounded in $\mathbb{B}_n$, where k is any positive integer greater than α. Let N be a sufficiently large positive integer such that the number β defined by

$$k - \alpha + \beta = s + N$$

has real part greater than -1. Then we use Proposition 9 to write

$$R^k f(z) = c \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k-\alpha+\beta}}.$$

Applying Proposition 5, we obtain a polynomial h such that

$$R^{s,\alpha} R^k f(z) = \int_{\mathbb{B}_n} \frac{g(w)h(\langle z, w \rangle)\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+\beta}}.$$

By Proposition 7, the function

$$\left(1 - |z|^2\right)^k R^{s,\alpha} R^k f(z) = \left(1 - |z|^2\right)^k R^k R^{s,\alpha}f(z)$$

is bounded in $\mathbb{B}_n$, so $R^{s,\alpha}f$ is in the Bloch space.

On the other hand, if $R^{s,\alpha}f$ is in the Bloch space, then by Lemma 1, the function $R_{s+\alpha,-\alpha}f$ is in the Bloch space. We fix a suffiently large positive integer N such that $\beta = N + s + \alpha$ has real part greater than -1. By part (d) of Theorem 3.4 in [**71**] (the result there was stated and proved for real β, it is clear that the complex case holds as well), there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that

$$R_{s+\alpha,-\alpha}f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}.$$

We apply the operator $R^{s+\alpha,-\alpha}$ to both sides and use Proposition 5 to obtain

$$f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle)g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\alpha}},$$

where p is a polynomial. An easy computation then shows that

$$R^k f(z) = \int_{\mathbb{B}_n} \frac{q(\langle z, w \rangle)g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+k-\alpha}},$$

where k is any positive integer greater than α and q is another polynomial. By Proposition 7, the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded in $\mathbb{B}_n$, namely, $f \in \Lambda_\alpha$. This completes the proof of the theorem. $\qquad\square$

More generally, if s is any complex number such that neither $n+s$ nor $n+s+\alpha-\beta$ is a negative integer, then the operator $R^{s,\alpha-\beta}$ is a bounded invertible operator from Λ_α onto Λ_β. See the proof of Theorem 12.

THEOREM 17. – *Suppose f is holomorphic in $\mathbb{B}_n$ and α is real. If $\operatorname{Re}\beta > -1$ and $n+\beta-\alpha$ is not a negative integer, then $f \in \Lambda_\alpha$ if and only if there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that, for $z \in \mathbb{B}_n$,*

$$(7) \qquad f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta-\alpha}}.$$

Proof. – If f admits the integral representation (7), then for any nonnegative integer $k > \alpha$ we have

$$R^k f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w\rangle)g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta+k-\alpha}},$$

where $p(z)$ is a certain polynomial of degree k. An application of Proposition 7 shows that the function $(1 - |z|^2)^{k-\alpha}R^k f(z)$ is bounded in $\mathbb{B}_n$.

On the other hand, if $f \in \Lambda_\alpha$, then by Theorem 16, the function $R^{\beta-\alpha,\alpha}f$ is in the Bloch space. According to the classical integral representation of functions in the Bloch space (see Choe [18] or part (d) of Theorem 3.4 in Zhu's book [71]), there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that

$$R^{\beta-\alpha,\alpha}f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+(\beta-\alpha)+\alpha}}.$$

Applying the operator $R_{\beta-\alpha,\alpha}$ to both sides and using Proposition 3, we conclude that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta-\alpha}}.$$

This completes the proof of the theorem. $\qquad\qquad\square$

Since the proof of Theorem 3.4 in Zhu [71] is constructive, it follows that there exists a bounded linear operator

$$L : \Lambda_\alpha \longrightarrow L^\infty(\mathbb{B}_n)$$

such that the integral representation in (7) can be given by choosing $g = L(f)$.

THEOREM 18. – *Suppose α is real and k is a nonnegative integer greater than α. Then a holomorphic function f in $\mathbb{B}_n$ belongs to Λ_α if and only if the functions*

$$\left(1 - |z|^2\right)^{k-\alpha}\partial^m f(z), \quad |m| = k,$$

are all bounded in $\mathbb{B}_n$.

Proof. – If $f \in \Lambda_\alpha$, we apply Theorem 17 to represent f in the form

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) \mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\alpha}},$$

where $g \in L^\infty(\mathbb{B}_n)$, $\beta > -1$, and $n + \beta - \alpha$ is not a negative integer. Differentiate under the integral sign and apply Proposition 7. We see that the functions $(1 - |z|^2)^{k-\alpha} \partial^m f(z)$, where $|m| = k$, are all bounded in $\mathbb{B}_n$.

Conversely, if the function $(1 - |z|^2)^{k-\alpha} \partial^m f(z)$ is bounded in $\mathbb{B}_n$ for every $|m| = k$, then it is easy to see that the function $(1 - |z|^2)^{k-\alpha} \partial^m f(z)$ is bounded in $\mathbb{B}_n$ for every $|m| \leq k$. Since R^k is a kth order linear partial differential operator on $H(\mathbb{B}_n)$ with polynomial coefficients (see Lemma 6), we see that the function $(1 - |z|^2)^{k-\alpha} R^k f(z)$ is bounded in $\mathbb{B}_n$, namely, $f \in \Lambda_\alpha$. $\qquad\square$

Various special cases (such as the Bloch space and the case $\alpha \in (0, 1)$) of the above theorem, as well as the next theorem, have been well known. See Aleksandrov [2], Choe [18], Nowark [40], Ouyang-Yang-Zhao [41], Pavlovic [42], Peloso [44], and Zhu [71] for related results.

THEOREM 19. – *Suppose α and t are real with $t > \alpha$. If s is a complex parameter such that neither $n+s$ nor $n+s+t$ is a negative integer, then a holomorphic function f in $\mathbb{B}_n$ belongs to Λ_α if and only if the function $(1 - |z|^2)^{t-\alpha} R^{s,t} f(z)$ is bounded in $\mathbb{B}_n$.*

Proof. – First assume that the function

$$g(z) = (1 - |z|^2)^{t-\alpha} R^{s,t} f(z)$$

is bounded in $\mathbb{B}_n$. By Proposition 9, there exists a positive constant c such that

$$R^{s,t} f(z) = c \int_{\mathbb{B}_n} \frac{g(w) \mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+t+\beta-\alpha}},$$

where β is a sufficiently large positive number with $\beta - \alpha = s + N$ for some positive integer N. If k is a nonnegative integer greater than α, it is easy to see that there exists a polynomial p of degree k such that

$$(8) \qquad R^k R^{s,t} f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle) g(w) \mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+t+\beta-\alpha}}.$$

We decompose

$$p(\langle z, w \rangle) = \sum_{j=0}^{k} c_j (1 - \langle z, w \rangle)^j,$$

apply the operator $R_{s,t}$ to both sides of (8), use Proposition 5, and combine the terms. The result is that

$$R_{s,t} R^k R^{s,t} f(z) = \int_{\mathbb{B}_n} \frac{h(z, w) g(w) \mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+\beta-\alpha}},$$

where h is a certain polynomial. Since all radial differential operators commute, we have

$$R_{s,t}R^k R^{s,t} = R^k.$$

This together with Proposition 7 shows that

$$\left|R^k f(z)\right| \leq \frac{C}{(1 - |z|^2)^{k-\alpha}}$$

for some constant $C > 0$, that is, $f \in \Lambda_\alpha$.

Next assume that $f \in \Lambda_\alpha$. Let N be a sufficiently large positive integer and write $\beta - \alpha = s + N$. By Theorem 17, there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta-\alpha}}.$$

According to Proposition 5, there exists a polynomial h such that

$$R^{s,t}f(z) = \int_{\mathbb{B}_n} \frac{h(z,w)g(w)\mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta-\alpha+t}}.$$

An application of Proposition 7 then shows that

$$\left|R^{s,t}f(z)\right| \leq \frac{C}{(1 - |z|^2)^{t-\alpha}}$$

for some constant $C > 0$, that is, the function $(1 - |z|^2)^{t-\alpha}R^{s,t}f(z)$ is bounded in $\mathbb{B}_n$. $\qquad\square$

All results in this chapter so far are in terms of a certain function being bounded in $\mathbb{B}_n$. We mention that these results remain true when the big oh conditions are replaced by the corresponding little oh conditions. More specifically, for each real number α, we let $\Lambda_{\alpha,0}$ denote the space of holomorphic functions f in $\mathbb{B}_n$ such that there exists a nonnegative integer $k > \alpha$ such that the function $(1 - |z|^2)^{k-\alpha}R^k f(z)$ is in $\mathbb{C}_0(\mathbb{B}_n)$. Here $\mathbb{C}_0(\mathbb{B}_n)$ denotes the space of continuous functions f in $\mathbb{B}_n$ with the property that

$$\lim_{|z|\to 1^-} f(z) = 0.$$

It can be shown that the definition of $\Lambda_{\alpha,0}$ is independent of the integer k used. The special case $\Lambda_{0,0}$ is denoted by $\mathscr{B}_0$ and is called the little Bloch space of $\mathbb{B}_n$. Clearly, $f \in \mathscr{B}_0$ if and only if

$$\lim_{|z|\to 1^-} \left(1 - |z|^2\right)Rf(z) = 0.$$

An alternative description of $\Lambda_{\alpha,0}$ is that it is the closure of the set of polynomials in Λ_α, or the closure in Λ_α of the set of functions holomorphic on the closed unit ball.

It is then clear how to state and prove the little oh analogues of all results of this chapter. It is also well known that when dealing with the little oh type results of this chapter, the space $\mathbb{C}_0(\mathbb{B}_n)$ can be replaced by $\mathbb{C}(\overline{\mathbb{B}}_n)$, the space of functions that are continuous on the closed unit ball. We leave out the routine details.

CHAPTER 7

POINTWISE ESTIMATES

We often need to know how fast a function in A_α^p grows near the boundary. Using results from the previous chapter, we obtain optimal pointwise estimates for functions in A_α^p.

THEOREM 20. – *Suppose $p > 0$ and $n+1+\alpha > 0$. Then there exists a constant $C > 0$ (depending on p and α) such that, for all $f \in A_\alpha^p$ and $z \in \mathbb{B}_n$,*

$$\bigl|f(z)\bigr| \le \frac{C\|f\|_{p,\alpha}}{(1 - |z|^2)^{(n+\alpha+1)/p}}.$$

Proof. – Suppose $f \in A_\alpha^p$. Then $R^N f \in A_{pN+\alpha}^p$, where $pN+\alpha > -1$. By Theorem 2.1 of [**71**],

$$\bigl(1 - |z|^2\bigr)^{(n+1+pN+\alpha)/p}\bigl|R^N f(z)\bigr| \le C\|f\|_{p,\alpha}$$

for some positive constant C (depending only on α). Since

$$\frac{n + 1 + pN + \alpha}{p} = N + \frac{n + 1 + \alpha}{p},$$

it follows from Lemma 15 that there exists a constant $C' > 0$ (depending on p and α) such that, for all $z \in \mathbb{B}_n$.,

$$\bigl(1 - |z|^2\bigr)^{(n+1+\alpha)/p}\bigl|f(z)\bigr| \le C'\|f\|_{p,\alpha} \qquad \square$$

In the case $\alpha > -1$ the above theorem can be found in numerous papers in the literature, including Beatrous-Burbea [**11**] and Vukotić [**60**].

It is not hard to see that the estimate given in Theorem 20 above is optimal, namely, the exponent $(n + \alpha + 1)/p$ cannot be improved. However, using polynomial approximations, we can show that

$$\lim_{|z|\to 1^-} \bigl(1 - |z|^2\bigr)^{(n+\alpha+1)/p} f(z) = 0$$

whenever $f \in A^p_\alpha$ with $n + 1 + \alpha > 0$. Also, if $\alpha > -1$, then the constant C can be taken to be 1; see Theorem 2.1 in [**71**].

THEOREM 21. – *Suppose $p > 0$ and $n + 1 + \alpha < 0$. Then every function in A^p_α is continuous on the closed unit ball and so is bounded in $\mathbb{B}_n$.*

Proof. – Given $f \in A^p_\alpha$, Theorem 10 tells us that we can find a function $g \in A^p$ such that $f = R^{s,\alpha/p}g$, where s is any real parameter such that neither $n + s$ nor $n + s + (\alpha/p)$ is a negative integer. By Theorem 20 and the remark following it, the function $(1 - |z|^2)^{(n+1)/p}g(z)$ is in $\mathbb{C}_0(\mathbb{B}_n)$, which, according to the little oh version of Lemma 15, is the same as $g \in \Lambda_{-(n+1)/p,0}$. Let β be a sufficiently large positive number such that

$$\beta + \frac{n+1}{p} = s + N$$

for some positive integer N. We first apply the little oh version of Theorem 17 to find a function $h \in \mathbb{C}_0(\mathbb{B}_n)$ such that

$$g(z) = \int_{\mathbb{B}_n} \frac{h(w)\,\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+(n+1)/p}}.$$

We then apply the operator $R^{s,\alpha/p}$ to both sides and make use of Proposition 5. The result is

$$f(z) = \int_{\mathbb{B}_n} \frac{p(z,w)h(w)\,\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+(n+1+\alpha)/p}},$$

where p is a polynomial. By part (a) of Proposition 7, the integral above converges uniformly for $z \in \mathbb{B}_n$ and so the function $f(z)$ is continuous on the closed unit ball. $\square$

When $n + 1 + \alpha < 0$, functions in A^p_α are actually much better than just being continuous on the closed unit ball. For example, it follows from Theorems 12, 19, and 20 that every function in A^p_α, $n + 1 + \alpha < 0$, actually belongs to a Lipschitz space Λ_β for some $\beta > 0$. See Corollary 5.5 of Beatrous-Burbea [**11**] for a slightly different version of this observation.

Theorems 20 and 21 also follow from Lemmas 5.4 and 5.6 of Beatrous-Burbea [**11**]. However, as our next result shows, the estimates in [**11**] for the remaining case $\alpha = -(n+1)$ do not seem to be optimal.

THEOREM 22. – *Suppose $n + 1 + \alpha = 0$ and $f \in A^p_\alpha$.*

 (a) *If $0 < p \leq 1$, then $f(z)$ is continuous on the closed unit ball. In particular, f is bounded in $\mathbb{B}_n$.*
 (b) *If $1 < p < \infty$ and $1/p + 1/q = 1$, then there exists a positive constant C (depending on p) such that, for all $z \in \mathbb{B}_n$,*

$$\left| f(z) \right| \leq C \left[\log \frac{2}{1 - |z|^2} \right]^{1/q}.$$

Proof. – Note that $A^p_{-(n+1)} = B_p$, the diagonal Besov spaces on $\mathbb{B}_n$; see Chapter 7 of [**71**]. If $0 < p \leq 1$, the Besov space B_p is contained in B_1 (this is well known, and follows easily from Theorem 32 and the fact that $l^p \subset l^1$ for $0 < p \leq 1$). Since B_1 is contained in the ball algebra (see Theorem 6.8 of [**71**] for example), we conclude that B_p is contained in the ball algebra whenever $0 < p \leq 1$.

If $p > 1$, we use Theorem 6.7 of [**71**] to find a function $g \in L^p(\mathbb{B}_n, \mathrm{d}\tau)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v(w)}{(1 - \langle z, w \rangle)^{n+1}},$$

where

$$\mathrm{d}\tau(z) = \frac{\mathrm{d}v(z)}{(1 - |z|^2)^{n+1}}$$

is the Möbius invariant measure on $\mathbb{B}_n$. Rewrite the above integral representation as

$$f(z) = \int_{\mathbb{B}_n} \left(\frac{1 - |w|^2}{1 - \langle z, w \rangle} \right)^{n+1} g(w)\mathrm{d}\tau(w)$$

and apply Hölder's inequality. We obtain

$$\left| f(z) \right| \leq \left[\int_{\mathbb{B}_n} |g(w)|^p \mathrm{d}\tau(w) \right]^{\frac{1}{p}} \left[\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{(n+1)(q-1)}}{|1 - \langle z, w \rangle|^{(n+1)q}} \mathrm{d}v(w) \right]^{\frac{1}{q}}.$$

An application of Proposition 7 to the last integral above yields the desired estimate for $f(z)$. $\qquad\square$

CHAPTER 8

DUALITY

A linear functional $F : A_\alpha^p \to \mathbb{C}$ is said to be bounded if there exists a positive constant C such that

$$(9) \qquad \big|F(f)\big| \le C\|f\|_{p,\alpha}$$

for all $f \in A_\alpha^p$. The dual space of A_α^p, denoted by $(A_\alpha^p)^*$, is the vector space of all bounded linear functionals on A_α^p. For any bounded linear functional F on A_α^p we use $\|F\|$ to denote the smallest constant C satisfying (9). It is then easy to check that $(A_\alpha^p)^*$ becomes a Banach space with this norm, regardless of $p \ge 1$ or $p < 1$.

By results of the previous chapter, the point evaluation at any $z \in \mathbb{B}_n$ is a bounded linear functional on A_α^p. Therefore, $(A_\alpha^p)^*$ is a nontrivial Banach space for all $p > 0$ and all real α.

Results of this chapter for the case $p > 1$ are motivated by the well-known duality relation $(A^p)^* = A^q$ for ordinary Bergman spaces under the ordinary volume integral pairing. Results of this chapter in the case $0 < p \le 1$ are motivated by and are generalizations of various special cases obtained in the papers Duren-Romberg-Shields [**24**], Rochberg [**46**], Shapiro [**51**], and Zhu [**68**]. We also mention that this chapter in spirit overlaps with Section 7 of Kaptanoglu [**34**].

THEOREM 23. – *Suppose $1 < p < \infty$, α is real, and β is real. If*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if s_1 and s_2 are complex parameters such that both $R_{s_1,\alpha/p}$ and $R_{s_2,\beta/q}$ are well-defined operators, then $(A_\alpha^p)^* = A_\beta^q$ (with equivalent norms) under the integral pairing

$$\langle f, g \rangle = \int_{\mathbb{B}_n} R_{s_1,\alpha/p} f \, \overline{R_{s_2,\beta/q}g} \, \mathrm{d}v,$$

where $f \in A_\alpha^p$ and $g \in A_\beta^q$.

Proof. – This follows from the identities

$$R^{s_1,\alpha/p}A^p = A^p_\alpha, \quad R^{s_2,\beta/q}A^q = A^q_\beta,$$

and the well-known duality $(A^p)^* = A^q$ under the integral pairing

$$\langle f, g \rangle = \int_{\mathbb{B}_n} f\,\bar{g}\mathrm{d}v,$$

where $f \in A^p_\alpha$ and $g \in A^q_\beta$. $\square$

If $\alpha > -1$ and $\beta > -1$, then the integral pairing

$$\int_{\mathbb{B}_n} R_{s_1,\alpha/p}f\,\overline{R_{s_2,\beta/q}g}\mathrm{d}v$$

can be replaced by the integral pairing

$$\int_{\mathbb{B}_n} f\,\bar{g}\mathrm{d}v_\gamma, \quad f \in A^p_\alpha, g \in A^q_\beta,$$

where

(10)
$$\gamma = \frac{\alpha}{p} + \frac{\beta}{q}.$$

See Theorem 2.12 of [**71**]. For arbitrary α and β, we can also use the integral pairing

$$\lim_{r \to 1^-} \int_{\mathbb{B}_n} R_{s,\gamma}f(rz)\,\overline{g(rz)}\mathrm{d}v(z), \quad f \in A^p_\alpha,\ g \in A^q_\beta,$$

where γ is defined by (10) and s is any complex parameter such that the operator $R_{s,\gamma}$ is well defined.

More generally, if γ is given by (10) and if k is a sufficiently large positive integer, then the duality $(A^p_\alpha)^* = A^q_\beta$ can be realized with the following integral pairing

$$\langle f, g \rangle_\gamma = f(0)\,\overline{g(0)} + \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^k R^k f(z)\,\overline{(1 - |z|^2)^k R^k g(z)}\mathrm{d}v_\gamma(z),$$

where $f \in A^p_\alpha$ and $g \in A^q_\beta$. Many other different, but equivalent, integral pairings are possible.

THEOREM 24. – *Suppose $0 < p \leq 1$, α is real, and β is real. If s_1 and s_2 are complex parameters such that the operators $R_{s_1,\alpha/p}$ and $R^{s_2,\beta}$ are well-defined, then $(A^p_\alpha)^* = \Lambda_\beta$ under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} R_{s_1,\alpha/p}f(rz)\,\overline{R^{s_2,\beta}g(rz)}\mathrm{d}v_\gamma(z),$$

where $f \in A^p_\alpha$, $g \in \Lambda_\beta$, and $\gamma = (n+1)(1/p - 1)$.

Proof. – This follows from the identities

$$R^{s_1,\alpha/p}A^p = A_\alpha^p, \quad R^{s_2,\beta}\Lambda_\beta = \mathcal{B},$$

and the well-known duality $(A^p)^* = \mathcal{B}$ under the integral pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(rz)}\mathrm{d}v_\gamma(z).$$

See Theorem 3.17 of [**71**]. $\qquad\square$

Once again, it is easy to come up with other different (but equivalent) duality pairings. We state two special cases.

COROLLARY 25. – *For any real α we have $(A_\alpha^1)^* = \Lambda_\alpha$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(rz)}\mathrm{d}v(z),$$

where $f \in A_\alpha^1$ and $g \in \Lambda_\alpha$.

Proof. – Simply choose $s_1 = s_2$, $\alpha = \beta$, and $p = 1$ in the theorem. $\qquad\square$

COROLLARY 26. – *Suppose α is real and s is any complex parameter such that $R^{s,\alpha}$ is well defined. Then $(A_\alpha^1)^* = \mathcal{B}$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} R_{s,\alpha}f(rz)\overline{g(rz)}\mathrm{d}v(z),$$

where $f \in A_\alpha^1$ and $g \in \mathcal{B}$.

Proof. – Simply choose $\beta = 0$ in the theorem. $\qquad\square$

THEOREM 27. – *Suppose α and β are real. If s_1 and s_2 are complex parameters such that the operators $R^{s_1,\alpha}$ and $R^{s_2,\beta}$ are both well defined, then $(\Lambda_{\beta,0})^* = A_\alpha^1$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} R_{s_1,\alpha}f(rz)\,\overline{R^{s_2,\beta}g(rz)}\mathrm{d}v(z),$$

where $f \in A_\alpha^1$ and $g \in \Lambda_{\beta,0}$.

Proof. – See the proof of Theorem 24. $\qquad\square$

We also mention two special cases.

COROLLARY 28. – *For any real α we have $(\Lambda_{\alpha,0})^* = A_\alpha^1$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(rz)}\mathrm{d}v(z),$$

where $f \in \Lambda_{\alpha,0}$ and $g \in A_\alpha^1$.

Proof. – See the proof of Corollary 25. □

COROLLARY 29. – *Suppose α is real and s is a complex parameter such that the operator $R^{s,\alpha}$ is well defined. Then $(\mathscr{B}_0)^* = A_\alpha^1$ (with equivalent norms) under the integral pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{B}_n} R_{s,\alpha} f(rz) \overline{g(rz)} \, \mathrm{d}v(z),$$

where $f \in \mathscr{B}_0$ and $g \in A_\alpha^1$.

Proof. – Just set $\beta = 0$ in the theorem. □

CHAPTER 9

INTEGRAL REPRESENTATIONS

In this chapter we focus on the case $1 \leq p < \infty$ and show that each space A_α^p is a quotient space of $L^p(\mathbb{B}_n, \mathrm{d}v_\beta)$. We do this using Bergman type projections.

Integral representations of functions in Bergman spaces of $\mathbb{B}_n$ started in Forelli-Rudin [25] and have seen several generalizations; see Choe [18], Kaptanoglu [34], and Zhu [71]. The next result appears to be new even in the case of unweighted Bergman spaces of the unit disk.

THEOREM 30. – *Suppose $p \geq 1$ and α is real. If γ and λ are complex parameters satisfying the two conditions,*

(a) $p(\operatorname{Re}\gamma + 1) > \operatorname{Re}\lambda + 1$,
(b) $n + \gamma + (\alpha - \lambda)/p$ *is not a negative integer,*

then a holomorphic function f in $\mathbb{B}_n$ belongs to A_α^p if and only if

$$(11) \qquad f(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_\gamma(w)}{(1 - \langle z, w \rangle)^{n+1+\gamma+(\alpha-\lambda)/p}}$$

for some $g \in L^p(\mathbb{B}_n, \mathrm{d}v_\lambda)$.

Proof. – Suppose that the parameters satisfy conditions (a) and (b). Let

$$\beta = \gamma - \frac{\lambda}{p}.$$

Then $\lambda = p(\gamma - \beta)$. Note that condition (a) is equivalent to $p(\operatorname{Re}\beta + 1) > 1$. In particular, $\operatorname{Re}\beta > -1$ and $n + \beta$ is not a negative integer. Also, condition (b) is equivalent to the condition that $n + (\alpha/p) + \beta$ is not a negative integer. So the operators $R^{\beta,\alpha/p}$ and $R_{\beta,\alpha/p}$ are well defined.

If $f \in A_\alpha^p$, then by Theorem 10, the function $R_{\beta,\alpha/p}f$ is in A^p. It follows from Theorem 2.11 of [**71**] (note that the result there was stated and proved for real parameters, the case of complex parameters is proved in exactly the same way) and the condition $p(\operatorname{Re}\beta + 1) > 1$ that there exists a function $h \in L^p(\mathbb{B}_n, \mathrm{d}v)$ such that

$$R_{\beta,\alpha/p}f(z) = \int_{\mathbb{B}_n} \frac{h(w)\,\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}.$$

Apply the operator $R^{\beta,\alpha/p}$ to both sides and use Proposition 3. Then

$$f(z) = \int_{\mathbb{B}_n} \frac{h(w)\,\mathrm{d}v_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+(\alpha/p)}}.$$

Let $g(w) = (1 - |w|^2)^{\beta-\gamma}h(w)$. Then $g \in L^p(\mathbb{B}_n, \mathrm{d}v_\lambda)$ and

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_\gamma(w)}{(1 - \langle z, w \rangle)^{n+1+\gamma+(\alpha-\lambda)/p}}.$$

The above arguments can be reversed. So any function represented by (11) is necessarily a function in A_α^p. This completes the proof of the theorem. $\square$

Once again, the proof of Theorem 2.11 of [**71**] is constructive. So there exists a bounded linear operator

$$L : A_\alpha^p \longrightarrow L^p(\mathbb{B}_n, \mathrm{d}v_\lambda)$$

such that the integral representation in (11) can be achieved with the choice $g = L(f)$.

If condition (b) above is not satisfied, then

$$\frac{1}{(1 - \langle z, w \rangle)^{n+1+\gamma+(\alpha-\lambda)/p}} = \left(1 - \langle z, w \rangle\right)^k$$

for some nonnegative integer k, and any function represented by (11) is a polynomial of degree less than or equal to k. In this case, the integral representation (11) cannot possibly give rise to all functions in A_α^p. This shows that condition (b) is essential for the theorem.

We can also show that condition (a) is essential. In fact, if every function g in $L^p(\mathbb{B}_n, \mathrm{d}v_\lambda)$ gives rise to a function f in A_α^p via the integral representation (11), then we can apply the operator $R^{\gamma+(\alpha-\lambda)/p,k}$ to both sides of (11) and use Theorem 14 to infer that the operator

$$Tg(z) = \left(1 - |z|^2\right)^k \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_\gamma(w)}{(1 - \langle z, w \rangle)^{n+1+k+\gamma+(\alpha-\lambda)/p}}$$

maps $L^p(\mathbb{B}_n, \mathrm{d}v_\lambda)$ boundedly into $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$, where k is any nonnegative integer such that $pk + \alpha > -1$. Write

$$g(w) = \left(1 - |w|^2\right)^{(\alpha-\lambda)/p}h(w).$$

Then $g \in L^p(\mathbb{B}_n, dv_\lambda)$ if and only if $h \in L^p(\mathbb{B}_n, dv_\alpha)$. It follows that the operator

$$Sh(z) = \left(1 - |z|^2\right)^k \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\gamma + (\alpha - \lambda)/p} h(w) dv(w)}{(1 - \langle z, w \rangle)^{n+1+k+\gamma+(\alpha-\lambda)/p}}$$

maps $L^p(\mathbb{B}_n, dv_\alpha)$ boundedly into $L^p(\mathbb{B}_n, dv_\alpha)$. By Proposition 8, the parameters must satisfy the conditions

$$-pk < \alpha + 1 < p \operatorname{Re}\left(\gamma + \frac{\alpha - \lambda}{p} + 1\right).$$

It is easy to see that these two conditions are the same as the two conditions

$$pk + \alpha > -1 \quad \text{and} \quad p(\operatorname{Re}\gamma + 1) > \operatorname{Re}\lambda + 1.$$

Therefore, the conditions in Theorem 30 above are best possible.

COROLLARY 31. – *Suppose $p \geq 1$ and α is real. If β is any complex parameter such that*

(a) *$p(\operatorname{Re}\beta + 1) > \alpha + 1$,*

(b) *$n + \beta$ is not a negative integer,*

then a holomorphic function f in $\mathbb{B}_n$ belongs to A_α^p if and only if

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) dv_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}$$

for some $g \in L^p(\mathbb{B}_n, dv_\alpha)$.

Proof. – Simply set $\gamma = \beta$ and $\lambda = \alpha$ in the theorem. $\qquad\square$

CHAPTER 10

ATOMIC DECOMPOSITION

Atomic decomposition for the Bergman spaces A_α^p was first obtained in Coifman-Rochberg [21] in the case $\alpha > -1$. This turns out to be a powerful theorem in the theory of Bergman spaces. We now generalize the result to all A_α^p. We will also obtain atomic decomposition for the generalized holomorphic Lipschitz spaces Λ_α.

THEOREM 32. – *Suppose $p > 0$, α is real, and b is real. If b is neither 0 nor a negative integer, and*

$$(12) \qquad b > n \max\left(1, \frac{1}{p}\right) + \frac{\alpha + 1}{p},$$

then there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that a holomorphic function f in $\mathbb{B}_n$ belongs to A_α^p if and only if, for some sequence $\{c_k\} \in \ell^p$,

$$(13) \qquad f(z) = \sum_{k=1}^\infty c_k \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}.$$

Proof. – Note that the condition in (12) implies that

$$b - \frac{\alpha}{p} > n \max\left(1, \frac{1}{p}\right) + \frac{1}{p} > n.$$

This, together with the assumption that b is neither 0 nor a negative integer, shows that the operators $R^{s,\alpha/p}$ and $R_{s,\alpha/p}$ are well defined, where s is determined by

$$b = n + 1 + s + \frac{\alpha}{p}.$$

Also note that the condition in (12) implies that

$$b' > n \max\left(1, \frac{1}{p}\right) + \frac{1}{p},$$

where $b' = b - (\alpha/p)$. By Theorem 2.30 of [**71**], there exists a sequence $\{a_k\}$ such that $f \in A^p$ if and only if, for some sequence $\{c_k\} \in \ell^p$,

$$(14) \qquad f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b' - (n+1)/p}}{(1 - \langle z, a_k \rangle)^{b'}}.$$

If f is given by (13), then

$$R_{s,\alpha/p} f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b - (n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^{b - \alpha/p}},$$

or

$$R_{s,\alpha/p} f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b' - (n+1)/p}}{(1 - \langle z, a_k \rangle)^{b'}}.$$

According to the previous paragraph, we have $R_{s,\alpha/p} f \in A^p$. Combining this with Theorem 10, we conclude that $f \in A^p_\alpha$.

The above arguments can be reversed, showing that every function $f \in A^p_\alpha$ admits an atomic decomposition (13). This completes the proof of the theorem. $\qquad\square$

Recall that when $\alpha = -(n+1)$, the resulting spaces A^p_α are nothing but the diagonal Besov spaces B_p. Atomic decompositions for Besov spaces have also been obtained in Frazier-Jawerth [**26**] and Peloso [**44**].

It can be shown that the assumptions on the parameters in the above theorem are optimal. It can also be shown that for $f \in A^p_\alpha$, we have

$$\|f\|^p_{p,\alpha} \sim \inf \sum_{k=1}^{\infty} |c_k|^p,$$

where the infimum is taken over all sequences $\{c_k\}$ satisfying the representation (13).

The atomic decomposition for functions in the Bloch space was first obtained in Rochberg [**46**]. As a consequence of atomic decomposition for the Bloch space, we now obtain an atomic decomposition for functions in the generalized Lipschitz spaces.

THEOREM 33. – *Suppose α and b are real parameters with the two properties:*

(a) *$b + \alpha > n$,*

(b) *b is neither 0 nor a negative integer.*

Then there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that a holomorphic function f in $\mathbb{B}_n$ belongs to Λ_α if and only if, for some sequence $\{c_k\} \in \ell^\infty$,

$$(15) \qquad f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b+\alpha}}{(1 - \langle z, a_k \rangle)^b}.$$

Proof. – Choose s so that $b = n + 1 + s$. Then the operators $R^{s,\alpha}$ and $R_{s,\alpha}$ are well defined. Let $b' = b + \alpha$. Then a function f is represented by (15) if and only if

$$R^{s,\alpha} f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b'}}{(1 - \langle z, a_k \rangle)^{b'}}$$

for some $\{c_k\} \in \ell^\infty$. Since $R^{s,\alpha} \Lambda_\alpha = \mathscr{B}$, the desired result then follows from the atomic decomposition for the Bloch space; see Theorem 3.23 of [**71**]. $\qquad\square$

Once again, the assumptions on the parameters b and α in the above theorem are best possible.

A little oh version of Theorem 33 also holds, giving the atomic decomposition for the space $\Lambda_{\alpha,0}$. The only adjustment to be made is to replace the sequence space ℓ^∞ by c_0 (consisting of sequences that tend to 0). We omit the details.

As a corollary of atomic decomposition, we prove the following embedding of weighted Bergman spaces which is well known and very useful in the special case $\alpha > -1$; see Aleksandrov [**2**], Beatrous-Burbea [**11**], Rochberg [**46**], and Lemma 2.15 of Zhu [**71**].

THEOREM 34. – *Suppose $0 < p \le 1$ and α is real. If*

$$\beta = \frac{n + 1 + \alpha}{p} - (n + 1),$$

then A_α^p is continuously contained in A_β^1.

Proof. – Suppose $0 < p \le 1$ and fix any positive integer b such that $b > (n+1+\alpha)/p$. If $f \in A_\alpha^p$, then there exists a sequence $\{c_k\} \in \ell^p \subset \ell^1$ such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b},$$

where $\{a_k\}$ is a certain sequence in $\mathbb{B}_n$. For any $k \ge 1$ write

$$f_k(z) = \frac{1}{(1 - \langle z, a_k \rangle)^b}.$$

Then

$$\|f\|_{1,\beta} \le \sum_{k=1}^{\infty} |c_k| \left(1 - |a_k|^2\right)^{b-(n+1+\alpha)/p} \|f_k\|_{1,\beta}.$$

An easy computation shows that

$$R^N f_k(z) = \frac{h(\langle z, a_k \rangle)}{(1 - \langle z, a_k \rangle)^{b+N}},$$

where N is the smallest nonnegative integer with $N + \beta > -1$ and h is a polynomial of degree N. It follows that

$$\int_{\mathbb{B}_n} (1 - |z|^2)^N |R^N f_k(z)| dv_\beta(z) \leq C \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{N+\beta} dv(z)}{|1 - \langle z, a_k \rangle|^{b+N}},$$

where C is a positive constant (independent of k). Estimating the second integral above by Proposition 7, we obtain

$$\int_{\mathbb{B}_n} (1 - |z|^2)^N |R^N f_k(z)| dv_\beta(z) \leq \frac{C'}{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}},$$

where C' is another positive constant independent of k. This shows that

$$\|f\|_{1,\beta} \leq C' \sum_{k=1}^{\infty} |c_k| < \infty,$$

completing the proof of the theorem.   □

The above theorem can also be proved without appealing to atomic decomposition. In fact, if k is a sufficiently large positive integer (such that $kp + \alpha > -1$ and $k + \beta > -1$), then the condition $f \in A_\alpha^p$, $0 < p \leq 1$, implies that $R^k f \in A_{\alpha'}^p$, where $\alpha' = kp + \alpha$. By Lemma 2.15 of [**71**], we have $R^k f \in A_{\beta'}^1$, where

$$\beta' = \frac{n+1+\alpha'}{p} - (n+1) = k + \frac{n+1+\alpha}{p} - (n+1) = k + \beta,$$

or equivalently, the function $(1 - |z|^2)^k R^k f(z)$ belongs to $L^1(\mathbb{B}_n, dv_\beta)$, that is, $f \in A_\beta^1$.

THEOREM 35. – *Suppose $p > 0$ and α is real. If q and r are positive numbers satisfying*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r},$$

then every function $f \in A_\alpha^p$ admits a decomposition

$$f(z) = \sum_{k=1}^{\infty} g_k(z) h_k(z),$$

where each g_k is in A_α^q and each h_k is in A_α^r. Furthermore, if $0 < p \leq 1$, then

$$\sum_{k=1}^{\infty} \|g_k\|_{q,\alpha} \|h_k\|_{r,\alpha} \leq C \|f\|_{p,\alpha},$$

where C is a positive constant independent of f.

Proof. – Consider the function

$$f(z) = \frac{1}{(1 - \langle z, a \rangle)^b},$$

where $a \in \mathbb{B}_n$ and b is the constant from Theorem 32. We can write $f = gh$, where

$$g(z) = \frac{1}{(1 - \langle z, a \rangle)^{bp/q}}, \quad h(z) = \frac{1}{(1 - \langle z, a \rangle)^{bp/r}}.$$

If k is a sufficiently large positive integer, then it follows from Proposition 7 that

$$\|f\|_{p,\alpha} \sim \Big[\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{pk} \mathrm{d}v_\alpha(z)}{|1 - \langle z, a \rangle|^{p(b+k)}} \Big]^{1/p}$$

$$\sim \Big[\frac{1}{(1 - |a|^2)^{pb - (n+1+\alpha)}} \Big]^{1/p} = \frac{1}{(1 - |a|^2)^{b - (n+1+\alpha)/p}}.$$

Similar computations show that

$$\|g\|_{q,\alpha} \sim \frac{1}{(1 - |a|^2)^{(bp-n-1-\alpha)/q}} \quad \text{and} \quad \|h\|_{r,\alpha} \sim \frac{1}{(1 - |a|^2)^{(bp-n-1-\alpha)/r}}.$$

It follows that

$$\|f\|_{p,\alpha} \sim \|g\|_{q,\alpha} \|h\|_{r,\alpha}.$$

The desired result then follows from Theorem 32 and the fact that

$$\sum_{k=1}^{\infty} |c_k| \le C \Big(\sum_{k=1}^{\infty} |c_k|^p \Big)^{1/p}$$

when $0 < p \le 1$. See the proof of Corollary 2.33 in [71] as well. $\qquad\square$

When $\alpha > -1$, the above theorem can be found in Coifman-Rochberg [21] and Rochberg [46].

CHAPTER 11

COMPLEX INTERPOLATION

In this chapter we determine the complex interpolation space of two generalized weighted Bergman spaces. We also determine the complex interpolation space between a weighted Bergman space and a Lipschitz space.

Throughout this chapter we let

$$S = \big\{z = x + iy \in \mathbb{C} : 0 < x < 1\big\}, \quad \overline{S} = \big\{z = x + iy \in \mathbb{C} : 0 \leq x \leq 1\big\}.$$

Thus S is an open strip in the complex plane and $\overline{S}$ is its closure. We denote the two boundary lines of S by

$$L(S) = \big\{z = x + iy \in \mathbb{C} : x = 0\big\}, \quad R(S) = \big\{z = x + iy \in \mathbb{C} : x = 1\big\}.$$

The complex method of interpolation is based on Hadamard's three lines theorem, which states that if f is a function that is continuous on $\overline{S}$, bounded on $\overline{S}$, and analytic in S, then for any $\theta \in (0, 1)$

$$\sup_{\mathrm{Re}\, z = \theta} \big|f(z)\big| \leq \big(\sup_{\mathrm{Re}\, z = 0} \big|f(z)\big| \big)^{1-\theta} \big(\sup_{\mathrm{Re}\, z = 1} \big|f(z)\big| \big)^{\theta}.$$

Let X and Y be two Banach spaces of holomorphic functions in $\mathbb{B}_n$. Then $X + Y$ becomes a Banach space with the norm

$$\|f\|_{X+Y} = \inf \big(\|g\|_X + \|h\|_Y \big), \quad f \in X + Y,$$

where the infimum is taken over all decompositions $f = g + h$ with $g \in X$ and $h \in Y$. If $\theta \in (0, 1)$, the complex interpolation space $[X, Y]_\theta$ consists of holomorphic functions f in $\mathbb{B}_n$ with the following properties:

1) There exists a function $\zeta \mapsto f_\zeta$ from $\overline{S}$ into the Banach space $X + Y$ that is analytic in S, continuous on $\overline{S}$, and bounded on $\overline{S}$.

2) $f_\theta = f$.

3) The function $\zeta \mapsto f_\zeta$ is bounded and continuous from $L(S)$ into X.

4) The function $\zeta \mapsto f_\zeta$ is bounded and continuous from $R(S)$ into Y.

The space $[X, Y]_\theta$ is a Banach space with the norm

$$\|f\|_\theta = \inf \max \big(\sup_{\mathrm{Re}\,\zeta=0} \|f_\zeta\|_X, \sup_{\mathrm{Re}\,\zeta=1} \|f_\zeta\|_Y \big),$$

where the infimum is taken over all f_ζ satisfying conditions 1) through 4) above. See Bergh-Löfström [14] and Bennett-Sharpley [13] for more information about complex interpolation.

The complex method of interpolation spaces is functorial in the sense that if

$$T : X + Y \longrightarrow X' + Y'$$

is a linear operator with the property that T maps X boundedly into X' and T maps Y boundedly into Y', then T also maps $[X, Y]_\theta$ boundedly into $[X', Y']_\theta$ for each $\theta \in (0, 1)$.

The most classical example of complex interpolation spaces concerns L^p spaces (over any measure space). More specifically, if $1 \le p_0 < p_1 \le \infty$ and

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$$

for some $0 < \theta < 1$, then

$$[L^{p_0}, L^{p_1}]_\theta = L^p$$

with equal norms.

More generally, if w_0 and w_1 are weight functions of a measure μ, and if $1 \le p_0 < p_1 < \infty$, then for any $\theta \in (0, 1)$ we have

$$\big[L^{p_0}(w_0), L^{p_1}(w_1)\big]_\theta = L^p(w)$$

with equal norms, provided that

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad w^{\frac{1}{p}} = w_0^{\frac{1-\theta}{p_0}} w_1^{\frac{\theta}{p_1}}.$$

This is usually referred to as the Stein-Weiss interpolation theorem. See Stein-Weiss [55].

THEOREM 36. – *Suppose α and β are real. If $1 \le p_0 \le p_1 < \infty$ and*

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$$

for some $\theta \in (0, 1)$, then

$$\big[A_\alpha^{p_0}, A_\beta^{p_1}\big]_\theta = A_\gamma^p$$

with equivalent norms, where γ is determined by

$$\frac{\gamma}{p} = \frac{\alpha}{p_0}(1 - \theta) + \frac{\beta}{p_1}\,\theta.$$

Proof. – It is clear that $1 \le p < \infty$. We fix a large positive number s such that

$$(16) \qquad p(s+1) > \gamma + 1, \quad p_0(s+1) > \alpha + 1, \quad p_1(s+1) > \beta + 1.$$

Then by Corollary 31, the integral operator

$$Tg(z) = \int_{\mathbb{B}_n} \frac{g(w)\mathrm{d}v_s(w)}{(1 - \langle z, w \rangle)^{n+1+s}}$$

maps $L^p(\mathbb{B}_n, \mathrm{d}v_\gamma)$ boundedly onto A^p_γ; it maps $L^{p_0}(\mathbb{B}_n, \mathrm{d}v_\alpha)$ boundedly onto $A^{p_0}_\alpha$; and it maps $L^{p_1}(\mathbb{B}_n, \mathrm{d}v_\beta)$ boundedly onto $A^{p_1}_\beta$. It follows from the functorial property of complex interpolation and the Stein-Weiss interpolation theorem that T maps the space

$$\left[L^{p_0}(\mathbb{B}_n, \mathrm{d}v_\alpha), L^{p_1}(\mathbb{B}_n, \mathrm{d}v_\beta) \right]_\theta = L^p(\mathbb{B}_n, \mathrm{d}v_\gamma)$$

boundedly into $\left[A^{p_0}_\alpha, A^{p_1}_\beta \right]_\theta$. Since $TL^p(\mathbb{B}_n, \mathrm{d}v_\gamma) = A^p_\gamma$, we conclude that

$$A^p_\gamma \subset \left[A^{p_0}_\alpha, A^{p_1}_\beta \right]_\theta,$$

and the inclusion is continuous.

On the other hand, if k is a sufficiently large positive integer, the operator L defined by

$$L(f)(z) = \left(1 - |z|^2\right)^k R^k f(z), \quad f \in H(\mathbb{B}_n),$$

maps $A^{p_0}_\alpha$ boundedly into $L^{p_0}(\mathbb{B}_n, \mathrm{d}v_\alpha)$; and it maps $A^{p_1}_\beta$ boundedly into $L^{p_1}(\mathbb{B}_n, \mathrm{d}v_\beta)$. By the functorial property of complex interpolation and the Stein-Weiss interpolation theorem, the operator L also maps $[A^{p_0}_\alpha, A^{p_1}_\beta]_\theta$ boundedly into $L^p(\mathbb{B}_n, \mathrm{d}v_\gamma)$. Equivalently, if $f \in [A^{p_0}_\alpha, A^{p_1}_\beta]_\theta$, then the function $(1 - |z|^2)^k R^k f(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\gamma)$, that is, $f \in A^p_\gamma$. We conclude that

$$\left[A^{p_0}_\alpha, A^{p_1}_\beta \right]_\theta \subset A^p_\gamma,$$

and the inclusion is continuous. This completes the proof of the theorem. $\square$

COROLLARY 37. – *Suppose α is real, β is real, $1 \le p < \infty$, and $0 < \theta < 1$. Then*

$$\left[A^p_\alpha, A^p_\beta \right]_\theta = A^p_\gamma,$$

where $\gamma = \alpha(1 - \theta) + \beta\theta$.

THEOREM 38. – *Suppose α and β are real. If $1 \le p < \infty$ and $0 < \theta < 1$, then*

$$\left[A^p_\alpha, \Lambda_\beta \right]_\theta = A^q_\gamma$$

with equivalent norms, where $q = p/(1 - \theta)$ and $\gamma = \alpha - q\beta\theta$.

Proof. – First we consider the linear operator

$$Tf(z) = \int_{\mathbb{B}_n} \frac{f(w)\,\mathrm{d}v_{s+\beta}(w)}{(1 - \langle z, w\rangle)^{n+1+s}},$$

where s is a fixed and sufficiently large positive number. By Theorem 17 and Theorem 30, the operator T maps $L^\infty(\mathbb{B}_n)$ boundedly onto Λ_β; and it maps $L^p(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta})$ boundedly onto A_α^p. Since

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{\infty},$$

it follows that T maps the space

$$L^q(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta}) = \left[L^p(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta}), L^\infty(\mathbb{B}_n)\right]_\theta$$

boundedly into $[A_\alpha^p, \Lambda_\beta]_\theta$. But we have $TL^q(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta}) = A_\gamma^q$ according again to Theorem 30. So

$$A_\gamma^q \subset \left[A_\alpha^p, \Lambda_\beta\right]_\theta,$$

and the inclusion is continuous.

Next we consider the linear operator

$$Lf(z) = \left(1 - |z|^2\right)^{k-\beta} R^k f(z), \quad f \in H(\mathbb{B}_n),$$

where k is a fixed and sufficiently large positive integer. The operator L maps A_α^p boundedly into $L^p(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta})$; and it maps Λ_β boundedly into $L^\infty(\mathbb{B}_n)$. Therefore, L also maps $[A_\alpha^p, \Lambda_\beta]_\theta$ boundedly into $L^q(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta})$, that is, $f \in [A_\alpha^p, \Lambda_\beta]_\theta$ implies that the function $(1 - |z|^2)^{k-\beta} R^k f(z)$ is in $L^q(\mathbb{B}_n, \mathrm{d}v_{\alpha+p\beta})$, which is the same as the function $(1 - |z|^2)^k R^k f(z)$ being in $L^q(\mathbb{B}_n, \mathrm{d}v_\gamma)$, or $f \in A_\gamma^q$. We conclude that

$$\left[A_\alpha^p, \Lambda_\beta\right]_\theta \subset A_\gamma^q,$$

and the inclusion is continuous. This completes the proof of the theorem. □

THEOREM 39. – *Suppose α is real, β is real, and $0 < \theta < 1$. Then*

$$\left[\Lambda_\alpha, \Lambda_\beta\right]_\theta = \Lambda_\gamma$$

with equivalent norms, where $\gamma = \alpha(1 - \theta) + \beta\,\theta$.

Proof. – Fix a sufficiently large positive number s. If $f \in \Lambda_\gamma$, there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_s(w)}{(1 - \langle z, w\rangle)^{n+1+s-\gamma}};$$

see Theorem 17. For any $\zeta \in \overline{S}$ we define

$$f_\zeta(z) = \int_{\mathbb{B}_n} \frac{g(w)(1 - |w|^2)^{\alpha(1-\zeta)+\beta\zeta-\gamma}\,\mathrm{d}v_s(w)}{(1 - \langle z, w\rangle)^{n+1+s-\gamma}}.$$

Let k be a sufficiently large positive integer. Then it follows easily from Proposition 7 that the map $\zeta \mapsto f_\zeta$ is a bounded continuous function from $\bar{S}$ into

$$\Lambda_\alpha + \Lambda_\beta = \Lambda_{\min(\alpha,\beta)},$$

and its restriction to S is analytic. Also, the map $\zeta \mapsto f_\zeta$ is a bounded continuous function from $L(S)$ into Λ_α, and from $R(S)$ into Λ_β. Since $f_\theta = f$, we conclude from the definition of complex interpolation that $f \in [\Lambda_\alpha, \Lambda_\beta]_\theta$. This shows that

$$\Lambda_\gamma \subset [\Lambda_\alpha, \Lambda_\beta]_\theta$$

and the inclusion is continuous.

On the other hand, if $f \in [\Lambda_\alpha, \Lambda_\beta]_\theta$, then there exists a family of functions f_ζ, where $\zeta \in \bar{S}$, such that

(a) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $\bar{S}$ into $\Lambda_{\min(\alpha,\beta)}$ whose restriction to S is analytic;

(b) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $L(S)$ into Λ_α;

(c) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $R(S)$ into Λ_β;

(d) $f_\theta = f$.

Let k be a positive integer with $k > \max(\alpha, \beta)$ and consider the functions

$$g_\zeta(z) = \left(1 - |z|^2\right)^{k-\alpha(1-\zeta)-\beta\zeta} R^k f_\zeta(z), \quad z \in \mathbb{B}_n, \ \zeta \in \bar{S}.$$

By conditions (b) and (c) of the previous paragraph, there exist finite positive constants M_0 and M_1 such that

$$(17) \qquad \sup_{\substack{z\in\mathbb{B}_n \\ \zeta\in L(S)}} \big|g_\zeta(z)\big| = M_0, \qquad \sup_{\substack{z\in\mathbb{B}_n \\ \zeta\in R(S)}} \big|g_\zeta(z)\big| = M_1.$$

For any fixed point $z \in \mathbb{B}_n$, it follows from condition (a) of the previous paragraph that the function $F(\zeta) = g_\zeta(z)$ is a bounded continuous function on $\bar{S}$ whose restriction to S is analytic. Moreover, it follows from (17) that $|F(\zeta)| \leq M_0$ for $\zeta \in L(S)$ and $|F(\zeta)| \leq M_1$ for $\zeta \in R(S)$. By Hadamard's three lines theorem, we must have

$$\big|F(\theta)\big| \leq M_0^{1-\theta}M_1^\theta \quad \text{or} \quad \left(1-|z|^2\right)^{k-\gamma}\big|R^k f(z)\big| \leq M_0^{1-\theta}M_1^\theta.$$

Since the constant on the right-hand side is independent of z, we have shown that $f \in \Lambda_\gamma$. Therefore,

$$[\Lambda_\alpha, \Lambda_\beta]_\theta \subset \Lambda_\gamma,$$

and the inclusion is continuous. This completes the proof of the theorem. $\qquad \square$

CHAPTER 12

REPRODUCING KERNELS

In this chapter we focus on the Hilbert space case $p = 2$. We are going to obtain a characterization of A_α^2 in terms of Taylor coefficients, and we are going to define a canonical inner product on A_α^2 so that the associated reproducing kernel can be calculated in closed form.

Reproducing kernels for A_α^2 are also calculated in Beatrous-Buebea [11] in terms of a certain family of hypergeometric functions. Our approach here is different. We wish to write the reproducing kernel of A_α^2 as something that is as close to $(1 - \langle z, w \rangle)^{-(n+1+\alpha)}$ as possible.

THEOREM 40. – *Suppose α is real and*

$$f(z) = \sum_m a_m z^m.$$

Then $f \in A_\alpha^2$ if and only if its Taylor coefficients satisfy the condition

$$(18) \qquad \sum_{|m|>0} \frac{m! \, \mathrm{e}^{|m|}}{|m|^{n+|m|+\alpha+\frac{1}{2}}} |a_m|^2 < \infty.$$

Proof. – Fix a positive integer k such that $2k + \alpha > -1$. If $f(z) = \sum_m a_m z^m$ is the Taylor series of f in $\mathbb{B}_n$, then

$$R^k f(z) = \sum_{|m|>0} a_m |m|^k z^m.$$

It follows that the integral

$$I_{k,\alpha}(f) = \int_{\mathbb{B}_n} \left| (1 - |z|^2)^k R^k f(z) \right|^2 \mathrm{d}v_\alpha(z)$$

is equal to

$$\sum_{|m|>0} |a_m|^2 \cdot |m|^{2k} \int_{\mathbb{B}_n} |z^m|^2 \left(1 - |z|^2\right)^{2k+\alpha} \mathrm{d}v(z).$$

By integration in polar coordinates (see 1.4.3 of Rudin [**47**] or Lemma 1.11 of Zhu [**71**]), there exists a constant $C > 0$ (independent of f) such that

$$I_{k,\alpha}(f) = C \sum_{|m|>0} \frac{|m|^{2k}\, m!}{\Gamma(n + |m| + 2k + \alpha + 1)}\, |a_m|^2.$$

Since n, k, and α are all constants, an application of Stirling's formula shows that

$$\Gamma(n + |m| + 2k + \alpha + 1) \sim |m|^{n+|m|+2k+\alpha+\frac{1}{2}}\, e^{-|m|}$$

as $|m| \to \infty$. We conclude that the integral $I_{k,\alpha}(f)$ is finite if and only if the condition in (18) holds, and the proof of the theorem is complete. $\qquad\square$

An immediate consequence of the condition in (18) is that the space A_α^2 is independent of the integer k used in the definition of A_α^2. Of course, we already knew this from Section 4.

THEOREM 41. – *Suppose $\alpha > -(n + 1)$. Then A_α^2 can be equipped with an inner product such that the associated reproducing kernel is given by*

(19) $$K_\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

Proof. – It follows from Stirling's formula again that the condition in (18) is equivalent to

(20) $$\sum_m \frac{m!\,\Gamma(n + 1 + \alpha)}{\Gamma(n + |m| + \alpha + 1)}\, |a_m|^2 < \infty.$$

Now define an inner product on A_α^2 as follows:

$$\langle f, g \rangle_\alpha = \sum_m \frac{m!\,\Gamma(n + 1 + \alpha)}{\Gamma(n + |m| + \alpha + 1)}\, a_m \bar{b}_m,$$

where

$$f(z) = \sum_m a_m z^m, \quad g(z) = \sum_m b_m z^m.$$

Then A_α^2 becomes a separable Hilbert space with the following functions forming an orthonormal basis:

$$e_m(z) = \sqrt{\frac{\Gamma(n + |m| + \alpha + 1)}{m!\,\Gamma(n + \alpha + 1)}}\, z^m,$$

where m runs over all n-tuples of nonnegative integers. It follows from the multinomial formula (1) that the reproducing kernel of A_α^2 is given by

$$K_\alpha(z, w) = \sum_m e_m(z)\overline{e_m(w)} = \sum_m \frac{\Gamma(n + |m| + \alpha + 1)}{m!\,\Gamma(n + \alpha + 1)} z^m \overline{w}^m$$

$$= \sum_{k=0}^\infty \frac{\Gamma(n + k + \alpha + 1)}{k!\,\Gamma(n + 1 + \alpha)} \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w}^m$$

$$= \sum_{k=0}^\infty \frac{\Gamma(n + k + 1 + \alpha)}{k!\,\Gamma(n + \alpha + 1)} \langle z, w \rangle^k = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

This proves the desired result. $\qquad\square$

When $\alpha > -1$, the reproducing kernel for A_α^2 is of course well known. See Rudin [47] or Zhu [71]. When $\alpha \leq -1$, the point here is that you need to use an appropriate inner product on A_α^2 so that its reproducing kernel is computable in closed form.

THEOREM 42. – *Suppose $\alpha = -(n + 1)$. Then A_α^2 can be equipped with an inner product such that the associated reproducing kernel is*

(21) $$K_{-(n+1)}(z, w) = 1 + \log \frac{1}{1 - \langle z, w \rangle}.$$

Proof. – If $\alpha = -(n + 1)$, then Theorem 40 tells us that a holomorphic function $f(z) = \sum_m a_m z^m$ in $\mathbb{B}_n$ belongs to $A_{-(n+1)}^2$ if and only if

$$\sum_{|m|>0} \frac{m!\,\mathrm{e}^{|m|}}{|m|^{|m|-\frac{1}{2}}} |a_m|^2 < \infty,$$

which, according to Stirling's formula, is equivalent to

$$\sum_{|m|>0} |m| \frac{m!}{|m|!} |a_m|^2 < \infty.$$

If we define an inner product on $A_{-(n+1)}^2$ by

(22) $$\langle f, g \rangle_{-(n+1)} = f(0)\overline{g(0)} + \sum_{|m|>0} |m| \frac{m!}{|m|!} a_m \overline{b}_m,$$

where

$$f(z) = \sum_m a_m z^m, \quad g(z) = \sum_m b_m z^m,$$

then $A_{-(n+1)}^2$ becomes a separable Hilbert space with the following functions forming an orthonormal basis:

$$1, \quad e_m(z) = \sqrt{\frac{|m|!}{m!\,|m|}}\, z^m,$$

where m runs over all n-tuples of nonnegative integers with $|m| > 0$. It follows from (1) that the reproducing kernel of $A^2_{-(n+1)}$ is given by

$$K_{-(n+1)}(z, w) = 1 + \sum_{|m|>0} \frac{|m|!}{m!|m|} z^m \overline{w}^m = 1 + \sum_{k=1}^{\infty} \frac{1}{k} \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w}^m$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\langle z, w \rangle^k}{k} = 1 + \log \frac{1}{1 - \langle z, w \rangle},$$

completing the proof of the theorem. $\square$

The space $A^2_{-(n+1)}$ can be thought of as the high dimensional analog of the classical Dirichlet space in the unit disk. It is the unique space of holomorphic functions in the unit ball that can be equipped with a semi-inner product that is invariant under the action of the automorphism group. See Zhu [67]. The formula in Theorem 42 above also appeared in Peloso [44] and Zhu [67].

THEOREM 43. – *Suppose $-N < n + 1 + \alpha < -N + 1$ for some positive integer N. Then for any polynomial*

$$Q(z, w) = \sum_{|m| \leq N} \omega_m z^m \overline{w}^m$$

with the property that

$$\omega_m > (-1)^{N+1} \frac{\Gamma(n + |m| + \alpha + 1)}{m! \, \Gamma(n + \alpha + 1)},$$

we can equip A^2_α with an inner product such that the associated reproducing kernel is given by

(23) $$K_\alpha(z, w) = Q(z, w) + \frac{(-1)^N}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

Proof. – By Theorem 40 and Stirling's formula again, a function $f(z) = \sum_m a_m z^m$ is in A^2_α if and only if

$$\sum_{|m|>0} \left| \frac{m! \, \Gamma(n + 1 + \alpha)}{\Gamma(n + |m| + \alpha + 1)} \right| |a_m|^2 < \infty.$$

If $-N < n + 1 + \alpha < -N + 1$, it follows from the identity

$$\frac{\Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)} = \frac{1}{(n + 1 + \alpha)(n + 2 + \alpha) \cdots (n + |m| + \alpha)}$$

that for any $|m| > N$ we have

$$\left| \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)} \right| = (-1)^N \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)}.$$

Therefore, for any positive coefficients c_m, where $|m| \leq N$, we can define an inner product on A_α^2 as follows:

$$(24) \qquad \langle f, g \rangle_\alpha = \sum_{|m| \leq N} c_m a_m \bar{b}_m + (-1)^N \sum_{|m| > N} \frac{m! \, \Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)} \, a_m \bar{b}_m,$$

where

$$f(z) = \sum_m a_m z^m, \quad g(z) = \sum_m b_m z^m.$$

Then A_α^2 becomes a separable Hilbert space with the following functions forming an orthonormal basis:

$$e_m(z) = \frac{1}{\sqrt{c_m}} \, z^m, \quad |m| \leq N,$$

and

$$e_m(z) = \sqrt{(-1)^N \frac{\Gamma(n + |m| + \alpha + 1)}{m! \, \Gamma(n + \alpha + 1)}} \, z^m, \quad |m| > N.$$

Using the mutinomial formula (1), we find that the corresponding reproducing kernel for A_α^2 is given by

$$
\begin{aligned}
K_\alpha(z, w) &= \sum_{|m| \leq N} e_m(z)\overline{e_m(w)} + \sum_{|m| > N} e_m(z)\overline{e_m(w)} \\
&= \sum_{|m| \leq N} \frac{1}{c_m} z^m \overline{w}^m + (-1)^N \sum_{|m| > N} \frac{\Gamma(n + |m| + \alpha + 1)}{m! \, \Gamma(n + \alpha + 1)} z^m \overline{w}^m \\
&= \sum_{|m| \leq N} \omega_m z^m \overline{w}^m + \frac{(-1)^N}{(1 - \langle z, w \rangle)^{n+1+\alpha}},
\end{aligned}
$$

where

$$\omega_m = \frac{1}{c_m} - (-1)^N \frac{\Gamma(n + |m| + \alpha + 1)}{m! \, \Gamma(n + \alpha + 1)} > (-1)^{N+1} \frac{\Gamma(n + |m| + \alpha + 1)}{m! \, \Gamma(n + \alpha + 1)}.$$

This completes the proof of the theorem. $\qquad \square$

The appearance of the sign $(-1)^N$ in (23) is a little peculiar; we do not know if there is any simple explanation for it. We also note in passing that the reproducing kernel given by (23) is bounded.

It remains for us to consider the case in which $n + 1 + \alpha = -N$ is a negative integer. The principal part of the reproducing kernel in this case will be shown to be the function

$$(\langle z, w \rangle - 1)^N \log \frac{1}{1 - \langle z, w \rangle}.$$

Thus for every positive integer N we consider the function

$$f_N(z) = (z - 1)^N \log \frac{1}{1 - z}, \quad z \in \mathbb{D}.$$

It is clear that each f_N is analytic in the unit disk $\mathbb{D}$ and

$$f'_{N+1}(z) = (N+1)f_N(z) - (z-1)^N.$$

In particular,

$$f_{N+1}^{(k+1)}(z) = (N+1)f_N^{(k)}(z), \quad k > N.$$

It follows from this and induction that $f_N^{(k)}(0) > 0$ for all $k > N$. Also observe that the Nth derivative of f_N is $-\log(1-z)$ plus a polynomial, so the Taylor coefficients of f_N has the property that, as $k \to \infty$,

$$\frac{f_N^{(k)}(0)}{k!} \sim \frac{1}{k^{N+1}}.$$

THEOREM 44. – *Suppose $n + 1 + \alpha = -N$ for some positive integer N and*

$$(z-1)^N \log \frac{1}{1-z} = \sum_{k=0}^{\infty} A_k z^k.$$

Then for any polynomial

$$Q(z,w) = \sum_{|m| \leq N} \omega_m z^m \overline{w}^m$$

with the property that

$$\omega_m > -\frac{|m|!\, A_{|m|}}{m!},$$

we can equip A_α^2 with an inner product such that the associated reproducing kernel is

$$(25) \qquad K_\alpha(z,w) = Q(z,w) + (\langle z,w \rangle - 1)^N \log \frac{1}{1 - \langle z,w \rangle}.$$

Proof. – It follows from (18) and Stirling's formula that a function $f(z) = \sum_m a_m z^m$ belongs to A_α^2 if and only if

$$\sum_m |m|^{N+1} \frac{m!}{|m|!} |a_m|^2 < \infty,$$

which is equivalent to

$$\sum_{|m|>N} \frac{m!}{|m|!\, A_{|m|}} |a_m|^2 < \infty.$$

If $c_m > 0$ for $|m| \leq N$, we can define an inner product on A_α^2 as follows:

$$(26) \qquad \langle f, g \rangle_\alpha = \sum_{|m| \leq N} c_m a_m \overline{b}_m + \sum_{|m|>N} \frac{m!}{|m|!\, A_{|m|}} a_m \overline{b}_m.$$

where

$$f(z) = \sum_m a_m z^m, \quad g(z) = \sum_m b_m z^m.$$

Then A_α^2 becomes a separable Hilbert space and the following functions form an orthonormal basis:

$$e_m(z) = \frac{1}{\sqrt{c_m}}\, z^m, \quad |m| \le N,$$

and

$$e_m(z) = \sqrt{\frac{|m|!\, A_{|m|}}{m!}}\, z^m, \quad |m| > N.$$

The associated reproducing kernel for A_α^2 is given by

$$\begin{aligned}
K_\alpha(z,w) &= \sum_{|m|\le N} e_m(z)\overline{e_m(w)} + \sum_{|m|>N} e_m(z)\overline{e_m(w)} \\
&= \sum_{|m|\le N} \frac{z^m \overline{w}^m}{c_m} + \sum_{|m|>N} \frac{|m|!\, A_{|m|}}{m!} z^m \overline{w}^m \\
&= \sum_{|m|\le N} \left(\frac{1}{c_m} - \frac{|m|!\, A_{|m|}}{m!} \right) z^m \overline{w}^m + \sum_{k=0}^{\infty} A_k \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w}^m \\
&= \sum_{|m|\le N} \omega_m z^m \overline{w}^m + \sum_{k=0}^{\infty} A_k \langle z, w\rangle^k \\
&= Q(z,w) + (\langle z,w\rangle - 1)^N \log \frac{1}{1 - \langle z,w\rangle},
\end{aligned}$$

where the coefficients of

$$Q(z,w) = \sum_{|m|\le N} \omega_m z^m \overline{w}^m$$

satisfy

$$\omega_m = \frac{1}{c_m} - \frac{|m|!\, A_{|m|}}{m!} > -\frac{|m|!\, A_{|m|}}{m!}.$$

This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Once again, the reproducing kernel in (25) is bounded on $\mathbb{B}_n \times \mathbb{B}_n$. Also notice that we can rewrite the kernel in (25) as

$$K_\alpha(z,w) = Q(z,w) + (-1)^N \big(1 - \langle z,w\rangle\big)^N \log \frac{1}{1 - \langle z,w\rangle},$$

which is probably a partial explanation for the sign $(-1)^N$ in (23).

It is clear that the reproducing kernel of a Hilbert space of holomorphic functions depends on the inner product used for the space. We close this chapter by examining the reproducing kernel of A_α^2 that corresponds to the following natural inner product which we have used in Chapter 7:

$$\langle f, g\rangle_\alpha = f(0)\overline{g(0)} + \int_{\mathbb{B}_n} R^k f(z)\overline{R^k g(z)}\, \mathrm{d}v_{2k+\alpha}(z),$$

where k is any nonnegative integer with $2k + \alpha > -1$. This inner product gives rise to the norm

$$\|f\|_{2,\alpha} = \left(|f(0)|^2 + \int_{\mathbb{B}_n} \left|R^k f(z)\right|^2 dv_{2k+\alpha}(z)\right)^{1/2}$$

for $f \in A_\alpha^2$. For this inner product we can show that the corresponding reproducing kernel for A_α^2 is

$$K_w^\alpha(z) = K_\alpha(z, w) = 1 + R^{-2k}\left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha+2k}}\right).$$

The result is a simple consequence of the identity,

$$f(0)\overline{g(0)} + \int_{\mathbb{B}_n} R^k f(z)\overline{R^{-k}g(z)} dv_{2k+\alpha}(z) = \int_{\mathbb{B}_n} f(z)\overline{g(z)} dv_{2k+\alpha}(z),$$

which can easily be proved by the use of Taylor expansions. We leave the details to the interested reader.

CHAPTER 13

CARLESON TYPE MEASURES

The purpose of this chapter is to study Carleson type measures for the Bergman spaces A_α^p. Unlike most other chapters of the paper, the results here depend very much on the various parameters.

The notion of Carleson measures was of course introduced by Carleson [15], [16] for the unit disk. Carleson's original definition works well in the theory of Hardy spaces, and this can easily be seen in such classics as Duren [23] and Garnett [27]. The characterization of Carleson measures for the Hardy spaces of the unit ball can be found in Hörmander [32] and Power [45].

Later, the notion of Carleson measures was extended to the context of Bergman spaces. Earlier papers in this direction include Cima-Wogen [20], Hastings [31], Luecking [38], Zhu [66]. Also, Carleson type measures have been studied for holomorphic Besov spaces (of which the Dirichlet space is a special case); see Arcozzi-Rochberg-Sawyer [7], Kaptanoglu [35], Stegenga [54], and Wu [61]. In particular, our results of this chapter contain several special cases that have been known before.

For any $\zeta \in \mathbb{S}_n$ and $r > 0$ let

$$Q_r(\zeta) = \big\{ z \in \mathbb{B}_n : |1 - \langle z, \zeta \rangle| < r \big\}.$$

These are the high dimensional analogues of Carleson squares in the unit disk. They are also called nonisotropic metric balls. See Rudin [47] or Zhu [71] for more information about the geometry of these nonisotropic balls.

THEOREM 45. – *Suppose $n + 1 + \alpha > 0$ and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following conditions are equivalent:*

(a) *There exists a constant $C > 0$ such that, for all $\zeta \in \mathbb{S}_n$ and all $r > 0$,*

$$(27) \qquad\qquad \mu\big(Q_r(\zeta)\big) \leq Cr^{n+1+\alpha}.$$

(b) *For each $s > 0$ there exists a constant $C > 0$ such that, for all $z \in \mathbb{B}_n$,*

$$(28) \qquad \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \, \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \leq C.$$

(c) *For some $s > 0$ there exists a constant $C > 0$ such that the inequality in (28) holds for all $z \in \mathbb{B}_n$.*

Proof. – It is obvious that condition (b) implies (c). Now assume that condition (c) holds, that is, there exist positive constants s and C such that the inequality in (28) holds. If $\zeta \in \mathbb{S}_n$ and $r \in (0, 1)$, then

$$(29) \qquad \int_{Q_r(\zeta)} \frac{(1 - |z|^2)^s \, \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \leq C$$

for all $z \in \mathbb{B}_n$. If we choose $z = (1 - r)\zeta$, then

$$1 - \langle z, w \rangle = (1 - r)\big(1 - \langle \zeta, w \rangle\big) + r$$

for all $w \in \mathbb{B}_n$, so

$$\big|1 - \langle z, w \rangle\big| \leq (1 - r)r + r < 2r$$

for all $w \in Q_r(\zeta)$, which gives

$$\frac{(1 - |z|^2)^s}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \geq \frac{r^s}{(2r)^{n+1+\alpha+s}} = \frac{2^{-(n+1+\alpha+s)}}{r^{n+1+\alpha}}$$

for all $w \in Q_r(\zeta)$. Combining this with (29), we conclude that

$$\mu\big(Q_r(\zeta)\big) \leq 2^{n+1+\alpha+s} C r^{n+1+\alpha}$$

for all $\zeta \in \mathbb{S}_n$ and all $r \in (0, 1)$. The case $r \geq 1$ can be disposed of very easily. This proves that condition (c) implies (a).

Next assume that condition (a) holds. In particular, μ is a finite measure, so

$$\sup_{|z| \leq \frac{3}{4}} \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \, \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} < \infty$$

for each $s > 0$. We fix an arbitrary positive number s and proceed to show that the inequality in (28) must hold for $\frac{3}{4} < |z| < 1$.

Fix some point $z \in \mathbb{B}_n$ with $\frac{3}{4} < |z| < 1$ and choose $\zeta = z/|z|$. For any nonnegative integer k let $r_k = 2^{k+1}(1 - |z|)$. We decompose the unit ball $\mathbb{B}_n$ into the disjoint union of the sets

$$E_0 = Q_{r_0}(\zeta), \quad E_k = Q_{r_k}(\zeta) - Q_{r_{k-1}}(\zeta), \quad 1 \leq k < \infty.$$

By condition (a), we have

$$\mu(E_k) \leq \mu(Q_{r_k}(\zeta)) \leq 2^{(k+1)(n+1+\alpha)}\big(1 - |z|\big)^{n+1+\alpha} C$$

for all $k \geq 0$. On the other hand, if $k \geq 1$ and $w \in E_k$, then

$$
\begin{aligned}
\big|1 - \langle z, w \rangle\big| &= \big|(1 - |z|) + |z|(1 - \langle \zeta, w \rangle)\big| \\
&\geq |z| \cdot \big|1 - \langle \zeta, w \rangle\big| - \big(1 - |z|\big) \\
&\geq \tfrac{3}{4} \times 2^k \big(1 - |z|\big) - \big(1 - |z|\big) \geq 2^{k-1}\big(1 - |z|\big).
\end{aligned}
$$

This holds for $k = 0$ as well, because

$$
\big|1 - \langle z, w \rangle\big| \geq 1 - |z| \geq \tfrac{1}{2}\big(1 - |z|\big).
$$

It follows that

$$
\begin{aligned}
\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} &= \sum_{k=0}^{\infty} \int_{E_k} \frac{(1 - |z|^2)^s \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \\
&\leq \sum_{k=0}^{\infty} \frac{(1 - |z|^2)^s \mu(E_k)}{(2^{k-1}(1 - |z|))^{n+1+\alpha+s}} \\
&\leq \sum_{k=0}^{\infty} \frac{2^{s+(k+1)(n+1+\alpha)}(1 - |z|)^{n+1+\alpha+s}C}{2^{(k-1)(n+1+\alpha+s)}(1 - |z|)^{n+1+\alpha+s}} = C' \sum_{k=0}^{\infty} \frac{1}{(2^s)^k} < \infty,
\end{aligned}
$$

where C' is a positive constant independent of z. This completes the proof of the theorem. $\qquad\square$

Our results are most complete when $0 < p \leq 1$. The following result settles the case $n + 1 + \alpha > 0$, and Proposition 49 deals with the cases $n + 1 + \alpha \leq 0$.

THEOREM 46. – *Suppose $\alpha > -(n+1)$, $0 < p \leq 1$, and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following two conditions are equivalent:*

(a) *There exists a constant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$
\text{(30)} \qquad \int_{\mathbb{B}_n} \big|f(w)\big|^p \mathrm{d}\mu(w) \leq C\|f\|_{p,\alpha}^p.
$$

(b) *There exists a constant $C > 0$ such that, for all $\zeta \in \mathbb{S}_n$ and all $r \in (0,1)$,*

$$
\text{(31)} \qquad \mu\big(Q_r(\zeta)\big) \leq Cr^{n+1+\alpha}.
$$

Proof. – First assume that condition (a) holds. We consider the function

$$
f(w) = \frac{1}{(1 - \langle w, z \rangle)^{(n+1+\alpha+s)/p}}, \quad w \in \mathbb{B}_n,
$$

where s is positive and $z \in \mathbb{B}_n$. If k is the smallest nonnegative integer such that $kp + \alpha > -1$, then an elementary calculation shows that

$$
R^k f(w) = \frac{Q(\langle w, z \rangle)}{(1 - \langle w, z \rangle)^{k+(n+1+\alpha+s)/p}},
$$

where Q is a polynomial of degree k. It follows from Proposition 7 that there exists a constant $C > 0$ (independent of z) such that

$$\int_{\mathbb{B}_n} \left|(1 - |w|^2)^k R^k f(w)\right|^p \mathrm{d}v_\alpha(w) \leq \frac{C}{(1 - |z|^2)^s}$$

for all $z \in \mathbb{B}_n$. Combining this with condition (a), we conclude that

$$\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \, \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \leq C$$

for all $z \in \mathbb{B}_n$, which, according to Theorem 45, is equivalent to condition (b).

Next assume that condition (b) holds. Then condition (b) of Theorem 45 holds. We proceed to prove the inequality in (30).

Given $f \in A_\alpha^p$, we use the atomic decomposition for A_α^p (see Theorem 32) to write

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b},$$

where b is a sufficiently large positive number and

$$\sum_{k=1}^{\infty} |c_k|^p \leq C \|f\|_{p,\alpha}^p$$

for some positive constant C independent of f. Since $0 < p \leq 1$, we have

$$\left|f(z)\right|^p \leq \sum_{k=1}^{\infty} |c_k|^p \frac{(1 - |a_k|^2)^{pb-(n+1+\alpha)}}{|1 - \langle z, a_k \rangle|^{pb}},$$

and so

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \leq \sum_{k=1}^{\infty} |c_k|^p (1 - |a_k|^2)^{pb-(n+1+\alpha)} \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu(z)}{|1 - \langle z, a_k \rangle|^{pb}}.$$

Apply condition (b) of Theorem 45 to the last integral above. We obtain $C' > 0$ (a constant independent of f) such that

$$\int_{\mathbb{B}_n} \left|f(z)\right|^p \mathrm{d}\mu(z) \leq C' \sum_{k=1}^{\infty} |c_k|^p \leq CC' \|f\|_{p,\alpha}^p.$$

This completes the proof of the theorem. $\qquad\square$

COROLLARY 47. – *If $\alpha > -1$ and $p > 0$, then the following two conditions are equivalent for a positive Borel measure μ on $\mathbb{B}_n$.*

(a) *There exists a constant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$\int_{\mathbb{B}_n} \left|f(z)\right|^p \mathrm{d}\mu(z) \leq C \int_{\mathbb{B}_n} \left|f(z)\right|^p \mathrm{d}v_\alpha(z).$$

(b) *There exists a constant $C > 0$ such that, for all $r > 0$ and $\zeta \in \mathbb{S}_n$,*

$$\mu\big(Q_r(\zeta)\big) \leq Cr^{n+1+\alpha}.$$

Proof. – That (a) implies (b) follows from the first part of the proof of Theorem 46. Theorem 46 also tells us that (b) implies (a) when $0 < p \leq 1$.

Now assume that condition (b) holds and $f \in A_\alpha^p$ for some $p > 1$. Then the function $g = f^N$ belongs to A_α^q, where N is any positive integer and $q = p/N$. We choose N large enough so that $0 < q < 1$. Then

$$\int_{\mathbb{B}_n} |g(z)|^q \mathrm{d}\mu(z) \leq C \int_{\mathbb{B}_n} |g(z)|^q \mathrm{d}v_\alpha(z),$$

where C is a positive constant independent of g. This is the same as

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \leq C \int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}v_\alpha(z),$$

and the corollary is proved. $\square$

Let $\beta(z, w)$ be the distance between z and w in the Bergman metric of $\mathbb{B}_n$. For any $R > 0$ and $a \in \mathbb{B}_n$ we write

$$D(a, R) = \{z \in \mathbb{B}_n : \beta(z, a) < R\}.$$

When $\alpha > -1$, the condition

$$\mu(Q_r(\zeta)) \leq C r^{n+1+\alpha}, \quad r > 0, \ \zeta \in \mathbb{S}_n,$$

is equivalent to the condition

$$\mu(D(a, R)) \leq C_R (1 - |a|^2)^{n+1+\alpha}, \quad a \in \mathbb{B}_n.$$

See Lemma 5.23 and Corollary 5.24 of [**71**] (note that the definition of $Q_r(\zeta)$ in [**71**] is different from its definition in this paper). It can be shown that these two conditions are no longer equivalent when $\alpha \leq -1$. In fact, if f is a function in the Bloch space that is not in BMOA, then the measure

$$\mathrm{d}\mu(z) = |Rf(z)|^2 (1 - |z|^2) \mathrm{d}v(z)$$

satisfies

$$\sup_{r, \zeta} \frac{\mu(Q_r(\zeta))}{r^n} = \infty \quad \text{and} \quad \sup_{a \in \mathbb{B}_n} \frac{\mu(D(a, R))}{(1 - |a|^2)^n} < \infty.$$

Recall that the Hardy space H^p, where $0 < p < \infty$, consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\|f\|_p^p = \sup_{0 < r < 1} \int_{\mathbb{S}_n} |f(r\zeta)|^p \mathrm{d}\sigma(\zeta) < \infty,$$

where $\mathrm{d}\sigma$ is the normalized surface area measure on $\mathbb{S}_n$. It is well known that every function $f \in H^p$ has a finite radial limit at almost every point on $\mathbb{S}_n$. If we write

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta), \quad \zeta \in \mathbb{S}_n,$$

then we actually have

$$\|f\|_p^p = \int_{\mathbb{S}_n} |f(\zeta)|^p \mathrm{d}\sigma(\zeta).$$

It is known that the following two conditions are equivalent for a positive Borel measure μ on $\mathbb{B}_n$; see Hörmander [32], Power [45], or Zhu [71].

(a) There exists a constant $C > 0$ such that, for all $f \in H^p$,

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \leq C \int_{\mathbb{S}_n} |f(\zeta)|^p \mathrm{d}\sigma(\zeta).$$

(b) There exists a constant $C > 0$ such that, for all $r > 0$ and $\zeta \in \mathbb{S}_n$,

$$\mu\big(Q_r(\zeta)\big) \leq Cr^n.$$

COROLLARY 48. – *Suppose $\alpha = -1$, $0 < p \leq 2$, and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following two conditions are equivalent.*

(a) *There exists a constant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \leq C\|f\|_{p,\alpha}^p.$$

(b) *There exists a constant $C > 0$ such that, for all $r > 0$ and $\zeta \in \mathbb{S}_n$,*

$$\mu\big(Q_r(\zeta)\big) \leq Cr^n.$$

Proof. – That (a) implies (b) follows from the first part of the proof of Theorem 46. To show that condition (b) implies (a), we notice that $A_{-1}^2 = H^2$, so the case $p = 2$ follows from the characterization of Carleson measures for Hardy spaces. The case $0 < p \leq 1$ follows from Theorem 46. The case of $1 \leq p \leq 2$ then follows from complex interpolation. $\square$

PROPOSITION 49. – *Let μ be a positive Borel measure on $\mathbb{B}_n$. If $n + 1 + \alpha < 0$ and $0 < p < \infty$, or if $n + 1 + \alpha = 0$ and $0 < p \leq 1$, then the following two conditions are equivalent.*

(a) *There exists a constant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z) \leq C\|f\|_{p,\alpha}^p.$$

(b) *The measure μ is finite.*

Proof. – Since A_α^p contains all constant functions, it is clear that condition (a) implies (b). On the other hand, if μ is a finite positive Borel measure, it follows from Theorems 21 and 22 that A_α^p is contained in $L^p(\mathbb{B}_n, \mathrm{d}\mu)$. By the closed graph theorem,

A_α^p is continuously contained in $L^p(\mathbb{B}_n, d\mu)$, so there exists a positive constant $C > 0$ such that, for all $f \in A_\alpha^p$,

$$\int_{\mathbb{B}_n} |f|^p d\mu \leq C\|f\|_{p,\alpha}^p. \qquad \square$$

As far as the condition

$$\int_{\mathbb{B}_n} |f(z)|^p d\mu(z) \leq C\|f\|_{p,\alpha}^p, \quad f \in A_\alpha^p,$$

is concerned, the most difficult case is probably when $\alpha = -(n+1)$ and $1 < p < \infty$. This case is considered in Arcozzi-Rochberg-Sawyer [7] and complete results are obtained in the range $1 < p < 2 + 1/(n-1)$. Earlier results dealing with the Besov spaces include Arcozzi [5], Arcozzi-Rochberg-Sawyer [6], Stegenga [54], and Wu [61].

THEOREM 50. – *Suppose $0 < p \leq q < \infty$, α is real, and μ is a positive Borel measure on $\mathbb{B}_n$. Then for any nonnegative integer k with $\alpha + kp > -1$ the following conditions are equivalent.*

(a) *There is a contant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$\int_{\mathbb{B}_n} |R^k f(w)|^q d\mu(w) \leq C\|f\|_{p,\alpha}^q.$$

(b) *For each (or some) $s > 0$ there is a constant $C > 0$ such that, for all $z \in \mathbb{B}_n$,*

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{s+(n+1+\alpha+kp)q/p}} d\mu(w) \leq C.$$

(c) *There is a constant $C > 0$ such that, for all $r > 0$ and $\zeta \in (0,1)$,*

$$\mu(Q_r(\zeta)) \leq Cr^{(n+1+\alpha+kp)q/p}.$$

(d) *For each (or some) $R > 0$ there exists a constant $C > 0$ such that, for all $a \in \mathbb{B}_n$,*

$$\mu(D(a,R)) \leq C(1-|a|^2)^{(n+1+\alpha+kp)q/p}.$$

Proof. – Suppose (a) holds. Applying (a) to the functions $f_k(z) = z_k$, $1 \leq k \leq n$, we see that μ is a finite measure. For a fixed $z \in \mathbb{B}_n$ let

$$h_z(w) = \frac{(1-|z|^2)^{s/q}}{(1-\langle w,z\rangle)^{s/q+(n+1+\alpha+k)/p}}$$

and let $f_z(w)$ be an analytic function on $\mathbb{B}_n$ such that

$$R^k f_z(w) = h_z(w) - h_z(0),$$

where $h_z(0) = (1-|z|^2)^{s/q} \leq 1$.

It follows from Proposition 7 that

$$\sup_{z\in\mathbb{B}_n} \|f_z\|_{p,\alpha} \leq C.$$

Applying (a) to f_z yields

$$\int_{\mathbb{B}_n} \left|h_z(w) - h_z(0)\right|^q \mathrm{d}\mu(w) \leq C.$$

It follows from the elementary inequality

$$\left|h_z(w)\right|^q \leq 2^q \left(|h_z(w) - h_z(0)|^q + |h_z(0)|^q\right)$$

that

$$\int_{\mathbb{B}_n} \left|h_z(w)\right|^q \mathrm{d}\mu(w) \leq 2^q \left(C + \mu(\mathbb{B}_n)\right),$$

which gives us (b).

Next assume that (b) holds. Recall that $D(z,r)$ is the Bergman metric ball at z with radius R. By Lemmas 2.24 and 2.20 of [**71**], we have

$$|R^k f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha+kp}} \int_{D(z,r)} \left|R^k f(w)\right|^p \mathrm{d}v_{\alpha+kp}(w)$$

$$\leq C \int_{D(z,r)} \frac{|R^k f(w)|^p (1-|w|^2)^{sp/q+kp}}{|1-\langle w,z\rangle|^{sp/q+n+1+\alpha+kp}} \mathrm{d}v_\alpha(w)$$

$$\leq C \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{sp/q} \mathrm{d}\lambda(w)}{|1-\langle z,w\rangle|^{sp/q+n+1+\alpha+kp}},$$

where

$$\mathrm{d}\lambda(w) = \left|R^k f(w)\right|^p \left(1-|w|^2\right)^{kp} \mathrm{d}v_\alpha(w)$$

is a finite measure on $\mathbb{B}_n$ whenever $f \in A_\alpha^p$. In fact, $\lambda(\mathbb{B}_n) \leq C\|f\|_{p,\alpha}^p$ for some constant independent of f.

If $p = q$, an application of Fubini's theorem to the estimate in the previous paragraph shows that (b) implies (a). If $p < q$, we write $p' = q/p$ and $1/p' + 1/q' = 1$, and apply Hölder's inequality to the estimate in the previous paragraph. The result is

$$\left|R^k f(z)\right|^p \leq C\Big[\int_{\mathbb{B}_n} \frac{(1-|w|^2)^s \mathrm{d}\lambda(w)}{|1-\langle z,w\rangle|^{s+(n+1+\alpha+kp)q/p}}\Big]^{1/p'} \left[\lambda(\mathbb{B}_n)\right]^{1/q'}.$$

It follows that

$$\left|R^k f(z)\right|^q \leq C\left(\lambda(\mathbb{B}_n)\right)^{p'/q'} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^s \mathrm{d}\lambda(w)}{|1-\langle z,w\rangle|^{s+(n+1+\alpha+kp)q/p}}.$$

We now integrate against the measure $d\mu$, apply Fubini's theorem, and use condition (b) to obtain

$$\int_{\mathbb{B}_n} \left|R^k f(z)\right|^q \mathrm{d}\mu(z) \leq C\left(\lambda(\mathbb{B}_n)\right)^{1+p'/q'}.$$

Since $\lambda(\mathbb{B}_n) \leq C\|f\|_{p,\alpha}^p$, we get

$$\int_{\mathbb{B}_n} \left|R^k f(z)\right|^q \mathrm{d}\mu(z) \leq C\|f\|_{p,\alpha}^q.$$

This shows that (b) implies (a).

The equivalence of (b) and (c) has already been proved in Theorem 45. Since

$$(n + 1 + \alpha + kp)q/p \ge n + 1 + \alpha + kp > n,$$

the equivalence of (c) and (d) follows from the remarks after Corollary 47. This completes the proof of the theorem. $\qquad\square$

A similar result can be obtained in terms of fractional radial differential operators $R^{s,t}$ instead of R^k above. We omit the details.

Once a certain result concerning Carleson measures is established, it is then relatively easy to formulate and prove its little oh version. For example, with the same assumptions in Theorem 50, we can show that the following four conditions are equivalent.

(a) If $\{f_j\}$ is a bounded sequence in A_α^p and $f_j(z) \to 0$ for every $z \in \mathbb{B}_n$, then

$$\lim_{j \to \infty} \int_{\mathbb{B}_n} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z) = 0.$$

(b) For every (or some) $s > 0$ we have

$$\lim_{|z| \to 1^-} \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{s + (n+1+\alpha+kp)q/p}} = 0.$$

(c) The following limit holds uniformly for $\zeta \in \mathbb{S}_n$:

$$\lim_{r \to 0^+} \frac{\mu(Q_r(\zeta))}{r^{(n+1+\alpha+kp)q/p}} = 0.$$

(d) For every (or some) $R > 0$ we have

$$\lim_{|a| \to 1^-} \frac{\mu(D(a, R))}{(1 - |a|^2)^{(n+1+\alpha+kp)q/p}} = 0.$$

The interested reader should have no trouble filling in the details.

As our next theorem shows, the assumption that $p \le q$ is essential for Theorem 50. To deal with the case $p > q$, we associate two functions to any positive Borel measure μ on $\mathbb{B}_n$. More specifically, for any real γ and s we define

$$B_{s,\gamma}(\mu)(z) = \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s \mathrm{d}\mu(w)}{|1 - \langle z, w \rangle|^{n+1+s+\gamma}}, \quad z \in \mathbb{B}_n,$$

and for any real γ and positive R we define

$$\widehat{\mu}_{R,\gamma}(z) = \frac{\mu(D(z, R))}{(1 - |z|^2)^{n+1+\gamma}}, \quad z \in \mathbb{B}_n.$$

If $\mathrm{d}\mu(z) = h(z)\mathrm{d}v_\gamma(z)$, we use the convention that

$$B_{s,\gamma}(h)(z) = B_{s,\gamma}(\mu), \quad \widehat{h}_{R,\gamma}(z) = \widehat{\mu}_{R,\gamma}(z).$$

It is clear that $B_{s,\gamma}(\mu)(z)$ and $\widehat{\mu}_{R,\gamma}(z)$ are certain averages of μ near the point z. The function $B_{s,\gamma}(\mu)$ is sometimes called a Berezin transform of μ.

LEMMA 51. – *Suppose μ is a positive Borel measure on $\mathbb{B}_n$. If γ is real, $p > 0$, and $R > 0$, then there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} |g(z)|^p \mathrm{d}\mu(z) \leq C \int_{\mathbb{B}_n} |g(z)|^p \widehat{\mu}_{R,\gamma}(z) \mathrm{d}v_\gamma(z)$$

for all holomorphic functions g in $\mathbb{B}_n$.

Proof. – It follows from Lemma 2.20 and Corollary 2.21 of [**71**] that

$$\left(1 - |z|^2\right)^{n+1} \sim \left(1 - |w|^2\right)^{n+1} \sim v\big(D(z,R)\big) \sim v\big(D(w,R)\big)$$

for $w \in D(z,R)$. We use Lemma 2.24 of [**71**] and Fubini's theorem to obtain

$$\begin{aligned}
\int_{\mathbb{B}_n} |g(z)|^p \mathrm{d}\mu(z) &\leq C_1 \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu(z)}{(1-|z|^2)^{n+1}} \int_{D(z,R)} |g(w)|^p \mathrm{d}v(w) \\
&\leq C_2 \int_{\mathbb{B}_n} \mathrm{d}\mu(z) \int_{D(z,R)} \frac{|g(w)|^p \mathrm{d}v_\gamma(w)}{(1-|w|^2)^{n+1+\gamma}} \\
&= C_2 \int_{\mathbb{B}_n} \frac{|g(w)|^p \mathrm{d}v_\gamma(w)}{(1-|w|^2)^{n+1+\gamma}} \int_{D(w,R)} \mathrm{d}\mu(z) \\
&= C_3 \int_{\mathbb{B}_n} |g(w)|^p \widehat{\mu}_{R,\gamma}(w) \mathrm{d}v_\gamma(w),
\end{aligned}$$

which proves the desired estimate. $\square$

LEMMA 52. – *Let μ be a positive Borel measure on $\mathbb{B}_n$. If γ is real, s is real, and $R > 0$, then there exists a constant $C > 0$ such that $B_{s,\gamma}(\mu) \leq C B_{s,\gamma}(\widehat{\mu}_{R,\gamma})$.*

Proof. – For $w \in \mathbb{B}_n$, apply Lemma 51 to the function

$$g(z) = \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{n+1+s}}$$

with $p = 1$. The desired result follows. $\square$

LEMMA 53. – *Let μ be a positive Borel measure on $\mathbb{B}_n$. If γ and s are real and R is positive, then there exists a positive constant C such that $\widehat{\mu}_{R,\gamma} \leq C B_{s,\gamma}(\mu)$.*

Proof. – Once again, we have $1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle|$ for $w \in D(z,R)$. It follows that

$$\widehat{\mu}_{R,\gamma}(z) = \frac{\mu(D(z,R))}{(1-|z|^2)^{n+1+\gamma}} \leq C \int_{D(z,R)} \frac{(1-|z|^2)^s \mathrm{d}\mu(w)}{|1 - \langle z,w \rangle|^{n+1+s+\gamma}} \leq C B_{s,\gamma}(\mu)(z),$$

proving the desired estimate. $\square$

THEOREM 54. – *Let $0 < q < p < \infty$ and α be any real number, and let μ be a positive Borel measure on $\mathbb{B}_n$. Then for any nonnegative integer k with $\alpha + kp > -1$ the following conditions are equivalent.*

(a) *There is a constant $C > 0$ such that, for all $f \in A_\alpha^p$,*

$$\int_{\mathbb{B}_n} \left| R^k f(w) \right|^q \mathrm{d}\mu(w) \le C \|f\|_{p,\alpha}^q.$$

(b) *For any bounded sequence $\{f_j\}$ in A_α^p with $f_j(z) \to 0$ for every $z \in \mathbb{B}_n$,*

$$\lim_{j \to \infty} \int_{\mathbb{B}_n} |R^k f_j(w)|^q \mathrm{d}\mu(w) = 0.$$

(c) *For any fixed $r > 0$ the function $\widehat{\mu}_{r,\gamma}$ is in $L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_\gamma)$, where $\gamma = \alpha + kp$.*

(d) *For any fixed $s > 0$ the function $B_{s,\gamma}(\mu)$ is in $L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_\gamma)$, where $\gamma = \alpha + kp$.*

Proof. – Let $s > 0$ satisfy $s + \alpha + kq > -1$. It follows from Lemmas 2.24 and 2.20 of [71] that

$$|R^k f(z)|^q \le \frac{C}{(1 - |z|^2)^{n+1+s+\alpha+kq}} \int_{D(z,r)} \left| R^k f(w) \right|^q \mathrm{d}v_{s+\alpha+kq}(w)$$

$$\le C \int_{D(z,r)} \frac{|R^k f(w)|^q (1 - |w|^2)^{s+kq}}{|1 - \langle w, z \rangle|^{n+1+s+\alpha+kq}} \mathrm{d}v_\alpha(w)$$

$$= C \int_{\mathbb{B}_n} \frac{|R^k f(w)|^q (1 - |w|^2)^{s+kq}}{|1 - \langle w, z \rangle|^{n+1+s+\alpha+kq}} \chi_{D(z,r)}(w) \mathrm{d}v_\alpha(w),$$

where $\chi_E(z)$ denotes the characteristic function of a set E. Integrate with respect to $\mathrm{d}\mu$, apply Fubini's theorem, and use Lemma 2.20 of [71]. We see that the integral

$$\int_{\mathbb{B}_n} \left| R^k f(z) \right|^q \mathrm{d}\mu(z)$$

is dominated by

$$\int_{\mathbb{B}_n} \frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kq}} \left| R^k f(w) \right|^q \left(1 - |w|^2 \right)^{kq} \mathrm{d}v_\alpha(w).$$

If condition (c) holds, then an application of Hölder's inequality yields

$$\int_{\mathbb{B}_n} |R^k f|^q \mathrm{d}\mu \le C \|f\|_{p,\alpha}^q \left[\int_{\mathbb{B}_n} \left(\frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kq}} \right)^{p/(p-q)} \mathrm{d}v_\alpha \right]^{1-q/p}$$

$$= C \|f\|_{p,\alpha}^q \left[\int_{\mathbb{B}_n} \left(\frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kp}} \right)^{p/(p-q)} \mathrm{d}v_{\alpha+kp} \right]^{1-q/p}$$

$$\le C \|f\|_{p,\alpha}^q.$$

This proves that (c) implies (a).

Since $1 - |z| \sim 1 - |w|$ for $z \in D(w,r)$ (see Lemma 2.20 of [71]), there exists a constant $\delta > 0$ such that

$$\delta^{-1} \le \frac{1 - |z|^2}{1 - |w|^2} \le \delta$$

for all $z \in D(w, r)$. For $0 < t < 1$ let

$$A_t = \{ z \in \mathbb{B}_n : 1 - |z|^2 < t \}.$$

Then the conditions $z \in A_t$ and $w \in D(z, r)$ imply that $w \in A_{\delta t}$.

Let $\{ f_j \}$ be a bounded sequence in A_α^p with $f_j(z) \to 0$ for every $z \in \mathbb{B}_n$. Then a normal family argument shows that $f_j(z) \to 0$ uniformly on every compact subset of $\mathbb{B}_n$. Using the estimate from the first paragraph of this proof together with Fubini's theorem, we see that the integral

$$\int_{A_t} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z)$$

is dominated by

$$\int_{\mathbb{B}_n} \left| R^k f_j(w) \right|^q \left(1 - |w|^2 \right)^{s+kq} \mathrm{d}v_\alpha(w) \int_{A_t} \frac{\chi_{D(w,r)} \, \mathrm{d}\mu(z)}{|1 - \langle z, w \rangle|^{n+1+s+\alpha+kq}}.$$

According to the previous paragraph,

$$\chi_{A_t \cap D(w,r)}(z) = 0, \quad z \in \mathbb{B}_n,$$

unless $w \in A_{\delta t}$. It follows that the integral

$$\int_{A_t} |R^k f_j(z)|^q \mathrm{d}\mu(z)$$

is dominated by

$$\int_{A_{\delta t}} \left| R^k f_j(w) \right|^q \left(1 - |w|^2 \right)^{s+kq} \mathrm{d}v_\alpha(w) \int_{A_t} \frac{\chi_{D(w,r)} \, \mathrm{d}\mu(z)}{|1 - \langle z, w \rangle|^{n+1+s+\alpha+kq}}.$$

Since $|1 - \langle z, w \rangle|$ is comparable to $1 - |w|^2$ whenever $z \in D(w, r)$, we get

$$\int_{A_t} |R^k f_j|^q \mathrm{d}\mu \leq C \int_{A_{\delta t}} \frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kq}} \left| R^k f_j(w) \right|^q \left(1 - |w|^2 \right)^{kq} \mathrm{d}v_\alpha.$$

By Hölder's inequality,

$$\int_{A_t} |R^k f_j|^q \mathrm{d}\mu \leq C \|f_j\|_{p,\alpha}^q \left[\int_{A_{\delta t}} \left(\frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kp}} \right)^{\frac{p}{p-q}} \mathrm{d}v_{\alpha+kp} \right]^{1-q/p}.$$

If the function

$$\widehat{\mu}_{r,\gamma}(z) = \frac{\mu(D(z,r))}{(1 - |z|^2)^{n+1+\alpha+kp}}$$

is in $L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_{\alpha+kp})$, then for any given $\epsilon > 0$ there is a $t \in (0, 1)$ such that

$$\int_{A_{\delta t}} \left(\frac{\mu(D(w,r))}{(1 - |w|^2)^{n+1+\alpha+kp}} \right)^{p/(p-q)} \mathrm{d}v_{\alpha+kp}(w) < \epsilon^{p/(p-q)}.$$

Thus for such t,

$$\int_{A_t} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z) \leq C\epsilon.$$

Since $\mathbb{B}_n \setminus A_t$ is a compact subset of $\mathbb{B}_n$ and $f_j \to 0$ uniformly on every compact subset of $\mathbb{B}_n$, we have

$$\lim_{j \to \infty} \int_{\mathbb{B}_n \setminus A_t} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z) = 0.$$

Combining this with an earlier estimate we get

$$\limsup_{j \to \infty} \int_{\mathbb{B}_n} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z) \leq C\epsilon.$$

Since ϵ is arbitrary, we must have

$$\lim_{j \to \infty} \int_{\mathbb{B}_n} \left| R^k f_j(z) \right|^q \mathrm{d}\mu(z) = 0.$$

This shows that (c) implies (b) as well.

The proof of that (b) implies (a) is standard. In fact, if (a) is not true, then there is a sequence $\{f_j\}$ in A_α^p such that $\|f_j\|_{p,\alpha} \leq 1$ and

$$(32) \qquad \lim_{j \to \infty} \int_{\mathbb{B}_n} \left| R^k f_j(w) \right|^q \mathrm{d}\mu(w) = \infty.$$

Since $\|f_j\|_{p,\alpha} \leq 1$, $\{f_j\}$ is uniformly bounded on compact subsets of $\mathbb{B}_n$. By Montel's Theorem, there is a subsequence of $\{f_j\}$, which we still denote by $\{f_j\}$, that converges uniformly on compact subsets of $\mathbb{B}_n$ to a holomorphic function f in $\mathbb{B}_n$. It follows from Fatou's lemma that $f \in A_\alpha^p$ with $\|f\|_{p,\alpha} \leq 1$. In particular,

$$\|f_j - f\|_{p,\alpha} \leq \max(2, 2^{1/p})$$

and $f_j - f \to 0$ uniformly on compact subsets of $\mathbb{B}_n$. If condition (b) holds, then

$$\lim_{j \to \infty} \int_{\mathbb{B}_n} \left| R^k f_j(w) - R^k f(w) \right|^q \mathrm{d}\mu(w) = 0,$$

which contradicts (32). This shows that (b) implies (a).

To prove that (a) implies (c), we follow the proof of Theorem 1 in Luecking [**39**]. Let $\{a_j\}$ be the sequence of points in $\mathbb{B}_n$ from Theorem 2.30 in [**71**]. Let b be a real number such that

$$b > n \max\left(1, \frac{1}{p}\right) + \frac{1+\alpha}{p}.$$

Let

$$g_j(z) = \frac{(1 - |a_j|^2)^{b - (n+1+\alpha)/p}}{(1 - \langle z, a_j \rangle)^{b+k}} = \frac{(1 - |a_j|^2)^{(b+k) - (n+1+\alpha+kp)/p}}{(1 - \langle z, a_j \rangle)^{b+k}}.$$

Let $\{c_j\} \in \ell^p$. Then by Theorem 2.30 of [**71**], we have

$$\sum_{j=1}^{\infty} c_j g_j(z) \in A_{\alpha+kp}^p.$$

Let

$$h_j(z) = R^{-k}\big(g_j(z) - g_j(0)\big) \quad and \quad f(z) = \sum_{j=1}^{\infty} c_j h_j(z).$$

Then

$$R^k f(z) = \sum_{j=1}^{\infty} c_j R^k h_j(z) = \sum_{j=1}^{\infty} c_j \big(g_j(z) - g_j(0)\big).$$

It is clear that $R^k f \in A^p_{\alpha+kp}$, and so $f \in A^p_\alpha$. Moreover,

$$\|f\|_{p,\alpha}^p \le C \sum_{j=1}^{\infty} |c_j|^p,$$

where C is a positive constant independent of f. If condition (a) holds, then

$$\int_{\mathbb{B}_n} \big|R^k f(z)\big|^q \mathrm{d}\mu(z) \le C\|f\|_{p,\alpha}^q \le C\Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{q/p}.$$

Therefore,

$$\int_{\mathbb{B}_n} \Big|\sum_{j=1}^{\infty} c_j g_j\Big|^q \mathrm{d}\mu$$

$$\le 2^q \Big[\int_{\mathbb{B}_n} \Big|\sum_{j=1}^{\infty} c_j g_j - \sum_{j=1}^{\infty} c_j g_j(0)\Big|^q \mathrm{d}\mu + \int_{\mathbb{B}_n} \Big|\sum_{j=1}^{\infty} c_j g_j(0)\Big|^q \mathrm{d}\mu\Big]$$

$$\le 2^q \int_{\mathbb{B}_n} \big|R^k f(z)\big|^q \mathrm{d}\mu(z) + 2^q \mu(\mathbb{B}_n)\Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{q/p} \le C\Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{q/p}$$

Let $r_j(t)$ be a sequence of Rademacher functions (see page 336 of Luecking [39]). If we replace c_j by $r_j(t)c_j$, the above inequality is still true, so

$$\int_{\mathbb{B}_n} \Big|\sum_{j=1}^{\infty} r_j(t) c_j g_j(z)\Big|^q \mathrm{d}\mu(z) \le C\Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{q/p}.$$

Integrating with respect to t from 0 to 1, applying Fubini's theorem, and invoking Khinchine's inequality (see Luecking [39]), we obtain

$$A_q \int_{\mathbb{B}_n} \Big(\sum_{j=1}^{\infty} |c_j|^2 |g_j(z)|^2\Big)^{q/2} \mathrm{d}\mu(z) \le C\Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{q/p},$$

where A_p is the constant that appears in Khinchine's inequality. The rest of the proof is exactly the same as the one in Luecking [39].

The condition in (d) first appeared in Choe-Koo-Yi [19], where it was used for the embedding of harmonic Bergman spaces into $L^q(d\mu)$. Our proof of the equivalence of (c) and (d) follows the method in [19]. In fact, if $\widehat{\mu}_{r,\gamma}$ is in $L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_\gamma)$, then an application of Proposition 8 shows that the function $B_{s,\gamma}(\widehat{\mu}_{r,\gamma})$ is also in $L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_\gamma)$. By Lemma 52, we must have $B_{s,\gamma}(\mu) \in L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_\gamma)$. This

proves that (c) implies (d). That (d) implies (c) is a direct consequence of Lemma 53. The proof of the theorem is now complete. □

CHAPTER 14

COEFFICIENT MULTIPLIERS

Recall from Theorem 12 that for $t = (\alpha - \beta)/p$, the operator $R_{s,t}$ maps A_α^p boundedly onto A_β^p. In terms of Taylor coefficients, we have

$$R_{s,t}\left(\sum_m a_m z^m\right) = \sum_m c_m a_m z^m,$$

where

$$c_m = \frac{\Gamma(n+1+s+t)\Gamma(n+1+|m|+s)}{\Gamma(n+1+s)\Gamma(n+1+|m|+s+t)}.$$

Therefore, the operator $R_{s,t}$ is just a coefficient multiplier on holomorphic functions in $\mathbb{B}_n$. When α and β are real, an application of Stirling's formula shows that

$$c_m \sim \frac{1}{|m|^t}$$

as $|m| \to \infty$. We are going to show that this result still holds if we replace the multiplier sequence $\{c_m\}$ above by the more explicit multiplier sequence $\{|m|^{(\beta-\alpha)/p}\}$. A similar result will be proved for the generalized Lipschitz spaces Λ_α.

We introduce two methods, one based on complex interpolation and the other based on atomic decomposition.

LEMMA 55. – *Suppose t is complex and k is a postive integer large enough so that $k + \operatorname{Re} t > 0$. There exists a constant c such that*

$$\int_0^1 R^k f(rz)\left(\log\frac{1}{r}\right)^{t+k-1}\frac{\mathrm{d}r}{r} = c\sum_{|m|>0}|m|^{-t}a_m z^m$$

for all holomorphic $f(z) = \sum_m a_m z^m$ in $\mathbb{B}_n$.

Proof. – Fix $z \in \mathbb{B}_n$. We want to evaluate the integral

$$I(f,z) = \int_0^1 R^k f(rz)\left(\log\frac{1}{r}\right)^{t+k-1}\frac{\mathrm{d}r}{r}$$

in terms of the Taylor expansion of f. If $f(z) = \sum_m a_m z^m$, then

$$R^k f(z) = \sum_{|m|>0} |m|^k a_m z^m,$$

so

$$I(f, z) = \sum_{|m|>0} |m|^k a_m z^m \int_0^1 r^{|m|-1} \Big(\log \frac{1}{r} \Big)^{t+k-1} \mathrm{d}r.$$

Making the change of variables $r = \mathrm{e}^{-s}$, we obtain

$$I(f, z) = \sum_{|m|>0} |m|^k a_m z^m \int_0^\infty \mathrm{e}^{-|m|s} s^{t+k-1} \mathrm{d}s.$$

Let $u = |m|s$. Then

$$I(f, z) = c \sum_{|m|>0} |m|^{-t} a_m z^m,$$

where $c = \int_0^\infty \mathrm{e}^{-u} u^{t+k-1} \mathrm{d}u$. $\qquad\qquad\square$

Given any real α and β, we are going to fix a sufficiently large positive integer k and consider operators on $H(\mathbb{B}_n)$ of the form

$$T_\zeta f(z) = \int_0^1 R^k f(rz) \Big(\log \frac{1}{r} \Big)^{(\alpha-\beta)(1-\zeta)+k-1} \frac{\mathrm{d}r}{r},$$

where $0 \leq \operatorname{Re} \zeta \leq 1$.

LEMMA 56. – *If $\operatorname{Re} \zeta = 0$, the operator T_ζ maps A_α^1 boundedly into A_β^1.*

Proof. – Let N be a sufficiently large positive integer. We have

$$R^N T_\zeta f(z) = \int_0^1 R^{N+k} f(rz) \Big(\log \frac{1}{r} \Big)^{(\alpha-\beta)(1-\zeta)+k-1} \frac{\mathrm{d}r}{r}.$$

If $\operatorname{Re} \zeta = 0$, it follows from Fubini's theorem that the integral

$$I = \int_{\mathbb{B}_n} |R^N T_\zeta f(z)| \big(1 - |z|^2\big)^{N+\beta} \mathrm{d}v(z)$$

does not exceed

$$\int_0^1 \Big(\log \frac{1}{r} \Big)^{\alpha-\beta+k-1} \frac{\mathrm{d}r}{r} \int_{\mathbb{B}_n} |R^{N+k} f(rz)| \big(1 - |z|^2\big)^{N+\beta} \mathrm{d}v(z).$$

Let $w = rz$ in the inner integral. Then I does not exceed the integral

$$\int_0^1 \Big(\log \frac{1}{r} \Big)^{\alpha-\beta+k-1} \frac{\mathrm{d}r}{r^{2n+1}} \int_{|w|<r} |R^{N+k} f(w)| \Big(1 - \frac{|w|^2}{r^2}\Big)^{N+\beta} \mathrm{d}v(w).$$

Since $1 - |w|^2/r^2 \leq 1 - |w|^2$ for all $|w| < r$, we have

$$I \leq \int_0^1 \Big(\log \frac{1}{r} \Big)^{\alpha-\beta+k-1} \frac{\mathrm{d}r}{r^{2n+1}} \int_{|w|<r} |R^{N+k} f(w)| \big(1 - |w|^2\big)^{N+\beta} \mathrm{d}v(w).$$

We interchange the order of integration and obtain

$$I \leq \int_{\mathbb{B}_n} |R^{N+k} f(w)| (1 - |w|^2)^{N+\beta} \, dv(w) \int_{|w|}^1 \left(\log \frac{1}{r} \right)^{\alpha - \beta + k - 1} \frac{dr}{r^{2n+1}}.$$

It is easy to see that there exists a constant $C > 0$ such that

$$\int_{|w|}^1 \left(\log \frac{1}{r} \right)^{\alpha - \beta + k - 1} \frac{dr}{r^{2n+1}} \leq C \frac{(1 - |w|^2)^{\alpha - \beta + k}}{|w|^{2n}}$$

for all $w \in \mathbb{B}_n$, so

$$I \leq C \int_{\mathbb{B}_n} |R^{N+k} f(w)| (1 - |w|^2)^{N+k+\alpha} \frac{dv(w)}{|w|^{2n}}.$$

Since $|R^{N+k} f(w)| \leq C|w|$ near the origin and

$$\int_{\mathbb{B}_n} \frac{dv(w)}{|w|^{2n-1}} < \infty$$

by polar coordinates, we can find another constant $C' > 0$, independent of f, such that

$$I \leq C \int_{\mathbb{B}_n} |R^{N+k} f(w)| (1 - |w|^2)^{N+k} \, dv_\alpha(w).$$

This completes the proof of the lemma. $\qquad \square$

LEMMA 57. – *If $\operatorname{Re} \zeta = 1$, the operator T_ζ is bounded on the Bloch space $\mathscr{B}$.*

Proof. – We have

$$RT_\zeta f(z) = \int_0^1 R^{k+1} f(rz) \left(\log \frac{1}{r} \right)^{(\alpha - \beta)(1 - \zeta) + k - 1} \frac{dr}{r}.$$

If $\operatorname{Re} \zeta = 1$ and $f \in \mathscr{B}$, then

$$|RT_\zeta f(z)| \leq \int_0^1 |R^{k+1} f(rz)| \left(\log \frac{1}{r} \right)^{k-1} \frac{dr}{r}$$

$$\leq C \int_0^1 (1 - r^2 |z|^2)^{-(k+1)} \left(\log \frac{1}{r} \right)^{k-1} dr \leq C' (1 - |z|^2)^{-1},$$

where C and C' are positive constants independent of z. This shows that $T_\zeta f$ is in the Bloch space. $\qquad \square$

LEMMA 58. – *Suppose $s > -1$, t is a positive integer, and*

$$I(z) = \int_0^1 \frac{(1 - x)^s \, dx}{(1 - xz)^{s+t+1}}, \quad z \in \mathbb{D}.$$

There exists a polynomial $p(z)$ such that

$$I(z) = \frac{p(z)}{(1 - z)^t}, \quad z \in \mathbb{D}.$$

Proof. – We compute the integral $I(z)$ with the help of Taylor expansion.

$$I(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+s+t+1)}{k!\,\Gamma(s+t+1)} z^k \int_0^1 x^k (1-x)^s \, \mathrm{d}x.$$

Since

$$\int_0^1 x^k (1-x)^s \, \mathrm{d}x = \frac{\Gamma(k+1)\Gamma(s+1)}{\Gamma(k+s+2)},$$

we have

$$I(z) = \frac{1}{s+1} \sum_{k=0}^{\infty} \frac{\Gamma(s+2)\Gamma(k+s+t+1)}{\Gamma(s+t+1)\Gamma(k+s+2)} z^k$$

$$= \frac{1}{s+1} R^{s,t-1} \sum_{k=0}^{\infty} z^k = \frac{1}{s+1} R^{s,t-1} \frac{1}{1-z}.$$

Since t is a positive integer, the operator $R^{s,t-1}$ is a linear differential operator of order $t-1$ on $H(\mathbb{D})$ with polynomial coefficients (see Proposition 4). It follows that there exists a polynomial $p(z)$ such that

$$I(z) = \frac{p(z)}{(1-z)^t}.$$

This completes the proof of the lemma.   □

We can now prove the first main result of the chapter.

THEOREM 59. – *Suppose α is real, β is real, and $p > 0$. Then the operator T defined on $H(\mathbb{B}_n)$ by*

$$Tf(z) = f(0) + \sum_{|m|>0} |m|^{(\beta-\alpha)/p} a_m z^m, \quad f(z) = \sum_m a_m z^m,$$

maps A_α^p boundedly onto A_β^p.

Proof. – By switching the roles of α and β, it is enough for us to show that the operator T maps A_α^p into A_β^p.

When $p = 1$, the desired result follows from Lemmas 55 and 56.

First suppose that $1 < p < \infty$ with $1/p + 1/q = 1$. Let $\theta = 1/q$. Then

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}.$$

Because the dual space of A_β^p can be identified with A_β^q under the integral pairing

$$\langle f, g \rangle = \int_{\mathbb{B}_n} (1-|z|^2)^N R^{s,N} f(z) \overline{(1-|z|^2)^N R^{s,N} g(z)} \, \mathrm{d}v_\beta(z),$$

where N is a sufficiently large positive number, it suffices for us to show that there exists a constant $C > 0$, independent of f and g, such that

(33) $$\left| \langle Tf, g \rangle \right| \leq C \|f\|_{p,\alpha} \|g\|_{q,\beta}$$

for all $f \in A_\alpha^p$ and $g \in A_\beta^q$.

Fix a unit vector f in A_α^p and fix a polynomial g that is a unit vector in A_β^q (recall that the polynomials are dense in A_β^q). It follows from the complex interpolation relation (see Theorems 38) $\left[A_\alpha^1, \mathscr{B}\right]_\theta = A_\alpha^p$ that there exist functions f_ζ, where $\zeta \in \bar{S}$, such that

(a) $f_\theta = f$;

(b) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $\bar{S}$ into $A_\alpha^1 + \mathscr{B}$ whose restriction to S is analytic;

(c) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $L(S)$ into A_α^1 with $\|f_\zeta\|_{1,\alpha} \leq C$;

(d) $\zeta \mapsto f_\zeta$ is a bounded continuous function from $R(S)$ into $\mathscr{B}$ with $\|f_\zeta\|_{\mathscr{B}} \leq C$.

Here C is a positive constant independent of f.

Consider the function

$$F(\zeta) = \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^N R^{s,N} T_\zeta f_\zeta(z) g_\zeta(z) \mathrm{d}v_\beta(z),$$

where $\zeta \in \bar{S}$ and

$$g_\zeta(z) = \frac{\overline{R^{s,N}g(z)}}{|R^{s,N}g(z)|} \left[(1 - |z|^2)^N |R^{s,N}g(z)|\right]^{q\zeta}.$$

Because g is a polynomial, the function F is bounded and continuous on $\bar{S}$ and its restriction to S is analytic. When $\zeta = \theta$, it follows from Lemma 55 that $F(\theta) = \langle Tf, g \rangle$.

When $\mathrm{Re}\,\zeta = 0$, it follows from Lemma 56 that T_ζ maps A_α^1 boundedly into A_β^1, so there exists a positive constant C_0 such that

$$\|T_\zeta f_\zeta\|_{1,\beta} \leq C_0 \|f_\zeta\|_{1,\alpha} \leq C_0 C$$

for all $\mathrm{Re}\,\zeta = 0$. Thus there exists a constant $M_0 > 0$ (independent of f, g, and ζ) such that, for all $\mathrm{Re}\,\zeta = 0$,

$$\left|F(\zeta)\right| \leq \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^N \left|R^{s,N} T_\zeta f_\zeta(z)\right| \mathrm{d}v_\beta(z) \leq M_0.$$

When $\mathrm{Re}\,\zeta = 1$, it follows from Lemma 57 that T_ζ is bounded on the Bloch space, so there exists a positive constant C_1 such that, for all $\mathrm{Re}\,\zeta = 1$,

$$\|T_\zeta f_\zeta\|_{\mathscr{B}} \leq C_1 \|f_\zeta\|_{\mathscr{B}} \leq C_1 C.$$

We can then find a positive constant M_1 (independent of f, g, and ζ) such that, for all $\mathrm{Re}\,\zeta = 1$,

$$\left|F(\zeta)\right| \leq C_1 \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{Nq} \left|R^{s,N}g(z)\right|^q \mathrm{d}v_\beta(z) \leq M_1.$$

It follows from Hadamard's three lines theorem that

$$\left|F(\theta)\right| \leq M_0^{1-\theta} M_1^\theta.$$

Since M_0 and M_1 are independent of f and g, this yields the estimate (33) and proves the theorem for $1 < p < \infty$.

Next assume that $0 < p \leq 1$. By Theorem 32, there exists a positive number b (we can choose b to be as large as we want) and a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that every function $f \in A_\alpha^p$ can be written as

$$f(z) = \sum_{k=1}^\infty c_k f_k(z),$$

with

$$\sum_{k=1}^\infty |c_k|^p \leq C\|f\|_{p,\alpha}^p,$$

where C is a positive constant independent of f and

$$f_k(z) = \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}.$$

By first considering finite sums and then taking a limit, we may assume that

$$Tf = \sum_{k=1}^\infty c_k Tf_k.$$

Since $0 < p \leq 1$, we must have

$$\|Tf\|_{p,\beta}^p \leq \sum_{k=1}^\infty |c_k|^p \|Tf_k\|_{p,\beta}^p.$$

Since the sequence $\{f_k\}$ is bounded in A_α^p, the proof of the theorem will be complete if we can show that there exists a constant $C > 0$ such that

$$\|Tf\|_{p,\beta} \leq C\|f\|_{p,\alpha}$$

for functions of the form

(34) $$f(z) = \frac{1}{(1 - \langle z, a \rangle)^b}, \quad a \in \mathbb{B}_n.$$

We fix a sufficiently large positive integer k and apply Lemma 55 to represent the operator T as

$$Tf(z) = f(0) + c \int_0^1 R^k f(rz) \left(\log \frac{1}{r}\right)^{k+(\alpha-\beta)/p-1} \frac{\mathrm{d}r}{r}.$$

Write $R^k = R^{k-1}R$ and take the factor R^{k-1} out of the integral sign. Then

$$Tf(z) = f(0) + cR^{k-1} \int_0^1 \frac{Rf(rz)}{r} \left(\log \frac{1}{r}\right)^{k+(\alpha-\beta)/p-1} \mathrm{d}r.$$

We assume that b is chosen so that $b - k - (\alpha - \beta)/p$ is a sufficiently large positive integer. It is easy to see that

$$\left(\frac{\log \frac{1}{r}}{1 - r}\right)^{k+(\alpha-\beta)/p-1} = 1 + \sum_{j=1}^{L} b_j(1 - r)^j + H(r),$$

where $H(r) = O((1 - r)^L)$ as $r \to 1$. It follows that

$$T = T_0 + T_1 + \cdots + T_L + T_{L+1},$$

where

$$T_0 f(z) = f(0) + cR^{k-1} \int_0^1 \frac{Rf(rz)}{r}(1 - r)^{k+(\alpha-\beta)-1} \mathrm{d}r,$$

and

$$T_j f(z) = cb_j R^{k-1} \int_0^1 \frac{Rf(rz)}{r}(1 - r)^{k+j+(\alpha-\beta)/p-1} \mathrm{d}r, \quad 1 \leq j \leq L,$$

and

$$T_{L+1} f(z) = c \int_0^1 R^k f(rz)(1 - r)^{k+(\alpha-\beta)/p-1} H(r)\frac{\mathrm{d}r}{r}.$$

It then follows from Lemma 58 that there exists a constant $C > 0$ such that

$$\|T_j f\|_{p,\beta} \leq C\|f\|_{p,\alpha}$$

for all $0 \leq j \leq L$ and all functions f given in (34). The same estimate holds for the operator T_{L+1} as well, except this time we do not use Lemma 58, but use the assumption that L is large enough so that

$$R^N T_{L+1} f(z) = c \int_0^1 R^{N+k} f(rz)(1 - r)^{k+(\alpha-\beta)/p-1} H(r)\frac{\mathrm{d}r}{r}$$

is bounded, where N is any nonnegative integer with $pN + \beta > -1$ and f is given by (34). This proves the case $0 < p \leq 1$ and completes the proof of the theorem. $\square$

As the second main result of this chapter we establish an isomorphism between Λ_α and Λ_β by a simple coefficient multiplier.

THEOREM 60. – *Suppose α and β are real. Then the operator T defined by*

$$f(z) = \sum_m a_m z^m \mapsto Tf(z) = f(0) + \sum_{|m|>0} a_m|m|^{\alpha-\beta} z^m$$

is an invertible operator from Λ_α onto Λ_β.

Proof. – By reversing the role of α and β, it suffices for us to show that the operator T maps Λ_α boundedly into Λ_β.

Given $f \in \Lambda_\alpha$, we fix a sufficiently large positive integer k and use Lemma 55 to write

$$Tf(z) = f(0) + c \int_0^1 R^k f(rz)\left(\log \frac{1}{r}\right)^{k-\alpha+\beta-1} \frac{\mathrm{d}r}{r}.$$

If N is another sufficiently large positive integer, then

$$R^N Tf(z) = \int_0^1 R^{N+k} f(rz) \Big(\log \frac{1}{r} \Big)^{k-\alpha+\beta-1} \frac{\mathrm{d}r}{r}.$$

Since $f \in \Lambda_\alpha$, it follows from Lemma 15 that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{N+k-\alpha} |R^{N+k} f(z)| < \infty.$$

But $R^{N+k} f(0) = 0$, we must also have

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{N+k-\alpha} \frac{|R^{N+k} f(z)|}{|z|} < \infty.$$

So there exists a constant $C > 0$ such that

$$\big| R^N Tf(z) \big| \leq C \int_0^1 \Big(\log \frac{1}{r} \Big)^{k-\alpha+\beta-1} \frac{\mathrm{d}r}{(1 - r^2|z|^2)^{N+k-\alpha}}.$$

Now the above integral clearly converges near $r = 0$. When r is away from 0, $\log \frac{1}{r}$ is comparable to $1 - r^2$. So there exists another constant $C > 0$ such that

$$\big| R^N Tf(z) \big| \leq C \int_0^1 \frac{(1 - r^2)^{k-\alpha+\beta-1} \mathrm{d}r}{(1 - r^2|z|^2)^{N+k-\alpha}}.$$

An elementary estimate then shows that

$$\big| R^N Tf(z) \big| \leq \frac{C}{(1 - |z|^2)^{N-\beta}}$$

for some constant $C > 0$ and all $z \in \mathbb{B}_n$. This shows that Tf is in Λ_β and completes the proof of the theorem. $\square$

We mention that, at least in the case $n = 1$, the theorem above also follows from Theorem 12 and the asymptotic expansion of a ratio of two gamma functions as given in Tricomi-Erdelyi [58]. In fact, in the one-dimensional case, it is easy to see that if

$$f(z) = \sum_{k=0}^\infty a_k z^k$$

is a function in Λ_α, then the sequence $\{k^\alpha a_k\}$ is bounded. It is also easy to show that if the sequence $\{k^{\alpha+1} a_k\}$ is bounded, then the function f is in Λ_α. This together with the main result of Tricomi-Erdelyi [58] easily gives Theorem 59 above. Coefficients of functions in Bloch and Lipschitz spaces are also studied in Bennet-Stegenga-Timoney [12].

CHAPTER 15

LACUNARY SERIES

One way to construct concrete examples in certain spaces of analytic functions is by using lacunary series. In this chapter we characterize lacunary series in weighted Bergman spaces and Lipschitz spaces.

We say that an analytic function f on $\mathbb{B}_n$ has a lacunary homogeneous expansion if its homogeneous expansion is of the form

$$f(z) = \sum_{k=1}^{\infty} f_{m_k}(z),$$

where m_k satisfies the condition

$$\inf_k \frac{m_{k+1}}{m_k} = \lambda > 1.$$

If $n = 1$, the lacunary homogeneous expansions are just lacunary series in the unit disk. When $n > 1$, we say a lacunary homogeneous expansion is a lacunary series if every homogeneous polynomial f_{m_k} consists of just one term.

Our first result characterizes a lacunary homogeneous expansion in A_α^p.

PROPOSITION 61. – *Let* $0 < p < \infty$, α *be any real number, and*

$$f(z) = \sum_{k=1}^{\infty} f_{m_k}(z)$$

be a lacunary homogeneous expansion. Then $f \in A_\alpha^p$ *if and only if*

$$\sum_{k=1}^{\infty} m_k^{-1-\alpha} \|f_{m_k}\|_{H^p}^p < \infty,$$

where

$$\|f\|_{H^p} = \left(\int_{\mathbb{S}_n} |f(\zeta)|^p \, d\sigma(\zeta) \right)^{1/p}$$

denotes the H^p*-norm of* f.

Proof. – By Proposition 3 in Yang-Ouyang [**64**], if

$$g(z) = \sum_{k=1}^{\infty} g_{m_k}(z)$$

is a lacunary homogeneous expansion, then $g \in A^p$ if and only if

$$\sum_{k=1}^{\infty} m_k^{-1} \|g_{m_k}\|_{H^p}^p < \infty.$$

Let $f \in A_{\alpha}^p$. By Theorem 10, if s is a real number such that neither $n + s$ nor $n + s + (\alpha/p)$ is a negative integer, then $f \in A_{\alpha}^p$ if and only if $R_{s,\alpha/p}f \in A^p$, which, by the above result, is equivalent to

$$\sum_{k=1}^{\infty} m_k^{-1} \|c_{m_k} f_{m_k}\|_{H^p}^p < \infty,$$

where

$$c_{m_k} = \frac{\Gamma(n+1+s+\alpha/p)\Gamma(n+1+m_k+s)}{\Gamma(n+1+s)\Gamma(n+1+m_k+s+\alpha/p)}.$$

It follows from Stirling's formula that

$$c_{m_k} \sim m_k^{-\alpha/p}$$

as $k \to \infty$. Thus the above condition is equivalent to

$$\sum_{k=1}^{\infty} m_k^{-1-\alpha} \|f_{m_k}\|_{H^p}^p < \infty.$$

The proof is complete. $\square$

The next result characterizes a lacunary series in A_{α}^p.

PROPOSITION 62. – *Let $0 < p < \infty$, α be any real number, and*

$$f(z) = \sum_{k=1}^{\infty} f_{m_k}(z)$$

be a lacunary series, where

$$f_{m_k}(z) = a_k z_1^{m_{k_1}} \cdots z_n^{m_{k_n}}.$$

Then $f \in A_{\alpha}^p$ if and ony if

$$\sum_{k=1}^{\infty} \frac{|a_k|^p \prod_{i=1}^{n} \Gamma(\frac{1}{2}m_{k_i}p + 1)}{m_k^{1+\alpha}\Gamma(\frac{1}{2}m_k p + n)} < \infty.$$

Proof. – Let $\zeta^m = \zeta_1^{m_1} \cdots \zeta_n^{m_n}$ and $|m| = m_1 + \cdots m_n$. An easy modification of the proof of Lemma 1.11 in [**71**] shows that

$$\|\zeta^m\|_{H^p}^p = \frac{(n-1)! \prod_{i=1}^n \Gamma(\frac{1}{2}m_i p + 1)}{\Gamma(\frac{1}{2}|m|p + n)}.$$

Combining this identity and Proposition 61, we get the desired result. $\square$

PROPOSITION 63. – *Let α be any real number, let*

$$f(z) = \sum_{k=1}^\infty f_{m_k}(z)$$

be a lacunary homogeneous expansion, and denote by

$$\|f_{m_k}\|_{H^\infty} = \sup_{\zeta \in \mathbb{S}_n} |f_{m_k}(\zeta)|.$$

Then

(a) $f \in \Lambda_\alpha$ *if and only if*

$$\sup_{k \geq 1} m_k^\alpha \|f_{m_k}\|_{H^\infty} < \infty.$$

(b) $f \in \Lambda_{\alpha,0}$ *if and only if*

$$\lim_{k \to \infty} m_k^\alpha \|f_{m_k}\|_{H^\infty} = 0.$$

Proof. – The results follow easily from Theorem 16, the corresponding result for $\Lambda_{\alpha,0}$, and Propositions 2 and 3 in Wulan-Zhu [**62**]. We leave the details to the interested reader. $\square$

PROPOSITION 64. – *Let α be any real number and*

$$f(z) = \sum_{k=1}^\infty f_{m_k}(z)$$

be a lacunary series, where $f_{m_k}(z) = a_k z_1^{m_{k_1}} \cdots z_n^{m_{k_n}}$. Then

(a) $f \in \Lambda_\alpha$ *if and only if*

$$\sup_{k \geq 1} m_k^\alpha |a_k| \sqrt{\frac{m_{k_1}^{m_{k_1}} \cdots m_{k_n}^{m_{k_n}}}{m_k^{m_k}}} < \infty.$$

(b) $f \in \Lambda_{\alpha,0}$ *if and only if*

$$\lim_{k \to \infty} m_k^\alpha |a_k| \sqrt{\frac{m_{k_1}^{m_{k_1}} \cdots m_{k_n}^{m_{k_n}}}{m_k^{m_k}}} = 0.$$

Proof. – The results follow directly from Proposition 63 and Lemma 4 in Wulan-Zhu [**62**]. $\square$

Several special cases of the main results of this chapter are known. For example, lacunary series in the Bloch space of the unit disk are described in Anderson-Clunie-Pommerenke [3], lacunary series in weighted Bergman spaces A_α^p of the unit ball, where $\alpha > -1$, are described in Stević [56], and lacunary series in Bloch and certain Lipschitz spaces of the unit ball are characterized in Wulan-Zhu [62].

CHAPTER 16

INCLUSION RELATIONS

In this chapter we study inclusion relations among weighted Bergman spaces and Lipschitz spaces. From the definition and Proposition 64 it is very easy to see that if $\alpha > \beta$ then $\Lambda_\alpha \subset \Lambda_\beta$, and the inclusion is strict.

The inclusion relations between weighted Bergman spaces are more complicated in general. Several embedding theorems have been known before, and our results here overlap with some of them; see Aleksandrov [2], Beatrous-Burbea [11], Graham [28], Luecking [39], and Rochberg [46]. We begin with the following simple case.

PROPOSITION 65. – *Let $0 < p < \infty$, and let α and β be any two real numbers satisfying $\alpha < \beta$. Then*

$$A^p_\alpha \subset A^p_\beta,$$

and the inclusion is strict.

Proof. – The inclusion is obvious. To prove that the inclusion is strict, we only need to test functions of the form $f_t(z) = (1 - z_1)^t$. See Yang-Ouyang [64] for a similar argument. $\square$

To better describe the inclusion relations of Bergman spaces, we introduce the notion of Lipschitz stretch first. More specifically, if X is a space of analytic functions, we define the *Lipschitz stretch* of X as follows:

$$\Lambda(X) = \inf \left\{ \beta - \alpha : \Lambda_{-\alpha} \subset X \subset \Lambda_{-\beta} \right\}.$$

We also call the constants

$$\alpha_0 = \sup \left\{ \alpha : \Lambda_{-\alpha} \subset X \right\}, \quad \beta_0 = \inf \left\{ \beta : X \subset \Lambda_{-\beta} \right\}$$

the lower and upper bounds of the Lipschitz stretch, respectively. A similar concept using Bloch type spaces was introduced in Zhao [65] for spaces of analytic functions in the unit disk.

THEOREM 66. – *Let $0 < p < \infty$ and let α be any real number. Then for any $\gamma < (1+\alpha)/p$ we have*

$$\Lambda_{-\gamma} \subset A_\alpha^p \subset \Lambda_{-(n+1+\alpha)/p}.$$

Both inclusions are strict and best possible, where "best possible" means that, for each p and α, the index γ of $\Lambda_{-\gamma}$ on the left-hand side cannot be replaced by a larger number, and the index $(n + 1 + \alpha)/p$ on the right-hand side cannot be replaced by a smaller one.

Proof. – Suppose $f \in A_\alpha^p$. Then $R^k f \in A_{pk+\alpha}^p$, where k is a nonnegative integer satisfying $pk + \alpha > -1$. By Theorem 20, there exists a positive constant C such that

$$\left(1 - |z|^2\right)^{k+(n+1+\alpha)/p} \left|R^k f(z)\right| \le C$$

for all $z \in \mathbb{B}_n$. This means $f \in \Lambda_{-(n+1+\alpha)/p}$, so $A_\alpha^p \subset \Lambda_{-(n+1+\alpha)/p}$.

Next suppose $\gamma < (1 + \alpha)/p$ and $f \in \Lambda_{-\gamma}$. Let k be a nonnegative integer such that $k + \gamma > 0$. Then $kp + \alpha > -1$ and $\alpha - p\gamma > -1$, so

$$\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{pk} \left|R^k f(z)\right|^p \mathrm{d}v_\alpha(z)$$
$$\le \sup_{z \in \mathbb{B}_n} \left((1 - |z|^2)^{k+\gamma} |R^k f(z)|\right)^p \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{-p\gamma} \mathrm{d}v_\alpha(z)$$
$$\le C \sup_{z \in \mathbb{B}_n} \left((1 - |z|^2)^{k+\gamma} |R^k f(z)|\right)^p.$$

Thus $\Lambda_{-\gamma} \subset A_\alpha^p$.

We only give a sketch of the rest of the proof since it is similar to the argument used in Yang-Ouyang [64]. For $t > 0$ let k be a nonnegative integer such that $k + \gamma > 0$. Since the radial derivative is an invertible operator on the space of holomorphic functions in $\mathbb{B}_n$ that vanish at the origin, we can define a holomorphic function f_t in $\mathbb{B}_n$ by

$$f_t(z) = R^{-k}\left[(1 - z_1)^{-t-k} - 1\right].$$

Then

$$R^k f_t(z) = (1 - z_1)^{-t-k} - 1,$$

so for any $z \in \mathbb{B}_n$ and $t \le \gamma$,

$$\left(1 - |z|^2\right)^{k+\gamma} \left|R^k f_t(z)\right| \le \left(1 - |z|^2\right)^{k+\gamma}\left(|1 - z_1|^{-t-k} + 1\right) \le C(1 - |z|^2)^{\gamma-t} \le C.$$

On the other hand, if $t > \gamma$, then we take $z = (x, 0, \dots, 0)$, where x is a real number between 0 and 1, to obtain

$$\left(1 - |z|^2\right)^{k+\gamma}\left|R^k f_t(z)\right| = (1 - x^2)^{k+\gamma}\left((1 - x)^{-t-k} - 1\right) \ge (1 - x)^{\gamma-t} \to \infty$$

as $x \to 1$. Thus

$$(35) \qquad\qquad f_t \in \Lambda_{-\gamma} \quad \text{if and only if} \quad t \le \gamma.$$

By a similar computation as used in Yang-Ouyang [**64**], we see that

(36)
$$f_t \in A_\alpha^p \quad \text{when} \quad t < \frac{n+1+\alpha}{p},$$

and

(37)
$$f_t \notin A_\alpha^p \quad \text{when} \quad t = \frac{n+1+\alpha}{p}.$$

For any $\varepsilon > 0$ let $t = (n+1+\alpha)/p - \varepsilon/2$. Then

$$(n+1+\alpha)/p - \varepsilon < t < (n+1+\alpha)/p.$$

By (36) and (35) we hav

$$f_t \in A_\alpha^p \quad \text{but} \quad f_t \notin \Lambda_{-((n+1+\alpha)/p-\varepsilon)}.$$

This shows that the inclusion $A_\alpha^p \subset \Lambda_{(n+1+\alpha)/p}$ is the best possible. At the same time it also shows that the inclusion $\Lambda_{-\gamma} \subset A_\alpha^p$ is strict, since

$$\Lambda_{-\gamma} \subset \Lambda_{-((n+1+\alpha)/p-\varepsilon)}$$

as $\varepsilon \le n/p$.

Let $t = (n+1+\alpha)/p$. Then by (37) and (35), $f_t \notin A_\alpha^p$ but $f_t \in \Lambda_{-(n+1+\alpha)/p}$, so the inclusion $A_\alpha^p \subset \Lambda_{-(n+1+\alpha)/p}$ is strict.

To show that the left inclusion is the best possible, we let

$$f_{p,\alpha}(z) = \sum_{k=1}^\infty f_{m_k}(z) = \sum_{k=1}^\infty 2^{k(1+\alpha)/p} W_{2^k}(z),$$

where $\{W_{2^k}\}$ is a sequence of polynomials with Hadamard gaps as in Theorem 1.2 of Ryll-Wojtaszczyk [**48**] and Corollary 1 of Ullrich [**59**] with the properties

$$\|W_{2^k}\|_{H^\infty} = 1, \quad \|W_{2^k}\|_{H^p} > C(n,p),$$

where $C(n,p)$ is a constant depending only on n and p.

From Proposition 63 and Proposition 61 we easily deduce that $f_{p,\alpha} \in \Lambda_{-(1+\alpha)/p}$ but $f_{p,\alpha} \notin A_\alpha^p$. Thus the inclusion $\Lambda_{-\gamma} \subset A_\alpha^p$ is best possible. The proof is complete. $\square$

As a direct consequence we obtain the Lipschitz stretch of A_α^p.

COROLLARY 67. – *Let $0 < p < \infty$ and let α be any real number. Then*

$$\Lambda(A_\alpha^p) = \frac{n}{p}$$

with lower bound $(1+\alpha)/p$ and upper bound $(n+1+\alpha)/p$.

COROLLARY 68. – *All weighted Bergman spaces are different, that is, $A_\alpha^p \ne A_\beta^q$ whenever $(p,\alpha) \ne (q,\beta)$.*

Proof. – If $p = q$ but $\alpha \neq \beta$, then by Proposition 65, A^p_α and A^q_β are different. If $p \neq q$, then Corollary 67 tells us that $\Lambda(A^p_\alpha) = n/p$, while $\Lambda(A^q_\beta) = n/q$. Thus A^p_α and A^q_β have different Lipschitz stretchs, so they must be different. $\qquad\square$

The following two theorems completely describe the inclusion relations between two weighted Bergman spaces.

THEOREM 69. – *Let $0 < p \leq q < \infty$. Then $A^p_\alpha \subset A^q_\beta$ if and only if*

$$\frac{n+1+\alpha}{p} \leq \frac{n+1+\beta}{q},$$

and in this case the inclusion is strict.

Proof. – Let $0 < p \leq q < \infty$ and $f \in A^p_\alpha$. Let k be a nonnegative integer such that $pk + \alpha > -1$ and $kq + \beta > -1$. It follows from the closed graph theorem that the inclusion $A^p_\alpha \subset A^q_\beta$ is equivalent to

$$(38) \qquad \int_{\mathbb{B}_n} \left| R^k f(z) \right|^q \mathrm{d}v_{kq+\beta}(z) \leq C \|f\|^q_{p,\alpha}.$$

Let $s > 0$ be a real number which is sufficiently large. By Theorem 50, the inequality in (38) is equivalent to

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1 - \langle z, w \rangle|^{s+(n+1+\alpha+kp)q/p}} \, \mathrm{d}v_{kq+\beta}(w) < \infty,$$

or

$$\sup_{z \in \mathbb{B}_n} (1-|z|^2)^s \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{kq+\beta}}{|1 - \langle z, w \rangle|^{s+kq+(n+1+\alpha)q/p}} \, \mathrm{d}v(w) < \infty.$$

By Proposition 7, the inequality above holds if and only if

$$c = s + kq + (n+1+\alpha)q/p - (n+1) - (kq+\beta) \leq s,$$

which is easily seen to be equivalent to

$$\frac{n+1+\alpha}{p} \leq \frac{n+1+\beta}{q}.$$

In view of Corollary 68 the proof is now complete. $\qquad\square$

THEOREM 70. – *Let $0 < q < p < \infty$. Then $A^p_\alpha \subset A^q_\beta$ if and only if*

$$\frac{1+\alpha}{p} < \frac{1+\beta}{q},$$

and in this case the inclusion is strict.

Proof. – Let $0 < q < p < \infty$ and $f \in A_\alpha^p$. Let k be a nonnegative integer such that $pk + \alpha > -1$ and $kq + \beta > -1$. Once again, the closed graph theorem tells us that the inclusion $A_\alpha^p \subset A_\beta^q$ is equivalent to

$$(39) \qquad \int_{\mathbb{B}_n} \left| R^k f(z) \right|^q \mathrm{d}v_{kq+\beta}(z) \le C \| f \|_{p,\alpha}^q.$$

Let $s > 0$ be a real number which is sufficiently large. By Theorem 54, the inequality in (39) is equivalent to

$$(40) \qquad B_{s,kp+\alpha}(v_{kq+\beta}) \in L^{p/(p-q)}(\mathbb{B}_n, \mathrm{d}v_{kp+\alpha}).$$

If s is large enough, then by Proposition 7,

$$B_{s,kp+\alpha}(v_{kq+\beta})(z) = \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s (1 - |w|^2)^{kq+\beta}}{|1 - \langle w, z \rangle|^{n+1+s+kp+\alpha}} \, \mathrm{d}v(w)$$
$$\sim \left(1 - |z|^2 \right)^{-k(p-q)-(\alpha-\beta)},$$

as $|z|$ approaches 1. Thus (40) is equivalent to

$$\int_{\mathbb{B}_n} (1 - |z|^2)^{-(k(p-q)+(\alpha-\beta))p/(p-q)+kp+\alpha} \, \mathrm{d}v(z) < \infty,$$

which is equivalent to

$$\frac{1 + \alpha}{p} < \frac{1 + \beta}{q}.$$

This along with Corollary 68 finishes the proof. $\qquad \square$

CHAPTER 17

FURTHER REMARKS

Unless $p = 2$, the space A^p_{-1} is not the same as the Hardy space H^p, although in many situations it is useful to think of H^p as the limit of A^p_α as α approaches -1. One particular problem here is to identify the complex interpolation space between H^p and A^p_α when $\alpha > -1$ and $p \geq 1$. It is also interesting to ask for the complex interpolation space between H^p and Λ_α.

The spaces A^p_α when α is a negative integer appear to be very special. It would be interesting to see some "singular properties" of these spaces.

One of the interesting problems left open in the paper is whether or not Theorem 46 remains valid when $1 < p < \infty$. This is certainly not an easy question, since an affirmative answer would characterize Carleson measures for the Hardy space H^2 as a special case, and it is well known that the characterization of Carleson measures for Hardy spaces is very technical. On the other hand, two special cases can be disposed of easily: Corollary 47 covers the case $\alpha > -1$ and $p > 0$, while Corollary 48 covers the case $\alpha = -1$ and $0 < p \leq 2$. In view of recent work by Arcozzi, Rochberg, and Sawyer (see [**5**], [**6**], and [**7**]) concerning Carleson measures for the standard Besov spaces $B_p = A^p_{-(n+1)}$, the extension of Theorem 46 to the case $p > 1$ when α is arbitrary is most likely a very challenging problem.

After the completion of this paper, several other interesting characterizations for Bergman spaces have appeared. See [**37**], [**43**], and [**63**].

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